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COMPACT LOCALLY HESSIAN MANIFOLDS

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Let M be a differentiable manifold with a locally flat linear connection D. A Riemannian metric g on M is said to be *locally Hessian* if for each point $p \in M$ there exists a C^∞-function φ defined on a neighbourhood of p such that $g = D^2\varphi$. Such a pair (D, g) is called a *locally Hessian structure* on M [4].

Let M be a differentiable manifold with a locally Hessian structure (D, g). Throughout this note the local expressions for the locally Hessian metric and the related concepts will be given in terms of affine local coordinate systems with respect to D. Let v be the volume element determined by the Riemannian metric g;

$$v = F dx^1 \wedge \cdots \wedge dx^n, \quad \text{where} \quad F = \sqrt{\det [g_{ij}]}\,.$$

We define the forms α and β by

$$\alpha_i = \frac{\partial \log F}{\partial x^i},$$

$$\beta_{ij} = \frac{\partial^2 \log F}{\partial x^i \partial x^j},$$

and call them the Koszul form and the canonical bilinear form respectively. These forms α and β play important roles in the study of locally Hessian manifolds [2] [3] [4] [5].

The following assertion is derived from a result of Koszul [3].

(a) *Let M be a compact connected differentiable manifold with a locally Hessian structure (D, g). If the canonical bilinear form β is positive definite on M, then the universal covering manifold of M with a locally Hessian structure induced by (D, β) is isomorphic to an open convex domain not containing any full straight line in a real affine space.*

In our viewpoint a theorem of Calabi [1] is stated as follows:

(b) *Let M be a domain in the n-dimensional real affine space and let φ be a C^∞-function on M such that $g = D^2\varphi$ is positive definite, where D is the natural flat linear connection on M (Thus $(D\ g)$ is a locally Hessian structure on M). If the Riemannian metric g on M is complete and if the Koszul form α vanishes identically* *on M, then the Riemannian metric g is locally flat.*

In this note we prove the following:

Theorem. *Let M be a compact orientable differentiable manifold with a locally Hessian structure* (D, g). *Then we have*

(i) $$\int_M \beta^i{}_i v = \int_M \alpha_i \alpha^i v \geqslant 0.$$ [1]

(ii) *If the equality* $\int_M \beta^i{}_i v = 0$ *holds, then the Riemannian metric g is locally flat.*

The proof of (ii) is based upon a technique of Calabi [1].

As an immediate consequence of this theorem we have:

Corollary. *Let M be a compact orientable differentiable manifold with a locally Hessian structure* (D, g). *Then we have*

(i) *The canonical bilinear form* β *can not be negative definite.*

(ii) *If the canonical bilinear form* β *is negative semi-definite, then the Riemannian metric g is locally flat.*

REMARK. In Corollary (i) the assumption that M is compact is necessary. For example, if $g = e^{-x^2} dxdx$ is a Riemannian metric on the real line $\boldsymbol{R}$, then g is a Hessian metric on $\boldsymbol{R}$ and the canonical bilinear form corresponding to g is $\beta = -dxdx$.

We shall express various tensors determined by g in terms of affine local coordinate system with respect to D.

1° The locally Hessian metric:

$$g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}.$$

A necessary and sufficient condition for g to be a locally Hessian metric is given by

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial g_{ik}}{\partial x^j}.$$

2° The Christoffel symbol:

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{is} \frac{\partial g_{sj}}{\partial x^k}.$$

We denote by the same letter $\Gamma^i{}_{jk}$ the tensor field induced by $\frac{1}{2} g^{is} \frac{\partial g_{sj}}{\partial x^k}$ and we have

1) Throughout this note we use Einstein's convention on indices.

$$\Gamma_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k},$$

$$\Gamma_{ijk} = \Gamma_{jik} = \Gamma_{ikj}.$$

3° The Koszul form:

$$\alpha_i = \Gamma^r{}_{ir}.$$

4° The Riemannian curvature tensor:

$$R_{ijkl} = -g^{rs}(\Gamma_{rjk}\Gamma_{sil} - \Gamma_{rjl}\Gamma_{sik}),$$
$$R^i{}_{jkl} = \Gamma^i{}_{rk}\Gamma^r{}_{jl} - \Gamma^i{}_{rl}\Gamma^r{}_{jk}.$$

5° The Ricci tensor:

$$R_{jk} = \Gamma^r{}_{js}\Gamma^s{}_{rk} - \Gamma^r{}_{jk}\Gamma^s{}_{rs}$$
$$= \Gamma^r{}_{js}\Gamma^s{}_{rk} - \alpha_r\Gamma^r{}_{jk}.$$

6° The scalar curvature:

$$R = \Gamma_{rst}\Gamma^{rst} - \alpha_r\alpha^r.$$

Let $\alpha_{i;j}$ denote the covariant derivative of α_i with respect to $\Gamma^i{}_{jk}$. Then we have

$$\alpha_{i;j} = \beta_{ij} - \alpha_r\Gamma^r{}_{ij},$$

and

$$\alpha^i{}_{;i} = \beta^i{}_i - \alpha_r\alpha^r.$$

Applying Green's theorem [6] we obtain

$$\int_M (\beta^i{}_i - \alpha_i\alpha^i)v = \int_M \alpha^i{}_{;i}v = 0.$$

Thus the integral formula of Theorem (i) is proved.

We shall prove Theorem (ii). If the equality $\int_M \beta^i{}_i v = 0$ holds, then by Theorem (i) the Koszul form α vanishes identically on M;

(1) $$\alpha_i = 0.$$

Therefore it follows from 5° and 6° that

(2) $$R_{jk} = \Gamma^r{}_{js}\Gamma^s{}_{kr},$$

(3) $$R = \Gamma_{rst}\Gamma^{rst}.$$

We compute the Laplacian ΔR of the scalar curvature R. Since the covariant derivative $\Gamma_{ijk;l}$ of Γ_{ijk} is

$$\Gamma_{ijk;l} = \frac{\partial \Gamma_{ijk}}{\partial x^l} - \Gamma_{rjk}\Gamma^r{}_{il} - \Gamma_{irk}\Gamma^r{}_{jl} - \Gamma_{ijr}\Gamma^r{}_{kl}$$
$$= \frac{1}{2}\frac{\partial^4 \varphi}{\partial x^i \partial x^j \partial x^k \partial x^l} - g^{rs}(\Gamma_{rjk}\Gamma_{sil} + \Gamma_{irk}\Gamma_{sjl} + \Gamma_{ijr}\Gamma_{skl}),$$

by 2° $\Gamma_{ijk;l}$ is symmetric in all pairs of indices;

$$\Gamma_{ijk;l} = \Gamma_{ljk;i}. \tag{4}$$

Using 3°, 4°, (1), (2), (4) and the Ricci formula, we obtain

$$\begin{aligned} g^{rs}\Gamma_{ijk;r;s} &= g^{rs}\Gamma_{rjk;i;s} = g^{rs}(\Gamma_{rjk;i;s} - \Gamma_{rjk;s;i}) + g^{rs}\Gamma_{rjk;s;i} \\ &= -g^{rs}(\Gamma_{pjk}R^p{}_{ris} + \Gamma_{rpk}R^p{}_{jis} + \Gamma_{rjp}R^p{}_{kis}) + g^{rs}\Gamma_{rsk;j;i} \\ &= \Gamma_{pjk}R^p{}_i - \Gamma^s{}_{pk}(\Gamma^p{}_{qi}\Gamma^q{}_{js} - \Gamma^p{}_{qs}\Gamma^q{}_{ji}) \\ &\quad - \Gamma^s{}_{jp}(\Gamma^p{}_{qi}\Gamma^q{}_{ks} - \Gamma^p{}_{qs}\Gamma^q{}_{ki}) + \alpha_{k;j;i} \\ &= \Gamma^{pqs}(\Gamma_{qsi}\Gamma_{pjk} + \Gamma_{spj}\Gamma_{qki} + \Gamma_{spk}\Gamma_{qji}) \\ &\quad - \Gamma^p{}_{qi}\Gamma^q{}_{sj}\Gamma^s{}_{pk} - \Gamma^p{}_{qi}\Gamma^q{}_{sk}\Gamma^s{}_{pj}. \end{aligned} \tag{5}$$

From 4°, (2), (3), and (5) we get

$$\begin{aligned} \frac{1}{2}\Delta R &= \Gamma^{ijk}g^{rs}\Gamma_{ijk;r;s} + \Gamma^{ijk;l}\Gamma_{ijk;l} \\ &= 3\Gamma^{ijk}\Gamma^{pqr}\Gamma_{qri}\Gamma_{pjk} - 2\Gamma^{ijk}\Gamma^p{}_{qi}\Gamma^q{}_{rj}\Gamma\Gamma^r{}_{pk} + \Gamma^{ijk;l}\Gamma_{ijk;l} \\ &= R_{ij}R^{ij} + R_{ijkl}R^{ijkl} + \Gamma_{ijk;l}\Gamma^{ijk;l} \\ &\geqslant 0. \end{aligned} \tag{6}$$

Therefore, applying the Bochner's lemma [6] we have

$$\Delta R = 0,$$

and in particular

$$R_{ijkl} = 0.$$

This means that g is a locally flat Riemannian metric. Thus the proof of Theorem (ii) is completed.

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