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REALIZATION OF HYPERELLIPTIC FAMILIES WITH THE HYPERELLIPTIC SEMISTABLE MONODROMIES

MIZUHO ISHIZAKA

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Abstract

Let Φ be an element of the mapping class group $\mathcal{M}_g$ of genus g (≥ 2) such that Φ is the isotopy class of a pseudo periodic map of negative twists. It is expected that, for each Φ which commutes with a hyperelliptic involution, there exists a hyperelliptic family whose monodromy is the conjugacy class of Φ in the mapping class group. In this paper, we give a partial solution for the conjecture in the case where Φ is a semistable element.

1. Introduction

Let $\phi\colon S \to \Delta$ be a proper surjective holomorphic map from a nonsingular complex surface S to a small disk $\Delta := \{t \in \mathbf{C} \mid |t| < \varepsilon\}$ such that $\phi^{-1}(t)$ is a nonsingular curve of genus $g \geq 2$ for each $t \in \Delta^* := \Delta \setminus \{0\}$. We call (ϕ, S, Δ) *a degeneration of curves or a family of curves* of genus g. If all $\phi^{-1}(t)$ ($t \in \Delta^*$) are hyperelliptic curves, we call (ϕ, S, Δ) a *hyperelliptic family*. We call $\phi^{-1}(0)$ *the special fiber* of S. Two degenerations (ϕ, S, Δ) and (ϕ', S', Δ') are said to be *topologically equivalent* if there exist orientation preserving homeomorphisms $\psi\colon S \to S'$ and $\overline{\psi}\colon \Delta \to \Delta'$ which satisfy $\phi' \circ \psi = \overline{\psi} \circ \phi$. For a topological equivalence class of a degeneration, we can uniquely determine the topological monodromy (called the monodromy, for short) as the conjugacy class of the isotopy class of a pseudo periodic map of negative twists in the mapping class group $\mathcal{M}_g$ of genus g (cf. [3]).

Let Σ_g be a compact real surface of genus g without boundary. It is well-known fact that $\mathcal{M}_g$ is generated by Dehn twists at simple closed curves on Σ_g (cf. [2]). We denote by $D_{C_i}^{n_i}$ the n_i-times right hand Dehn twists at a simple closed curve C_i on Σ_g. An involution I of Σ_g is called *a hyperelliptic involution* if it has $2g+2$ fixed points. We call Φ *a hyperelliptic element with* I if there exists a homeomorphism $\widetilde{\Phi}$ whose isotopy class is Φ satisfying $I \circ \widetilde{\Phi} = \widetilde{\Phi} \circ I$ as map. We denote by $[\Phi]$ the conjugacy class of Φ in $\mathcal{M}_g$. An element Φ of $\mathcal{M}_g$ is called *semistable* if there exists a disjoint union of simple closed curves $\mathcal{C} := \{C_i\}_{i=1,2,\dots,r}$ and positive integers $\{n_i\}_{i=1,2,\dots,r}$ satisfying $\Phi = D_{C_1}^{n_1} \cdots D_{C_r}^{n_r}$. We call $\mathcal{C}$ *an admissible system* of Φ if any two simple closed curves are not homotopic to each other.

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Let Φ be the isotopy class of a pseudo-periodic map of negative twists. It is expected that if Φ is a hyperelliptic element, there exists a hyperelliptic family of curves of genus g with monodromy $[\Phi]$. In this paper, we give a partial solution for the conjecture in the case where Φ is a semistable element. Thus, our main theorem is the following;

Theorem 1.1. *Let Φ be a hyperelliptic semistable element. Then, there exists a hyperelliptic family with monodromy* $[\Phi]$.

To prove Theorem 1.1, for each Φ, we construct a hyperelliptic family using a double covering of $\mathbf{P}^1 \times \Delta$. Thus, for each hyperelliptic semistable element Φ, we give not only the proof of the existence but also an algorithm to construct a hyperelliptic family with monodromy $[\Phi]$.

2. Hyperelliptic semistable monodromy

Let $\langle I \rangle$ be the cyclic group generated by a hyperelliptic involution I. We denote by $\Pi\colon \Sigma_g \to \Sigma_g/\langle I \rangle \simeq S^2$ the canonical projection from Σ_g to the quotient of Σ_g by $\langle I \rangle$. Let $\mathcal{P} := \{P_1, \dots, P_{2g+2}\}$ be the set of the branch points of Π. To prove the main proposition (Lemma 2.4) in this section, we need to observe simple closed curves on Σ_g and their images on $\Sigma_g/\langle I \rangle$ by Π.

Lemma 2.1. *Let $\Phi = D_{C_1}^{n_1} \cdots D_{C_r}^{n_r}$ be a hyperelliptic semistable element with I, where $\{C_i\}_{i=1,\dots,r}$ is an admissible system of Φ. Then, for each i, there exists j $(1 \leq j \leq r)$ such that $I(C_i)$ is homotopic to C_j with $n_i = n_j$.*

Proof. Let $\widetilde{D}_C$ and $\widetilde{D}_{I(C)}$ be homeomorphisms whose isotopy classes are D_C and $D_{I(C)}$, respectively. Since I is homeomorphism of Σ_g, we see that $I \circ \widetilde{D}_C$ is isotopic to $\widetilde{D}_{I(C)} \circ I$ (cf. [2], Lemma 1). Since Φ is a hyperelliptic element, we obtain

$$D_{I(C_1)}^{n_1} \cdots D_{I(C_r)}^{n_r} = D_{C_1}^{n_1} \cdots D_{C_r}^{n_r}. \tag{1}$$

Let U_{C_i} be a small annular open neighbourhood of C_i such that $U_{C_i} \cap U_{C_j} = \emptyset$ for all i, j. Note that there exists a homeomorphism whose isotopy class is Φ such that the restriction to $\Sigma_g \setminus \left\{\bigcup U_{C_i}\right\}$ is the identity. Thus from the equation (1), we see that each $I(C_i)$ does not intersect properly some C_j. So, $I(C_i)$ is homotopic to some C_j with $n_i = n_j$ or $I(C_i)$ is homotopic to a curve contained in $\Sigma_g \setminus \left\{\bigcup U_{C_i}\right\}$. In the latter case, the restriction map of any homeomorphisms in the isotopy class of $D_{I(C_1)}^{n_1} \cdots D_{I(C_r)}^{n_r}$ to $\Sigma_g \setminus \left\{\bigcup U_{C_i}\right\}$ are not identity, a contradiction. □

Let $c\colon [0,1] \to \Sigma_g$ be a simple closed curve on Σ_g such that the curve $\Pi \circ c$ is the composite $c_2 \circ c_1$ of curves c_1 and c_2, where $c_1\colon [0,1] \to S^2$ is a simple closed curve rounding only one branch point P_i. Taking another curve homotopic to c if necessary, we may assume that $\Pi \circ c$ intersects itself transversally at the initial point of c_1. We also assume that any branch points are not on $\Pi \circ c$. We denote by D_i the disk with boundary c_1 that contains P_i in its inside.

Lemma 2.2. *Notation is as above. There exists a simple closed curve $\widetilde{c}$ homotopic to c satisfying $\Pi \circ \widetilde{c} = c_2 \circ c_1^{-1}$. Then, any simple closed curve on Σ_g is homotopic to a lift of a curve on S^2 which has no subloop rounding only one branch point of Π.*

Proof. Let $\overline{c_1}$ and $\overline{c_2}$ be curves on Σ_g such that $\Pi \circ \overline{c_i} = c_i$ $(i = 1, 2)$ and the composite $\overline{c_2} \circ \overline{c_1}$ is c. Since c_1 rounds only one branch point P_i, $\Pi^{-1}(D_i)$ is a disk on Σ_g. Thus, there exists a curve $\widetilde{c_1}$ on Σ_g such that $\Pi \circ \widetilde{c_1} = c_1$ and the composite $\overline{c}_1 \circ \widetilde{c_1}$ is the boundary of $\Pi^{-1}(D_i)$. Then, we see that the curve $\widetilde{c} := \overline{c_2} \circ \widetilde{c_1}^{-1}$ is homotopic to c and $\Pi \circ \widetilde{c} = c_2 \circ c_1^{-1}$. Since the configuration of $\Pi \circ \widetilde{c}$ near P_i is as shown in Fig. 1 (1), we can find a curve on S^2 homotopic to $c_2 \circ c_1^{-1}$ such that it does not have a subloop rounding only P_i (see, Fig. 1 (2)). □

REMARK 2.3. Assume that c is a curve on Σ_g such that the configuration of $\Pi \circ c$ near P_i is as shown in Fig. 1 (3). Then c is not a simple closed curve on Σ_g.

Let Φ be a hyperelliptic semistable element with I. For each simple closed curve C_i in an admissible system of Φ, we denote it by $\overrightarrow{C_i}$ when we emphasize its orientation. By Lemma 2.1, we can classify the curves in an admissible system into the following three types;

(Type A′) $I\left(\overrightarrow{C_i}\right)$ is homotopic to $\overrightarrow{C_i^{-1}}$.

(Type B′) $I\left(\overrightarrow{C_i}\right)$ is homotopic to $\overrightarrow{C_i}$.

(Type C′) There exists j $(\neq i)$ such that $I\left(\overrightarrow{C_i}\right)$ is homotopic to $\overrightarrow{C_j}$ or $\overrightarrow{C_j^{-1}}$.

We consider the case where $I(C_i) \neq C_i$ and $I(C_i) \cap C_i \neq \emptyset$. We may assume that $I(C_i)$ intersects C_i transversally and there exist no branch points of Π on $\Pi(C_i)$. Moreover, by Lemma 2.2, we may assume that $\Pi(C_i)$ does not have a subloop rounding only one branch point of Π. Let $\overrightarrow{\zeta_1}$ be an oriented subcurve of $\overrightarrow{C_i}$ such that $\overrightarrow{\zeta_1} \cap I\left(\overrightarrow{C_i}\right) = \{Q_1, Q_2\}$, where Q_1 and Q_2 are the initial and end points of $\overrightarrow{\zeta_1}$, respectively. We assume that there exists a subcurve $\overrightarrow{\zeta_2}$ of $I(C_i)$ (or $I(C_i)^{-1}$) such that the composite $\overrightarrow{\zeta_2} \circ \overrightarrow{\zeta_1}$ is homotopic to zero and $\overrightarrow{\zeta_1} \cap \overrightarrow{\zeta_2} = \{Q_1, Q_2\}$. We denote by $\mathcal{D}_{Q_1Q_2}$ the disk on Σ_g with boundary $\overrightarrow{\zeta_2} \circ \overrightarrow{\zeta_1}$. If $I(Q_1) \neq Q_2$, then $I(\mathcal{D}_{Q_1Q_2}) \cap \mathcal{D}_{Q_1Q_2} = \emptyset$. Thus, we see that $\Pi(\mathcal{D}_{Q_1Q_2})$ is a disk on S^2 containing no branch points and the configuration of $\Pi(C_i)$ near $\Pi(Q_1)$ and $\Pi(Q_2)$ is as shown in Fig. 2 (1). A lift $\widetilde{C_i}$ of $\Pi(C_i)'$ as shown in Fig. 2 (2) is homotopic to C_i. Replacing C_i to $\widetilde{C_i}$, we can obtain more simpler admissible system. Repeating this, we may assume that $I(Q_1) = Q_2$.

Moreover, by Lemma 2.2, we may assume that $I\left(\overrightarrow{\zeta_1}\right) \neq \overrightarrow{\zeta_2}$. Let $\overrightarrow{\zeta_1^c}$ be the subcurve of $\overrightarrow{C_i}$ satisfying $\overrightarrow{\zeta_1^c} \circ \overrightarrow{\zeta_1} = \overrightarrow{C_i}$. Since $I\left(\overrightarrow{\zeta_1}\right) \neq \overrightarrow{\zeta_2}$, we see that $\overrightarrow{\zeta_2^{-1}} = I\left(\overrightarrow{\zeta_1^c}\right)$, namely, $I\left(\overrightarrow{\zeta_1^c}\right)$ is homotopic to $\overrightarrow{\zeta_1}$. Thus, $\overrightarrow{\zeta_1^c}$ is homotopic to $I\left(\overrightarrow{\zeta_1}\right)$. Set $C_i' := I\left(\overrightarrow{\zeta_1}\right) \circ \overrightarrow{\zeta_1}$. By the argument above, we see that C_i' is homotopic to C_i and $I\left(\overrightarrow{C_i'}\right) = \overrightarrow{C_i'}$. In this case, C_i' is of Type B'. Repeating this process, we obtain a new admissible system $\{C_i'\}$ satisfying $C_i \cap I(C_i) = \emptyset$ or $C_i = I(C_i)$.

Lemma 2.4. *Let Φ be a hyperelliptic semistable element with I. Then, we can find an admissible system $\{C_i\}_{i=1,\dots,r}$ of Φ such that each C_i satisfies one of the following conditions*:
(Type A) $I\left(\overrightarrow{C_i}\right) = \overrightarrow{C_i^{-1}}$.
(Type B) $I\left(\overrightarrow{C_i}\right) = \overrightarrow{C_i}$.
(Type C) *There exists j $(\neq i)$ such that $I(C_i) = C_j$.*

Proof. Let $\{C_i'\}$ be an admissible system of Φ obtained from an admissible system by repeating the above process. For each C_i', we find a curve C_i homotopic to C_i' satisfying one of the three conditions in Lemma 2.4. In the case where $C_i' = I(C_i')$, we set $C_i := C_i'$. In the case where $C_i' \cap I(C_i') = \emptyset$, C_i' is not of Type B' because $\Pi(C_i')$ is a simple closed curve on S^2. Thus, we may assume that C_i' is of Type A or Type C.

Assume that C_i' is of Type A'. We see that $\Pi(C_i')$ rounds the two branch points of Π. Let c_i be a simple path connecting the two branch points satisfying $c_i \cap \Pi(C_i') = \emptyset$. We see that $C_i := \Pi^{-1}(c_i \circ c_i^{-1})$ is of Type A and homotopic to C_i'. Assume that C_i' is of Type C'. Let C_j' be a curve that is homotopic to $I(C_i')$. In this case, we set $C_i := C_i'$ and $C_j := I(C_i)$. We obtain an admissible system $\{C_i\}$ that we want. □

DEFINITION 2.5. An admissible system of Φ is called *simple* if each simple closed curve in the admissible system satisfies one of the conditions Type A, B, or C in Lemma 2.4.

REMARK 2.6. If C_i is of Type B, there exists a simple closed curve c on S^2 such that $c \circ c = \Pi(C_i)$. Moreover, we see that the number of the branch points of Π contained in the disk with boundary c is odd. Thus, without fear of confusions, we consider that $\Pi(C_i)$ is a simple closed curve on S^2.

We describe the configuration of the special fiber of a family of curves whose monodromy is the conjugacy class of a semistable element (cf. [3]).

Let $\{(C_i, n_i)\}_{1\leq i\leq r}$ be pairs of simple closed curves on Σ_g and positive integers. We assume that $\{C_i\}$ be a disjoint union of simple closed curves. For each C_i, we choose an open neighbourhood U_{C_i} of C_i satisfying; (I) U_{C_i} is homeomorphic to an open annulus, (II) $\overline{U_{C_i}} \bigcap \overline{U_{C_j}} = \emptyset$ $(i \neq j)$, where $\overline{U_{C_i}}$ is the closure of U_{C_i}. We denote by $\partial^1_{C_i}$ and $\partial^2_{C_i}$ the connected components of the boundary of $\Sigma_g \setminus U_{C_i}$.

Let $R(C_i)_{n_i} := L_{C_i,0} \cup L_{C_i,1} \cup \cdots \cup L_{C_i,n_i-1} \cup L_{C_i,n_i}$ be a union of two closed disks and $n_i - 1$ spheres satisfying the following;

(A) $L_{C_i,0}$ and L_{C_i,n_i} are disks with boundaries $\partial L_{C_i,0}$ and $\partial L_{C_i,n_i}$, respectively.

(B) $L_{C_i,j}$ intersects $L_{C_i,j+1}$ at a point and $L_{C_i,j} \cap L_{C_i,k} = \emptyset$ when $|j-k| > 1$.

(C) $L_{C_i,0} \cap L_{C_i,1}$ and $L_{C_i,n_i-1} \cap L_{C_i,n_i}$ are inner points of $L_{C_i,0}$ and L_{C_i,n_i}, respectively.

Identifying $\partial^1_{C_i}$ with $\partial L_{C_i,0}$ and $\partial^2_{C_i}$ with $\partial L_{C_i,n_i}$, we obtain the topological space

$$X_{\{(C_i,n_i)_{1\le i\le r}\}} := \left(\Sigma_g \setminus \bigcup U_{C_i}\right) \cup \left(\bigcup R(C_i)_{n_i}\right)$$

called the chorizo space (cf. [3]). An irreducible component of $X_{\{(C_i,n_i)_{1\le i\le r}\}}$ which is not contained in any $R(C_i)_{n_i}$ is called *a body component* of $X_{\{(C_i,n_i)_{1\le i\le r}\}}$. We call the sub chorizo space $L_{C_i,1} \cup \cdots \cup L_{C_i,n_i-1}$ of $X_{\{(C_i,n_i)_{1\le i\le r}\}}$ *the core chain at* C_i. We call a union of spheres satisfying the condition (B) *a* $\mathbf{P}^1$*-chain*. We call a point at which two components intersect *a double point*. When $g = 0$, we can also define the chorizo space, similarly. It is well-known fact that the special fiber of a family of curves is homeomorphic to $X_{\{(C_i,n_i)_{1\le i\le r}\}}$ if the monodromy of the family is the conjugacy class $[\Phi]$ of a semistable element $\Phi = D^{n_1}_{C_1} \cdots D^{n_r}_{C_r}$. Conversely, the monodromy of a family with the special fiber $X_{\{(C_i,n_i)_{1\le i\le r}\}}$ is $[\Phi]$. We set $X_{[\Phi]} := X_{\{(C_i,n_i)_{1\le i\le r}\}}$, for short.

EXAMPLE 2.7. Let $\left\{C_i, \overline{C}_j, C'_k, C''_k\right\}_{1\le i,k\le 3, 1\le j\le 5}$ be a set of simple closed curves on Σ_{12} as shown in Fig. 3. If the monodromy of a family is the conjugacy class of

$$\Phi = D_{C_1} D_{C_2} D^3_{C_3} D^4_{\overline{C}_1} D_{\overline{C}_2} D^2_{\overline{C}_3} D^2_{\overline{C}_4} D^2_{\overline{C}_5} D^3_{C'_1} D^3_{C''_1} D_{C'_2} D_{C''_2} D^2_{C'_3} D^2_{C''_3},$$

the configuration of the special fiber is as shown in Fig. 4.

3. Proof of Theorem 1.1

3.1. Canonical resolution of double covering. In this section, we review the canonical resolution for double coverings introduced by Horikawa (cf. [1]). For a positive small real number ε, we set $\Delta_\varepsilon := \{t \in \mathbf{C} \mid |t| < \varepsilon\}$ and $W_0 := \mathbf{P}^1 \times \Delta_\varepsilon$. Let $\pi_0 \colon W_0 \to \Delta_\varepsilon$ be the second projection, $\left(\widetilde{Z}_0 : \widetilde{Z}_1\right)$ a homogeneous coordinates of $\mathbf{P}^1$ and t a parameter of Δ_ε. Let $F\left(\widetilde{Z}_0, \widetilde{Z}_1, t\right) \in \mathbf{C}\left[\widetilde{Z}_0, \widetilde{Z}_1, t\right]$ be a polynomial satisfying the following conditions; (a) F is a homogeneous polynomial of degree $2g+2$ with respect to $\left(\widetilde{Z}_0 : \widetilde{Z}_1\right)$, (b) the equation $F\left(\widetilde{Z}_0, 1, t\right) = 0$ has $2g+2$ distinct roots for each $t \in \Delta_\varepsilon \setminus \{0\}$. Let B_0 and $[B_0]$ be the divisor defined by $F\left(\widetilde{Z}_0, \widetilde{Z}_1, t\right) = 0$ and the associated line bundle on W_0, respectively. By (a), $[B_0]$ is even, namely, there exists a line bundle F_0 satisfying $[B_0] \simeq F_0^{\otimes 2}$. Thus, there exists a morphism $\psi_0 \colon S_0 \to W_0$ of degree two branched along the divisor B_0. By (b), the fibers $(\pi_0 \circ \psi_0)^{-1}(t)$ $(t \in \Delta_\varepsilon \setminus \{0\})$ are smooth hyperelliptic curves. We set $\Gamma_t := \pi_0^{-1}(t)$.

We define τ_i, $\widetilde{\tau}_i$, π_i, B_i, F_i, E_i and ψ_i inductively as follows: We choose a singular point p_{i-1} of B_{i-1}. Let $\tau_i\colon W_i \to W_{i-1}$ be the blowing-up at p_{i-1}. We denote the multiplicity of B_{i-1} at p_{i-1} by $m_{p_{i-1}}$. Let E_i be the exceptional set of τ_i. We set $B_i := \tau_i^* B_{i-1} - 2[m_{p_{i-1}}/2]E_i$ and $F_i := \tau_i^* F_{i-1} - [m_{p_{i-1}}/2]E_i$, where $[m_{p_{i-1}}/2]$ is the greatest integer not exceeding $m_{p_{i-1}}/2$. Since $[B_i] \simeq F_i^{\otimes 2}$, we can take a double covering $\psi_i\colon S_i \to W_i$ branched along B_i and naturally define a bimeromorphic map $\widetilde{\tau}_i\colon S_i \to S_{i-1}$ (cf. [1, §2]). We set $\pi_i := \pi_{i-1} \circ \tau_i$. Repeating this process, we obtain a sequence of blowing-ups $W_r \stackrel{\tau_r}{\to} \cdots \to W_1 \stackrel{\tau_1}{\to} W_0$ satisfying that B_r is nonsingular. Since the set of singular points of S_i coincides with the inverse image of the set of the singular points of B_i by ψ_i, we see that S_r is nonsingular. We obtain the relatively minimal model $\phi\colon S \to \Delta_\varepsilon$ by the composite of the blowing-downs of suitable (-1)-curves successively on S_r. We call the above process *Horikawa's canonical resolution* (the canonical resolution, for short).

Note that if a component E of $(\tau_1 \circ \cdots \circ \tau_r)^*\Gamma_0$ is a component of B_r, the multiplicity of $\psi_r^*(E)$ is $2n_i$, and n_i, otherwise.

3.2. Construction of hyperelliptic families. In this section, we prove Theorem 1.1, namely, for any hyperelliptic semistable element Φ, we construct a hyperelliptic family with monodromy $[\Phi]$. Set

$$\Phi = D_{C_1}^{n_1} \cdots D_{C_k}^{n_k} D_{\overline{C}_1}^{\overline{n}_1} \cdots D_{\overline{C}_m}^{\overline{n}_m} D_{C_1'}^{n_1'} D_{I(C_1')}^{n_1'} \cdots D_{C_s'}^{n_s'} D_{I(C_s')}^{n_s'},$$

where $\mathcal{C}_A := \{C_i\}_{1 \le i \le k}$, $\mathcal{C}_B := \left\{\overline{C}_j\right\}_{1 \le j \le m}$ and $\mathcal{C}_C := \{C_l', I(C_l')\}_{1 \le l \le s}$ are the sets of simple closed curves of Type A, Type B and Type C, respectively.

Since the monodromy of a family is $[\Phi]$ if and only if the special fiber of the family is homeomorphic to $X_{[\Phi]}$, we construct a hyperelliptic family whose special fiber is $X_{[\Phi]}$. We would obtain such a family as the nonsingular minimal model of a double covering $\psi_0\colon S_0 \to W_0 := \mathbf{P}^1 \times \Delta$ introduced in Section 3.1. Our strategy is as follows;

In Step 1, we construct the chorizo spaces $\widetilde{X}_{[\Phi]}$ and $X_{[\Phi,\Pi]}$ and an involution $\tilde{I}$ on $\widetilde{X}_{[\Phi]}$. There exists a surjective map $\Pi_{[\Phi]}\colon \widetilde{X}_{[\Phi]} \to X_{[\Phi,\Pi]}$ of degree two such that $\Pi_{[\Phi]}$ is induced from the natural map $\widetilde{X}_{[\Phi]} \to \widetilde{X}_{[\Phi]}/\langle \tilde{I} \rangle$, where $\widetilde{X}_{[\Phi]}\big/\langle \tilde{I} \rangle$ is the quotient by the group $\langle \tilde{I} \rangle$ generated by $\tilde{I}$. In Step 2, we give a symbol to each irreducible component of $X_{[\Phi,\Pi]}$ and each point at which two components intersect for convenience. In Step 3, we give the defining equation of the branch locus B_0 on W_0 using the symbols defined in Step 2 and observe the canonical resolution. Let $W_r \stackrel{\tau_r}{\to} \cdots \to W_1 \stackrel{\tau_1}{\to} W_0$ be the subsequence of the blowing-ups obtained by the canonical resolution satisfying that S_r admits only rational double points of type A_n. We can easily see that $X_{[\Phi,\Pi]}$ is homeomorphic to $(\tau_r \circ \cdots \circ \tau_1)^*\Gamma_0$ and $\widetilde{X}_{[\Phi]}$ is homeomorphic to the singular fiber of S_r. Finally, we show that the special fiber of the nonsingular minimal model of S_r is homeomorphic to $X_{[\Phi]}$.

STEP 1 We first consider $\Sigma_g/\langle I\rangle \simeq S^2$ with the data $\left(\Pi(C_i), \Pi\left(\overline{C}_j\right), \Pi(C'_l), n_i, \overline{n}_j, n'_l\right)$ and the set of the branch points $\mathcal{P} = \{P_1, P_2, \dots P_{2g+2}\}$ of Π. Let $\mathcal{P}'$ be the subset of $\mathcal{P}$ such that each point in $\mathcal{P}'$ is not on $\Pi(\mathcal{C}_A)$.

Since $\mathcal{C}_A \cup \mathcal{C}_{\mathcal{B}} \cup \mathcal{C}_C$ is a simple admissible system, the set $\left\{\Pi\left(\overline{C_j}\right)\right\}_{1\le j\le m} \cup \left\{\Pi(C'_l)\right\}_{1\le l\le s}$ is the disjoint union of simple closed curves (cf. Remark 2.6). Thus, we can consider the chorizo space

$$X_{[\Phi,\Pi]} := X_{\left\{\left(\Pi\left(\overline{C}_j\right), 2\overline{n}_j\right)_{1\le j\le m}, \left(\Pi(C'_l), n'_l\right)_{1\le l\le s}\right\}}$$

defined in Section 2. With this chorizo space, we consider the data $\{(\Pi(C_i), n_i)\}_{1\le i\le k}$ and $\mathcal{P}$. For example, in the case where Φ is an element in Example 2.7, S^2 with the data is as shown in Fig. 5 and $X_{[\Phi,\Pi]}$ with the data is as shown in Fig. 6. For each C_i, we take a point on $\Pi(C_i)$ and denote it by P_{C_i}.

We use the same notations as in Section 2. For each simple closed curve C in $\mathcal{C}_A \cup \mathcal{C}_B \cup \mathcal{C}_C$, we denote a small annular open neighbourhood by U_C satisfying $I(U_C) = U_{I(C)}$ and $U_C \cap \Pi^{-1}(\mathcal{P}') = \emptyset$. We assume that they do not intersect each other. Moreover, we assume that $I\left(\partial^1_{C'_l}\right) = \partial^1_{I(C'_l)}$ and $I\left(\partial^2_{C'_l}\right) = \partial^2_{I(C'_l)}$. Let $\mathcal{U}$ be the union of all annular neighbourhoods defined as above. Set

$$\widetilde{X}_{[\Phi]} := X_{\left\{(C_i,1)_{1\le i\le k}, \left(\overline{C}_j, 2\overline{n}_j\right)_{1\le j\le m}, \left(C'_l, n'_l\right)_{1\le l\le s}, \left(I(C'_l), n'_l\right)_{1\le l\le s}\right\}} \tag{2}$$

We define an orientation preserving homeomorphism $\tilde{I}\colon \widetilde{X}_{[\Phi]} \to \widetilde{X}_{[\Phi]}$ induced from I as follows; Set $\mathcal{B}o := \Sigma_g \setminus \mathcal{U}$. We decompose $\widetilde{X}_{[\Phi]}$ as $\widetilde{X}_{[\Phi]} = \mathcal{B}o \cup \mathcal{C}h_A \cup \mathcal{C}h_B \cup \mathcal{C}h_C$, where $\mathcal{C}h_A := \bigcup R(C_i)_1$, $\mathcal{C}h_B := \bigcup R\left(\overline{C}_j\right)_{2\overline{n}_j}$, and $\mathcal{C}h_C := \bigcup\left(R(C'_l)_{n'_l} \cup R(I(C'_l))_{n'_l}\right)$. For $\mathcal{B}o$ and each member of $\mathcal{C}h_A \cup \mathcal{C}h_B \cup \mathcal{C}h_C$, we define an orientation preserving homeomorphism satisfying suitable conditions in the following way.

We can naturally define $I_{\mathcal{B}o}\colon \mathcal{B}o \to \mathcal{B}o$ by the restriction of I to $\Sigma_g \setminus \mathcal{U}$. Note that $\Pi^{-1}(\mathcal{P}')$ is the set of fixed points of $I_{\mathcal{B}o}$. Thus, we can consider that $\Pi^{-1}(\mathcal{P}')$ is the set of points on $\widetilde{X}_{[\Phi]}$.

For each $R(C_i)_1 = L_{C_i,0} \cup L_{C_i,1}$, we can define a homeomorphism $I_{C_i}\colon R(C_i)_1 \to R(C_i)_1$ of order two such that I_{C_i} coincides with $I_{\mathcal{B}o}$ on $\partial^1_{C_i} \cup \partial^2_{C_i}$ (we identify $\partial L_{C_i,0}$ with $\partial^1_{C_i}$, and $\partial L_{C_i,1}$ with $\partial^2_{C_i}$). Note that $I_{C_i}(L_{C_i,0}) = L_{C_i,1}$ and the fixed point is $L_{C_i,0} \cap L_{C_i,1}$.

For each $R\left(\overline{C}_j\right)_{2\overline{n}_j}$, we define a homeomorphism $I_{\overline{C}_j}\colon R\left(\overline{C}_j\right)_{2\overline{n}_j} \to R\left(\overline{C}_j\right)_{2\overline{n}_j}$ of order two such that $I_{\overline{C}_j}$ coincides with $I_{\mathcal{B}o}$ on $\partial^1_{\overline{C}_j} \cup \partial^2_{\overline{C}_j}$ and the fixed locus is $\bigcup_{d=1}^{\overline{n}_j} L_{\overline{C}_j,2d-1}$. Thus, the restriction map $I_{\overline{C}_j}\big|_{L_{\overline{C}_j,2d}}$ $(d = 2, \dots, \overline{n}_j - 1)$ is a homeomorphism of $L_{\overline{C}_j,2d}$ of order two with fixed points $L_{\overline{C}_j,2d} \cap L_{\overline{C}_j,2d+1}$ and $L_{\overline{C}_j,2d} \cap L_{\overline{C}_j,2d-1}$.

For $R(C'_l)_{n'_l} \cup R(I(C'_l))_{n'_l}$, we can define a homeomorphism $I_{C'_l}\colon R(C'_l)_{n'_l} \cup R(I(C'_l))_{n'_l} \to R(C'_l)_{n'_l} \cup R(I(C'_l))_{n'_l}$ of order two such that $I_{C'_l}\left(L_{C'_l,d}\right) = L_{I(C'_l),d}$ and $I_{C'_l}$ coincides with $I_{\mathcal{B}o}$ on $\partial^1_{C'_l} \cup \partial^2_{C'_l} \cup \partial^1_{I(C'_l)} \cup \partial^2_{I(C'_l)}$.

By gluing these maps, we obtain a homeomorphism $\tilde{I}$ of $\widetilde{X}_{[\Phi]}$. Since we see that $\tilde{I}$ is an involution, we can consider the quotient map $\widetilde{\Pi}\colon \widetilde{X}_{[\Phi]} \to \widetilde{X}_{[\Phi]}/\langle \tilde{I} \rangle$ of degree two. From the construction, there exists a natural homeomorphism $\Theta\colon \widetilde{X}_{[\Phi]}/\langle \tilde{I} \rangle \to X_{[\Phi,\Pi]}$ such that $\Theta\left(\widetilde{\Pi}(\Pi^{-1}(P_i))\right) = P_i$ $(P_i \in \mathcal{P}')$ and $\Theta\left(\widetilde{\Pi}(L_{C_i,0} \cap L_{C_i,1})\right) = P_{C_i}$.

Then we can consider the surjective map $\Pi_{[\Phi]}\colon \widetilde{X}_{[\Phi]} \to X_{[\Phi,\Pi]}$ of degree two such that the branch locus is $\bigcup\left(\bigcup L_{\overline{C}_j,2d-1}\right)$ and the set of isolated branch points is $\mathcal{P}' \cup \{P_{C_i}\}$. Note that $\Pi_{[\Phi]}^{-1}(P_i)$ $(P_i \in \mathcal{P}')$ is not a double point and $\Pi_{[\Phi]}^{-1}(P_{C_i})$ is a double point.

STEP 2 A component of a chorizo space which intersects only one component is called *a terminal component*. Since the dual graph of $X_{[\Phi,\Pi]}$ is a tree, there exists at least one terminal component. For later use, we give a symbol to each component of $X_{[\Phi,\Pi]}$ by the following way (see Fig. 7, for example, in Fig. 7, we give a symbol to each component of $X_{[\Phi,\Pi]}$ appearing in Example 2.7. The lines mean irreducible components of $X_{[\Phi,\Pi]}$); Choose a terminal component of $X_{[\Phi,\Pi]}$ and denote it by Z_0. If $X_{[\Phi,\Pi]}$ has another terminal component, then Z_0 intersects only one component of $X_{[\Phi,\Pi]}$. We denote it by $Z_{0,1}^1$. If there exist components of $X_{[\Phi,\Pi]} \setminus Z_0$ which intersect $Z_{0,1}^1$, choose a component among them and denote it by $Z_{0,2}^1$. Inductively, if there exist components of $X_{[\Phi,\Pi]} \setminus Z_{0,i-1}^1$ intersecting $Z_{0,i}^1$, choose such a component and denote it by $Z_{0,i+1}^1$. Finally, we obtain a $\mathbf{P}^1$-chain $Z_0 \cup Z_{0,1}^1 \cup \cdots \cup Z_{0,k_\xi}^1$ such that Z_{0,k_ξ}^1 is a terminal component of $X_{[\Phi,\Pi]}$.

Let $Z_{0,j}^1$ be a component which is not a terminal component of $X_{[\Phi,\Pi]} \setminus \left\{Z_{0,j-1}^1\right\}$. If $j = 1$, we set $Z_{0,0}^1 := Z_0$. We denote by $Z_{0,j,1}^{1,1}, Z_{0,j,1}^{1,2}, \dots, Z_{0,j,1}^{1,d}$ the components of $X_{[\Phi,\Pi]} \setminus \left\{Z_{0,j-1}^1, Z_{0,j+1}^1\right\}$ that intersect $Z_{0,j}^1$. For each $Z_{0,j,1}^{1,i}$ which is not a terminal component of $X_{[\Phi,\Pi]}$, choose a component of $X_{[\Phi,\Pi]} \setminus Z_{0,j}^1$ intersecting $Z_{0,j,1}^{1,i}$ and denote it by $Z_{0,j,2}^{1,i}$. Inductively, if $Z_{0,j,j'}^{1,i}$ is not a terminal component of $X_{[\Phi,\Pi]}$, choose a component of $X_{[\Phi,\Pi]} \setminus Z_{0,j,j'-1}^{1,i}$ intersecting $Z_{0,j,j'}^{1,i}$, and denote it by $Z_{0,j,j'+1}^{1,i}$. Finally, we obtain a $\mathbf{P}^1$-chain $Z_{0,j,1}^{1,i} \cup \cdots \cup Z_{0,j,\eta}^{1,i}$ such that $Z_{0,j,\eta}^{1,i}$ is a terminal component of $X_{[\Phi,\Pi]}$.

By the same way, we give symbols to all components of $X_{[\Phi,\Pi]}$; For simplicity, we denote a sequence $1, i_1, \dots, i_{l-1}$ by I_l and a sequence $0, j_1, \dots, j_{l-1}$ by J_l. Let $Z_{J_l,j_l}^{I_l}$ be a component which is not a terminal component of $X_{[\Phi,\Pi]} \setminus Z_{J_l,j_l-1}^{I_l}$. We denote by $Z_{J_l,j_l,1}^{I_l,1}, Z_{J_l,j_l,1}^{I_l,2}, \dots, Z_{J_l,j_l,1}^{I_l,i_l}$ the components of $X_{[\Phi,\Pi]} \setminus \left\{Z_{J_l,j_l-1}^{I_l}, Z_{J_l,j_l+1}^{I_l}\right\}$ which intersect $Z_{J_l,j_l}^{I_l}$. For each $Z_{J_l,j_l,1}^{I_l,\alpha}$, choose a subchorizo space $Z_{J_l,j_l,1}^{I_l,\alpha} \cup Z_{J_l,j_l,2}^{I_l,\alpha} \cup \cdots \cup Z_{J_l,j_l,j_{l+1}}^{I_l,\alpha}$ of $X_{[\Phi,\Pi]}$ such that $Z_{J_l,j_l,j_{l+1}}^{I_l,\alpha}$ is a terminal component of $X_{[\Phi,\Pi]}$.

We also give a symbol to each point at which two components intersect. We denote by $a_{J_l,j_l}^{I_l}$ the point at which $Z_{J_l,j_l}^{I_l}$ intersects $Z_{J_l,j_l-1}^{I_l}$ when $j_l \neq 1$. We denote by $a_{J_l,j_l,1}^{I_l,\alpha}$ the point at which $Z_{J_l,j_l,1}^{I_l,\alpha}$ intersects $Z_{J_l,j_l}^{I_l}$. We set

$$\mathcal{I}_\Phi := \left\{(I_l, J_l, j_l) \;\middle|\; a_{J_l,j_l}^{I_l} \in X_{[\Phi,\Pi]}\right\}.$$

When $\theta = (I_l, J_l, j_l) \in \mathcal{I}_\Phi$, we sometimes write a_θ and Z_θ instead of writing $a^{I_l}_{J_l, j_l}$ and $Z^{I_l}_{J_l, j_l}$, for simplicity.

STEP 3 Let $P_\xi \in \mathcal{P}'$ be a point on $Z^{1, i_1, \dots i_{l-1}}_{0, j_1, \dots, j_l}$. We define the polynomial $f_{P_\xi}\left(\widetilde{Z}_0, t, a_\theta, P_\xi\right)$ associated to P_ξ as follows;

$$f_{P_\xi}\left(\widetilde{Z}_0, t, a_\theta, P_\xi\right) := \widetilde{Z}_0 - \left(\sum_{j=1}^{j_1} a^1_{0,j} t^{j-1} + \sum_{j=1}^{j_2} a^{1,i_1}_{0,j_1,j} t^{j_1+j-1} + \cdots \right.$$
$$\left. + \sum_{j=1}^{j_l} a^{1,i_1,\dots,i_{l-1}}_{0,j_1,\dots,j_{l-1},j} t^{j_1+j_2+\cdots+j_{l-1}+j-1} + P_\xi t^{j_1+j_2+\cdots+j_l} \right).$$

If P_ξ is on Z_0, we set $f_{P_\xi} := \widetilde{Z}_0 - P_\xi$.

Let $\Pi(C_i)$ be the image of a curve of Type A by Π on $Z^{1, i_1, \dots i_{l-1}}_{0, j_1, \dots, j_l}$. We define the polynomial $g_{C_i}\left(\widetilde{Z}_0, t, a_\theta, P_{C_i}\right)$ associated to $\Pi(C_i)$ as following;

$$g_{C_i}(\widetilde{Z}_0, t, a_\theta, P_{C_i}) := \left\{ \widetilde{Z}_0 - \left(\sum_{j=1}^{j_1} a^1_{0,j} t^{j-1} + \sum_{j=1}^{j_2} a^{1,i_1}_{0,j_1,j} t^{j_1+j-1} + \cdots \right.\right.$$
$$\left.\left. + \sum_{j=1}^{j_l} a^{1,i_1,\dots,i_{l-1}}_{0,j_1,\dots,j_{l-1},j} t^{j_1+j_2+\cdots+j_{l-1}+j-1} + P_{C_i} t^{j_1+j_2+\cdots+j_l} \right)\right\}^2$$
$$- t^{n_i + 2(j_1+\cdots+j_l)}.$$

If $\Pi(C_i)$ is on Z_0, we set $g_{C_i}\left(\widetilde{Z}_0, t, P_{C_i}\right) := \left(\widetilde{Z}_0 - P_{C_i}\right)^2 - t^{n_i}$. Set

$$F\left(\widetilde{Z}_0, t, \{P_{C_i}\}, \{P_\xi\}, \{a_\theta\}\right) := \Pi_{P_\xi \in \mathcal{P}'} \Pi_{C_i \in \mathcal{C}_A} f_{P_\xi}\left(\widetilde{Z}_0, t, a_\theta, P_\xi\right) g_{C_i}\left(\widetilde{Z}_0, t, a_\theta, P_{C_i}\right).$$

Fix $\{[P_{C_i}], [P_\xi], [a_\theta]\}$ a set of mutually distinct complex non-zero numbers and consider the polynomial $F\left(\widetilde{Z}_0, t\right) := F\left(\widetilde{Z}_0, t, \{[P_{C_i}]\}, \{[P_\xi]\}, \{[a_\theta]\}\right)$. Note that the degree of $F\left(\widetilde{Z}_0, t\right)$ with respect to Z_0 is $2g + 2$. Moreover, since $\{[P_{C_i}], [P_\xi], [a_\theta]\}$ is a set of mutually distinct complex numbers, the roots of $F\left(\widetilde{Z}_0, t\right) = 0$ is mutually distinct when $t \neq 0$ and $|t|$ is sufficiently small. Let ε be the small positive real number such that the roots of $F\left(\widetilde{Z}_0, t\right) = 0$ is mutually distinct. We set $\Delta := \{t \in \mathbf{C} \mid |t| < \varepsilon\}$. Let $\widetilde{F}\left(\widetilde{Z}_0, \widetilde{Z}_1, t\right)$ be the homogeneous polynomial of degree $2g + 2$ with respect to $\left(\widetilde{Z}_0 : \widetilde{Z}_1\right)$ satisfying $\widetilde{F}\left(\widetilde{Z}_0, 1, t\right) = F\left(\widetilde{Z}_0, t\right)$.

Let $\psi_0 \colon S_0 \to W_0 := \mathbf{P}^1 \times \Delta$ be the double covering branched along B_0; $\widetilde{F}\left(\widetilde{Z}_0, \widetilde{Z}_1, t\right) = 0$, where $\left(\widetilde{Z}_0 : \widetilde{Z}_1\right)$ is a homogeneous coordinates of $\mathbf{P}^1$. Since the divisor defined by $\widetilde{Z}_1 = 0$ does not intersect B_0, it is sufficient to observe B_0 on $\widetilde{Z}_1 \neq 0$ defined by $F\left(\widetilde{Z}_0, t\right) = 0$. We observe the canonical resolution of the family $\pi_0 \circ \psi_0 \colon S_0 \to \Delta$. In the case where $X_{[\Phi,\Pi]}$ has only one component, the assertion is clear because each simple closed curve in a simple admissible system of Φ is of Type A.

Let $\tau_1 \colon W_1 \to W_0$ be the blowing-up at $\widetilde{Z}_0 - \left[a^1_{0,1}\right] = t = 0$. Let $\overline{Z}^1_{0,1}$ be the exceptional set of τ_1. We denote by $\widetilde{Z}^1_{0,1}$ an affine coordinates of the exceptional set satisfying $\widetilde{Z}_0 - \left[a^1_{0,1}\right] = t\widetilde{Z}^1_{0,1}$.

If $(1;0,2) \in \mathcal{I}_\Phi$, we blow up at $\widetilde{Z}^1_{0,1} - \left[a^1_{0,2}\right] = t = 0$ and denote by $\overline{Z}^1_{0,2}$ the exceptional set of this blowing-up. We denote by $\widetilde{Z}^1_{0,2}$ an affine coordinates satisfying $\widetilde{Z}^1_{0,1} - \left[a^1_{0,2}\right] = t\widetilde{Z}^1_{0,2}$. Similarly, if $(1,d;0,1,1) \in \mathcal{I}_\Phi$, we blow up at $\widetilde{Z}^1_{0,1} - \left[a^{1,d}_{0,1,1}\right] = t = 0$. We denote by $\overline{Z}^{1,d}_{0,1,1}$ the exceptional set of this blowing-up. We denote by $\widetilde{Z}^{1,d}_{0,1,1}$ an affine coordinates of the exceptional set satisfying $\widetilde{Z}^1_{0,1} - \left[a^{1,d}_{0,1,1}\right] = t\widetilde{Z}^{1,d}_{0,1,1}$.

Inductively, we blow up and give the symbols to the exceptional sets of the blowing-ups in the way similar to the above; If $(I_l; J_l, j_l + 1) \in \mathcal{I}_\Phi$, we blow up at $\widetilde{Z}^{I_l}_{J_l,j_l} - \left[a^{I_l}_{J_l,j_l+1}\right] = t = 0$ and denote by $\overline{Z}^{I_l}_{J_l,j_l+1}$ the exceptional set of this blowing-up. We denote by $\widetilde{Z}^{I_l}_{J_l,j_l+1}$ an affine coordinates of the exceptional set satisfying $\widetilde{Z}^{I_l}_{J_l,j_l} - \left[a^{I_l}_{J_l,j_l+1}\right] = t\widetilde{Z}^{I_l}_{J_l,j_l+1}$. If there exists $d \in \mathbf{Z}$ such that $(I_l, d; J_l, j_l, 1) \in \mathcal{I}_\Phi$, we blow up at $\widetilde{Z}^{I_l}_{J_l,j_l} - \left[a^{I_l,d}_{J_l,j_l,1}\right] = t = 0$ and give the symbol $\overline{Z}^{I_l,d}_{J_l,j_l,1}$ to the exceptional set of this blowing-up. We denote by $\widetilde{Z}^{I_l,d}_{J_l,j_l,1}$ an affine coordinates satisfying $\widetilde{Z}^{I_l}_{J_l,j_l} - \left[a^{I_l,d}_{J_l,j_l,1}\right] = t\widetilde{Z}^{I_l,d}_{J_l,j_l,1}$.

Let $W_r \stackrel{\tau_r}{\to} W_{r-1} \stackrel{\tau_{r-1}}{\to} \cdots \stackrel{\tau_1}{\to} W_0$ be the sequence of the blowing-ups obtained by the process above. Then, we obtain the chorizo space $(\tau_1 \circ \cdots \circ \tau_r)^*\Gamma_0 = \bigcup_{\theta\in\mathcal{I}_\Phi} \overline{Z}_\theta$. Here, we use the same symbol for the exceptional set of each blowing-up $\tau_{r'} \colon W_{r'} \to W_{r'-1}$ $(r' \leq r)$ and its strict transform by $\tau_{r'+1}\circ\cdots\circ\tau_r$. Note that the multiplicity of each component of $\overline{Z}_\theta$ is one. For each r', we can define the double covering $\psi_{r'} \colon S_{r'} \to W_{r'}$ branched along $B_{r'}$ and bimeromorphic map $\widetilde{\tau_{r'}} \colon S_{r'} \to S_{r'-1}$ introduced in the previous section. Since $\overline{Z}^{I_l}_{J_l,j_l+1}$ intersects $\overline{Z}^{I_l}_{J_l,j_l}$ at $\widetilde{Z}^{I_l}_{J_l,j_l} = \left[a^{I_l}_{J_l,j_l+1}\right]$ and $\overline{Z}^{I_l,\alpha}_{J_l,j_l,1}$ intersects $\overline{Z}^{I_l}_{J_l,j_l}$ at $\widetilde{Z}^{I_l}_{J_l,j_l} = \left[a^{I_l,\alpha}_{J_l,j_l,1}\right]$, there exists a natural homeomorphism between $\bigcup \overline{Z}_\theta$ to $X_{[\Phi,\Pi]}$ that sends each exceptional set $\overline{Z}_\theta$ $(\theta \in \mathcal{I}_\Phi)$ to the irreducible component Z_θ of $X_{[\Phi,\Pi]}$. Then, we can identify $X_{[\Phi,\Pi]}$ with $(\tau_1 \circ \cdots \circ \tau_r)^*\Gamma_0$. Moreover, if $P_\xi \in Z_\theta$, the strict transform of $f_{P_\xi} = 0$ on W_r intersects the exceptional set $\overline{Z}_\theta$ at $\widetilde{Z}_\theta = [P_\xi]$, transversally. Thus, we can identify the point $P_\xi \in \mathcal{P}'$ on Z_θ with the point on the component $\overline{Z}_\theta$ defined by $\widetilde{Z}_\theta = [P_\xi]$. If $\Pi(C_i) \subset Z_\theta$, the strict transform of $g_{C_i} = 0$ on W_r intersects the exceptional set $\overline{Z}_\theta$ at $\widetilde{Z}_\theta = [P_{C_i}]$. Then we identify naturally the point $P_{C_i} \in \mathcal{P}'$ on Z_θ of $X_{[\Phi,\Pi]}$ with the point on $\overline{Z}_\theta$ defined by $\widetilde{Z}_\theta - [P_{C_i}] = t = 0$.

We can easily see that the defining equation of the strict transform of $g_{C_i} = 0$ on W_r near $\widetilde{Z}_\theta = [P_{C_i}]$ is $\left(\widetilde{Z}_\theta - [P_{C_i}]\right)^2 = t^{n_i}$. Thus, if $\overline{Z}_\theta$ is not a component of B_r, the singular point on S_r over $\widetilde{Z}_\theta - [P_{C_i}] = t = 0$ is a rational double point of type A_{n_i-1}. In the proof of Claim 3.2, we show that a exceptional set corresponding to a body component of $X_{[\Phi,\Pi]}$ is not a component of B_r.

CLAIM 3.1. Let $\tau_{r'} \colon W_{r'} \to W_{r'-1}$ be the blowing-up at Q; $\widetilde{Z}_\theta - [a_{\theta'}] = t = 0$. Then, the strict transform $\widetilde{B}_{r'-1}$ of the divisor B_0 by $\overline{\tau}_{r'-1} = \tau_1 \circ \cdots \circ \tau_{r'-1}$ is singular at Q.

Proof of Claim 3.1. Note that the strict transform of $f_{P_\xi} = 0$ or $g_{C_i} = 0$ by $\overline{\tau}_{r'-1}$ contains Q, if and only if f_{P_ξ} or g_{C_i} include a monomial whose coefficient is $[a_{\theta'}]$. Assume that $\widetilde{B}_{r'-1}$ is nonsingular at $\widetilde{Z}_\theta - [a_{\theta'}] = t = 0$. Then, there exists unique irreducible component D of $\widetilde{B}_{r'-1}$ that contains Q. Let D' be the irreducible component of B_0 such that the strict transform of D' by $\widetilde{\tau}_{r'-1}$ is D. If the defining equation of D' is $g_{C_i} = 0$, we see that $\Pi(C_i)$ is on the component Z_θ of $X_{[\Phi,\Pi]}$ because if $\Pi(C_i)$ is not on Z_θ, the strict transform of D' by $\overline{\tau}_{r'-1}$ is singular. Then, the strict transform of $g_{C_i} = 0$ intersects $\overline{Z}_\theta$ at $\widetilde{Z}_\theta = [P_{C_i}]$. It contradicts that $\{[P_{C_i}], [P_\xi], [a_\theta]\}$ is a set of mutually distinct complex numbers. Thus, there exists a branch point $P_\xi \in Z_{\widetilde{\theta}}$ of Π such that the defining equation of D' is $f_{P_\xi} = 0$. If $\theta = \widetilde{\theta}$, it contradicts the fact $[P_\xi] \neq [a_\theta]$. If $\theta \neq \widetilde{\theta}$, we see that there exists no $\Pi(C_j)$ and no branch points but P_ξ on $Z_{\widetilde{\theta}}$. Moreover, we see that $Z_{\widetilde{\theta}}$ is a terminal component of $X_{[\Phi,\Pi]}$. It contradicts the assumption that $\mathcal{C}_A \cup \mathcal{C}_B \cup \mathcal{C}_C$ is an admissible system of Φ. □

CLAIM 3.2. Let $\overline{Z}_{\theta_1} \cup \cdots \cup \overline{Z}_{\theta_N}$ be a set of the exceptional sets corresponding to the core chain at the image of a curve of Type B or Type C by Π. Let $\overline{Z}_{\theta_0}$ and $\overline{Z}_{\theta_{N+1}}$ are the exceptional sets corresponding to the body components such that $\overline{Z}_{\theta_0}$ and $\overline{Z}_{\theta_{N+1}}$ intersect $\overline{Z}_{\theta_1}$ and $\overline{Z}_{\theta_N}$, respectively. Then, if $\overline{Z}_{\theta_1} \cup \cdots \cup \overline{Z}_{\theta_N}$ is a set of the exceptional sets corresponding to the core chain at the image of a curve of Type C, each $\overline{Z}_{\theta_i}$ is not a component of B_r. If $\overline{Z}_{\theta_1} \cup \cdots \cup \overline{Z}_{\theta_N}$ is corresponding to the core chain at the image of a curve of Type B by Π, then each $\overline{Z}_{\theta_i}$ is a component of B_r when i is odd and not a component of B_r when i is even.

Proof of Claim 3.2. Let $\tau_{r'} \colon W_{r'} \to W_{r'-1}$ be the blowing-up whose exceptional set is $\overline{Z}_{\theta_1}$. Without loss of generality, we can assume that $\overline{Z}_{\theta_2}, \ldots, \overline{Z}_{\theta_N}$ are not the exceptional sets of $\overline{\tau}_{r'-1}$. Thus, we can consider that $\tau_{r'}$ is the blowing-up at Q; $\widetilde{Z}_{\theta_0} - [a_{\theta_1}] = t = 0$.

Let $\{\theta'_1, \ldots \theta'_w\}$ be the subset of $\mathcal{I}_\Phi$ such that each $Z_{\theta'_i}$ is contracted to Q by $\tau_{r'} \circ \cdots \circ \tau_r$. By the definition of f_{P_ξ} and g_{C_i}, we see that each strict transform of $f_{P_\xi} = 0$ (resp. $g_{C_i} = 0$) by $\overline{\tau}_{r'-1}$ contains Q if and only if there exists θ'_i such that f_{P_ξ} (resp. g_{C_i}) includes a monomial whose coefficient is $[a_{\theta'_i}]$. The multiplicities of the strict transform of $f_{P_\xi} = 0$ and $g_{C_i} = 0$ at Q are one and two, respectively if they contain Q. Thus, the multiplicity of the strict transform $\widetilde{B}_{r'-1}$ of B_0 at Q by $\overline{\tau}_{r'-1}$ coincides with the number of the branch points of Π that are on the components $Z_{\theta'_1}, \ldots, Z_{\theta'_w}$ of $X_{[\Phi,\Pi]}$. Then, we see that the multiplicity of $\widetilde{B}_{r'-1}$ at Q is odd if $\overline{Z}_{\theta_1} \cup \cdots \cup \overline{Z}_{\theta_N}$ is the core chain at the image of a curve of Type B, and even if not. If $\overline{Z}_{\theta_0}$ is not a component of B_r, the assertion is clear because N is odd when $\overline{Z}_{\theta_1} \cup \cdots \cup \overline{Z}_{\theta_N}$ is the core chain at the image of a curve of Type B. Though, since the strict transform of Γ_0 is a component corresponding to a body component and not a component of B_r, we see that all components corresponding to body components are not components of B_r. $\square$

Let $\overline{P}_i$ and $\overline{P}_{C_i}$ be points on $(\tau_1 \circ \cdots \circ \tau_r)^*\Gamma_0$ corresponding to P_i and P_{C_i}, respectively. Let $\widetilde{r}\colon \widetilde{S}_r \to S_r$ be the minimal resolution of all singular points of type A_n on S_r and $\widetilde{S}_r \to S$ the blowing-downs of suitable (-1)-curves successively on $\widetilde{S}_r$ such that S has no (-1)-curve. Let $\widetilde{X}$ be the singular fiber of $\pi_r \circ \psi_r\colon S_r \to \Delta$. By Claim 3.2, $\psi_r|_{\widetilde{X}}\colon \widetilde{X} \to (\tau_1 \circ \cdots \circ \tau_r)^*\Gamma_0 \simeq X_{[\Phi,\Pi]}$ is a double cover branched along the components corresponding to $\bigcup\left(\bigcup L_{\overline{C}_j,2d-1}\right)$ and branched at the points corresponding to $\mathcal{P}' \cup \{P_{C_i}\}$. Moreover, $\psi_r|_{\widetilde{X}}^{-1}\left(\overline{P}_{C_i}\right)$ is a double point of $\widetilde{X}$ and $\psi_r|_{\widetilde{X}}^{-1}\left(\overline{P}_i\right)$ is nonsingular point of $\widetilde{X}$. Thus we see that $\psi_r|_{\widetilde{X}}$ satisfies the same conditions as $\Pi_{[\Phi]}$ and $\widetilde{X}$ is homeomorphic to $\widetilde{X}_{[\Phi]}$. Since $\psi_r^{-1}\left(\overline{P}_{C_i}\right)$ is a rational double point of type A_{n_i-1}, the singular fiber of $\pi_r \circ \psi_r \circ \widetilde{r}\colon \widetilde{S}_r \to \Delta$ is homeomorphic to

$$X_{\left\{(C_i,n_i)_{i\le k},\left(\overline{C}_j,2\overline{n}_j\right)_{j\le m},(C'_l,n_l)_{l\le s},(I(C'_l),n_l)_{l\le s}\right\}}$$

because $\widetilde{X}_{[\Phi]}$ is given by (2).

By the proof of Claim 3.2, we see that $\overline{Z}_\theta$ is a component of B_r if and only if $\overline{Z}_\theta$ corresponds to a component of $\bigcup\left(\bigcup L_{\overline{C}_j,2d-1}\right)$. Since the multiplicity of $\psi_r^*\left(\overline{Z}_\theta\right)$ is two when $\overline{Z}_\theta \subset B_r$, $\psi_r^*\left(\overline{Z}_\theta\right)$ is a (-1)-curve. Moreover, we see that $\psi_r^*\left(\overline{Z}_\theta\right)$ is not a (-1)-curve when $\overline{Z}_\theta$ does not correspond to a component of $\bigcup\left(\bigcup L_{\overline{C}_j,2d-1}\right)$ by Claim 3.1. Thus, we see that the special fiber of $\phi\colon S \to \Delta$ is homeomorphic to $X_{[\Phi]}$. We complete the proof of Theorem 1.1. $\square$

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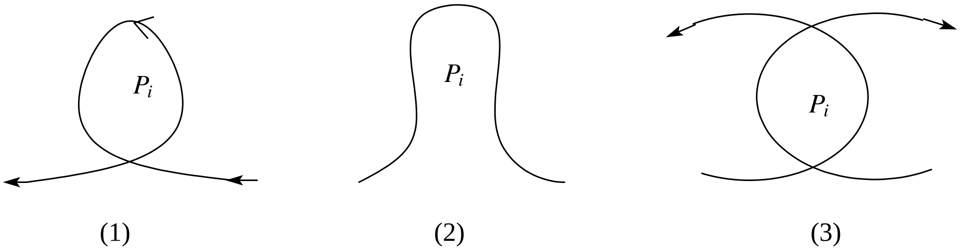


Fig. 1.

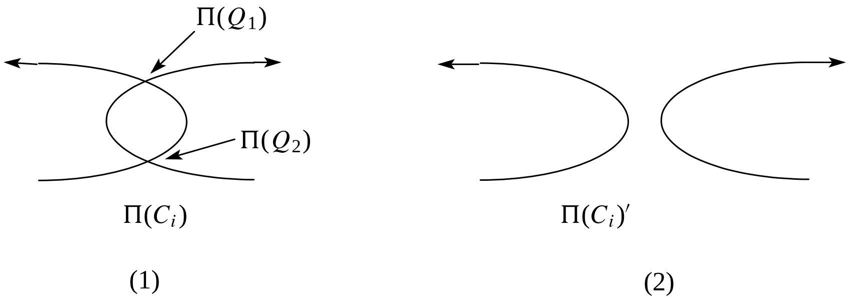


Fig. 2.

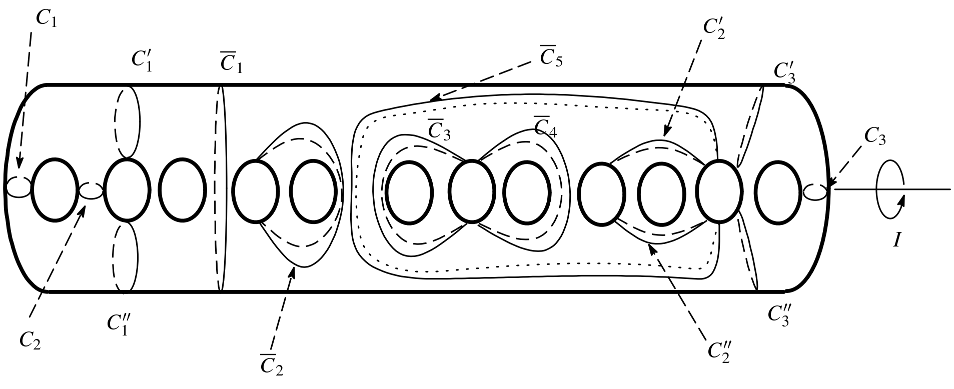
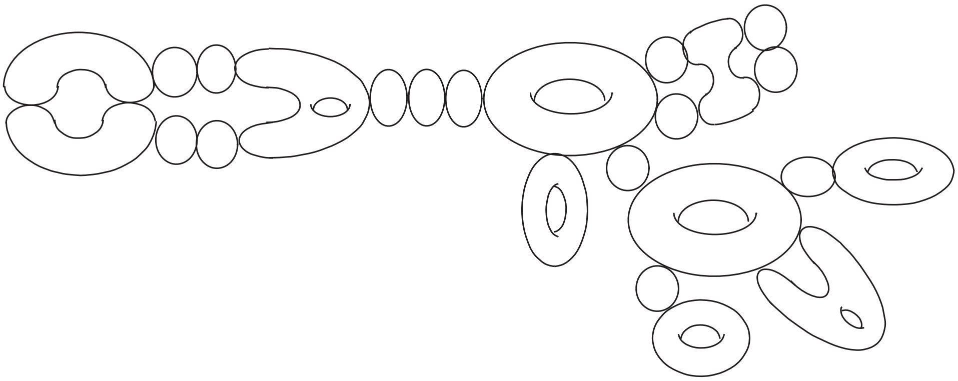


Fig. 3.



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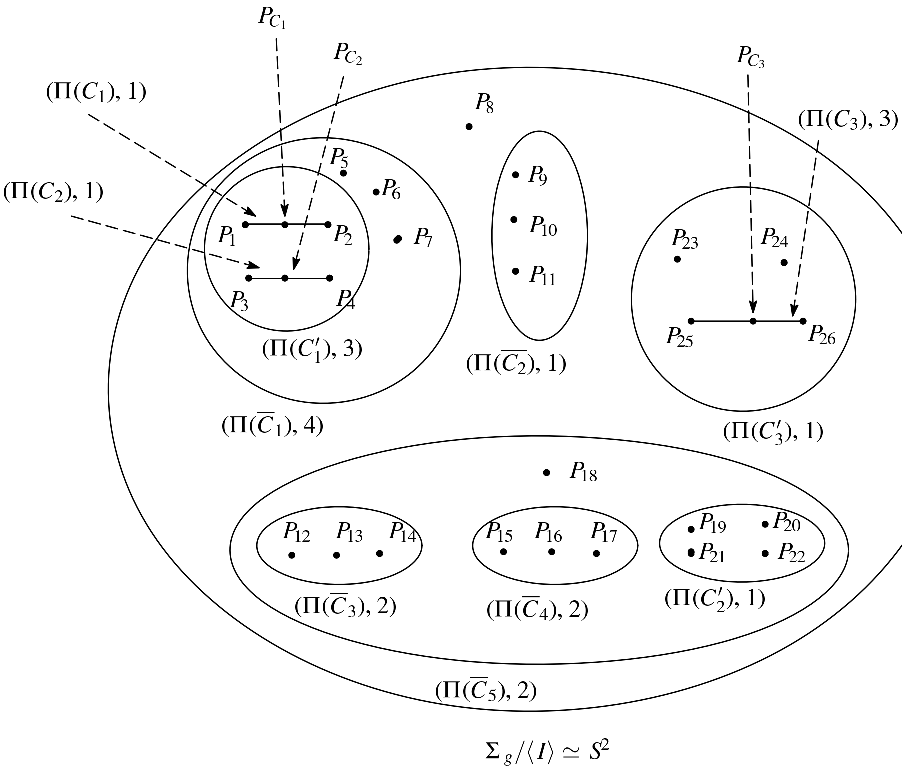
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P_{C_3}
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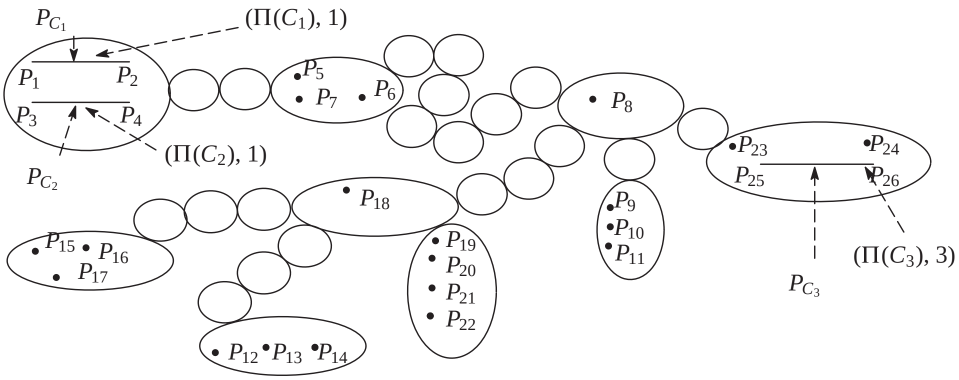
$(\Pi(C_2), 1)$
$(\Pi(C_3), 3)$
P_1
P_2
P_3
P_4
P_5
P_6
P_7
P_8
P_9
P_{10}
P_{11}
P_{12}
P_{13}
P_{14}
P_{15}
P_{16}
P_{17}
P_{18}
P_{19}
P_{20}
P_{21}
P_{22}
P_{23}
P_{24}
P_{25}
P_{26}
$(\Pi(C_1'), 3)$
$(\Pi(\overline{C_2}), 1)$
$(\Pi(\overline{C}_1), 4)$
$(\Pi(C_3'), 1)$
$(\Pi(\overline{C}_3), 2)$
$(\Pi(\overline{C}_4), 2)$
$(\Pi(C_2'), 1)$
$(\Pi(\overline{C}_5), 2)$
$\Sigma_g/\langle I\rangle \simeq S^2$

Fig. 5.



P_{C_1}
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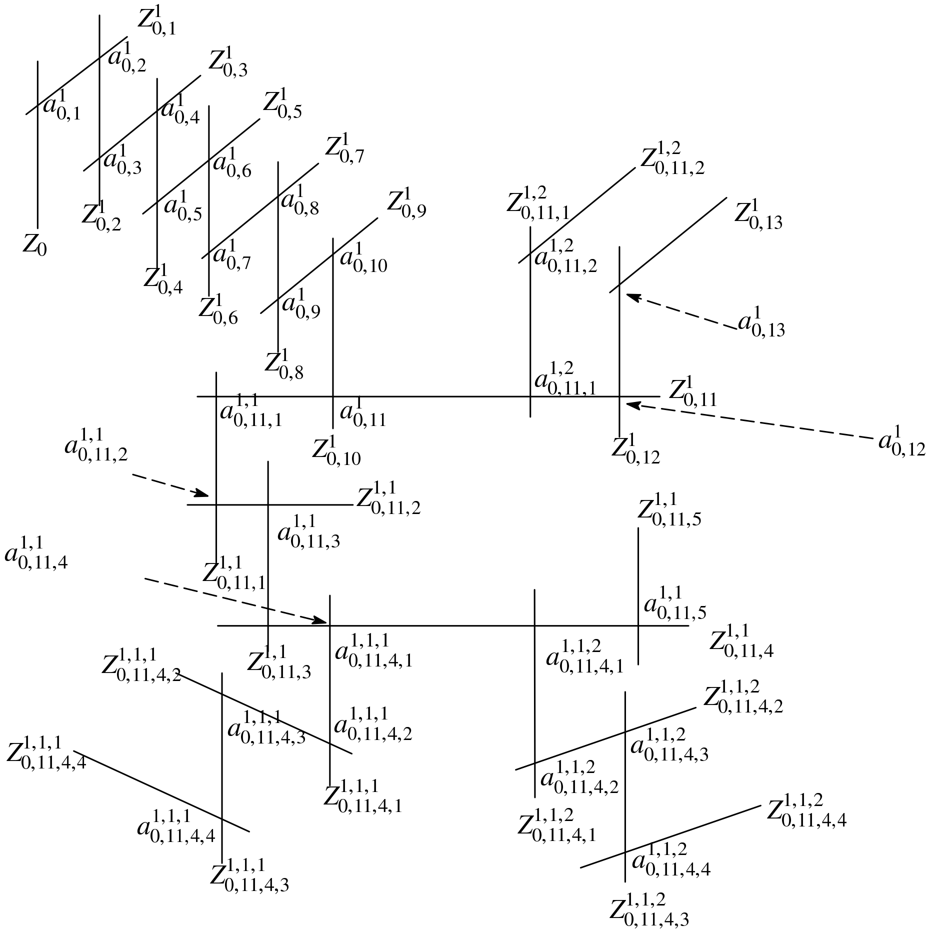
P_1
P_2
P_3
P_4
$(\Pi(C_2), 1)$
P_{C_2}
P_5
P_7
P_6
P_8
P_{23}
P_{24}
P_{25}
P_{26}
$(\Pi(C_3), 3)$
P_{C_3}
P_9
P_{10}
P_{11}
P_{18}
P_{15}
P_{16}
P_{17}
P_{19}
P_{20}
P_{21}
P_{22}
P_{12} P_{13} P_{14}

Fig. 6.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$a^1_{0,6}$
$Z^1_{0,7}$
$Z^1_{0,2}$
$a^1_{0,5}$
$a^1_{0,8}$
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$a^1_{0,7}$
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$Z^{1,1,2}_{0,11,4,3}$

Fig. 7.

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Department of Mathematics
School of Science and Engineering
Waseda University
3-4-1, Okubo Shinjuku
Tokyo 169 8050, Japan
e-mail: ishizakamizuho@aoni.waseda.jp