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Author(s)	Shima, Hirohiko
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COMPACT LOCALLY HESSIAN MANIFOLDS

HIROHIKO SHIMA

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Let M be a differentiable manifold with a locally flat linear connection D . A Riemannian metric g on M is said to be *locally Hessian* if for each point $p \in M$ there exists a C^∞ -function φ defined on a neighbourhood of p such that $g = D^2\varphi$. Such a pair (D, g) is called a *locally Hessian structure* on M [4].

Let M be a differentiable manifold with a locally Hessian structure (D, g) . Throughout this note the local expressions for the locally Hessian metric and the related concepts will be given in terms of affine local coordinate systems with respect to D . Let v be the volume element determined by the Riemannian metric g ;

$$v = F dx^1 \wedge \cdots \wedge dx^n, \quad \text{where} \quad F = \sqrt{\det [g_{ij}]}$$

We define the forms α and β by

$$\alpha_i = \frac{\partial \log F}{\partial x^i},$$
$$\beta_{ij} = \frac{\partial^2 \log F}{\partial x^i \partial x^j},$$

and call them the Koszul form and the canonical bilinear form respectively. These forms α and β play important roles in the study of locally Hessian manifolds [2] [3] [4] [5].

The following assertion is derived from a result of Koszul [3].

(a) *Let M be a compact connected differentiable manifold with a locally Hessian structure (D, g) . If the canonical bilinear form β is positive definite on M , then the universal covering manifold of M with a locally Hessian structure induced by (D, β) is isomorphic to an open convex domain not containing any full straight line in a real affine space.*

In our viewpoint a theorem of Calabi [1] is stated as follows:

(b) *Let M be a domain in the n -dimensional real affine space and let φ be a C^∞ -function on M such that $g = D^2\varphi$ is positive definite, where D is the natural flat linear connection on M (Thus (D, g) is a locally Hessian structure on M). If the Riemannian metric g on M is complete and if the Koszul form α vanishes identically*

on M , then the Riemannian metric g is locally flat.

In this note we prove the following:

Theorem. *Let M be a compact orientable differentiable manifold with a locally Hessian structure (D, g) . Then we have*

$$(i) \quad \int_M \beta^i;v = \int_M \alpha; \alpha^i v \geq 0.^{1)}$$

(ii) *If the equality $\int_M \beta^i;v = 0$ holds, then the Riemannian metric g is locally flat.*

The proof of (ii) is based upon a technique of Calabi [1].

As an immediate consequence of this theorem we have:

Corollary. *Let M be a compact orientable differentiable manifold with a locally Hessian structure (D, g) . Then we have*

(i) *The canonical bilinear form β can not be negative definite.*

(ii) *If the canonical bilinear form β is negative semi-definite, then the Riemannian metric g is locally flat.*

REMARK. In Corollary (i) the assumption that M is compact is necessary. For example, if $g = e^{-x^2} dx dx$ is a Riemannian metric on the real line \mathbf{R} , then g is a Hessian metric on \mathbf{R} and the canonical bilinear form corresponding to g is $\beta = -dx dx$.

We shall express various tensors determined by g in terms of affine local coordinate system with respect to D .

1° The locally Hessian metric:

$$g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}.$$

A necessary and sufficient condition for g to be a locally Hessian metric is given by

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial g_{ik}}{\partial x^j}.$$

2° The Christoffel symbol:

$$\Gamma^i_{jk} = \frac{1}{2} g^{is} \frac{\partial g_{sj}}{\partial x^k}.$$

We denote by the same letter Γ^i_{jk} the tensor field induced by $\frac{1}{2} g^{is} \frac{\partial g_{sj}}{\partial x^k}$ and we have

1) Throughout this note we use Einstein's convention on indices.

$$\Gamma_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k},$$

$$\Gamma_{ijk} = \Gamma_{jik} = \Gamma_{ikj}.$$

3° The Koszul form:

$$\alpha_i = \Gamma_{ir}^r.$$

4° The Riemannian curvature tensor:

$$R_{ijkl} = -g^{rs}(\Gamma_{rjk}\Gamma_{sil} - \Gamma_{rjl}\Gamma_{sik}),$$

$$R^i_{jhl} = \Gamma^i_{rk}\Gamma^r_{jl} - \Gamma^i_{rl}\Gamma^r_{jk}.$$

5° The Ricci tensor:

$$\begin{aligned} R_{jk} &= \Gamma^r_{js}\Gamma^s_{rk} - \Gamma^r_{jk}\Gamma^s_{rs} \\ &= \Gamma^r_{js}\Gamma^s_{rk} - \alpha_r\Gamma^r_{jk}. \end{aligned}$$

6° The scalar curvature:

$$R = \Gamma_{rst}\Gamma^{rst} - \alpha_r\alpha^r.$$

Let $\alpha_{i;j}$ denote the covariant derivative of α_i with respect to Γ^i_{jk} . Then we have

$$\alpha_{i;j} = \beta_{ij} - \alpha_r\Gamma^r_{ij},$$

and

$$\alpha^i_{;i} = \beta^i_i - \alpha_r\alpha^r.$$

Applying Green's theorem [6] we obtain

$$\int_M (\beta^i_i - \alpha_i\alpha^i)v = \int_M \alpha^i_{;i}v = 0.$$

Thus the integral formula of Theorem (i) is proved.

We shall prove Theorem (ii). If the equality $\int_M \beta^i_{;i}v=0$ holds, then by Theorem (i) the Koszul form α vanishes identically on M ;

$$(1) \quad \alpha_i = 0.$$

Therefore it follows from 5° and 6° that

$$(2) \quad R_{jk} = \Gamma^r_{js}\Gamma^s_{kr},$$

$$(3) \quad R = \Gamma_{rst}\Gamma^{rst}.$$

We compute the Laplacian ΔR of the scalar curvature R . Since the covariant derivative $\Gamma_{ijk;l}$ of Γ_{ijk} is

$$\begin{aligned}\Gamma_{ijk;l} &= \frac{\partial \Gamma_{ijk}}{\partial x^l} - \Gamma_{rjk} \Gamma_{il}^r - \Gamma_{irk} \Gamma_{jl}^r - \Gamma_{ijr} \Gamma_{kl}^r \\ &= \frac{1}{2} \frac{\partial^4 \varphi}{\partial x^i \partial x^j \partial x^k \partial x^l} - g^{rs} (\Gamma_{rjk} \Gamma_{sil} + \Gamma_{irk} \Gamma_{sjl} + \Gamma_{ijr} \Gamma_{skl}),\end{aligned}$$

by 2° $\Gamma_{ijk;l}$ is symmetric in all pairs of indices;

$$(4) \quad \Gamma_{ijk;l} = \Gamma_{ljk;i}.$$

Using 3°, 4°, (1), (2), (4) and the Ricci formula, we obtain

$$\begin{aligned}(5) \quad g^{rs} \Gamma_{ijk;rs} &= g^{rs} \Gamma_{rjk;is} = g^{rs} (\Gamma_{rjk;is} - \Gamma_{rjk;si}) + g^{rs} \Gamma_{rjk;si} \\ &= -g^{rs} (\Gamma_{\rho jk} R_{ris}^{\rho} + \Gamma_{r \rho k} R_{jis}^{\rho} + \Gamma_{rj \rho} R_{kis}^{\rho}) + g^{rs} \Gamma_{rsk;ji} \\ &= \Gamma_{\rho jk} R_{i}^{\rho} - \Gamma_{\rho k}^s (\Gamma_{qi}^{\rho} \Gamma_{js}^q - \Gamma_{qs}^{\rho} \Gamma_{ji}^q) \\ &\quad - \Gamma_{j \rho}^s (\Gamma_{qi}^{\rho} \Gamma_{ks}^q - \Gamma_{qs}^{\rho} \Gamma_{ki}^q) + \alpha_{k;ji} \\ &= \Gamma^{\rho qs} (\Gamma_{qsi} \Gamma_{\rho jk} + \Gamma_{s \rho j} \Gamma_{qki} + \Gamma_{s \rho k} \Gamma_{qji}) \\ &\quad - \Gamma_{qi}^{\rho} \Gamma_{sj}^q \Gamma_{\rho k}^s - \Gamma_{qi}^{\rho} \Gamma_{sk}^q \Gamma_{\rho j}^s.\end{aligned}$$

From 4°, (2), (3), and (5) we get

$$\begin{aligned}(6) \quad \frac{1}{2} \Delta R &= \Gamma^{ijk} g^{rs} \Gamma_{ijk;rs} + \Gamma^{ijk;l} \Gamma_{ijk;l} \\ &= 3 \Gamma^{ijk} \Gamma^{\rho qr} \Gamma_{qri} \Gamma_{\rho jk} - 2 \Gamma^{ijk} \Gamma_{qi}^{\rho} \Gamma_{rj}^q \Gamma_{\rho k}^r + \Gamma^{ijk;l} \Gamma_{ijk;l} \\ &= R_{ij} R^{ij} + R_{ijkl} R^{ijkl} + \Gamma_{ijk;l} \Gamma^{ijk;l} \\ &\geq 0.\end{aligned}$$

Therefore, applying the Bochner's lemma [6] we have

$$\Delta R = 0,$$

and in particular

$$R_{ijkl} = 0.$$

This means that g is a locally flat Riemannian metric. Thus the proof of Theorem (ii) is completed.

YAMAGUCHI UNIVERSITY

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