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ON REALIZATION OF KIRBY-SIEBENMANN'S OBSTRUCTIONS BY 6-MANIFOLDS

SHIGEO ICHIRAKU

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1. Introduction

Let \( M^n \) be a closed topological manifold. By Kirby-Siebenmann ([5], [6]), an obstruction to triangulate \( M^n \) is defined as an element of \( H^4(M^n : \mathbb{Z}_2) \), provided \( n \geq 5 \). We will denote this obstruction by \( k(M) \). In this paper, we will consider the following problem.

Problem. Let \( M^n \) be a closed PL manifold. For a given non-zero element \( \eta \in H^4(M^n : \mathbb{Z}_2) \), do there exist a nontriangulable manifold \( M^n \) and a homotopy equivalence \( f : M^n \rightarrow M^n \) such that \( f^*k(M^n) = \eta \)? Here, \( f^* : H^4(M^n : \mathbb{Z}_2) \rightarrow H^4(M^n : \mathbb{Z}_2) \) is the isomorphism induced by \( f \).

Since there exists a non-triangulable manifold \( M^n \) which is homotopy equivalent to \( S^4 \times S^1 \) ([5], Introduction p.v), this problem for \( M^n = S^4 \times S^1 \) has an affirmative answer. In some cases, however, the problem has a negative answer. For example, Dr. S. Fukuhara has proved the following ([3]); let \( M^5 \) be a closed (possibly non-triangulable) topological manifold which is homotopy equivalent to \( S^4 \times S^1 \), then \( M^5 \) is really homeomorphic to \( S^4 \times S^1 \).

When \( M^n \) is a closed manifold with \( \pi_1(M^n) \) is free and \( H^4(M^n : \mathbb{Z}_2) = 0 \), the problem will be answered affirmatively. And the problem for \( M^n = S^4 \times S^{n-4} \) will be solved, provided \( n \geq 9 \). (See Corollary 2.)

The method of this paper can be found in [5] and [9]. The author wishes to express his hearty thanks to Professor K. Kawakubo who showed him a construction of non-triangulable manifold having the homotopy type of \( CP^n \).

2. Six-dimensional case

In dimension six, our results are as follow.

Theorem 1. Let \( M^n \) be a closed PL 6-manifold with \( H^4(M^n : \mathbb{Z}_2) = 0 \) and \( \eta \) a non-zero element of \( H^4(M^n : \mathbb{Z}_2) \) whose Poincaré dual \( \eta \) is spherical. Then there exist a non-triangulable manifold \( M^n \) and a homotopy equivalence \( f : M^n \rightarrow M^n \) such that \( f^*k(M) = \eta \), where \( f^* : H^4(M^n : \mathbb{Z}_2) \rightarrow H^4(M^n : \mathbb{Z}_2) \) is the isomorphism
Corollary 1. Let $M^6_0$ be a closed PL 6-manifold. Suppose $H_3(\pi_1(M^6_0) : \mathbb{Z}) = 0$ and $H^3(M^6_0 : \mathbb{Z}) = 0$. Then, for any non-zero element $\eta$ in $H^3(M^6_0 : \mathbb{Z})$, there exist a non-triangulable manifold $M^6$ and a homotopy equivalence $f : M^6_0 \to M^6$ such that $f^*(\eta) = \eta$, where $f^* : H^*(M^6 : \mathbb{Z}) \to H^*(M^6_0 : \mathbb{Z})$ is the isomorphism induced by $f$.

In Theorem 1, we cannot drop the assumption that the Poincaré dual $\eta$ of $\eta$ is spherical. Hence, in Corollary 1, we cannot drop the assumption about the fundamental group of $M^6_0$. The following proposition shows both.

Proposition 1. Let $M^6$ be a closed topological manifold. Suppose $M^6$ has the same homotopy type of $S^4 \times S^1 \times S^3$, then $M^6$ is triangulable.

First, we prove Corollary 1 assuming Theorem 1.

Proof of Corollary 1. By the theorem of Hopf (see [1], p. 356), the fact that $H_2(\pi_1(M^6) : \mathbb{Z}) = 0$ implies that any element of $H_2(M^6 : \mathbb{Z})$ is spherical. This reduces Corollary 1 to Theorem 1.

To prove Theorem 1, we need some lemmas. The following is proved in [5].

Lemma 1. Let $E^{n-1}$ be a closed simply-connected PL manifold such that $H^i(E^{n-1} : \mathbb{Z}) = 0$ and that the Bockstein homomorphism $\beta : H^i(E^{n-1} : \mathbb{Z}) \to H^i(E^{n-1} : \mathbb{Z})$ is trivial. If $n \geq 6$, then there exists a homeomorphism $h_\Theta : E^{n-1} \to E^{n-1}$ which is homotopic to the identity but never isotopic to a PL homeomorphism.

For completeness, we supply the proof of Lemma 1.

Proof of Lemma 1. Since $H^i(E^{n-1} : \mathbb{Z}) = 0$ and $n \geq 6$, there exists a PL structure $\Theta$ on $E^{n-1}$ which is not isotopic to the original PL structure on $E^{n-1}$ ([5], [6]). Since $E^{n-1}$ is simply-connected and the Bockstein homomorphism $\beta : H^i(E^{n-1} : \mathbb{Z}) \to H^i(E^{n-1} : \mathbb{Z})$ is trivial, there exists a PL homeomorphism $g : E^{n-1} \to E^{n-1}$ which is homotopic to the identity by D. Sullivan ([7], [10]). Put $h_\Theta = \text{"identity"} \circ g$, where "identity" $: E^{n-1} \to E^{n-1}$ is a homeomorphism defined by "identity" $(x) = x$. Then clearly $h_\Theta$ is homotopic to the identity. If $h_\Theta$ is isotopic to a PL homeomorphism, then "identity" $: E^{n-1} \to E^{n-1}$ is also isotopic to a PL homeomorphism, for $g$ is a PL homeomorphism. This is a contradiction to the choice of $\Theta$. Therefore $h_\Theta$ is never isotopic to a PL homeomorphism. This proves the lemma.

Lemma 2. Let $E^{n-1}$ be a PL manifold which is a fibration with fibre $S^3$ over a simply-connected closed manifold $N^{n-1}$ such that $H^i(N^{n-1} : \mathbb{Z}) = H^i(N^{n-1} : \mathbb{Z})$
If $n \geq 6$, then there exists a homeomorphism $h_0 : E^{n-1} \to E^{n-1}$ which is homotopic to the identity but never isotopic to a PL homeomorphism.

REMARK. If we put $h = h_0 \times \text{id.} : E^{n-1} \times R \to E^{n-1} \times R$, then $h$ is also never isotopic to a PL homeomorphism by stability $\pi_3(\text{TOP}_m, \text{PL}_m) = \pi_3(\text{TOP}|\text{PL})$ ([5], [6]).

Proof of Lemma 2. Note that $E^{n-1}$ is simply-connected. By Lemma 1, we need only prove that $\varphi^*E^\sim_1$ is nontrivial and that the Bockstein homomorphism $\beta : H^4(E^{n-1} : \mathbb{Z}_2) \to H^4(E^{n-1} : \mathbb{Z})$ is trivial.

Applying the generalized Gysin cohomology exact sequence to the fibration $E^{n-1} \to N^{n-4}$ with fibre $S^3$, we obtain the following exact sequence:

$$H^*(E^{n-1} : G) \to H^*(N^{n-4} : G) \to H^*(N^{n-4} : G) \to H^4(N^{n-4} : \mathbb{Z})$$

where the coefficient group $G$ is $\mathbb{Z}$ or $\mathbb{Z}_2$. By hypothesis, $H^4(N^{n-4} : \mathbb{Z}) = H^4(N^{n-4} : \mathbb{Z}_2) = 0$ and $H^4(N^{n-4} : \mathbb{Z}) = \text{Hom}(H_* N^{n-4} : \mathbb{Z}, \mathbb{Z}) = 0$. Therefore, $H^4(E^{n-1} : \mathbb{Z})$ is non-trivial and $H^4(E^{n-1} : \mathbb{Z})$ is trivial. This proves the lemma.

Proof of Theorem 1. Since $\eta$ is spherical, there exists a continuous map $S^2 \to M^6_0$ representing $\eta \in H_3(M^6_0 : \mathbb{Z}_2)$. By general position, we can assume that this $S^2$ is PL embedded in $M^6_0$. By Haefliger-Wall [4], $S^2$ has a normal PL disk bundle $D(v)$.

Clearly, $\text{Int } D(v) - S^2$ is PL homeomorphic to $\partial D(v) \times R$. Put $\partial D(v) = E^6$, then by Lemma 2 and Remark we can find a homeomorphism $h : E^6 \times R \to E^6 \times R$ which is homotopic to the identity but never isotopic to a PL homeomorphism. Clearly $M^6_0 - S^2$ contains $E^6 \times R$ as an open PL collar of the end at $S^2$. Then $M^6_0$ can be written obviously as $(M^6_0 - S^2) \cup \text{Int } D(v)$.

Let $M^4$ be a topological manifold $(M^6_0 - S^2) \cup \text{Int } D(v)$ obtained by pasting $\text{Int } D(v)$ to $M^6_0 - S^2$ by the above homeomorphism $h : E^6 \times R \to E^6 \times R$. Let $H_0 : E^6 \times I \to E^6$ be a homotopy connecting $h_0$ to the identity. Put $H = h_0 \times \text{id.} : (E^6 \times R) \times I \to E^6 \times R$. Consider the adjunction space $\mathcal{W} = ((M^6_0 - S^2) \times I) \cup \text{Int } D(v)$ obtained by pasting $(M^6_0 - S^2) \times I$ to $\text{Int } D(v)$ by the continuous map $H : (E^6 \times R) \times I \to E^6 \times R$. Then, clearly, $\mathcal{W}$ is homeomorphic to the adjunction space $(M^6_0 - \text{Int } D(v)) \times I \cup D(v)$ obtained by pasting together $(M^6_0 - \text{Int } D(v)) \times I$ and $D(v)$ by the continuous map $H_0 : E^6 \times I \to E^6$. Then, we can see that $\mathcal{W}$ has both $M^6_0$ and $M^4$ as deformation retracts. (see [8], p. 21, Adjunction Lemma.) Define a homotopy equivalence $f : M^6_0 \to M^4$ to be the composition of the following maps.
Next, we will show that $M^6$ is non-triangulable. Suppose $M^6$ is triangulable. Both $\{M^6 - S^2\}$ and $\text{Int}D(\nu)$ are open $PL$ submanifolds of $M^6$. We denote these submanifolds with induced $PL$ structures from $M^6$ by $(M^6 - S^2)_\alpha$ and $(\text{Int} D(\nu))_\beta$. Then the composition of

\[
\text{"identity"} : (E^6 \times R)_{\alpha\mid E^6 \times R} \to E^6 \times R,
\]
\[
h : E^6 \times R \to E^6 \times R \quad \text{and}
\]
\[
\text{"identity"} : E^6 \times R \to (E^6 \times R)_{\beta\mid E^6 \times R}
\]
is a $PL$ homeomorphism. On the other hand, by the following diagram, we see that $H^3(M^6 - S^2 : \mathbb{Z}_2) = 0$.

\[
\begin{array}{c}
\begin{array}{c}
H^3(M^6 : \mathbb{Z}_2) \\
\downarrow \quad \quad \downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
H^3(M^6 - S^2 : \mathbb{Z}_2) \\
\downarrow \quad \quad \downarrow
\end{array}
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\begin{array}{c}
\begin{array}{c}
H^3(M^6 : \mathbb{Z}_2) \\
\downarrow \quad \quad \downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
H^3(M^6 - S^2 : \mathbb{Z}_2) \\
\downarrow \quad \quad \downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k(M) \\
\downarrow \quad \quad \downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\]

where the horizontal sequence is exact and the vertical maps are Poincaré and Alexander dualities. Therefore, $\alpha$ is concordant to the original $PL$ structure on $M^6 - S^2$ and hence $\alpha\mid E^6 \times R$ is concordant to the original $PL$ structure on $E^6 \times R$ ([5], [6]). This means that "identity" : $(E^6 \times R)_{\alpha\mid E^6 \times R} \to E^6 \times R$ is isotopic to a $PL$ homeomorphism. In a similar way, we have that "identity" : $E^6 \times R \to (E^6 \times R)_{\beta\mid E^6 \times R}$ is isotopic to a $PL$ homeomorphism. Then $h$ itself is isotopic to a $PL$ homeomorphism which is a contradiction. Therefore $M^6$ must be non-triangulable.

Note that $M^6 - S^2 = M^6 - S^2$ is triangulable. Then the naturality of Kirby-Siebenmann's obstruction with respect to inclusion maps of open submanifolds and the following commutative diagram imply that $S^2$ in $M^6$ represents the Poincaré dual of $k(M)$ in $H^3(M^6 : \mathbb{Z}_2)$.

\[
\begin{array}{c}
\begin{array}{c}
H^3(S^2 : \mathbb{Z}_2) \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
H^3(M^6 : \mathbb{Z}_2) \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
H^3(M^6 - S^2 : \mathbb{Z}_2) \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k(M) \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array}
\]

where the horizontal sequences are exact and the vertical isomorphisms are Poincaré and Alexander dualities. Now, it is clear that $f^*k(M^6) = \eta$, this proves the theorem.

Proof of Proposition 1. By virtue of a topological version ([8]) of fibering theorem due to F.T. Farrell [2], $M^6$ is a fibering over a circle, since $\text{Wh}(\pi_1(M^6))$
Therefore there exists a submanifold $N^5$ of $M^6$ and a homeomorphism $g : N^5 \to N^5$ such that the mapping torus of $g$ is homeomorphic to $M^6$. Since $N^5$ has the homotopy type of $S^4 \times S^1$, $N^5$ is really homeomorphic to $S^4 \times S^1$ by S. Fukuwara [3]. Since $H^4(S^4 \times S^1 : Z_2) = 0$, any homeomorphism of $S^4 \times S^1$ onto itself is isotopic to a PL homeomorphism ([5], [6]). Therefore $M^6$ is triangulable. This proves the proposition.

3. Higher dimensional case

In higher dimensional case, we can only obtain a weaker result.

**Theorem 2.** Let $M^n_0$ be a closed PL manifold of dimension $n \geq 6$ with $H^i(M^n_0 : Z_2) = 0$. Suppose $\eta$ is a non-zero element of $H^i(M^n_0 : Z_2)$ whose Poincaré dual $\eta$ in $H_{n-4}(M^n_0 : Z_2)$ is represented by a simply-connected $(n-4)$-submanifold $N^{n-4}$ with $H^i(N^{n-4} : Z_2) = H^i(N^n : Z_2) = H^i(N^{n-4} : Z_2) = 0$. Then there exist a non-triangulable manifold $M^n$ and a homotopy equivalence $f : M^n_0 \to M^n$ such that $f^*k(M^n) = \eta$.

As an application of Theorem 2, we can obtain a number of non-triangulable manifolds which are homotopy equivalent to some PL manifolds.

**Corollary 2.** Let $N^{n-4}$ be a closed 4-connected PL manifold and $L^4$ a simply-connected 4-manifold. If $n \geq 9$, then there exists a non-triangulable manifold which has the homotopy type of $L^4 \times N^{n-4}$.

Proof of Theorem 2. By the assumption, there exists a $(n-4)$-submanifold $N^{n-4}$ of $M^n_0$ representing $\eta$. Let $D(\nu)$ be a normal block bundle of $N^{n-4}$ in $M^n_0$. Put $E^{n-1} = \partial D(\nu)$, then by Lemma 2 and Remark, there exists a homeomorphism $h : E^{n-1} \times R \to E^{n-1} \times R$ which is homotopic to the identity but never isotopic to a PL homeomorphism. As before, put $M^n = (M^n_0 - N^{n-4}) \cup \text{Int} D(\nu)$. Then the rest of the proof is exactly same as that of Theorem 1.

Proof of Corollary 2. By the preceding arguments, we have only to show that $H^i(L^4 \times N^{n-4} : Z_2) = 0$. By the Künneth formula and the Poincaré duality, we have the following:

$$H^i(L^4 \times N^{n-4} : Z_2) = H^i(N^{n-4} : Z_2) \oplus [H^i(L^4 : Z) \otimes H^i(N^{n-4} : Z_2)] \oplus [H^i(L^4 : Z) \otimes H^i(N^{n-4} : Z_2)]$$

$$= 0$$

This proves the corollary.

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References