

| Title        | Realization of hyperelliptic families with the hyperelliptic semistable monodromies |
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| Author(s)    | Ishizaka, Mizuho  |
| Citation     | Osaka Journal of Mathematics. 2006, 43(1), p.<br>103–119                            |
| Version Type | VoR   |
| URL          | https://doi.org/10.18910/8801   |
| rights       |   |
| Note         |   |

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# REALIZATION OF HYPERELLIPTIC FAMILIES WITH THE HYPERELLIPTIC SEMISTABLE MONODROMIES

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(Received June 28, 2004, revised April 20, 2005)

### Abstract

Let  $\Phi$  be an element of the mapping class group  $\mathcal{M}_g$  of genus  $g \geq 2$  such that  $\Phi$  is the isotopy class of a pseudo periodic map of negative twists. It is expected that, for each  $\Phi$  which commutes with a hyperelliptic involution, there exists a hyperelliptic family whose monodromy is the conjugacy class of  $\Phi$  in the mapping class group. In this paper, we give a partial solution for the conjecture in the case where  $\Phi$  is a semistable element.

# 1. Introduction

Let  $\phi: S \to \Delta$  be a proper surjective holomorphic map from a nonsingular complex surface *S* to a small disk  $\Delta := \{t \in \mathbb{C} \mid |t| < \varepsilon\}$  such that  $\phi^{-1}(t)$  is a nonsingular curve of genus  $g \ge 2$  for each  $t \in \Delta^* := \Delta \setminus \{0\}$ . We call  $(\phi, S, \Delta)$  a degeneration of curves or a family of curves of genus g. If all  $\phi^{-1}(t)$   $(t \in \Delta^*)$  are hyperelliptic curves, we call  $(\phi, S, \Delta)$  a hyperelliptic family. We call  $\phi^{-1}(0)$  the special fiber of S. Two degenerations  $(\phi, S, \Delta)$  and  $(\phi', S', \Delta')$  are said to be topologically equivalent if there exist orientation preserving homeomorphisms  $\psi: S \to S'$  and  $\overline{\psi}: \Delta \to \Delta'$  which satisfy  $\phi' \circ \psi = \overline{\psi} \circ \phi$ . For a topological equivalence class of a degeneration, we can uniquely determine the topological monodromy (called the monodromy, for short) as the conjugacy class of the isotopy class of a pseudo periodic map of negative twists in the mapping class group  $\mathcal{M}_g$  of genus g (cf. [3]).

Let  $\Sigma_g$  be a compact real surface of genus g without boundary. It is well-known fact that  $\mathcal{M}_g$  is generated by Dehn twists at simple closed curves on  $\Sigma_g$  (cf. [2]). We denote by  $D_{C_i}^{n_i}$  the  $n_i$ -times right hand Dehn twists at a simple closed curve  $C_i$  on  $\Sigma_g$ . An involution I of  $\Sigma_g$  is called a hyperelliptic involution if it has 2g + 2 fixed points. We call  $\Phi$  a hyperelliptic element with I if there exists a homeomorphism  $\tilde{\Phi}$  whose isotopy class is  $\Phi$  satisfying  $I \circ \tilde{\Phi} = \tilde{\Phi} \circ I$  as map. We denote by  $[\Phi]$  the conjugacy class of  $\Phi$  in  $\mathcal{M}_g$ . An element  $\Phi$  of  $\mathcal{M}_g$  is called *semistable* if there exists a disjoint union of simple closed curves  $\mathcal{C} := \{C_i\}_{i=1,2,...,r}$  and positive integers  $\{n_i\}_{i=1,2,...,r}$  satisfying  $\Phi = D_{C_1}^{n_1} \cdots D_{C_r}^{n_r}$ . We call  $\mathcal{C}$  an admissible system of  $\Phi$  if any two simple closed curves are not homotopic to each other.

<sup>2000</sup> Mathematics Subject Classification. Primary 14D06; Secondary 14H45, 14H15, 57M99, 30F99.

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Let  $\Phi$  be the isotopy class of a pseudo-periodic map of negative twists. It is expected that if  $\Phi$  is a hyperelliptic element, there exists a hyperelliptic family of curves of genus g with monodromy [ $\Phi$ ]. In this paper, we give a partial solution for the conjecture in the case where  $\Phi$  is a semistable element. Thus, our main theorem is the following;

**Theorem 1.1.** Let  $\Phi$  be a hyperelliptic semistable element. Then, there exists a hyperelliptic family with monodromy  $[\Phi]$ .

To prove Theorem 1.1, for each  $\Phi$ , we construct a hyperelliptic family using a double covering of  $\mathbf{P}^1 \times \Delta$ . Thus, for each hyperelliptic semistable element  $\Phi$ , we give not only the proof of the existence but also an algorithm to construct a hyperelliptic family with monodromy  $[\Phi]$ .

# 2. Hyperelliptic semistable monodromy

Let  $\langle I \rangle$  be the cyclic group generated by a hyperelliptic involution *I*. We denote by  $\Pi: \Sigma_g \to \Sigma_g / \langle I \rangle \simeq S^2$  the canonical projection from  $\Sigma_g$  to the quotient of  $\Sigma_g$  by  $\langle I \rangle$ . Let  $\mathcal{P} := \{P_1, \ldots, P_{2g+2}\}$  be the set of the branch points of  $\Pi$ . To prove the main proposition (Lemma 2.4) in this section, we need to observe simple closed curves on  $\Sigma_g$  and their images on  $\Sigma_g / \langle I \rangle$  by  $\Pi$ .

**Lemma 2.1.** Let  $\Phi = D_{C_1}^{n_1} \cdots D_{C_r}^{n_r}$  be a hyperelliptic semistable element with I, where  $\{C_i\}_{i=1,...,r}$  is an admissible system of  $\Phi$ . Then, for each i, there exists j  $(1 \le j \le r)$  such that  $I(C_i)$  is homotopic to  $C_j$  with  $n_i = n_j$ .

Proof. Let  $\widetilde{D}_C$  and  $\widetilde{D}_{I(C)}$  be homeomorphisms whose isotopy classes are  $D_C$  and  $D_{I(C)}$ , respectively. Since I is homeomorphism of  $\Sigma_g$ , we see that  $I \circ \widetilde{D}_C$  is isotopic to  $\widetilde{D}_{I(C)} \circ I$  (cf. [2], Lemma 1). Since  $\Phi$  is a hyperelliptic element, we obtain

(1) 
$$D_{I(C_1)}^{n_1} \cdots D_{I(C_r)}^{n_r} = D_{C_1}^{n_1} \cdots D_{C_r}^{n_r}.$$

Let  $U_{C_i}$  be a small annular open neighbourhood of  $C_i$  such that  $U_{C_i} \cap U_{C_j} = \emptyset$  for all i, j. Note that there exists a homeomorphism whose isotopy class is  $\Phi$  such that the restriction to  $\Sigma_g \setminus \{\bigcup U_{C_i}\}$  is the identity. Thus from the equation (1), we see that each  $I(C_i)$  does not intersect properly some  $C_j$ . So,  $I(C_i)$  is homotopic to some  $C_j$  with  $n_i = n_j$  or  $I(C_i)$  is homotopic to a curve contained in  $\Sigma_g \setminus \{\bigcup U_{C_i}\}$ . In the latter case, the restriction map of any homeomorphisms in the isotopy class of  $D_{I(C_1)}^{n_1} \cdots D_{I(C_r)}^{n_r}$  to  $\Sigma_g \setminus \{\bigcup U_{C_i}\}$  are not identity, a contradiction.

Let  $c: [0, 1] \to \Sigma_g$  be a simple closed curve on  $\Sigma_g$  such that the curve  $\Pi \circ c$  is the composite  $c_2 \circ c_1$  of curves  $c_1$  and  $c_2$ , where  $c_1: [0, 1] \to S^2$  is a simple closed curve rounding only one branch point  $P_i$ . Taking another curve homotopic to c if necessary,

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we may assume that  $\Pi \circ c$  intersects itself transversally at the initial point of  $c_1$ . We also assume that any branch points are not on  $\Pi \circ c$ . We denote by  $D_i$  the disk with boundary  $c_1$  that contains  $P_i$  in its inside.

**Lemma 2.2.** Notation is as above. There exists a simple closed curve  $\tilde{c}$  homotopic to c satisfying  $\Pi \circ \tilde{c} = c_2 \circ c_1^{-1}$ . Then, any simple closed curve on  $\Sigma_g$  is homotopic to a lift of a curve on  $S^2$  which has no subloop rounding only one branch point of  $\Pi$ .

Proof. Let  $c_1$  and  $c_2$  be curves on  $\Sigma_g$  such that  $\Pi \circ c_i = c_i$  (i = 1, 2) and the composite  $c_2 \circ c_1$  is c. Since  $c_1$  rounds only one branch point  $P_i$ ,  $\Pi^{-1}(D_i)$  is a disk on  $\Sigma_g$ . Thus, there exists a curve  $\tilde{c_1}$  on  $\Sigma_g$  such that  $\Pi \circ \tilde{c_1} = c_1$  and the composite  $\overline{c_1} \circ \tilde{c_1}$  is the boundary of  $\Pi^{-1}(D_i)$ . Then, we see that the curve  $\tilde{c} := c_2 \circ \tilde{c_1}^{-1}$  is homotopic to c and  $\Pi \circ \tilde{c} = c_2 \circ c_1^{-1}$ . Since the configuration of  $\Pi \circ \tilde{c}$  near  $P_i$  is as shown in Fig. 1 (1), we can find a curve on  $S^2$  homotopic to  $c_2 \circ c_1^{-1}$  such that it does not have a subloop rounding only  $P_i$  (see, Fig. 1 (2)).

REMARK 2.3. Assume that c is a curve on  $\Sigma_g$  such that the configuration of  $\Pi \circ c$  near  $P_i$  is as shown in Fig. 1 (3). Then c is not a simple closed curve on  $\Sigma_g$ .

Let  $\Phi$  be a hyperelliptic semistable element with *I*. For each simple closed curve  $C_i$  in an admissible system of  $\Phi$ , we denote it by  $\overrightarrow{C_i}$  when we emphasize its orientation. By Lemma 2.1, we can classify the curves in an admissible system into the following three types;

(Type A')  $I\left(\overrightarrow{C_i}\right)$  is homotopic to  $\overrightarrow{C_i^{-1}}$ . (Type B')  $I\left(\overrightarrow{C_i}\right)$  is homotopic to  $\overrightarrow{C_i}$ .

(Type C') There exists  $j \ (\neq i)$  such that  $I\left(\overrightarrow{C_i}\right)$  is homotopic to  $\overrightarrow{C_j}$  or  $\overrightarrow{C_j^{-1}}$ .

We consider the case where  $I(C_i) \neq C_i$  and  $I(C_i) \cap C_i \neq \emptyset$ . We may assume that  $I(C_i)$  intersects  $C_i$  transversally and there exist no branch points of  $\Pi$  on  $\Pi(C_i)$ . Moreover, by Lemma 2.2, we may assume that  $\Pi(C_i)$  does not have a subloop rounding only one branch point of  $\Pi$ . Let  $\vec{\zeta}_1$  be an oriented subcurve of  $\vec{C}_i$  such that  $\vec{\zeta}_1 \cap I\left(\vec{C}_i\right) = \{Q_1, Q_2\}$ , where  $Q_1$  and  $Q_2$  are the initial and end points of  $\vec{\zeta}_1$ , respectively. We assume that there exists a subcurve  $\vec{\zeta}_2$  of  $I(C_i)$  (or  $I(C_i)^{-1}$ ) such that the composite  $\vec{\zeta}_2 \circ \vec{\zeta}_1$  is homotopic to zero and  $\vec{\zeta}_1 \cap \vec{\zeta}_2 = \{Q_1, Q_2\}$ . We denote by  $\mathcal{D}_{Q_1Q_2}$ the disk on  $\Sigma_g$  with boundary  $\vec{\zeta}_2 \circ \vec{\zeta}_1$ . If  $I(Q_1) \neq Q_2$ , then  $I(\mathcal{D}_{Q_1Q_2}) \cap \mathcal{D}_{Q_1Q_2} = \emptyset$ . Thus, we see that  $\Pi(\mathcal{D}_{Q_1Q_2})$  is a disk on  $S^2$  containing no branch points and the configuration of  $\Pi(C_i)$  near  $\Pi(Q_1)$  and  $\Pi(Q_2)$  is as shown in Fig. 2 (1). A lift  $\tilde{C}_i$  of  $\Pi(C_i)'$  as shown in Fig. 2 (2) is homotopic to  $C_i$ . Replacing  $C_i$  to  $\tilde{C}_i$ , we can obtain more simpler admissible system. Repeating this, we may assume that  $I(Q_1) = Q_2$ . Moreover, by Lemma 2.2, we may assume that  $I\left(\overrightarrow{\zeta_1}\right) \neq \overrightarrow{\zeta_2}$ . Let  $\overrightarrow{\zeta_1^c}$  be the subcurve of  $\overrightarrow{C_i}$  satisfying  $\overrightarrow{\zeta_1^c} \circ \overrightarrow{\zeta_1} = \overrightarrow{C_i}$ . Since  $I\left(\overrightarrow{\zeta_1}\right) \neq \overrightarrow{\zeta_2}$ , we see that  $\overrightarrow{\zeta_2^{-1}} = I\left(\overrightarrow{\zeta_1^c}\right)$ , namely,  $I\left(\overrightarrow{\zeta_1^c}\right)$  is homotopic to  $\overrightarrow{\zeta_1}$ . Thus,  $\overrightarrow{\zeta_1^c}$  is homotopic to  $I\left(\overrightarrow{\zeta_1}\right)$ . Set  $C'_i := I\left(\overrightarrow{\zeta_1}\right) \circ \overrightarrow{\zeta_1}$ . By the argument above, we see that  $C'_i$  is homotopic to  $C_i$  and  $I\left(\overrightarrow{C'_i}\right) = \overrightarrow{C'_i}$ . In this case,  $C'_i$  is of Type B'. Repeating this process, we obtain a new admissible system  $\{C'_i\}$ satisfying  $C_i \cap I(C_i) = \emptyset$  or  $C_i = I(C_i)$ .

**Lemma 2.4.** Let  $\Phi$  be a hyperelliptic semistable element with *I*. Then, we can find an admissible system  $\{C_i\}_{i=1,...,r}$  of  $\Phi$  such that each  $C_i$  satisfies one of the following conditions:

(Type A)  $I\left(\overrightarrow{C_i}\right) = \overrightarrow{C_i^{-1}}$ . (Type B)  $I\left(\overrightarrow{C_i}\right) = \overrightarrow{C_i}$ . (Type C) There exists  $j \ (\neq i)$  such that  $I(C_i) = C_j$ .

Proof. Let  $\{C'_i\}$  be an admissible system of  $\Phi$  obtained from an admissible system by repeating the above process. For each  $C'_i$ , we find a curve  $C_i$  homotopic to  $C'_i$  satisfying one of the three conditions in Lemma 2.4. In the case where  $C'_i = I(C'_i)$ , we set  $C_i := C'_i$ . In the case where  $C'_i \cap I(C'_i) = \emptyset$ ,  $C'_i$  is not of Type B' because  $\Pi(C'_i)$  is a simple closed curve on  $S^2$ . Thus, we may assume that  $C'_i$  is of Type A or Type C.

Assume that  $C'_i$  is of Type A'. We see that  $\Pi(C'_i)$  rounds the two branch points of  $\Pi$ . Let  $c_i$  be a simple path connecting the two branch points satisfying  $c_i \cap \Pi(C'_i) = \emptyset$ . We see that  $C_i := \Pi^{-1}(c_i \circ c_i^{-1})$  is of Type A and homotopic to  $C'_i$ . Assume that  $C'_i$  is of Type C'. Let  $C'_j$  be a curve that is homotopic to  $I(C'_i)$ . In this case, we set  $C_i := C'_i$  and  $C_j := I(C_i)$ . We obtain an admissible system  $\{C_i\}$  that we want.

DEFINITION 2.5. An admissible system of  $\Phi$  is called *simple* if each simple closed curve in the admissible system satisfies one of the conditions Type A, B, or C in Lemma 2.4.

REMARK 2.6. If  $C_i$  is of Type B, there exists a simple closed curve c on  $S^2$  such that  $c \circ c = \Pi(C_i)$ . Moreover, we see that the number of the branch points of  $\Pi$  contained in the disk with boundary c is odd. Thus, without fear of confusions, we consider that  $\Pi(C_i)$  is a simple closed curve on  $S^2$ .

We describe the configuration of the special fiber of a family of curves whose monodromy is the conjugacy class of a semistable element (cf. [3]).

Let  $\{(C_i, n_i)\}_{1 \le i \le r}$  be pairs of simple closed curves on  $\Sigma_g$  and positive integers. We assume that  $\{C_i\}$  be a disjoint union of simple closed curves. For each  $C_i$ , we choose an open neighbourhood  $U_{C_i}$  of  $C_i$  satisfying; (I)  $U_{C_i}$  is homeomorphic to an open annulus, (II)  $\overline{U_{C_i}} \cap \overline{U_{C_j}} = \emptyset$   $(i \neq j)$ , where  $\overline{U_{C_i}}$  is the closure of  $U_{C_i}$ . We denote by  $\partial_{C_i}^1$  and  $\partial_{C_i}^2$  the connected components of the boundary of  $\Sigma_g \setminus U_{C_i}$ .

Let  $R(C_i)_{n_i} := L_{C_i,0} \cup L_{C_i,1} \cup \cdots \cup L_{C_i,n_i-1} \cup L_{C_i,n_i}$  be a union of two closed disks and  $n_i - 1$  spheres satisfying the following;

- (A)  $L_{C_i,0}$  and  $L_{C_i,n_i}$  are disks with boundaries  $\partial L_{C_i,0}$  and  $\partial L_{C_i,n_i}$ , respectively.
- (B)  $L_{C_{i,j}}$  intersects  $L_{C_{i,j+1}}$  at a point and  $L_{C_{i,j}} \cap L_{C_{i,k}} = \emptyset$  when |j k| > 1.
- (C)  $L_{C_i,0} \cap L_{C_i,1}$  and  $L_{C_i,n_i-1} \cap L_{C_i,n_i}$  are inner points of  $L_{C_i,0}$  and  $L_{C_i,n_i}$ , respectively. Identifying  $\partial_{C_i}^1$  with  $\partial L_{C_i,0}$  and  $\partial_{C_i}^2$  with  $\partial L_{C_i,n_i}$ , we obtain the topological space

$$X_{\{(C_i,n_i)_{1\leq i\leq r}\}} := \left(\Sigma_g \setminus \bigcup U_{C_i}\right) \cup \left(\bigcup R(C_i)_{n_i}\right)$$

called the chorizo space (cf. [3]). An irreducible component of  $X_{\{(C_i,n_i)_{1\leq i\leq r}\}}$  which is not contained in any  $R(C_i)_{n_i}$  is called *a body component* of  $X_{\{(C_i,n_i)_{1\leq i\leq r}\}}$ . We call the sub chorizo space  $L_{C_i,1} \cup \cdots \cup L_{C_i,n_i-1}$  of  $X_{\{(C_i,n_i)_{1\leq i\leq r}\}}$  the core chain at  $C_i$ . We call a union of spheres satisfying the condition (B) a  $\mathbf{P}^1$ -chain. We call a point at which two components intersect *a double point*. When g = 0, we can also define the chorizo space, similarly. It is well-known fact that the special fiber of a family of curves is homeomorphic to  $X_{\{(C_i,n_i)_{1\leq i\leq r}\}}$  if the monodromy of the family is the conjugacy class  $[\Phi]$  of a semistable element  $\Phi = D_{C_1}^{n_1} \cdots D_{C_r}^{n_r}$ . Conversely, the monodromy of a family with the special fiber  $X_{\{(C_i,n_i)_{1\leq i\leq r}\}}$  is  $[\Phi]$ . We set  $X_{\{\Phi\}} := X_{\{(C_i,n_i)_{1\leq i\leq r}\}}$ , for short.

EXAMPLE 2.7. Let  $\{C_i, \overline{C}_j, C'_k, C''_k\}_{1 \le i,k \le 3, 1 \le j \le 5}$  be a set of simple closed curves on  $\Sigma_{12}$  as shown in Fig. 3. If the monodromy of a family is the conjugacy class of

$$\Phi = D_{C_1} D_{C_2} D_{C_3}^3 D_{\overline{C}_1}^4 D_{\overline{C}_2} D_{\overline{C}_3}^2 D_{\overline{C}_4}^2 D_{\overline{C}_5}^2 D_{C_1}^3 D_{C_1''}^3 D_{C_2''} D_{C_2''}^2 D_{C_3''}^2 D_{C_3''}^2,$$

the configuration of the special fiber is as shown in Fig. 4.

## 3. Proof of Theorem 1.1

**3.1.** Canonical resolution of double covering. In this section, we review the canonical resolution for double coverings introduced by Horikawa (cf. [1]). For a positive small real number  $\varepsilon$ , we set  $\Delta_{\varepsilon} := \{t \in \mathbb{C} \mid |t| < \varepsilon\}$  and  $W_0 := \mathbb{P}^1 \times \Delta_{\varepsilon}$ . Let  $\pi_0: W_0 \to \Delta_{\varepsilon}$  be the second projection,  $(\widetilde{Z}_0: \widetilde{Z}_1)$  a homogeneous coordinates of  $\mathbb{P}^1$  and t a parameter of  $\Delta_{\varepsilon}$ . Let  $F(\widetilde{Z}_0, \widetilde{Z}_1, t) \in \mathbb{C}[\widetilde{Z}_0, \widetilde{Z}_1, t]$  be a polynomial satisfying the following conditions; (a) F is a homogeneous polynomial of degree 2g + 2 with respect to  $(\widetilde{Z}_0: \widetilde{Z}_1)$ , (b) the equation  $F(\widetilde{Z}_0, 1, t) = 0$  has 2g + 2 distinct roots for each  $t \in \Delta_{\varepsilon} \setminus \{0\}$ . Let  $B_0$  and  $[B_0]$  be the divisor defined by  $F(\widetilde{Z}_0, \widetilde{Z}_1, t) = 0$  and the associated line bundle on  $W_0$ , respectively. By (a),  $[B_0]$  is even, namely, there exists a line bundle  $F_0$  satisfying  $[B_0] \simeq F_0^{\otimes 2}$ . Thus, there exists a morphism  $\psi_0: S_0 \to W_0$  of degree two branched along the divisor  $B_0$ . By (b), the fibers  $(\pi_0 \circ \psi_0)^{-1}(t)$  ( $t \in \Delta_{\varepsilon} \setminus \{0\}$ ) are smooth hyperelliptic curves. We set  $\Gamma_t := \pi_0^{-1}(t)$ .

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We define  $\tau_i$ ,  $\tilde{\tau}_i$ ,  $\pi_i$ ,  $B_i$ ,  $F_i$ ,  $E_i$  and  $\psi_i$  inductively as follows: We choose a singular point  $p_{i-1}$  of  $B_{i-1}$ . Let  $\tau_i: W_i \to W_{i-1}$  be the blowing-up at  $p_{i-1}$ . We denote the multiplicity of  $B_{i-1}$  at  $p_{i-1}$  by  $m_{p_{i-1}}$ . Let  $E_i$  be the exceptional set of  $\tau_i$ . We set  $B_i := \tau_i^* B_{i-1} - 2[m_{p_{i-1}}/2]E_i$  and  $F_i := \tau_i^* F_{i-1} - [m_{p_{i-1}}/2]E_i$ , where  $[m_{p_{i-1}}/2]$  is the greatest integer not exceeding  $m_{p_{i-1}}/2$ . Since  $[B_i] \simeq F_i^{\otimes 2}$ , we can take a double covering  $\psi_i: S_i \to W_i$  branched along  $B_i$  and naturally define a bimeromorphic map  $\tilde{\tau}_i: S_i \to S_{i-1}$  (cf. [1, §2]). We set  $\pi_i := \pi_{i-1} \circ \tau_i$ . Repeating this process, we obtain a sequence of blowing-ups  $W_r \xrightarrow{\tau_r} \cdots \to W_1 \xrightarrow{\tau_1} W_0$  satisfying that  $B_r$  is nonsingular. Since the set of singular points of  $S_i$  coincides with the inverse image of the set of the singular points of  $B_i$  by  $\psi_i$ , we see that  $S_r$  is nonsingular. We obtain the relatively minimal model  $\phi: S \to \Delta_{\varepsilon}$  by the composite of the blowing-downs of suitable (-1)-curves successively on  $S_r$ . We call the above process *Horikawa's canonical resolution* (the canonical resolution, for short).

Note that if a component E of  $(\tau_1 \circ \cdots \circ \tau_r)^* \Gamma_0$  is a component of  $B_r$ , the multiplicity of  $\psi_r^*(E)$  is  $2n_i$ , and  $n_i$ , otherwise.

**3.2.** Construction of hyperelliptic families. In this section, we prove Theorem 1.1, namely, for any hyperelliptic semistable element  $\Phi$ , we construct a hyperelliptic family with monodromy  $[\Phi]$ . Set

$$\Phi = D_{C_1}^{n_1} \cdots D_{C_k}^{n_k} D_{\overline{C}_1}^{\overline{n}_1} \cdots D_{\overline{C}_m}^{\overline{n}_m} D_{C_1}^{n_1'} D_{I(C_1')}^{n_1'} \cdots D_{C_s'}^{n_s'} D_{I(C_s')}^{n_s'},$$

where  $C_A := \{C_i\}_{1 \le i \le k}$ ,  $C_B := \{\overline{C}_j\}_{1 \le j \le m}$  and  $C_C := \{C'_l, I(C'_l)\}_{1 \le l \le s}$  are the sets of simple closed curves of Type A, Type B and Type C, respectively.

Since the monodromy of a family is  $[\Phi]$  if and only if the special fiber of the family is homeomorphic to  $X_{[\Phi]}$ , we construct a hyperelliptic family whose special fiber is  $X_{[\Phi]}$ . We would obtain such a family as the nonsingular minimal model of a double covering  $\psi_0: S_0 \to W_0 := \mathbf{P}^1 \times \Delta$  introduced in Section 3.1. Our strategy is as follows;

In Step 1, we construct the chorizo spaces  $\widetilde{X}_{[\Phi]}$  and  $X_{[\Phi,\Pi]}$  and an involution  $\widetilde{I}$  on  $\widetilde{X}_{[\Phi]}$ . There exists a surjective map  $\Pi_{[\Phi]}: \widetilde{X}_{[\Phi]} \to X_{[\Phi,\Pi]}$  of degree two such that  $\Pi_{[\Phi]}$  is induced from the natural map  $\widetilde{X}_{[\Phi]} \to \widetilde{X}_{[\Phi]}/\langle \widetilde{I} \rangle$ , where  $\widetilde{X}_{[\Phi]}/\langle \widetilde{I} \rangle$  is the quotient by the group  $\langle \widetilde{I} \rangle$  generated by  $\widetilde{I}$ . In Step 2, we give a symbol to each irreducible component of  $X_{[\Phi,\Pi]}$  and each point at which two components intersect for convenience. In Step 3, we give the defining equation of the branch locus  $B_0$  on  $W_0$  using the symbols defined in Step 2 and observe the canonical resolution. Let  $W_r \xrightarrow{\tau_1} \cdots \to W_1 \xrightarrow{\tau_1} W_0$  be the subsequence of the blowing-ups obtained by the canonical resolution satisfying that  $S_r$  admits only rational double points of type  $A_n$ . We can easily see that  $X_{[\Phi,\Pi]}$  is homeomorphic to  $(\tau_r \circ \cdots \circ \tau_1)^* \Gamma_0$  and  $\widetilde{X}_{[\Phi]}$  is homeomorphic to the singular fiber of  $S_r$ . Finally, we show that the special fiber of the nonsingular minimal model of  $S_r$  is homeomorphic to  $X_{[\Phi]}$ .

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STEP 1 We first consider  $\Sigma_g/\langle I \rangle \simeq S^2$  with the data  $(\Pi(C_i), \Pi(\overline{C}_j), \Pi(C'_l), n_i, \overline{n}_j, n'_l)$  and the set of the branch points  $\mathcal{P} = \{P_1, P_2, \dots, P_{2g+2}\}$  of  $\Pi$ . Let  $\mathcal{P}'$  be the subset of  $\mathcal{P}$  such that each point in  $\mathcal{P}'$  is not on  $\Pi(\mathcal{C}_A)$ .

Since  $C_A \cup C_B \cup C_C$  is a simple admissible system, the set  $\{\Pi(\overline{C_j})\}_{1 \le j \le m} \cup \{\Pi(C'_l)\}_{1 \le l \le s}$  is the disjoint union of simple closed curves (cf. Remark 2.6). Thus, we can consider the chorizo space

$$X_{\left[\Phi,\Pi\right]} := X_{\left\{\left(\Pi\left(\overline{C}_{j}\right), 2\overline{n}_{j}\right)_{1 \leq j \leq m}, \left(\Pi(C'_{l}), n'_{l}\right)_{1 \leq l \leq s}\right\}}$$

defined in Section 2. With this chorizo space, we consider the data  $\{(\Pi(C_i), n_i)\}_{1 \le i \le k}$ and  $\mathcal{P}$ . For example, in the case where  $\Phi$  is an element in Example 2.7,  $S^2$  with the data is as shown in Fig. 5 and  $X_{[\Phi,\Pi]}$  with the data is as shown in Fig. 6. For each  $C_i$ , we take a point on  $\Pi(C_i)$  and denote it by  $P_{C_i}$ .

We use the same notations as in Section 2. For each simple closed curve *C* in  $C_A \cup C_B \cup C_C$ , we denote a small annular open neighbourhood by  $U_C$  satisfying  $I(U_C) = U_{I(C)}$  and  $U_C \cap \Pi^{-1}(\mathcal{P}') = \emptyset$ . We assume that they do not intersect each other. Moreover, we assume that  $I\left(\partial_{C_i}^1\right) = \partial_{I(C_i)}^1$  and  $I\left(\partial_{C_i}^2\right) = \partial_{I(C_i)}^2$ . Let  $\mathcal{U}$  be the union of all annular neighbourhoods defined as above. Set

(2) 
$$\widetilde{X}_{\left[\Phi\right]} := X_{\left\{(C_i,1)_{1 \le i \le k}, \left(\overline{C}_j, 2\overline{n}_j\right)_{1 \le j \le m}, \left(C'_i, n'_i\right)_{1 \le l \le s}, \left(I(C'_l), n'_i\right)_{1 \le l \le s}\right\}}$$

We define an orientation preserving homeomorphism  $\tilde{I}: \widetilde{X}_{[\Phi]} \to \widetilde{X}_{[\Phi]}$  induced from I as follows; Set  $\mathcal{B}o := \Sigma_g \setminus \mathcal{U}$ . We decompose  $\widetilde{X}_{[\Phi]}$  as  $\widetilde{X}_{[\Phi]} = \mathcal{B}o \cup \mathcal{C}h_A \cup \mathcal{C}h_B \cup \mathcal{C}h_C$ , where  $\mathcal{C}h_A := \bigcup R(C_i)_1$ ,  $\mathcal{C}h_B := \bigcup R(\overline{C}_j)_{2\overline{n}_j}$ , and  $\mathcal{C}h_C := \bigcup (R(C'_l)_{n'_l} \cup R(I(C'_l))_{n'_l})$ . For  $\mathcal{B}o$  and each member of  $\mathcal{C}h_A \cup \mathcal{C}h_B \cup \mathcal{C}h_C$ , we define an orientation preserving homeomorphism satisfying suitable conditions in the following way.

We can naturally define  $I_{\mathcal{B}o}: \mathcal{B}o \to \mathcal{B}o$  by the restriction of I to  $\Sigma_g \setminus \mathcal{U}$ . Note that  $\Pi^{-1}(\mathcal{P}')$  is the set of fixed points of  $I_{\mathcal{B}o}$ . Thus, we can consider that  $\Pi^{-1}(\mathcal{P}')$  is the set of points on  $\widetilde{X}_{[\Phi]}$ .

For each  $R(C_i)_1 = L_{C_i,0} \cup L_{C_i,1}$ , we can define a homeomorphism  $I_{C_i} : R(C_i)_1 \rightarrow R(C_i)_1$  of order two such that  $I_{C_i}$  coincides with  $I_{\mathcal{B}o}$  on  $\partial_{C_i}^1 \cup \partial_{C_i}^2$  (we identify  $\partial L_{C_i,0}$  with  $\partial_{C_i}^1$ , and  $\partial L_{C_i,1}$  with  $\partial_{C_i}^2$ ). Note that  $I_{C_i}(L_{C_i,0}) = L_{C_i,1}$  and the fixed point is  $L_{C_i,0} \cap L_{C_i,1}$ .

For each  $R(\overline{C}_j)_{2\overline{n}_j}$ , we define a homeomorphism  $I_{\overline{C}_j}: R(\overline{C}_j)_{2\overline{n}_j} \to R(\overline{C}_j)_{2\overline{n}_j}$ of order two such that  $I_{\overline{C}_j}$  coincides with  $I_{\mathcal{B}_0}$  on  $\partial_{\overline{C}_j}^1 \cup \partial_{\overline{C}_j}^2$  and the fixed locus is  $\bigcup_{d=1}^{\overline{n}_j} L_{\overline{C}_j,2d-1}$ . Thus, the restriction map  $I_{\overline{C}_j}|_{L_{\overline{C}_j,2d}}$   $(d = 2, \ldots, \overline{n}_j - 1)$  is a homeomorphism of  $L_{\overline{C}_j,2d}$  of order two with fixed points  $L_{\overline{C}_j,2d} \cap L_{\overline{C}_j,2d+1}$  and  $L_{\overline{C}_j,2d} \cap L_{\overline{C}_j,2d-1}$ .

For  $R(C'_l)_{n'_l} \cup R(I(C'_l))_{n'_l}$ , we can define a homeomorphism  $I_{C'_l} : R(C'_l)_{n'_l} \cup R(I(C'_l))_{n'_l} \to R(C'_l)_{n'_l} \cup R(I(C'_l))_{n'_l}$  of order two such that  $I_{C'_l} (L_{C'_l,d}) = L_{I(C'_l),d}$  and  $I_{C'_l}$  coincides with  $I_{\mathcal{B}_o}$  on  $\partial^1_{C'_l} \cup \partial^2_{C'_l} \cup \partial^1_{I(C'_l)} \cup \partial^2_{I(C'_l)}$ .

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By gluing these maps, we obtain a homeomorphism  $\tilde{I}$  of  $\widetilde{X}_{[\Phi]}$ . Since we see that  $\tilde{I}$  is an involution, we can consider the quotient map  $\widetilde{\Pi}: \widetilde{X}_{[\Phi]} \to \widetilde{X}_{[\Phi]}/\langle \tilde{I} \rangle$  of degree two. From the construction, there exists a natural homeomorphism  $\Theta: \widetilde{X}_{[\Phi]}/\langle \tilde{I} \rangle \to X_{[\Phi,\Pi]}$  such that  $\Theta(\widetilde{\Pi}(\Pi^{-1}(P_i))) = P_i \ (P_i \in \mathcal{P}')$  and  $\Theta(\widetilde{\Pi}(L_{C_i,0} \cap L_{C_i,1})) = P_{C_i}$ .

Then we can consider the surjective map  $\Pi_{[\Phi]}: \widetilde{X}_{[\Phi]} \to X_{[\Phi,\Pi]}$  of degree two such that the branch locus is  $\bigcup \left(\bigcup L_{\overline{C}_{j},2d-1}\right)$  and the set of isolated branch points is  $\mathcal{P}' \cup \{P_{C_i}\}$ . Note that  $\Pi_{[\Phi]}^{-1}(P_i)$   $(P_i \in \mathcal{P}')$  is not a double point and  $\Pi_{[\Phi]}^{-1}(P_{C_i})$  is a double point.

STEP 2 A component of a chorizo space which intersects only one component is called *a terminal component*. Since the dual graph of  $X_{[\Phi,\Pi]}$  is a tree, there exists at least one terminal component. For later use, we give a symbol to each component of  $X_{[\Phi,\Pi]}$  by the following way (see Fig. 7, for example, in Fig. 7, we give a symbol to each component of  $X_{[\Phi,\Pi]}$  appearing in Example 2.7. The lines mean irreducible components of  $X_{[\Phi,\Pi]}$ ); Choose a terminal component of  $X_{[\Phi,\Pi]}$  and denote it by  $Z_0$ . If  $X_{[\Phi,\Pi]}$  has another terminal component, then  $Z_0$  intersects only one component of  $X_{[\Phi,\Pi]}$ . We denote it by  $Z_{0,1}^1$ . If there exist components of  $X_{[\Phi,\Pi]} \setminus Z_0$  which intersect  $Z_{0,1}^1$ , choose a component among them and denote it by  $Z_{0,2}^1$ . Inductively, if there exist components of  $X_{[\Phi,\Pi]} \setminus Z_{0,i-1}^1$  intersecting  $Z_{0,i}^1$ , choose such a component and denote it by  $Z_{0,i+1}^1$ . Finally, we obtain a  $\mathbf{P}^1$ -chain  $Z_0 \cup Z_{0,1}^1 \cup \cdots \cup Z_{0,k_{\xi}}^1$  such that  $Z_{0,k_{\xi}}^1$  is a terminal component of  $X_{[\Phi,\Pi]}$ .

Let  $Z_{0,j}^{1}$  be a component which is not a terminal component of  $X_{[\Phi,\Pi]} \setminus \{Z_{0,j-1}^{1}\}$ . If j = 1, we set  $Z_{0,0}^{1} := Z_{0}$ . We denote by  $Z_{0,j,1}^{1,1}, Z_{0,j,1}^{1,2}, \ldots, Z_{0,j,1}^{1,d}$  the components of  $X_{[\Phi,\Pi]} \setminus \{Z_{0,j-1}^{1}, Z_{0,j+1}^{1}\}$  that intersect  $Z_{0,j}^{1}$ . For each  $Z_{0,j,1}^{1,i}$  which is not a terminal component of  $X_{[\Phi,\Pi]}$ , choose a component of  $X_{[\Phi,\Pi]} \setminus Z_{0,j,1}^{1,i}$  and denote it by  $Z_{0,j,2}^{1,i}$ . Inductively, if  $Z_{0,j,j'}^{1,i}$  is not a terminal component of  $X_{[\Phi,\Pi]} \setminus Z_{0,j,j'}^{1,i}$  intersecting  $Z_{0,j,j'}^{1,i}$ , choose a component of  $X_{[\Phi,\Pi]} \setminus Z_{0,j,j'+1}^{1,i}$ . Finally, we obtain a  $\mathbf{P}^{1}$ -chain  $Z_{0,j,1}^{1,i} \cup \cdots \cup Z_{0,j,\eta}^{1,i}$  such that  $Z_{0,j,\eta}^{1,i}$  is a terminal component of  $X_{[\Phi,\Pi]}$ .

By the same way, we give symbols to all components of  $X_{[\Phi,\Pi]}$ ; For simplicity, we denote a sequence  $1, i_1, \ldots, i_{l-1}$  by  $I_l$  and a sequence  $0, j_1, \ldots, j_{l-1}$  by  $J_l$ . Let  $Z_{J_l,j_l}^{I_l}$  be a component which is not a terminal component of  $X_{[\Phi,\Pi]} \setminus Z_{J_l,j_{l-1}}^{I_l}$ . We denote by  $Z_{J_l,j_1,1}^{I_l,1}, Z_{J_l,j_1,1}^{I_l,2}, \ldots, Z_{J_l,j_{l-1}}^{I_l,i_l}$  the components of  $X_{[\Phi,\Pi]} \setminus \{Z_{J_l,j_{l-1}}^{I_l}, Z_{J_l,j_{l+1}}^{I_l}\}$  which intersect  $Z_{J_l,j_l}^{I_l,\alpha}$ . For each  $Z_{J_l,j_{l-1}}^{I_l,\alpha}$ , choose a subchorizo space  $Z_{J_l,j_{l-1}}^{I_l,\alpha} \cup Z_{J_l,j_{l-2}}^{J_l,\alpha} \cup \cdots \cup$  $Z_{J_l,j_{l-1},j_{l-1}}^{I_l,\alpha}$  of  $X_{[\Phi,\Pi]}$  such that  $Z_{J_l,j_{l-1},j_{l+1}}^{I_l,\alpha}$  is a terminal component of  $X_{[\Phi,\Pi]}$ .

We also give a symbol to each point at which two components intersect. We denote by  $a_{J_l,j_l}^{I_l}$  the point at which  $Z_{J_l,j_l}^{I_l}$  intersects  $Z_{J_l,j_l-1}^{I_l}$  when  $j_l \neq 1$ . We denote by  $a_{J_l,j_l,1}^{I_l,\alpha}$  the point at which  $Z_{J_l,j_l,1}^{I_l,\alpha}$  intersects  $Z_{J_l,j_l}^{I_l}$ . We set

$$\mathcal{I}_{\Phi} := \left\{ (I_l, J_l, j_l) \mid a_{J_l, j_l}^{I_l} \in X_{[\Phi, \Pi]} \right\}.$$

When  $\theta = (I_l, J_l, j_l) \in \mathcal{I}_{\Phi}$ , we sometimes write  $a_{\theta}$  and  $Z_{\theta}$  instead of writing  $a_{J_l, j_l}^{I_l}$  and  $Z_{J_l, j_l}^{I_l}$ , for simplicity.

STEP 3 Let  $P_{\xi} \in \mathcal{P}'$  be a point on  $Z_{0,j_1,\ldots,j_l}^{1,i_1,\ldots,i_{l-1}}$ . We define the polynomial  $f_{P_{\xi}}(\widetilde{Z}_0, t, a_{\theta}, P_{\xi})$  associated to  $P_{\xi}$  as follows;

$$f_{P_{\xi}}\left(\widetilde{Z}_{0}, t, a_{\theta}, P_{\xi}\right) \coloneqq \widetilde{Z}_{0} - \left(\sum_{j=1}^{j_{1}} a_{0,j}^{1} t^{j-1} + \sum_{j=1}^{j_{2}} a_{0,j_{1},j}^{1,i_{1}} t^{j_{1}+j-1} + \cdots + \sum_{j=1}^{j_{l}} a_{0,j_{1},\dots,j_{l-1},j}^{1,i_{1},\dots,i_{l-1}} t^{j_{1}+j_{2}+\dots+j_{l-1}+j-1} + P_{\xi} t^{j_{1}+j_{2}+\dots+j_{l}}\right).$$

If  $P_{\xi}$  is on  $Z_0$ , we set  $f_{P_{\xi}} := \widetilde{Z}_0 - P_{\xi}$ .

Let  $\Pi(C_i)$  be the image of a curve of Type A by  $\Pi$  on  $Z_{0,j_1,...,j_l}^{1,i_1,...,i_{l-1}}$ . We define the polynomial  $g_{C_i}(\widetilde{Z}_0, t, a_{\theta}, P_{C_i})$  associated to  $\Pi(C_i)$  as following;

$$g_{C_{i}}(\widetilde{Z}_{0}, t, a_{\theta}, P_{C_{i}}) \coloneqq \left\{ \widetilde{Z}_{0} - \left( \sum_{j=1}^{j_{1}} a_{0,j}^{1} t^{j-1} + \sum_{j=1}^{j_{2}} a_{0,j_{1},j}^{1,i_{1}} t^{j_{1}+j-1} + \cdots + \sum_{j=1}^{j_{i}} a_{0,j_{1},\dots,j_{l-1},j}^{1,i_{1},\dots,i_{l-1}} t^{j_{1}+j_{2}+\dots+j_{l-1}+j-1} + P_{C_{i}} t^{j_{1}+j_{2}+\dots+j_{l}} \right) \right\}^{2} - t^{n_{i}+2(j_{1}+\dots+j_{l})}.$$

If  $\Pi(C_i)$  is on  $Z_0$ , we set  $g_{C_i}\left(\widetilde{Z}_0, t, P_{C_i}\right) := \left(\widetilde{Z}_0 - P_{C_i}\right)^2 - t^{n_i}$ . Set

$$F\left(\widetilde{Z}_{0},t,\{P_{C_{i}}\},\{P_{\xi}\},\{a_{\theta}\}\right) := \prod_{P_{\xi}\in\mathcal{P}'}\prod_{C_{i}\in\mathcal{C}_{A}}f_{P_{\xi}}\left(\widetilde{Z}_{0},t,a_{\theta},P_{\xi}\right)g_{C_{i}}\left(\widetilde{Z}_{0},t,a_{\theta},P_{C_{i}}\right).$$

Fix  $\{[P_{C_i}], [P_{\xi}], [a_{\theta}]\}\$  a set of mutually distinct complex non-zero numbers and consider the polynomial  $F(\widetilde{Z}_0, t) := F(\widetilde{Z}_0, t, \{[P_{C_i}]\}, \{[P_{\xi}]\}, \{[a_{\theta}]\})\)$ . Note that the degree of  $F(\widetilde{Z}_0, t)$  with respect to  $Z_0$  is 2g + 2. Moreover, since  $\{[P_{C_i}], [P_{\xi}], [a_{\theta}]\}\)$  is a set of mutually distinct complex numbers, the roots of  $F(\widetilde{Z}_0, t) = 0$  is mutually distinct when  $t \neq 0$  and |t| is sufficiently small. Let  $\varepsilon$  be the small positive real number such that the roots of  $F(\widetilde{Z}_0, t) = 0$  is mutually distinct. We set  $\Delta := \{t \in \mathbb{C} \mid |t| < \varepsilon\}$ . Let  $\widetilde{F}(\widetilde{Z}_0, \widetilde{Z}_1, t)$  be the homogeneous polynomial of degree 2g + 2 with respect to  $(\widetilde{Z}_0 : \widetilde{Z}_1)$  satisfying  $\widetilde{F}(\widetilde{Z}_0, 1, t) = F(\widetilde{Z}_0, t)$ . Let  $\psi_0: S_0 \rightarrow W_0 := \mathbb{P}^1 \times \Delta$  be the double covering branched along  $B_0$ ;  $\widetilde{F}(\widetilde{Z}_0, \widetilde{Z}_1, t)$ 

Let  $\psi_0: S_0 \to W_0 := \mathbf{P}^1 \times \Delta$  be the double covering branched along  $B_0$ ;  $\widetilde{F}(\widetilde{Z}_0, \widetilde{Z}_1, t) = 0$ , where  $(\widetilde{Z}_0: \widetilde{Z}_1)$  is a homogeneous coordinates of  $\mathbf{P}^1$ . Since the divisor defined by  $\widetilde{Z}_1 = 0$  does not intersect  $B_0$ , it is sufficient to observe  $B_0$  on  $\widetilde{Z}_1 \neq 0$  defined by  $F(\widetilde{Z}_0, t) = 0$ . We observe the canonical resolution of the family  $\pi_0 \circ \psi_0: S_0 \to \Delta$ . In the case where  $X_{[\Phi,\Pi]}$  has only one component, the assertion is clear because each simple closed curve in a simple admissible system of  $\Phi$  is of Type A.

Let  $\tau_1: W_1 \to W_0$  be the blowing-up at  $\widetilde{Z}_0 - [a_{0,1}^1] = t = 0$ . Let  $\overline{Z}_{0,1}^1$  be the exceptional set of  $\tau_1$ . We denote by  $\widetilde{Z}_{0,1}^1$  an affine coordinates of the exceptional set satisfying  $\widetilde{Z}_0 - [a_{0,1}^1] = t \widetilde{Z}_{0,1}^1$ .

If  $(1;0,2) \in \mathcal{I}_{\Phi}$ , we blow up at  $\widetilde{Z}_{0,1}^{1} - [a_{0,2}^{1}] = t = 0$  and denote by  $\overline{Z}_{0,2}^{1}$  the exceptional set of this blowing-up. We denote by  $\widetilde{Z}_{0,2}^{1}$  an affine coordinates satisfying  $\widetilde{Z}_{0,1}^{1} - [a_{0,2}^{1}] = t \widetilde{Z}_{0,2}^{1}$ . Similarly, if  $(1, d; 0, 1, 1) \in \mathcal{I}_{\Phi}$ , we blow up at  $\widetilde{Z}_{0,1}^{1} - [a_{0,1,1}^{1,d}] = t = 0$ . We denote by  $\overline{Z}_{0,1,1}^{1,d}$  the exceptional set of this blowing-up. We denote by  $\widetilde{Z}_{0,1,1}^{1,d}$  an affine coordinates of the exceptional set satisfying  $\widetilde{Z}_{0,1}^{1} - [a_{0,1,1}^{1,d}] = t \widetilde{Z}_{0,1,1}^{1,d}$ .

Inductively, we blow up and give the symbols to the exceptional sets of the blowing-ups in the way similar to the above; If  $(I_l; J_l, j_l + 1) \in \mathcal{I}_{\Phi}$ , we blow up at  $\widetilde{Z}_{J_l, j_l}^{I_l} - [a_{J_l, j_l+1}^{I_l}] = t = 0$  and denote by  $\overline{Z}_{J_l, j_l+1}^{I_l}$  the exceptional set of this blowing-up. We denote by  $\widetilde{Z}_{J_l, j_l+1}^{I_l}$  an affine coordinates of the exceptional set satisfying  $\widetilde{Z}_{J_l, j_l}^{I_l} - [a_{J_l, j_l+1}^{I_l}] = t \widetilde{Z}_{J_l, j_l+1}^{I_l}$ . If there exists  $d \in \mathbb{Z}$  such that  $(I_l, d; J_l, j_l, 1) \in \mathcal{I}_{\Phi}$ , we blow up at  $\widetilde{Z}_{J_l, j_l}^{I_l} - [a_{J_l, j_l}^{I_l, j_l}] = t = 0$  and give the symbol  $\overline{Z}_{J_l, j_l, 1}^{I_l, d}$  to the exceptional set of this blowing-up. We denote by  $\widetilde{Z}_{J_l, j_l, 1}^{I_l, d}$  an affine coordinates satisfying  $\widetilde{Z}_{J_l, j_l}^{I_l} - [a_{J_l, j_l, 1}^{I_l, d}] = t \widetilde{Z}_{J_l, j_l, 1}^{I_l, d}$ .

Let  $W_r \xrightarrow{\tau_r} W_{r-1} \xrightarrow{\tau_{r-1}} \cdots \xrightarrow{\tau_1} W_0$  be the sequence of the blowing-ups obtained by the process above. Then, we obtain the chorizo space  $(\tau_1 \circ \cdots \circ \tau_r)^* \Gamma_0 = \bigcup_{\theta \in \mathcal{I}_{\Phi}} \overline{Z}_{\theta}$ . Here, we use the same symbol for the exceptional set of each blowing-up  $\tau_{r'} \colon W_{r'} \to W_{r'-1}$  $(r' \leq r)$  and its strict transform by  $\tau_{r'+1} \circ \cdots \circ \tau_r$ . Note that the multiplicity of each component of  $\overline{Z}_{\theta}$  is one. For each r', we can define the double covering  $\psi_{r'} \colon S_{r'} \to W_{r'}$ branched along  $B_{r'}$  and bimeromorphic map  $\tilde{\tau}_{r'} \colon S_{r'} \to S_{r'-1}$  introduced in the previous section. Since  $\overline{Z}_{J_i, j_i+1}^{I_i}$  intersects  $\overline{Z}_{J_i, j_i}^{I_i}$  at  $\widetilde{Z}_{J_i, j_i}^{I_i} = [a_{J_i, j_i+1}^{I_i}]$  and  $\overline{Z}_{J_i, j_i}^{I_i, \alpha}$  intersects  $\overline{Z}_{J_i, j_i}^{I_i}$  at  $\widetilde{Z}_{J_i, j_i}^{I_i} = [a_{J_i, j_i}^{I_i, \alpha}]$ , there exists a natural homeomorphism between  $\bigcup \overline{Z}_{\theta}$  to  $X_{[\Phi, \Pi]}$  that sends each exceptional set  $\overline{Z}_{\theta}$  ( $\theta \in \mathcal{I}_{\Phi}$ ) to the irreducible component  $Z_{\theta}$ of  $X_{[\Phi,\Pi]}$ . Then, we can identify  $X_{[\Phi,\Pi]}$  with  $(\tau_1 \circ \cdots \circ \tau_r)^* \Gamma_0$ . Moreover, if  $P_{\xi} \in Z_{\theta}$ , the strict transform of  $f_{P_{\xi}} = 0$  on  $W_r$  intersects the exceptional set  $\overline{Z}_{\theta}$  at  $\widetilde{Z}_{\theta} = [P_{\xi}]$ , transversally. Thus, we can identify the point  $P_{\xi} \in \mathcal{P}'$  on  $Z_{\theta}$  with the point on the component  $\overline{Z}_{\theta}$  defined by  $\widetilde{Z}_{\theta} = [P_{\xi}]$ . If  $\Pi(C_i) \subset Z_{\theta}$ , the strict transform of  $g_{C_i} = 0$ on  $W_r$  intersects the exceptional set  $\overline{Z}_{\theta}$  at  $\widetilde{Z}_{\theta} = [P_{C_i}]$ . Then we identify naturally the point  $P_{C_i} \in \mathcal{P}'$  on  $Z_{\theta}$  of  $X_{[\Phi,\Pi]}$  with the point on  $\overline{Z}_{\theta}$  defined by  $\widetilde{Z}_{\theta} - [P_{C_i}] = t = 0$ .

We can easily see that the defining equation of the strict transform of  $g_{C_i} = 0$  on  $W_r$  near  $\widetilde{Z}_{\theta} = [P_{C_i}]$  is  $(\widetilde{Z}_{\theta} - [P_{C_i}])^2 = t^{n_i}$ . Thus, if  $\overline{Z}_{\theta}$  is not a component of  $B_r$ , the singular point on  $S_r$  over  $\widetilde{Z}_{\theta} - [P_{C_i}] = t = 0$  is a rational double point of type  $A_{n_i-1}$ . In the proof of Claim 3.2, we show that a exceptional set corresponding to a body component of  $X_{[\Phi,\Pi]}$  is not a component of  $B_r$ .

CLAIM 3.1. Let  $\tau_{r'}: W_{r'} \to W_{r'-1}$  be the blowing-up at  $Q; \widetilde{Z}_{\theta} - [a_{\theta'}] = t = 0$ . Then, the strict transform  $\widetilde{B}_{r'-1}$  of the divisor  $B_0$  by  $\overline{\tau}_{r'-1} = \tau_1 \circ \cdots \circ \tau_{r'-1}$  is singular at Q.

Proof of Claim 3.1. Note that the strict transform of  $f_{P_{\xi}} = 0$  or  $g_{C_i} = 0$  by  $\overline{\tau}_{r'-1}$ contains Q, if and only if  $f_{P_{\xi}}$  or  $g_{C_i}$  include a monomial whose coefficient is  $[a_{\theta'}]$ . Assume that  $\widetilde{B}_{r'-1}$  is nonsingular at  $\widetilde{Z}_{\theta} - [a_{\theta'}] = t = 0$ . Then, there exists unique irreducible component D of  $\widetilde{B}_{r'-1}$  that contains Q. Let D' be the irreducible component of  $B_0$  such that the strict transform of D' by  $\overline{\tau}_{r'-1}$  is D. If the defining equation of D' is  $g_{C_i} = 0$ , we see that  $\Pi(C_i)$  is on the component  $Z_{\theta}$  of  $X_{[\Phi,\Pi]}$  because if  $\Pi(C_i)$ is not on  $Z_{\theta}$ , the strict transform of D' by  $\overline{\tau}_{r'-1}$  is singular. Then, the strict transform of  $g_{C_i} = 0$  intersects  $\overline{Z}_{\theta}$  at  $\widetilde{Z}_{\theta} = [P_{C_i}]$ . It contradicts that  $\{[P_{C_i}], [P_{\xi}], [a_{\theta}]\}$  is a set of mutually distinct complex numbers. Thus, there exists a branch point  $P_{\xi} \in Z_{\widetilde{\theta}}$  of  $\Pi$  such that the defining equation of D' is  $f_{P_{\xi}} = 0$ . If  $\theta = \widetilde{\theta}$ , it contradicts the fact  $[P_{\xi}] \neq [a_{\theta}]$ . If  $\theta \neq \widetilde{\theta}$ , we see that there exists no  $\Pi(C_j)$  and no branch points but  $P_{\xi}$ on  $Z_{\widetilde{\theta}}$ . Moreover, we see that  $Z_{\widetilde{\theta}}$  is a terminal component of  $X_{[\Phi,\Pi]}$ . It contradicts the assumption that  $C_A \cup C_B \cup C_C$  is an admissible system of  $\Phi$ .

CLAIM 3.2. Let  $\overline{Z}_{\theta_1} \cup \cdots \cup \overline{Z}_{\theta_N}$  be a set of the exceptional sets corresponding to the core chain at the image of a curve of Type B or Type C by  $\Pi$ . Let  $\overline{Z}_{\theta_0}$  and  $\overline{Z}_{\theta_{N+1}}$ are the exceptional sets corresponding to the body components such that  $\overline{Z}_{\theta_0}$  and  $\overline{Z}_{\theta_{N+1}}$ intersect  $\overline{Z}_{\theta_1}$  and  $\overline{Z}_{\theta_N}$ , respectively. Then, if  $\overline{Z}_{\theta_1} \cup \cdots \cup \overline{Z}_{\theta_N}$  is a set of the exceptional sets corresponding to the core chain at the image of a curve of Type C, each  $\overline{Z}_{\theta_i}$  is not a component of  $B_r$ . If  $\overline{Z}_{\theta_1} \cup \cdots \cup \overline{Z}_{\theta_N}$  is corresponding to the core chain at the image of a curve of Type B by  $\Pi$ , then each  $\overline{Z}_{\theta_i}$  is a component of  $B_r$  when *i* is odd and not a component of  $B_r$  when *i* is even.

Proof of Claim 3.2. Let  $\tau_{r'}: W_{r'} \to W_{r'-1}$  be the blowing-up whose exceptional set is  $\overline{Z}_{\theta_1}$ . Without loss of generality, we can assume that  $\overline{Z}_{\theta_2}, \ldots, \overline{Z}_{\theta_N}$  are not the exceptional sets of  $\overline{\tau}_{r'-1}$ . Thus, we can consider that  $\tau_{r'}$  is the blowing-up at Q;  $\widetilde{Z}_{\theta_0} - [a_{\theta_1}] = t = 0$ .

Let  $\{\theta'_1, \ldots, \theta'_w\}$  be the subset of  $\mathcal{I}_{\Phi}$  such that each  $Z_{\theta'_i}$  is contracted to Q by  $\tau_{r'} \circ \cdots \circ \tau_r$ . By the definition of  $f_{P_{\xi}}$  and  $g_{C_i}$ , we see that each strict transform of  $f_{P_{\xi}} = 0$  (resp.  $g_{C_i} = 0$ ) by  $\overline{\tau}_{r'-1}$  contains Q if and only if there exists  $\theta'_i$  such that  $f_{P_{\xi}}$  (resp.  $g_{C_i}$ ) includes a monomial whose coefficient is  $[a_{\theta'_i}]$ . The multiplicities of the strict transform of  $f_{P_{\xi}} = 0$  and  $g_{C_i} = 0$  at Q are one and two, respectively if they contain Q. Thus, the multiplicity of the strict transform  $\widetilde{B}_{r'-1}$  of  $B_0$  at Q by  $\overline{\tau}_{r'-1}$  coincides with the number of the branch points of  $\Pi$  that are on the components  $Z_{\theta'_1}, \ldots, Z_{\theta'_w}$  of  $X_{[\Phi,\Pi]}$ . Then, we see that the multiplicity of  $\widetilde{B}_{r'-1}$  at Q is odd if  $\overline{Z}_{\theta_0} \cup \cdots \cup \overline{Z}_{\theta_N}$  is the core chain at the image of a curve of Type B, and even if not. If  $\overline{Z}_{\theta_0}$  is not a component of  $B_r$ , the assertion is clear because N is odd when  $\overline{Z}_{\theta_1} \cup \cdots \cup \overline{Z}_{\theta_N}$  is the core chain at the image of a curve of Type B. Though, since the strict transform of  $\Gamma_0$  is a component corresponding to a body component and not a component of  $B_r$ , we see that all components corresponding to body components are not components.

of  $B_r$ .

Let  $\overline{P}_i$  and  $\overline{P}_{C_i}$  be points on  $(\tau_1 \circ \cdots \circ \tau_r)^* \Gamma_0$  corresponding to  $P_i$  and  $P_{C_i}$ , respectively. Let  $\widetilde{r} \colon \widetilde{S}_r \to S_r$  be the minimal resolution of all singular points of type  $A_n$  on  $S_r$  and  $\widetilde{S}_r \to S$  the blowing-downs of suitable (-1)-curves successively on  $\widetilde{S}_r$  such that S has no (-1)-curve. Let  $\widetilde{X}$  be the singular fiber of  $\pi_r \circ \psi_r \colon S_r \to \Delta$ . By Claim 3.2,  $\psi_r|_{\widetilde{X}} \colon \widetilde{X} \to (\tau_1 \circ \cdots \circ \tau_r)^* \Gamma_0 \simeq X_{[\Phi,\Pi]}$  is a double cover branched along the components corresponding to  $\bigcup (\bigcup L_{\overline{C}_i, 2d-1})$  and branched at the points corresponding to  $\mathcal{P}' \cup \{P_{C_i}\}$ . Moreover,  $\psi_r|_{\widetilde{X}}^{-1}(\overline{P}_{C_i})$  is a double point of  $\widetilde{X}$  and  $\psi_r|_{\widetilde{X}}^{-1}(\overline{P}_i)$  is nonsingular point of  $\widetilde{X}$ . Thus we see that  $\psi_r|_{\widetilde{X}}$  satisfies the same conditions as  $\Pi_{[\Phi]}$  and  $\widetilde{X}$  is homeomorphic to  $\widetilde{X}_{[\Phi]}$ . Since  $\psi_r^{-1}(\overline{P}_{C_i})$  is a rational double point of type  $A_{n_i-1}$ , the singular fiber of  $\pi_r \circ \psi_r \circ \widetilde{r} \colon \widetilde{S}_r \to \Delta$  is homeomorphic to

$$X_{\left\{(C_i,n_i)_{i\leq k}, \left(\overline{C}_j, 2\overline{n}_j\right)_{j\leq m}, (C'_l,n_l)_{l\leq s}, (I(C'_l),n_l)_{l\leq s}\right\}}$$

because  $\widetilde{X}_{[\Phi]}$  is given by (2).

By the proof of Claim 3.2, we see that  $\overline{Z}_{\theta}$  is a component of  $B_r$  if and only if  $\overline{Z}_{\theta}$  corresponds to a component of  $\bigcup \left(\bigcup L_{\overline{C}_j,2d-1}\right)$ . Since the multiplicity of  $\psi_r^*(\overline{Z}_{\theta})$  is two when  $\overline{Z}_{\theta} \subset B_r$ ,  $\psi_r^*(\overline{Z}_{\theta})$  is a (-1)-curve. Moreover, we see that  $\psi_r^*(\overline{Z}_{\theta})$  is not a (-1)-curve when  $\overline{Z}_{\theta}$  does not correspond to a component of  $\bigcup \left(\bigcup L_{\overline{C}_j,2d-1}\right)$  by Claim 3.1. Thus, we see that the special fiber of  $\phi: S \to \Delta$  is homeomorphic to  $X_{[\Phi]}$ . We complete the proof of Theorem 1.1.

ACKNOWLEDGEMENT. The author thanks Professors Tadashi Ashikaga, Kazuhiro Konno, Tatsuya Arakawa and Shigeru Takamura, for their useful advice and valuable discussions. He also wishes to express his gratitude to Professors Tadao Oda, Masanori Ishida, Takeshi Kajiwara and Nobuo Hara for their continuous encouragement. This research is partially supported by Waseda University Grant for Special Research Project: 2005B-216.



Fig. 1.



Fig. 2.



Fig. 3.







Fig. 5.



Fig. 6.



Fig. 7.

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