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EIGENFUNCTION EXPANSIONS FOR THE SCHRÖDINGER OPERATORS WITH LONG-RANGE POTENTIALS $Q(y)=O(|y|^{-\varepsilon})$, $\varepsilon>0$

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1. Introduction

The present paper is devoted to developing an eigenfunction expansion theory for the Schrödinger operator

$$S=-\Delta+Q(y) \qquad (y\in \boldsymbol{R}^N) \tag{1.1}$$

with a long-range potential $Q(y)=O(|y|^{-\varepsilon})$, $\varepsilon>0$, as $|y|\to\infty$. This work is a direct continuation of [12] and we shall make use of the results of [12] as main tools throughout this work. Thus, as in [12], in place of the Schrödinger operator S we shall consider the differential operator L with operator-valued coefficients

$$L=-\frac{d^2}{dr^2}+B(r)+C(r) \qquad (r\in I=(0,\infty)) \tag{1.2}$$

with

$$\begin{cases} B(r)=r^{-2}\Big(-\Lambda_N+\dfrac{(N-1)(N-3)}{4}\Big), \\ C(r)=Q(r\omega)\times \qquad (\omega\in S^{N-1}), \end{cases} \tag{1.3}$$

S^{N-1} being the $(N-1)$-sphere and Λ_N denoting the Laplace-Beltrami operator on S^{N-1}. L can be considered as an operator in $L_2(I, X)$, where $X=L_2(S^{N-1})$ and $L_2(I, X)$ is the Hilbert space of all X-valued functions $f(r)$ on I such that $|f(r)|_X$ is square integrable over I ($|\ |_X$ is the norm of X). Since L is represented as

$$L=USU^{-1} \tag{1.4}$$

by the use of a unitary operator U

$$U\colon L_2(\boldsymbol{R}^N)\ni F(y)\mapsto r^{(N-1)/2}F(r\omega)\in L_2(I, X) \qquad (r=|y|,\ \omega=y/r\in S^{N-1}) \tag{1.5}$$

from $L_2(\boldsymbol{R}^N)$ onto $L_2(I, X)$, L and S are unitarily equivalent, and hence all the results obtained for L can be applied to S with the aid of the unitary operator U.

The investigation of the operator (1.2) with a self-adjoint operator $B(r)$ and a symmetric operator $C(r)$ in a Hilbert space X has been originated by W. Jäger. His paper [6] develops an eigenfunction expansion theory for L which can be applied to the Schrödniger operator S with $Q(y)=O(|y|^{-(3/2)-\varepsilon})$. Saitō [8]~[11] have extended the results of [6] to apply the Schrödinger operator. In [10] an eigenfunction expansion formula for S in $\boldsymbol{R}^N$ $(N\geqq 3)$ with $Q(y)=O(|y|^{-(1/2)-\varepsilon})$ is given. On the other hand Ikebe [4], [5] have treated the Schrödinger operator S directly and given a spectral representation formula for S with $Q(y)=O(|y|^{-(1/2)-\varepsilon})$ by using essentially the same idea as the above works.

Now let us state the conditions imposed on the potential $Q(y)$

Assumption 1.1.

(Q) $Q(y)$ can be decomposed as $Q(y)=Q_0(y)+Q_1(y)$ such that Q_0 and Q_1 are real-valued functions on $\boldsymbol{R}^N$, N being an integer such that $N\geqq 2$.

(Q_0) There exist constants $C>0$ and $0<\varepsilon\leqq 1/2$ such that $Q_0\in C^m(\boldsymbol{R}^N)$ and

$$|D^jQ_0(y)|\leqq C(1+|y|)^{-j-\varepsilon} \qquad (y\in\boldsymbol{R}^N, j=0, 1, \cdots, m), \tag{1.6}$$

where D^j denotes an arbitrary derivative of j-th order and

$$m=\begin{cases}[2/\varepsilon] & (\text{if } 2/\varepsilon \text{ is an integer}),\\ [2/\varepsilon]+1 & (\text{otherwise}),\end{cases} \tag{1.7}$$

$[a]$ denoting the greatest integer n such that $n\leqq a$. Further, we have

$$Q_0(y)=0 \qquad (|y|\leqq 1). \tag{1.8}$$

(Q_1) $Q_1\in C^0(\boldsymbol{R}^N)$ and

$$|Q_1(y)|\leqq C(1+|y|)^{-1-\varepsilon} \qquad (y\in\boldsymbol{R}^N) \tag{1.9}$$

with the same C, ε as in (Q_0).

Let $Q(y)$ satisfy all the conditions of Assumption 1.1 except for (1.8). Then, by replacing Q_0 and Q_1 by αQ_0 and $(1-\alpha)Q_0+Q_1$, respectively, where $\alpha(y)$ is a real-valued C^∞ function such that $\alpha(y)=0$ $(|y|\leqq 1)$, $=1$ $(|y|\geqq 2)$, $Q(y)$ with the new Q_0 and Q_1 is considered to satisfy all the conditions of Assumption 1.1. Hence (1.8) is a trivial condition.

In §2 we shall introduce the Green kernel $G(r, s, k)$ which will be useful in constructing the eigenoperator $\eta(r, k)$ in §3. In §3 and §4, in addition to Assumption 1.1, $Q_1(y)$ will be assumed to satisfy a stronger condition

$$|Q_1(y)|\leqq C(1+|y|)^{-2} \qquad (y\in\boldsymbol{R}^N). \tag{1.10}$$

Under these conditions an expansion theorem will be shown in §4. In §5 we shall discuss the general case where we assume Assumption 1.1 only.

As to the notations we shall follow [12]. The list of the notations is given at the beginning of [12], §2.

2. The Green kernel

Let L be as in (1.2). We shall define the Green kernel $G(r, s, k)$ ($r, s \in \bar{I} = [0, \infty)$, $k \in \boldsymbol{C}^+$) and investigate some properties of it. Let $s \in \bar{I}$, $x \in X = L_2(S^{N-1})$ and let $\ell[s, x]$ be an anti-linear functional on $H_0^{1,B}(I, X) = UH_1(\boldsymbol{R}^N)$[1] defined by

$$\langle \ell[s, x], \phi \rangle = (x, \phi(s))_X \qquad (\phi \in H_0^{1,B}(I, X)), \tag{2.1}$$

$(\ ,\ )_X$ denoting the inner product of X. Then it follows from Lemma 5.1 of [12] that we have

$$|\phi(s)|_X \leqq \|\phi\|_B \qquad (\phi \in H_0^{1,B}(I, X)), \tag{2.2}$$

and hence $\ell[s, x] \in F_\gamma(I, X)$ for any positive number γ and the estimate

$$|||\ell[s, x]|||_\gamma \leqq (1+s)^\gamma |x|_X \qquad (s \in \bar{I},\ x \in X,\ \gamma \geqq 0) \tag{2.3}$$

is valid.[2]. Denote by $v = v(\cdot, k, s, x)$ the radiative function for $\{L, k, \ell[s, x]\}$, whose existence is guaranteed by the limiting absorption principle (Theorems 2.2 and 5.3 of [12]). Then, by the use of (2.2), (2.3), the interior estimate (Lemma 3.1 of [9] and Lemma 5.2 of [12]) and the limiting absorption principle ((2.4) and (5.12) of [12]), we can easily show

$$|v(r)|_X \leqq C(1+s)^\delta |x|_X \qquad (C = C(R),\ r \in [0, R],\ s \in \bar{I},\ x \in X), \tag{2.4}$$

δ being a fixed constant such that $1/2 < \delta < 1/2 + \varepsilon/4$, whence follows that a bounded linear operator $G(r, s, k)$ on X is well-defined by

$$G(r, s, k)x = v(r, s, k, x). \tag{2.5}$$

DEFINITION 2.1 (the Green kernel). The bounded linear operator $G(r, s, k)$ ($r, s \in \bar{I}$, $k \in \boldsymbol{C}^+$) will be called the *Green kernel* for L.

The linearity of the operator $G(r, s, k)$ directly follows from the linearity of $\ell[s, x]$ with respect to x. Roughly speaking, $G(r, s, k)$ satisfies

$$(L - k^2)G(r, s, k) = \delta(r - s), \tag{2.6}$$

1) For $u = U\psi$, $v = U\phi \in H_0^{1,B}(I, X)$ ($\psi, \phi \in H_1(\boldsymbol{R}^N)$) the inner product $(u, v)_B$ of $H_0^{1,B}(I, X)$ is defind by $(u, v)_B = (\ , \phi)_1$, $(\ ,\ )_1$ being the inner product of $H_1(\boldsymbol{R}^N)$. If $N \geqq 3$, then $(u, v)_B$ is equal to $(u', v')_0 + (u, v)_0 + (B^{1/2}u, B^{1/2})_0$, where $(\ ,\ )_0$ denotes the inner product of $L_2(I, X)$.

2) For the definition of $F_\gamma(I, X)$ and $|||\ |||_\gamma$ see the list of the notations of [12], §2.

the right-hand side denoting the δ-function. The following properties of the Green kernel $G(r, s, k)$ will be made use of further on.

Proposition 2.2. *Let Assumption* 1.1 *be satisfied.*

(i) *Then* $G(\cdot, s, k)x$ *is an* $L_{2,-\delta}(I, X)$*-valued continuous function on* $\bar{I}\times \boldsymbol{C}^{+}\times X$. *Further,* $G(\cdot, s, k)x$ *is an* X*-valued continuous function on* $\bar{I}\times\bar{I}\times \boldsymbol{C}^{+}\times X$, *too.*

(ii) $G(0, r, k)=G(r, 0, k)=0$ *for any pair* $(r, k)\in\bar{I}\times \boldsymbol{C}^{+}$.

(iii) *Let* $(s, k, x)\in\bar{I}\times \boldsymbol{C}^{+}\times X$ *and let* J *be an arbitrary compact interval in* $I-\{s\}$. *Then* $v(r)=G(r, s, k)x$ *is an* X*-valued* C^1 *function on* J *with its derivative* v'. $v'(r)$ *is a strongly absolutely continuous* X*-valued function on* J *and is differentiable almost everywhere on* J *with its derivative* $v''\in L_2(J, X)$. *We have* $v(r)\in D$ *and* $(L-k^2)v(r)=0$ *for almost all* $r\in J$[3].

(iv) *Let* $R>0$ *and let* K *be a compact set o*ᶠ $\boldsymbol{C}^{+}$. *Then there exists* $C=C(R, K)$ *such that*

$$\|G(r, s, k)\|\leqq C \qquad (0\leqq r, s\leqq R, k\in K), \tag{2.7}$$

where $\|\ \|$ *means the operator norm.*

(v) *We have for any triple* $(r, s, k)\in\bar{I}\times\bar{I}\times \boldsymbol{C}^{+}$

$$G(r, s, k)^* = G(s, r, -\bar{k}), \tag{2.8}$$

$G(r, s, k)^*$ *denoting the adjoint of* $G(r, s, k)$.

Proof. (i), (ii), (iv) and (v) can be directly obtained by proceeding as in the proof of Proposition 1.3 of [10]. Next let us show (iii). Let $\rho(r)$ be a real-valued smooth function on I such that the support of $\rho(r)$ is compact in $I-\{s\}$ and $\rho(r)=1$ on J. Set $u(r)=\rho(r)v(r)=\rho(r)G(r, s, k)x$. Then u satisfies the equation $(u, (L-k^2)\phi)_0=(f, \phi)_0$ $(\phi\in UC_0^\infty(\boldsymbol{R}^N))$ with $f=-2\rho'v'-\rho''v\in L_2(I, X)_{loc}$. Thus (iii) follows from Lemma 2.3 of [12]. Q.E.D.

3. The eigenoperator

The main purpose of this section is to construct the eigenoperator $\eta(r, k)$ $(r\in\bar{I}, k\in\boldsymbol{R}-\{0\})$ by the use of the Green kernel $G(r, s, k)$ which was defined in §2. In this and the following sections $Q(y)$ will be assumed to satisfy both Assumption 1.1 and (1.10) which enable us to apply the results of [12]. Using Theorems 2.5 and 5.4 of [12], we shall first show some more properties of the Green kernel in addition to Proposition 2.2.

Proposition 3.1. *Let Assumption* 1.1 *and* (1.10) *be satisfied. Then we have*

3) D is the domain of the Laplace-Beltrami operator Λ_N as in [12].

$$\|G(r, s, k)\| \leqq \begin{cases} C_1(k) & (k\in \boldsymbol{C}^+,\ \mathrm{Im}\, k>0,\ r,\ s\in \bar{I}), \\ C_2(k) \min\{(1+r)^{1+\delta-\varepsilon},\ (1+s)^{1+\delta-\varepsilon}\} & \\ & (k\in \boldsymbol{R}-\{0\},\ r,\ s\in \bar{I}), \end{cases} \tag{3.1}$$

where $C_1(k)$ $(C_2(k))$ *is bounded when* k *moves in a compact set in* $\{k\in \boldsymbol{C}^+/\mathrm{Im}\, k>0\}$ $(\boldsymbol{R}-\{0\})$. *Further,*

$$v(r, k, \ell[f]) = \int_I G(r, s, k) f(s) ds \qquad \text{in } X\ (r\in \bar{I}) \tag{3.2}$$

holds for any radiative function $v(\cdot, k, \ell[f])$ *for* $\{L, k, \ell[f]\}$, *where* $k\in \boldsymbol{C}^+$, $f\in L_{2,\delta}(I, X)$ *and the definition of* $\ell[f]$ *is given in* (2.6) *of* [12].

Proof. Let us assume that $k\in \boldsymbol{C}^+$ with $\mathrm{Im}\, k>0$. Then it follows from Lemma 1.7 of [9][4] that $v=G(\cdot, s, k)x$ $(x\in X, s\in \bar{I})$ belongs to $H_0^{1,B}(I, X)$. The first estimate of (3.1) is obtained from (2.2), (2.3) with $\gamma=0$ and Lemma 1.7 of [9]. Next let us show the second estimate of (3.1). Applying Theorems 2.5 and 5.4 of [12], and using (2.3) with $\gamma=1+\delta-\varepsilon$, we have $|G(r, s, k)x|_X \leqq C(1+s)^{1+\delta-\varepsilon}|x|_X$ $(r, s\in \bar{I}, x\in X, k\in \boldsymbol{R}-\{0\})$ with $C=C(k)$, which implies that

$$\|G(r, s, k)\| \leqq C(1+s)^{1+\delta-\varepsilon} \qquad (r,\ s\in \bar{I},\ k\in \boldsymbol{R}-\{0\}). \tag{3.3}$$

The second estimate of (3.1) is obtained from (3.3) and the relation $G(r, s, k)^* = G(s, r, -\bar{k})$ ((v) of Proposition 2.2). Finally let us show (3.2). If $k\in \boldsymbol{C}^+$ with $\mathrm{Im}\, k>0$, then $u=G(\cdot, r, \bar{k})x\in H_0^{1,B}(I, X)$ satisfies

$$b_k(u, \phi) = (u, \phi)_B + ((C(\cdot)-1-k^2)u, \phi)_0 = (x, \phi(r))_X \tag{3.4}$$

for all $\phi\in H_0^{1,B}(I, X)$. Set in (3.4) $\phi=v(\cdot, k, \ell[f])$, where $f\in L_2(I, X)$ with compact support in $\bar{I}$. Then it follows that

$$\begin{aligned} (x, v(r))_X &= \overline{b_k(v, u)} = \overline{(f, u)_0} = (u, f)_0 \\ &= (x, \int_I G(r, s, k) f(s) ds)_X . \end{aligned} \tag{3.5}$$

Since $x\in X$ is arbitrary, we arrive at (3.2). If $k\in \boldsymbol{R}-\{0\}$, then we can approximate k by $\{k_n\}$ $(k_n\in \boldsymbol{C}^+, \mathrm{Im}\, k_n>0)$ to obtain (3.2), where we have made use of the continuity of the radiative function $v(\cdot, k_n, \ell[f])$ with respect to k_n and the estimate (2.7) in Proposition 2.2. Thus (3.2) has been established for $k\in \boldsymbol{C}^+$ and $f\in L_2(I, X)$ with compact support in $\bar{I}$. Approximate $f\in L_{2,\delta}(I, X)$ by $\{f_n\}$,

4) Note that in the case of $N=2$ the result of Lemma 1.7 of [9] is valid by the following modification: $\cdots v_0$ in $H_0^{1,B}(I, X)\cap L_{2,\gamma}(I, X)$ and

$$\|\mathrm{grad}\, \phi\|_{L_{2,\gamma}(\boldsymbol{R}^2)} + \|\phi\|_{L_{2,\gamma}(\boldsymbol{R}^2)} \leqq C|||l|||_\gamma$$

with $\phi=r^{-(1/2)}v_0$ and $C=C(k_0, \gamma)$, where $\|\ \|_{L_{2,\gamma}(\boldsymbol{R}^2)}$ means the norm of $L_{2,\gamma}(\boldsymbol{R}^2) = L_2(\boldsymbol{R}^2, (1+|y|)^2 dy)$. $\cdots$.

where $f_n \in L_2(I, X)$ with compact support in $\bar{I}$, and take note of (3.1). Then (3.2) will be proved completely. Q.E.D.

Let $s \in \bar{I}$, $x \in X$ and let $\rho(t) = \rho_s(t)$ be a real-valued, smooth function on $\bar{I}$ such that $\rho(t) = 0$ $(t \leqq s+1)$, $=1$ $(t \geqq s+2)$. Let $\mu(y, k)$ be as in (2.12) of [12]. Then Theorems 2.4 and 5.5 of [12] can be applied to $v = \rho(r)G(r, s, k)x$ $(x \in X, k \in \boldsymbol{R} - \{0\})$ to show that there exists the strong limit

$$\alpha(r, k, x) = \mathrm{s}-\lim_{r \to \infty} e^{-i\mu(r\cdot,k)} G(r, r, k)x \qquad \text{in } X. \tag{3.6}$$

Here it should be noted that v is the radiative function for $\{L, k, \ell[f]\}$ with $f = -2\rho' v' - \rho'' v$. It can be easily shown that $\alpha(r, k, x)$ is linear with respect to x. On the other hand the estimate $|\alpha(r, k, x)|_X \leqq C(1+s)^{1+\delta-\varepsilon} |x|_X$ follows from Proposition 3.1. Therefore the bounded linear operator $\eta(r, k)$ is well-defined by

$$\eta(r, k)x = \mathrm{s}-\lim_{t \to \infty} e^{-i\mu(t\cdot,k)} G(t, r, k)x \tag{3.7}$$

Definition 3.2. The bounded linear operator $\eta(r, k)$ $(r \in \bar{I}, k \in \boldsymbol{R} - \{0\})$ defined by (3.7) will be called the *eigenoperator* associated with L.

The appropriateness of this naming will be justified in the remainder of this section (especially in Theorem 3.5).

Proposition 3.3. *Let Assumption* 1.1 *and* (1.10) *be satisfied. Then we have*

$$s-\lim_{s \to \infty} G(r, s, -k) e^{i\mu(s\cdot,k)} x = \eta^*(r, k)x \qquad \text{in } X \tag{3.8}$$

for any triple $(r, k, x) \in \bar{I} \times (\boldsymbol{R} - \{0\}) \times X$, *where* $\eta^*(r, k)$ *is the adjoint of* $\eta(r, k)$ *and* $\mu(y, k)$ *is given by* (2.12) *of* [12].

Proof. Let us first note that $G(r, s, -k)e^{i\mu(s\cdot,k)}x$ converges weakly to $\eta^*(r, k)x$ as $s \to \infty$. Suppose that there exist $r_0 > 0$, $k_0 \in \boldsymbol{R} - \{0\}$, $x_0 \in X$ and a sequence $\{s_n\}$ such that $|v_n(r_0) - \eta^*(r_0, -k_0)x_0|_X \geqq \delta_0$ holds for all $n = 1, 2, \cdots$ with some $\delta_0 > 0$, where we set $v_n(r) = G(r, s_n, -k_0)e^{i\mu(s_n\cdot,k_0)}x_0$. By using the interior estimate (Lemma 4.1 of Jäger [6] or Lemma 3.1 of [10]) and using (3.1) it can be seen that the sequence $\|v_n\|_{B,(0,R)}$ is bounded for each $R > 0$. Since the imbedding

$$H_0^{1,B}(I, X)_{loc} \to L_2(I, X)_{loc} \tag{3.9}$$

is compact by the Rellich theorem, there exists a subsequence of $\{v_n\}$, which is denoted again by $\{v_n\}$ for the sake of simplicity, such that $\{v_n\}$ is a Cauchy sequence in $L_2(I, X)_{loc}$. Make use of the interior estimate again. Then we can show that $\{v_n\}$ is a Cauchy sequence in $H_0^{1,B}(I, X)_{loc}$. Therefore the estimate (2.2) can be applied to see that $\{v_n(r_0)\}$ is a Cauchy sequence in X. Thus $v_n(r_0)$ converges strongly to $\eta^*(r_0, k_0)x_0$, which is a contradiction. Q.E.D.

Let us summarize these results in the following

Theorem 3.4. *Let Assumption* 1.1 *and* (1.10) *be satisfied.*
(i) *Then*

$$\eta(r, k)x = s\text{-}\lim_{s\to\infty} e^{-i\mu(s\cdot,k)}G(s, r, k)x \qquad in\ X \tag{3.10}$$

and

$$\eta^*(r, k)x = s\text{-}\lim_{s\to\infty} G(r, s, -k)e^{i\mu(s\cdot,k)}x \quad in\ X \tag{3.11}$$

for any triple $(r, k, x)\in\bar{I}\times(\boldsymbol{R}-\{0\})\times X$.
(ii) *The relation*

$$2ik(\eta(s, k)x, \eta(r, k)x')_X=(\{G(r, s, k)-G(r, s, -k)\}x, x')_X \tag{3.12}$$

holds for any $x, x'\in X$ *and any* $r, s\in\bar{I}$. $\eta(r, k)x$ *is a strongly continuous X-valued function on* $\bar{I}\times(\boldsymbol{R}-\{0\})\times X$.

(iii) $v=\eta^*(\cdot, k)x\in H_0^{1,B}(I, X)_{loc}$ *and v satisfies the condition* (1)～(3) *given in Lemma* 2.3 *of* [12] *and*

$$(L-k^2)v(r) = 0 \qquad a.e.\ r\in\bar{I}\,, \tag{3.13}$$

where $r\in\bar{I}$, $k\in R-\{0\}$, $x\in X$. $\eta^*(r, k)x$ *is a strongly continuous X-valued function on* $\bar{I}\times(\boldsymbol{R}-\{0\})\times X$.

(iv) *We have the estimates*

$$||\eta(r, k)|| = ||\eta^*(r, k)|| \leqq C(1+r)^{(1+\delta-\varepsilon)/2} \qquad (r\in\bar{I})\,, \tag{3.14}$$

where $||\ ||$ *means the operator norm and* $C=C(k)$ *is bounded when k moves in a compact set in* $\boldsymbol{R}-\{0\}$.

Proof. (i) follows from (3.7) and Proposition 3.3. (ii) can be obtained in quite the same way as in the proof of Theorem 2.9 of [10]. By proceeding as in the proof of Lemma 2.8 of [10] and the proof of Theorem 2.9 of [10] we can show (iii). Finally (iv) can be obtained from (3.12) and (3.1) with $s=r$, $x'=x$. Q.E.D.

Theorems 2.4 and 5.5 of [12] can be also used to define one more important operator from $L_{2,\delta}(I, X)$ into X. For any fixed $k\in\boldsymbol{R}-\{0\}$ let us define a linear operator $\mathcal{F}(k)$ from $L_{2,1+\delta-\varepsilon}(I, X)$ into X by

$$\mathcal{F}(k)f = s-\lim_{r\to\infty} e^{-i\mu(r\cdot,k)}v(r, k, \ell[f]) \qquad \text{in } X\,, \tag{3.15}$$

where $v=v(\cdot, k, \ell[f])$ is the radiative function for $\{L, k, \ell[f]\}$. Let $\{r_n\}$ be a sequence which satisfies $v'(r_n)-ikv(r_n)\to 0$ in X. It follows from the Green formula that

$$(f, v)_{0,(0,r_n)} - (v, f)_{0,(0,r_n)} = (v(r_n), v'(r_n) - ikv(r_n))_X - (v'(r_n) - ikv(r_n), v(r_n))_X - 2ik|e^{-i\mu(r_n)}v(r)|_X^2 \tag{3.16}$$

Letting $n \to \infty$, and taking note of Theorems 2.5 and 5.4 of [12], we arrive at

$$|\mathcal{F}(k)f|_X^2 \leqq |k|^{-1}|\mathrm{Im}(f, v)_0| \leqq |k|^{-1}C(k)\|f\|_\delta^2, \tag{3.17}$$

and hence $\mathcal{F}(k)$ can be uniquly extended to a bounded linear operator on $L_{2,\delta}(I, X)$.

DEFINITION 3.5. We denote again by $\mathcal{F}(k)$ the above bounded linear extension of $\mathcal{F}(k)$.

The operator norm $\|\mathcal{F}(k)\|$ of $\mathcal{F}(k)$ is bounded when k moves in a compact set contained in $\boldsymbol{R} - \{0\}$. The following formula, which will be used in §4, can be easily obtained by starting with (3.16).

Proposition 3.6. *Let $f \in L_{2,\delta}(I, X)$ and let $v(\cdot, k, \ell[f]) = v$ be the radiative function for $\{L, k, \ell[f]\}$ with $k \in \boldsymbol{R} - \{0\}$. Then we have*

$$(v, f)_0 - (f, v)_0 = 2ik|\mathcal{F}(k)f|_X^2 . \tag{3.18}$$

The following Theorem gives a relation between $\eta^*(r, k)$ and $\mathcal{F}(k)$.

Theorem 3.7. *Let Assumption* 1.1 *and* (1.10) *be satisfied.*

(i) *Let $k \in \boldsymbol{R} - \{0\}$. Then $\eta^*(\cdot, k)x \in L_{2,-\delta}(I, X)$ for any $x \in X$ with the estimate*

$$\|\eta^*(\cdot, k)x\|_{-\delta} \leqq C|x|_X \qquad (x \in X), \tag{3.19}$$

where $C = C(k)$ is bounded when k moves in a compact set in $\boldsymbol{R} - \{0\}$.

(ii) *The relation*

$$(\eta^*(\cdot, k)x, f)_0 = (x, \mathcal{F}(k)f)_X \tag{3.20}$$

holds for any triple $(k, x, f) \in (\boldsymbol{R} - \{0\}) \times X \times L_{2,\delta}(I, X)$.

This theorem can be proved in the very same way as in the proof of Proposition 4.3 of [10], and hence the proof will be omitted.

Finally we shall show a theorem which gives a relation between $\mathcal{F}(k)$ and $\eta(r, k)$.

Theorem 3.8. *Let Assumption* 1.1 *and* (1.10) *be satisfied. Then we have*

$$\mathcal{F}(k)f = \int_I \eta(r, k)f(r)dr \tag{3.21}$$

for any $f \in L_{2,\beta}(I, X)$ with $\beta > (2 + \delta - \varepsilon)/2$ and any $k \in \boldsymbol{R} - \{0\}$, $\mathcal{F}(k)$ being given in Definition 3.5.

Proof. Let us first assume that f belongs to $UC_0^\infty(\boldsymbol{R}^N)$. Then the change of the order of integration in (3.20) enables us to obtain

$$\left(x, \int_I \eta((r, k)f(r)dr\right)_X = (x, \mathcal{F}(k)f)_X, \tag{3.22}$$

by which (3.21) is implied. Let us next consider the general case. Then, noting that $\|\eta(r, k)\| \leqq C(1+r)^{(1+\delta-\varepsilon)/2}$ ((3.14)), we can approximate f in $L_{2,\beta}(I, X)$ $(\beta > (2+\delta-\varepsilon)/2)$ by a sequence $\{f_n\} \subset UC_0^\infty(\boldsymbol{R}^N)$ to obtain (3.21). Q.E.D.

4. Expansion theorem

In this section we shall assume, as in the preceding section, that $Q(y)$ satisfies (1.10) in addition to Assumption 1.1. Now let us show an eigenfunction expansion theorem associated with a self-adjoint realization of the operator L in $L_2(I, X)$.

As is well known, under Assumption 1.1, the Schrödinger operator S restricted to $C_0^\infty(\boldsymbol{R}^N)$ is essentially self-adjoint in $L_2(\boldsymbol{R}^N)$[5]. Its unique self-adjoint extension will be denoted by M. Then we have

$$\begin{cases} M\varphi = S\varphi \\ \mathcal{D}(M) = H_2(\boldsymbol{R}^N)^{6)}. \end{cases} \tag{4.1}$$

Now let us define a self-adjoint operator T in $L_2(I, X)$ by $T = UMU^{-1}$, i.e.

$$\begin{cases} T\phi = L\phi \\ \mathcal{D}(T) = UH_2(\boldsymbol{R}^N). \end{cases} \tag{4.2}$$

Set $R(z; T) = (T-z)^{-1}$ and denote by $E(\cdot; T)$ the spectral measure associated with T. It can be easily shown that T is bounded below and the essential spectrum $\sigma_e(T)$ of T is equal to $[0, \infty)$. We can also show by the limiting absorption principle that the spectrum of T is absolutely continuous on $(0, \infty)$ (cf. Proposition 1.5 of [10]).

Lemma 4.1. *Let* Δ *be a compact interval in* $(0, \infty)$ *and let* $f \in L_{2,\beta}(I, X)$ *with* $\beta > (2+\delta-\varepsilon)/2$. *Then*

$$\begin{aligned}(E(\Delta; T)f, f)_0 &= \int_{\sqrt{\Delta}} \frac{2k^2}{\pi} \left| \int_I \eta(r, k)f(r)dr \right|_X^2 dk \\ &= \int_{\sqrt{\Delta}} \frac{2k^2}{\pi} \left| \int_I \eta(r, -k)f(r)dr \right|_X^2 dk,\end{aligned} \tag{4.3}$$

where $\sqrt{\Delta} = \{k>0/k^2 \in \Delta\}$ *and* $\eta(r, k)$ *is as in* §3.

5) See, for example, Kato [7].

6) $\mathcal{D}(W)$ denotes the domain of W.

Proof. Let us first note that $R(z; T)g=v(\cdot, \sqrt{z}, \ell[g])$ for $z\in \boldsymbol{C}-\boldsymbol{R}$ and $g\in L_{2,\delta}(I, X)$. Here $\sqrt{z}$ is the square root of z with $\mathrm{Im}\sqrt{z}\geqq 0$ and $v(\cdot, \sqrt{z}, \ell[g])$ is the radiative function for $\{L, \sqrt{z}, \ell[g]\}$. In fact this follows from the uniqueness of the radiative function and the fact that $R(z; T)g\in UH_2(\boldsymbol{R}^N)$. Moreover let us note that

$$\lim_{b\downarrow 0} R(k^2\pm ib; T)g = v(\cdot, \pm|k|, \ell[g]) \quad \text{in } L_{2,-\delta}(I, X) \quad (k\in \boldsymbol{R}-\{0\}, g\in L_{2,\delta}(I, X)), \tag{4.4}$$

which follows from the continuity of the radiative function. Then from the well-known formula

$$\begin{aligned}(E(\Delta; T)f, f)_0 &= (2\pi i)^{-1}\lim_{b\downarrow 0}\int_\Delta \{(R(a+ib; T)f, f)_0-(f, R(a+ib; T)f)_0\}\,da \\ &= (2\pi i)^{-1}\lim_{b\downarrow 0}\int_\Delta \{(f, R(-a-ib; T)f)_0-(R(a-ib; T)f, f)_0\}\,da\end{aligned} \tag{4.5}$$

we obtain, setting $v(\cdot, k, \ell[f])=v(k)$,

$$\begin{aligned}(E(\Delta; T)f, f)_0 &= (2\pi i)^{-1}\int_\Delta \{(v(\sqrt{a})_0, f-(f, v(\sqrt{a}))_0\}\,da \\ &= (2\pi i)^{-1}\int_\Delta \{(f, v(-\sqrt{a}))_0-(v(-\sqrt{a}), f)_0\}\,da\,,\end{aligned} \tag{4.6}$$

and hence by the use of Proposition 3.6 and Theorem 3.8 we arrive at

$$\begin{aligned}(E(\Delta; T)f, f)_0 &= \int_\Delta \frac{\sqrt{a}}{\pi}\left|\int_I \eta(r, \sqrt{a})f(r)\,dr\right|_X^2 da \\ &= \int_\Delta \frac{\sqrt{a}}{\pi}\left|\int_I \eta(r, -\sqrt{a})f(r)\,dr\right|_X^2 da\,,\end{aligned} \tag{4.7}$$

whence (4.3) easily follows. Q.E.D.

Starting with the Lemma 4.1, we can proceed quite similarly as in §3 of [10] to show an eigenfunction expansion theorem (or to be more exact, an eigenoperator expansion theorem). Set

$$\eta_\pm(r, k) = \pm\sqrt{\frac{2}{\pi}}\, ik\eta(r, \pm k) \qquad (r\in \bar{I}, k>0)\,. \tag{4.8}$$

$\eta_\pm^*(r, k)$ denote the adjoints of $\eta_\pm(r, k)$, respectively, i.e.,

$$\eta_\pm^*(r, k) = \mp\sqrt{\frac{2}{\pi}}\, ik\eta^*(r, \pm k)\,. \tag{4.9}$$

Using these operators we can define the "generalized Fourier transforms" $\mathcal{F}_\pm$ from $L_2(I, X, dr)$ into $L_2(I, X, dk)$ by

$$(4.10) \qquad (\mathcal{F}_\pm f)(k) = \underset{R\to\infty}{\text{l.i.m.}} \int_0^R \eta_\pm(r, k) f(r)\, dr \qquad \text{in } L_2(I, X, dk).$$

$\mathcal{F}_\pm$ are bounded operators and their adjoint $\mathcal{F}_\pm^*$ have the forms

$$(4.11) \qquad (\mathcal{F}_\pm^* F)(r) = \underset{K\to\infty}{\text{l.i.m.}} \int_{K^{-1}}^K \eta_\pm^*(r, k) F(k)\, dk \qquad \text{in } L_2(I, X, dr)$$

(see Lemma 3.3 of [10]). Thus we obtain

Theorem 4.2 (expansion theorem). *Let Assumption* 1.1 *be satisfied. Let* B *be an arbitrary Borel srt in* $(0, \infty)$. *Then*

$$(4.12) \qquad E(B; T) = \mathcal{F}_\pm^* \chi_{\sqrt{B}} \mathcal{F}_\pm,$$

where $\chi_{\sqrt{B}}$ *is the characteristic function of* B, *i.e.,* $\chi_{\sqrt{B}}(k)=1\ (k^2\in B),\ =1\ (k^2\notin B)$. *Especially we have*

$$(4.13) \qquad E((0, \infty); T) = \mathcal{F}_\pm^* \mathcal{F}_\pm.$$

Since the proof is quite the same as the proof of Theorem 3.4 of [10], it will be omitted.

Now we shall show a key lemma to the proof of the orthogonality of $\mathcal{F}_\pm$.

Lemma 4.3. *Let* $k\in \boldsymbol{R}-\{0\}$, $x\in D$. *Let* $\xi(r)$ *be a real-valued, smooth function on* $[0, \infty)$ *such that* $(r)=0\ (r\leqq 1),\ =1\ (r\geqq 2)$ *and set* $f=(L-k^2)(\xi e^{i\mu} x)$. *Then* $\mathcal{F}(k)f=x$.

Proof. As is easily seen from (3.8) of [12], f belongs to $L_{2,\delta}(I, X)$. It can be easily checked that $v_0=\xi e^{j\mu}x$ is the radiative function for $\{L, k, \ell[f]\}$. For each positive integer m v_m and u_m denote theradiative functions for $\{L, k, \ell[f]\}$ and $\{L, k, \ell[g_m]\}$, respectively, where $f_m=\chi_m f$, $g_m=(\chi_m-1)f$ and $\chi_m(r)$ is the characteristic function of the interval $(0, m)$. Then, by the relation $v_m=v_0+u_m$ and the definition of $\mathcal{F}(k)f_m$, we have

$$(4.14) \qquad \mathcal{F}(k)f_m = s-\lim_{r\to\infty} \{e^{-i\mu(r,\cdot,k)} v_0(r) + e^{-i\mu(r,\cdot,k)} u_m(r)\}.$$

The first term of the right-hand side of (4.14) is equal to x, and hence the second term has the limit x_m, i.e., we obtain

$$(4.15) \qquad \mathcal{F}(k)f_m = x + x_m \qquad (m = 1, 2, \cdots,).$$

On the other hand, setting $f=g_n$, $v=u_m$ in (3.16), letting m to infinity and using the limiting absorption principle, we arrive at

$$|x_m|_X \leqq C(k)\|g_m\|_\delta\,, \tag{4.16}$$

which, together with (4.15), implies

$$|\mathcal{F}(k)f_m - x|_X \leqq C(k)\|g_m\|_\delta\,. \tag{4.17}$$

Since $\mathcal{F}(k)$ is a bounded linear operator on $L_{2,\delta}(I, X)$, we let m to infinity to arrive at $\mathcal{F}(k)f=x$, where we should note that f_m and g_m converge to f and 0 in $L_{2,\delta}(I, X)$ as $m\to\infty$, respectively. Q.E.D.

Theorem 4.4 (the orthogonality of $\mathcal{F}_\pm$). *Let Assumption* 1.1 *and* (1.10) *be satisfied. Then $\mathcal{F}_\pm$ transform $L_2(I, X, dr)$ onto $L_2(I, X, dk)$.*

Proof. We can proceed as in the proof of Theorem 4.1 of [10]. It suffices to show the following: If $F\in L_2(I, X, dk)$ and $\mathcal{F}_+F=0$ ($\mathcal{F}_-F=0$) in $L_2(I, X, dr)$, then $F=0$. Let us assume that $\mathcal{F}_+F=0$. Then there exists a null set e such that $\eta^*(r, k)F(k)=0$ for $(r, k)\in\bar{I}\times((0, \infty)-e)$ (see Lemma 4.5 of [10] and the proof of Theorem 4.1 of [10]). Therefore, taking account of Theorem 3.7, (ii), we obtain

$$(F(k), \mathcal{F}(k)f)_X = 0 \qquad (k\in(0, \infty)-e) \tag{4.18}$$

for any $f\in L_{2,\delta}(I, X)$. Take f as in Lemma 4.3. Then we obtain $(F(k), x)_X=0$ for any $x\in D$ and any $k\notin e$ It follows from the densness of D in X that $F(k)=0$ for almost all $k\in(0, \infty)$, and hence $F=0$. The case of $\mathcal{F}_-$ can be treated quite similarly. Q.E.D.

5. The case that $Q_1(y)=O(|y|^{-1-\varepsilon})$

In §3～§4 we have assumed that the potential $Q(y)=Q_0(y)+Q_1(y)$ satisfies not only Assumption 1.1 but also (1.10). In this section, however, we shall construct the eigenoperators and show the expansion theorem under Assumption 1.1 only. The fundamental idea is to approximate $Q_1(y)$ by a sequence $\{Q_{1n}(y)\}$ of short-range potentials, where $Q_{1n}(y)$ satisfies the condition $|Q_{n1}(y)| \leqq C(1+|y|)^{-2}$ $(y\in\boldsymbol{R}^N)$ uniformly for $n=1, 2, \cdots$. We may take, for example,

$$Q_{1n}(y)=\rho_n(|y|)Q_1(y) \qquad (n = 1, 2, \cdots)\,, \tag{5.1}$$

where $\rho_n(r)=\rho(r-n)$ and $\rho(t)$ is a real-valued, smooth function on $(-\infty, \infty)$ such that $\rho(t)=1$ $(t\leqq0)$, $=0$ $(t\geqq1)$ and $0\leqq\rho(t)\leqq1$. We set

$$\begin{cases} S_n = -\Delta+Q_0(y)+Q_{1n}(y)\,, \\ L_n = -\dfrac{d^2}{dr^2}+B(r)+C_0(r)+C_{1n}(r) \qquad (C_{1n}(r)=Q_{1n}(r\omega))\,. \end{cases} \tag{5.2}$$

The main tool in this section is Theorem 4.1 of [9] which gives a uniform estimate for the radiative function $v_n(\cdot, k, \ell)$ for $\{L_n, k, \ell\}$[7]. Since all the results of the preceding sections can be applied to L_n, we can define bounded linear operators $\mathcal{F}_n(k)$ $(k\in \boldsymbol{R}-\{0\})$ from $L_{2,\delta}(I, X)$ into X. The eigenoperators $\eta_n(r, k)$ $((r, k)\in \bar{I}\times(\boldsymbol{R}-\{0\})$ on X are also well-defined.

Proposition 5.1. *Let Assumption* 1.1 *be satisfied and let* $\eta_n(r, k)$ *as above. Then the operator norm* $\|\eta_n(r, k)\|$ $(=\|\eta_n^*(r, k)\|)$ *is uniformly bounded when* $n=1, 2, \cdots$, *and* r *and* k *move in a bounded set in* $\bar{I}$ *and compact set in* $\boldsymbol{R}-\{0\}$, *respectively. For each pair* $(r, k)\in\bar{I}\times(\boldsymbol{R}-\{0\})$ *there exists a bounded linear operator* $\eta(r, k)$ *on* X *such that*

$$s-\lim_{n\to\infty} \eta_n(r, k)x = \eta(r, k)x \qquad \text{in } X \tag{5.3}$$

for any $x\in X$, *and* (*ii*) *of Theorem* 3.4 *is satisfied.*

Proof. Using the Green formula and proceeding as in the proof of Lemma 3.3 of [8], we have

$$\begin{aligned}(\{G_n(r, s, k)-G_m(r, s, -k)\}x, x')_X \\ +\int_I (\{C_{1n}(t)-C_{1m}(t)\{G_n(t, s, k)x, G_m(t, r, k)x')_X dt \\ =2ik(\eta_n(s, k)x, \eta_m(r, k)x')_X \\ (m, n = 1, 2, \cdots, k\in\boldsymbol{R}-\{0\}, rk, s\in\bar{I}),\end{aligned} \tag{5.4}$$

where $G_n(r, s, k)$ is the Green kernel for L_n. Since the left-hand side of (5.4) tends to $(\{G(r, s, k)-G(r, s, -k)\}x, x')_X$ as $m, n\to\infty$ by Theorem 4.1 of [9], it can be easily shown by setting $s=r$ and $x'=x$ in (5.4) that $\{\eta_n(r, k)x\}$ is a Cauchy sequence in X and that $\|\eta_n(r, k)\|$ is uniformly bounded. By the use of these facts we can prove the existence of $\eta(r, k)$ which satisfies (5.3). At the same time the relation (3.12) is obtained, whence follows the continuity of $\eta(r, k)x$, too. Q.E.D.

Proposition 5.2. *Let Assumption* 1.1 *be satisfied and let* $\mathcal{F}_n(k)$ *be as above. Then the operator norm* $\|\mathcal{F}_n(k)\|$ *is uniformly bounded when* $n=1, 2, \cdots$ *and* k *moves in a compact set in* $\boldsymbol{R}-\{0\}$. *For each* $k\in\boldsymbol{R}-\{0\}$ *there exists a bounded linear operator* $\mathcal{F}(k)$ *from* $L_{2,\delta}(I, X)$ *into* X *such that*

$$s-\lim_{n\to\infty} \mathcal{F}_n(k)f = \mathcal{F}(k)f \qquad \text{in } X \tag{5.5}$$

for any $f\in L_{2,\delta}(I, X)$ *and* (3.18) *holds good.*

Proof. Denote by v_n the radiative function for $\{L_n, k, \ell[f]\}$ with

7) In the case of $N=2$ we have to modify the proof of Theorem 4.1 of [9]. But we shall not find any difficulty in the modification (cf. §5 of [12]).

$f \in L_{2,1+\delta-\varepsilon}(I, X)$. Then by the Green formula and the definition of $\mathcal{F}_n(k)$ we have

$$(5.6) \qquad (v_n, f)_0 - (f, v_m)_0 + ((C_{1n} - C_{1m})v_n, v_m)_0 = 2ik(\mathcal{F}_n(k)f, \mathcal{F}_m(k)f)_X .$$

From the boundedness of the operators $\mathcal{F}_n(k)$, $\mathcal{F}_m(k)$ and the continuity of the radiative function it follows that (5.6) is valid for all $f \in L_{2,\delta}(I, X)$. As in the proof of Proposition 5.1, by starting with (5.6) and making use of Theorem 4.1 of [9], $\{\mathcal{F}_n(k)f\}$ can be shown to be a Cauchy sequence in X and (3.18) is seen to hold good. Q.E.D.

Proposttion 5.3. *Let Assumption* 1.1 *be satisfied and let* $\eta_n^*(r, k)$ *be as above. Then, denoting by* $\eta^*(r, k)$ *the adjoint of* $\eta(r, k)$, *we have*

$$(5.7) \qquad \|\eta_n^*(\cdot, k)x\|_{-\delta} \leqq C(k)|x|_X \qquad (k \in \boldsymbol{R} - \{0\},\ x \in X,\ n = 1, 2, \cdots),$$

$C(k)$ being bounded when k moves in a compact set in $\boldsymbol{R} - \{0\}$. $\eta_n^*(\cdot, k)x$ converges to $\eta^*(\cdot, k)x$ in $L_{2,-\delta}(I, X) \cap H_0^{1,B}(I, X)_{loc}$ for any pair $(k, x) \in (\boldsymbol{R} - \{0\}) \times X$. $\eta^*(r, k)$ satifies (iii) of Theorem 3.4. Further, we have

$$(5.8) \qquad \eta^*(\cdot, k)x = \eta_0^*(\cdot, k)x - v(\cdot, -k, \ell[g]) \qquad (x \in X,\ k \in \boldsymbol{R} - \{0\}),$$

where $\eta_0(r, k)$ denotes the eigenoperator for $L_0 = -\frac{d^2}{dr^2} + B(r) + C_0(r)$ and $v(\cdot, -k, \ell[g])$ is the radiative function for $\{L, -k, \ell[g]\}$ with $g = C_1\eta_0^*(\cdot, k)x$[8].

Proof. Set $w_n = G_0(\cdot, s, -k) + x - G_n(\cdot, s, -k)x$, $G_0(r, s, k)$ being the Green kernel for L_0. Then w_n is the radiative function for $\{L_n, -k, \ell[h]\}$, $h_n = C_{1n}G_0(\cdot, s, -k)x$, i.e., $G_n(r, s, -k)x = G_0(\cdot, s, -k)x - v_n(\cdot, -k, \ell[h_n])$. If we replace x by $e^{i\mu(s,k)}x$ and let $s \to \infty$, then we obtain from Proposition 4.3 $\eta_n^*(r, k)x = \eta_0^*(r, k)x_n - v(\cdot, -k, \ell[g_n])$ with $g_n = C_{1n}\eta_n^*(\cdot, k)x$. Further, let $n \to \infty$. Then by Theorem 4.1 of [9] $\eta_n^*(\cdot, k)x$ converges to $\eta^*(\cdot, k)x$ in $L_{2,\delta}(I, X) \cap H_0^{1,B}(I, X)_{loc}$ and the relation (5.8) is valid. The rest of the statement can be easily justified by using (5.8). Q.E.D.

Now that the eigenoperator $\eta(r, k)$ has been constructed and the propoerties of $\eta(r, k)$ have been investigated, the expansion theorem (Theorem 4.2) and the orthogonality of the generalized Fourier transforms (Theorem 4.4) can be easily shown.

Theorem 5.4. *Let the potential $Q(y)$ satisfy Assumption* 1.1. *Then all*

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8) If we denote the radiative function $v(\cdot, k, f)$ by $(L - k^2)^{-1}f$, then (5.8) can be represented as

$$\eta^*(\cdot, k)x = \{I - (L - (-k)^2)^{-1}C_1\}\eta_0^*(\cdot, k)x ,$$

where I means the identity operator.

the results of Theorems 4.2 *nad* 4.4 *hold good, i.e., the generalized Fourier transforms* $\mathcal{F}_\pm$ *can be well-defined by* (4.10). (4.11) *and* (4.12) *hold.* $\mathcal{F}_\pm$ *transforms* $L_2(I, X, dr)$ *onto* $L_2(I, X, dk)$.

Proof. Let $f \in L_2(I, X)$ with compact support in $\bar{I}$. Combine (3.18) with

$$\mathcal{F}(k)f = \int_I \eta(r, k)f(r)dr \qquad (k \in \boldsymbol{R}-\{0\}), \tag{5.9}$$

which is obtained by letting $n \to \infty$ in (3.31) with $\mathcal{F}(k)$ and $\eta(r, k)$ replaced by $\mathcal{F}_n(k)$ and $\eta_n(r, k)$, respectively. Then it is easy to see that (4.3) in Lemma 4.1 is valid for any $f \in L_2(I, X)$ with compact support in $\bar{I}$. Thus we can show the expansion formula. Let us show the orthogonality of $\mathcal{F}_\pm$. The essential point of the proof is to show the following: if $\eta^*(r, k)x=0$ for all $r \in I$, then $x=0$. In view of (5.8) it follows from the relation $\eta^*(r, k)x=0$ $(r \in I)$ that $\eta_0^*(r, k)x = v(r, -k, \ell[g])$, and hence $\eta_0^*(\cdot, k)x$ is the radiative function for $\{L_0, -k, 0\}$. Here we should note that $\eta_0^*(\cdot, k)x$ satisfies the equation $(L_0-k^2)v=0$. Because of the uniqueness of the radiative function we have $\eta_0^*(r, k)x=0$ for all $r \in I$. To $\eta_0(r, k)$ we can apply the same argument as in the proof of Theorem 4.4. Thus we have $x=0$. Q.E.D.

6. Concluding remarks

1° The expansion theorem for the Schrödinger operator. The expansion theorem (Theorems 4.2 and 5.4) for the operator T can be directly translated into the case of the self-adjoint realization M of the Schrödinger operator S as follows (cf. Theorem 5.10 of [10]): Let us define the generalized Fourier trans- $\tilde{\mathcal{F}}_\pm$ from $L_2(\boldsymbol{R}^N, dy)$ onto $L_2(\boldsymbol{R}^N, d\xi)$ by $\tilde{\mathcal{F}}_\pm = U_k^{-1}\mathcal{F}_\pm U$ with the unitary operator $U_k = k^{(N-1)/2}$ from $L_2(\boldsymbol{R}^N, d\xi)$ onto $L_2(I, X, dk)$ $(k=|\xi|)$. If the bounded operators $\tilde{\eta}_\pm(r, k)$ $(r \in \bar{I}, k>0)$ on $X=L_2(S^{N-1})$ are defined by $\tilde{\eta}_\pm(r, k) = r^{-(N-1)/2}k^{-(N-1)/2}\eta_\pm(r, k)$, then we have

$$\begin{cases} (\tilde{\mathcal{F}}_\pm F)(\xi) = \underset{R\to\infty}{\text{l.i.m.}} \displaystyle\int_0^R (\tilde{\eta}_\pm(r, k)F(r\cdot))(\omega')r^{N-1}dr & \text{in } L_2(\boldsymbol{R}^N, d\xi)\,, \\ (\tilde{\mathcal{F}}_\pm G)(y) = \underset{K\to\infty}{\text{l.i.m.}} \displaystyle\int_{K^{-1}}^K (\tilde{\eta}_\pm^*(r, k)G(k\cdot))(\omega)k^{N-1}dk & \text{in } L_2(\boldsymbol{R}^N, dy)\,, \end{cases} \tag{6.1}$$

where $y=r\omega$ and $\xi=k\omega'$. Further the relations $E(B, M) = \tilde{\mathcal{F}}_\pm^* \chi_{\sqrt{B}} \tilde{\mathcal{F}}_\pm$ hold good for an arbitrary Borel set B in $(0, \infty)$, $E(\cdot, M)$ being the spectral measure associated with M and $\chi_{\sqrt{B}}$ being as in Theorem 4.2. As has been shown in (5.52) of [10], $\mathcal{F}_\pm$ are essentially the usual Fourier transforms when $Q(y)=0$.

2° Let us note that

$$(Z(y)+\xi'(r)\psi(\omega))^2+\varphi(y, \lambda) = (\text{grad}\, \lambda(y))^2\,, \tag{6.2}$$

$\varphi(y)$ and $\lambda(y)$ being as in (3.6) and (2.13) of [12], respectively. Hence the second relation of [12], (3.17) can be rewritten as

$$\left|2k\frac{\partial\lambda}{\partial|y|}-Q_0(y)-(\text{grad}\,\lambda(y))^2\right|\leqq C(1+|y|)^{-2}. \tag{6.3}$$

3° The modified wave operators. The time-dependent modified wave operators $W_{D,\pm}$ for the Schrödinger operator with a long-range potential were defined by Alsholm-Kato [2], Alsholm [1] and Buslaev-Matveev [3] as

$$W_{D,\pm} = \text{s}-\lim_{t\to\pm\infty} e^{itM}e^{-itM_0-iX_t}, \tag{6.4}$$

where M_0 is the closure of $-\Delta$ and X_t is a function of M_0. On the other hand from the viewponit of the stationary method the sattionary wave operators $\tilde{W}_{D,\pm}$ should be defined by

$$\tilde{W}_{D,\pm} = \mathcal{F}_{\pm}^{*}\mathcal{F}_{0,\pm}, \tag{6.5}$$

$\mathcal{F}_{0,\pm}$ being the the generalized Fourier transforms for M_0. From the orthogonality of the generalized Fourier transforms (Theorems 4.4 and 5.4) we can easily see that $\tilde{W}_{D,\pm}$ are complete. Recently the relation $W_{D,\pm}=\tilde{W}_{D,\pm}$ is shown by H. Kitada [13], [14] and T. Ikebe-H. Isozaki [15], whence follows the completeness of the time-dependent modified wave operators $W_{D,\pm}$.

4° In this paper and [12] we have treated the Schrödinger operator S by transforming S into the differntial operator

$$L = -\frac{d^2}{dr^2}+B(r)+C(r) \tag{6.6}$$

with operator-valued coefficients. Of course, as in Jäger [6] and Saitō [8]~[11], we can start with the operator (6.4) and apply the resuts obtained to the Schrödinger operator. Then, however, the conditions imposed on $B(r)$ become rather complicated than in Jäger [6] and Saitō [8]~[11].

5° The case that the potential $Q(y)$ has singularities can be treated in essentially the same way. For example, we may replace the condition (Q_1) by (Q) $Q_1\in Q_{\alpha,loc}$ with some $\alpha>0$ and there exists $R_0>0$ such that

$$|Q_1(y)|\leqq C(1+|y|)^{-1-\varepsilon} \qquad (|y|\geqq R_0). \tag{6.7}$$

Here $Q_{\alpha,log}$ denotes the class of locally L_2 functions $p(y)$ such that

$$M_p(y) = \int_{|y-z|\leqq 1}\frac{|p(z)|^2}{|y-z|^{N-4+\alpha}}dz \tag{6.8}$$

is locally bounded in $\boldsymbol{R}^N$.

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