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THE RECURRENCE TIME FOR IRRATIONAL ROTATIONS

DONG HAN KIM

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Abstract

Let $T$ be a measure preserving transformation on $X \subset \mathbb{R}^d$ with a Borel measure $\mu$ and $R_E$ be the first return time to a subset $E$. If $(X, \mu)$ has positive pointwise dimension for almost every $x$, then for almost every $x$

$$\limsup_{r \to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log \mu(B(x,r))} \leq 1,$$

where $B(x,r)$ the the ball centered at $x$ with radius $r$. But the above property does not hold for the neighborhood of the 'skewed' ball. Let $B(x,r:s) = (x-rs, x+rs)$ be an interval for $s > 0$. For arbitrary $\alpha \geq 1$ and $\beta \geq 1$, there are uncountably many irrational numbers whose rotation satisfy that

$$\limsup_{r \to 0^+} \frac{\log R_{B(x,r,s)}(x)}{-\log \mu(B(x,r:s))} = \alpha \quad \text{and} \quad \liminf_{r \to 0^+} \frac{\log R_{B(x,r,s)}(x)}{-\log \mu(B(x,r:s))} = \frac{1}{\beta}$$

for some $s$.

1. Introduction

Let $\mu$ be a probability measure on $X$ and $T : X \to X$ be a $\mu$-preserving transformation. For a measurable subset $E \subset X$ with $\mu(E) > 0$ and a point $x \in E$ which returns to $E$ under iteration by $T$, we define the first return time $R_E$ on $E$ by

$$R_E(x) = \min \{ j \geq 1 : T^j x \in E \}.$$

Kac’s lemma [5] states that

$$\int_E R_E(x) d\mu \leq 1.$$

If $T$ is ergodic, then the equality holds.

For a decreasing sequence of subsets $\{E_n\}$ containing $x$, $R_{E_n}$ is an increasing sequence. The asymptotic behavior between $R_{E_n}$ and the measure of $E_n$ has been studied after Wyner and Ziv’s work [13] for ergodic processes. Let $\mathcal{P}$ be a partition of $X$ and $\{\mathcal{P}_n\}$ be a sequence of partitions of $X$ obtained by $\mathcal{P}_n = \mathcal{P} \lor T^{-1} \mathcal{P} \lor \cdots \lor T^{-n+1} \mathcal{P}$.
where $\mathcal{P} \vee \mathcal{Q} = \{ P \cap Q : P \in \mathcal{P}, \ Q \in \mathcal{Q} \}$. Ornstein and Weiss [9] showed that if $T$ is ergodic, then

$$\lim_{n \to \infty} \frac{\log R_{P(x)}(x)}{n} = h(T, \mathcal{P}) \text{ a.e.,}$$

where $P_n(x)$ is the element in $\mathcal{P}_n$ containing $x$. Therefore, by the Shannon-McMillan-Brieman theorem, if the entropy with respect to a partition $\mathcal{P}$, $h(T, \mathcal{P})$ is positive, then we have

$$\lim_{n \to \infty} \frac{\log R_{P_n(x)}(x)}{-\log \mu(P_n(x))} = 1 \text{ a.e.}$$

Let $(X, d)$ be a metric space and $B(x, r) = \{ y : d(x, y) < r \}$. Define the upper and lower pointwise dimension of $\mu$ at $x$ by

$$\overline{d}_\mu(x) = \limsup_{r \to 0^+} \frac{\log \mu(B(x, r))}{\log r}, \quad \underline{d}_\mu(x) = \liminf_{r \to 0^+} \frac{\log \mu(B(x, r))}{\log r}.$$  

Now we have another recurrence theorem for the decreasing sequence of balls.

**Theorem 1.1.** Let $T : X \to X$ be a Borel measurable transformation on a measurable set $X \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$ and $\mu$ be a $T$-invariant probability measure on $X$. If $\underline{d}_\mu(x) > 0$ for $\mu$-almost every $x$, then we have

$$\limsup_{r \to 0^+} \frac{\log R_{B(x, r)}(x)}{-\log \mu(B(x, r))} \leq 1$$

for $\mu$-almost every $x$.

This theorem is a modified version of Barreira and Saussol’s result [1] which states that

$$\limsup_{r \to 0^+} \frac{\log R_{B(x, r)}(x)}{-\log r} \leq \overline{d}_\mu(x), \quad \liminf_{r \to 0^+} \frac{\log R_{B(x, r)}(x)}{-\log r} \leq \underline{d}_\mu(x).$$

See also [2], [3], [7], and [11] for the transformations which satisfy that

$$\lim_{r \to 0^+} \frac{\log R_{B(x, r)}(x)}{-\log r} = \text{dimension of } \mu.$$  

Note that for some irrational rotations the limit does not exist [4].

So one might expect that if we choose a decreasing sequence of sets $E_n$ as ‘good’ neighborhoods of $x$

$$\limsup_{n} \frac{\log R_{E_n}(x)}{-\log \mu(E_n)} \leq 1.$$
However, we show that even for interval $E_n$’s on $X$ the limsup can be larger than 1 for some irrational rotations.

For $t \in \mathbb{R}$ we define $\| \cdot \|$ and $\lfloor \cdot \rfloor$ by

$$\|t\| = \min_{n \in \mathbb{Z}} |t - n|, \quad \lfloor t \rfloor = t - |t|,$$

i.e., the distance to the nearest integer and the nearest integer which is less than or equal to $t$, respectively.

An irrational number $\theta$, $0 < \theta < 1$, is said to be of type $\eta$ if

$$\eta = \sup \left\{ t > 0 : \lim_{j \to \infty} j^t \|j\theta\| = 0 \right\}.$$

Every irrational number is of type $\eta \geq 1$. The set of irrational numbers of type 1 has measure 1 and includes the set of irrational numbers with bounded partial quotients, which is of measure 0. There exist numbers of type $\infty$, called Liouville numbers. Here we introduce a new definition on type of irrational numbers:

**DEFINITION 1.2.** An irrational number $\theta$, $0 < \theta < 1$, is said to be of type $(\alpha, \beta)$ if

$$\alpha = \sup \left\{ t > 0 : \lim_{j \to \infty} j^t \{-j\theta\} = 0 \right\},$$

$$\beta = \sup \left\{ t > 0 : \lim_{j \to \infty} j^t \{j\theta\} = 0 \right\}.$$

For example, if the partial quotients of an irrational number $\theta$ is $a_{2k} = 2^k$ for $k \geq 1$ and $a_{2k+1} = 1$ for $k \geq 0$, then $\theta$ is of type $(2, 1)$. Note that $\alpha, \beta \geq 1$ and $\eta = \max \{\alpha, \beta\}$. For each $\alpha, \beta > 1$ there are uncountably many (but measure zero) $\theta$’s which are of type $(\alpha, \beta)$.

Let $0 < \theta < 1$ be an irrational number and $T : [0, 1) \to [0, 1)$ an irrational rotation, i.e.,

$$Tx = x + \theta \pmod{1}.$$

Then $T$ preserves the Lebesgue measure $\mu$ on $X = [0, 1)$.

Let $B(x, r; s)$ be an interval $(x - r^x, x + r)$, $s > 0$ and put $B(x, r ; \infty) = [x, x + r)$.

**Theorem 1.3.** If $\theta$ is of type $(\alpha, \beta)$, then for $1 \leq s \leq \infty$ and any $x \in [0, 1)$, we have

$$\limsup_{r \to 0^+} \frac{\log R_{B(x, r; s)}(x)}{-\log \mu(B(x, r; s))} = \min \{\alpha, s\}, \quad \liminf_{r \to 0^+} \frac{\log R_{B(x, r; s)}(x)}{-\log \mu(B(x, r; s))} = \min \left\{ \frac{1}{\beta}, \frac{s}{\alpha} \right\}.$$
and for $0 < s < 1$ and any $x \in [0, 1)$, we have
\[
\limsup_{r \to 0^+} \frac{\log R_{\beta(x,r)}(x)}{- \log \mu(B(x,r))} = \min \left\{ \beta, \frac{1}{s} \right\}, \quad \liminf_{r \to 0^+} \frac{\log R_{\beta(x,r)}(x)}{- \log \mu(B(x,r))} = \min \left\{ \frac{1}{\alpha'}, \frac{1}{s \beta} \right\}.
\]

By the symmetry, we have
\[
\limsup_{r \to 0^+} \frac{\log R_{\beta(x,r)}(x)}{- \log \mu(B(x,r))} = \beta, \quad \liminf_{r \to 0^+} \frac{\log R_{\beta(x,r)}(x)}{- \log \mu(B(x,r))} = \frac{1}{\alpha'}.
\]

Note that if $s = 1$ the theorem is reduced to
\[
\limsup_{r \to 0^+} \frac{\log R_{\beta(x,r)}(x)}{- \log \mu(B(x,r))} = 1, \quad \liminf_{r \to 0^+} \frac{\log R_{\beta(x,r)}(x)}{- \log \mu(B(x,r))} = \frac{1}{\eta},
\]
which was shown in [4].

2. Return time for measure space

In this section we prove Theorem 1.1. Let $X \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$. Define
\[
\overline{Q}_n = \{ [i_1 2^{-n}, (i_1 + 1)2^{-n}) \times \cdots \times [i_d 2^{-n}, (i_d + 1)2^{-n}) : (i_1, \ldots, i_d) \in \mathbb{Z}^d \}
\]
to be the dyadic partition of $\mathbb{R}^d$ and $Q_n = \{X \cap A : A \in \overline{Q}_n\}$. Let $Q_n(x)$ as the element of $Q_n$ containing $x$.

In order to prove Theorem 1.1 we need a lemma, which is a slight modification of the weakly diametrically regularity in [1].

**Lemma 2.1.** Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$. For $\mu$-almost every $x$ we have
\[
\mu \left( B(x, 2^{-n}) \right) \leq n^2 \mu(Q_n(x))
\]
for sufficiently large $n$.

**Proof.** Let
\[
E_n = \{ x : \mu \left( B(x, 2^{-n}) \right) > n^2 \mu(Q_n(x)) \}.
\]
For each $A \in Q_n$ with $A \cap E_n \neq \emptyset$ choose one $x \in A \cap E_n$ and let $F$ be a set of
such $x$’s. Then we have

$$E_n \subset \bigcup_{x \in F} Q_n(x)$$

and

$$\mu(E_n) \leq \sum_{x \in F} \mu(Q_n(x)) < \sum_{x \in F} n^{-2} \mu(B(x, 2^{-n})) .$$

There is a constant $D$ depending on $d$ such that for each $y \in \mathbb{R}^d$, there are at most $D$ $x$’s in $F$ such that $x \in B(y, 2^{-n})$. Therefore, we have

$$\sum_{x \in F} \mu(B(x, 2^{-n})) \leq D \cdot \mu(\mathbb{R}^d) = D$$

and

$$\mu(E_n) < \sum_{x \in F} n^{-2} \mu(B(x, 2^{-n})) \leq Dn^{-2} .$$

Since

$$\sum_n \mu(E_n) < D \sum_n n^{-2} < \infty ,$$

the first Borel-Cantelli lemma completes the proof.

Proposition 2.2. Let $T : X \to X$ be a Borel measurable transformation on a measurable set $X \subset \mathbb{R}^d$ and $\mu$ be a $T$-invariant probability measure on $X$. If $d_{\mu}(x) > 0$ for $\mu$-almost every $x$, then

$$\limsup_{n \to \infty} \frac{\log R_{Q_n(x)}(x)}{-\log \mu(Q_n(x))} \leq 1$$

for $\mu$-almost every $x$.

Proof. Choose an arbitrary $\epsilon > 0$. For an $A \in \mathcal{Q}_n$, we have by Markov’s inequality

$$\mu \left( \left\{ x \in A : R_A(x) \geq \frac{2^{n\epsilon}}{\mu(A)} \right\} \right) \leq \mu(A)2^{-n\epsilon} \int_A R_A(x) \, d\mu .$$

By Kac’s lemma we have

$$\mu \left( \left\{ x \in A : R_A(x) \geq \frac{2^{n\epsilon}}{\mu(A)} \right\} \right) \leq \mu(A)2^{-n\epsilon} .$$
Hence we have
\[
\mu\left(\left\{ x \in X : R_{Q_n(x)}(x) \geq \frac{2^{-\varepsilon}}{\mu(Q_n(x))} \right\}\right) \leq \sum_{A \in Q_n} \mu(A)2^{-\varepsilon} \leq 2^{-\varepsilon}
\]
and
\[
\sum_{n=1}^{\infty} \mu\left(\left\{ x \in X : R_{Q_n(x)}(x) \geq \mu(Q_n(x))^{-\frac{1}{2}2^{-\varepsilon}} \right\}\right) < \infty.
\]

By the first Borel-Cantelli lemma, for almost every \( x \) we have
\[
R_{Q_n(x)}(x) < \frac{2^{-\varepsilon}}{\mu(Q_n(x))}
\]
eventually. Thus for almost every \( x \)
\[
\limsup_{n \to \infty} \frac{\log R_{Q_n(x)}(x)}{-\log \mu(Q_n(x))} \leq 1 + \varepsilon \cdot \limsup_{n \to \infty} \frac{-n \log 2}{\log \mu(Q_n(x))} = 1 + \varepsilon \cdot \limsup_{n \to \infty} \frac{-n \log 2}{\log \mu(B(x, 2^{-n}))} \leq 1 + \varepsilon \cdot \limsup_{r \to 0} \frac{-n \log 2}{\log \mu(B(x, r))}
\]
since \( Q_n(x) \subset B(x, 2^{-n}) \). Hence we have
\[
\limsup_{n \to \infty} \frac{\log R_{Q_n(x)}(x)}{-\log \mu(Q_n(x))} \leq 1 + \frac{\varepsilon}{d_{\mu}(x)}.
\]

By the assumption of \( d_{\mu}(x) > 0 \) for almost every \( x \), we have
\[
\limsup_{n \to \infty} \frac{\log R_{Q_n(x)}(x)}{-\log \mu(Q_n(x))} \leq 1
\]
for almost every \( x \). \( \square \)

Proof of Theorem 1.1. By Lemma 2.1 we have \( \log \mu(B(x, 2^{-n})) \leq \log \mu(Q_n(x)) + 2 \log n \) and \( \log R_{B(x, 2^{-n})}(x) \leq \log R_{Q_n(x)}(x) \) from \( Q_n(x) \subset B(x, 2^{-n}) \). Therefore,
\[
\frac{\log R_{B(x, 2^{-n})}(x)}{-\log \mu(B(x, 2^{-n}))} \leq \frac{\log R_{Q_n(x)}(x)}{-\log \mu(Q_n(x)) - 2 \log n}
\]
for sufficiently large \( n \). By Proposition 2.2

\[
\limsup_{n \to \infty} \frac{\log R_{B(x,2^{-n})}(x)}{-\log \mu(B(x,2^{-n}))} \leq \limsup_{n \to \infty} \left( \frac{\log R_{Q_n(x)}(x)}{-\log \mu(Q_n(x))} \cdot \frac{\log \mu(Q_n(x))}{\log \mu(Q_n(x)) + 2 \log n} \right) \leq \limsup_{n \to \infty} \frac{1}{1 + 2 \log n / \log \mu(Q_n(x))}.
\]

Since

\[
d_n(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \leq \limsup_{n \to \infty} \frac{\log \mu(B(x,2^{-n}))}{-n \log 2} \leq \liminf_{n \to \infty} \frac{\log \mu(Q_n(x))}{-n \log 2},
\]

for large \( n \) we see

\[
\log \mu(Q_n(x)) < -\frac{n}{2} d_n(x) \log 2.
\]

Hence we have

\[
\limsup_{n \to \infty} \frac{\log R_{B(x,2^{-n})}(x)}{-\log \mu(B(x,2^{-n}))} \leq \limsup_{n \to \infty} \left( 1 - \frac{4 \log n}{n d_n(x) \log 2} \right)^{-1} = 1. \quad \Box
\]

3. Return time for irrational rotations

In this section we prove Theorem 1.3.

We need some properties on diophantine approximations. For more details, consult [6] and [10]. For an irrational number \( 0 < \theta < 1 \), we have a unique continued fraction expansion;

\[
\theta = [a_1, a_2, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}
\]

if \( a_i \geq 1 \) for all \( i \geq 1 \). Put \( p_0 = 0 \) and \( q_0 = 1 \). Choose \( p_i \) and \( q_i \) for \( i \geq 1 \) such that \((p_i, q_i) = 1\) and

\[
\frac{p_i}{q_i} = [a_1, a_2, \ldots, a_i] = \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_i}}}
\]

We call each \( a_i \) the \( i \)-th partial quotient and \( p_i/q_i \) the \( i \)-th convergent. Then the denominator \( q_i \) and the numerator \( p_i \) of the \( i \)-th convergent satisfy the following properties: \( q_{i+2} = a_{i+2}q_{i+1} + q_i \) and \( p_{i+2} = a_{i+2}p_{i+1} + p_i \) and

\[
\frac{1}{2q_i+1} < \frac{1}{q_i+1} < \|q_i\theta\| < \frac{1}{q_i+1}
\]

for \( i \geq 1 \).
It is well known [6] that \( \| j \theta \| \geq \| q_i \theta \| \) for \( 0 < j < q_{i+1} \) and \( \theta - p_i/q_i \) is positive if and only if \( i \) is even. Thus, by the definition of type \((\alpha, \beta)\) in Definition 1.2, we have
\[
\eta = \sup \left\{ t > 0 : \liminf_{i \to \infty} q^*_i \| q_i \theta \| = 0 \right\},
\]
\[
\alpha = \sup \left\{ t > 0 : \liminf_{i \to \infty} q_{2i+1} \| q_{2i+1} \theta \| = 0 \right\},
\]
\[
\beta = \sup \left\{ t > 0 : \liminf_{i \to \infty} q_{2i} \| q_{2i} \theta \| = 0 \right\}.
\]

And we have the following lemma:

**Lemma 3.1.** For any \( \epsilon > 0 \) and \( C > 0 \), we have (i)
\[
q^*_{2i+1} \| q_{2i+1} \theta \| > C \quad \text{and} \quad q^*_{2i} \| q_{2i} \theta \| > C.
\]
for sufficiently large integer \( i \), and (ii) there are infinitely many odd \( i \)'s such that
\[
q^*_{2i-1} \| q_i \theta \| < C \quad \text{and even} \ i \)'s such that
\[
q^*_{2i-2} \| q_i \theta \| < C.
\]

It is known that the first return time \( R_E \) of an irrational rotation \( T \) has at most three values if \( E \) is an interval [12]. For the proof consult [8].

**FACT 3.2.** Let \( T \) be an irrational rotation and \( b \in (0, \| \theta \|) \) a fixed real number. Moreover let \( i \geq 0 \) be an integer such that \( \| q_i \theta \| < b \leq \| q_{i-1} \theta \| \) and put
\[
K = \max\{k \geq 0 : k \| q_i \theta \| + \| q_{i+1} \theta \| < b\}.
\]
If \( i \) is even, then
\[
R_{(0,b)}(x) = \begin{cases} q_i, & 0 < x < b - \| q_i \theta \|, \\ q_{i+1} - (K-1)q_i, & b - \| q_i \theta \| \leq x \leq K \| q_i \theta \| + \| q_{i+1} \theta \|, \\ q_{i+1} - Kq_i, & K \| q_i \theta \| + \| q_{i+1} \theta \| < x < b. \end{cases}
\]
If \( i \) is odd, then
\[
R_{(0,b)}(x) = \begin{cases} q_i - Kq_i, & 0 < x < b - K \| q_i \theta \| - \| q_{i+1} \theta \|, \\ q_{i+1} - (K-1)q_i, & b - K \| q_i \theta \| - \| q_{i+1} \theta \| \leq x \leq \| q_i \theta \|, \\ q_i, & \| q_i \theta \| < x < b. \end{cases}
\]
And we have \( R_{(0,b)}(0) = q_i \) for even \( i \) and \( R_{(0,b)}(0) = q_{i+1} - Kq_i \) for odd \( i \).

Note that the value at the middle interval is the sum of the other two values and \( 0 \leq K \leq q_{i+1} - 1 \) since \( \| q_{i-1} \theta \| = a_{i+1} \| q_i \theta \| + \| q_{i+1} \theta \| \).
Lemma 3.4. Let $i$ be an integer such that $\|q_i\| < \mu(B(x, r; s)) \leq \|q_i+1\|$. Put $K = \max\{k \geq 0: k \|q_i\| + \|q_i+1\| < \mu(B(x, r; s))\}$ as in Fact 3.2. Then

(i) if $i$ is even, then $R_{B(x, r; s)}(x) = q_i$ for $r > \|q_i\|$ and $R_{B(x, r; s)}(x) \geq q_{i+1} - Kq_i$ for $r \leq \|q_i\|$, 

(ii) if $i$ is odd, then $R_{B(x, r; s)}(x) = q_i$ for $r^* > \|q_i\|$ and $R_{B(x, r; s)}(x) \geq q_{i+1} - Kq_i$ for $r^* \leq \|q_i\|$.

Proof. Put $b = \mu(B(x, r; s)) = r^* + r$ and apply Fact 3.2. Then $R_{\mu(B(x, r; s))}(x) = R_{(0,b)}(r^*)$ for $s < \infty$ and $R_{\mu(B(x, r; s))}(x) = R_{(0,b)}(0)$ for $s = \infty$. □

By the symmetry, we only consider the case $s \geq 1$.

Proposition 3.5.

$$\liminf_{r \to 0^+} \frac{\log R_{B(x, r; s)}(x)}{-\log \mu(B(x, r; s))} \geq \min \left\{ \frac{1}{\beta}, \frac{s}{\alpha} \right\}.$$

Proof. If $\|q_{2i}\| < \mu(B(x, r; s)) \leq \|q_{2i-1}\|$, then for any $C > 0$ and $\epsilon > 0$ by Lemma 3.4 (i) and Lemma 3.1 (i) we have

$$R_{B(x, r; s)}(x) \geq q_{2i} > \frac{C^{1/(\beta+\epsilon)}}{\|q_{2i}\|^{1/(\beta+\epsilon)}} > \frac{C^{1/(\beta+\epsilon)}}{\mu(B(x, r; s))^{1/(\beta+\epsilon)}}.$$

If $\|q_{2i+1}\| < \mu(B(x, r; s)) \leq \|q_{2i}\|$ and $r^* \geq \|q_{2i+1}\|$, then

$$R_{B(x, r; s)}(x) = q_{2i+1} > \frac{C^{1/(\alpha+\epsilon)}}{\|q_{2i+1}\|^{1/(\alpha+\epsilon)}} > \frac{C^{1/(\alpha+\epsilon)}}{\mu(B(x, r; s))^{1/(\alpha+\epsilon)}}.$$

If $\|q_{2i+1}\| < \mu(B(x, r; s)) \leq \|q_{2i}\|$ and $r^* \leq \|q_{2i+1}\|$, then by Remark 3.3

$$R_{B(x, r; s)}(x) \geq q_{2i+2} - Kq_{2i+1} > \frac{1}{2} (q_{2i+2} - (K - 1)q_{2i+1}) > \frac{1}{2\mu(B(x, r; s))}. □$$

Proposition 3.6.

$$\limsup_{r \to 0^+} \frac{\log R_{B(x, r; s)}(x)}{-\log \mu(B(x, r; s))} \leq \min\{\alpha, s\}.$$

Proof. Suppose $\|q_{2i+1}\| < \mu(B(x, r; s)) \leq \|q_{2i}\|$. If $r^* > \|q_{2i+1}\|$, then

$$R_{B(x, r; s)}(x) = q_{2i+1} < \frac{1}{\|q_{2i}\|} \leq \frac{1}{\mu(B(x, r; s))}.$$
If $r^s \leq \|q_{2i+1}\|$, then
\[
\mu(B(x, r; s)) \leq \|q_{2i+1}\| + \|q_{2i+1}\|^s \leq 2\|q_{2i+1}\|^s,
\]
so we have
\[
(1) \quad R_{B(x, r; s)}(x) \leq q_{2i+2} + q_{2i+1} < 2q_{2i+1} < \frac{2}{\|q_{2i+1}\|^s} \leq \frac{2 \cdot 2^s}{\mu(B(x, r; s))^s}.
\]
Also by Lemma 3.1 (i) for any $C > 0$ and $\epsilon > 0$ we have
\[
(2) \quad R_{B(x, r; s)}(x) < \frac{2}{\|q_{2i+1}\|^s} < \frac{2q_{2i+1}}{C \|q_{2i+1}\|^s} \leq \frac{2}{C \mu(B(x, r; s))}\mu(B(x, r; s))^{s+\epsilon}.
\]
Suppose $\|q_{2i}\| < \mu(B(x, r; s)) \leq \|q_{2i+1}\|$. If $r > \|q_{2i}\|$, then
\[
R_{B(x, r; s)}(x) = q_{2i} < \frac{1}{\|q_{2i}\|} < \frac{1}{\mu(B(x, r; s))}.
\]
If $r \leq \|q_{2i}\|$, then
\[
R_{B(x, r; s)}(x) \leq q_{2i+1} + q_{2i} < 2q_{2i+1} < \frac{2}{\|q_{2i}\|} \leq \frac{2}{r} \leq \frac{4}{\mu(B(x, r; s))}.
\]
Since $\alpha \geq 1$ and $s \geq 1$, by (1) and (2), we have
\[
\limsup_{r \to 0^+} \frac{\log R_{B(x, r; s)}(x)}{-\log \mu(B(x, r; s))} \leq \min\{\alpha, s\}. \quad \square
\]

**Proposition 3.7.**

\[
\liminf_{r \to 0^+} \frac{\log R_{B(x, r; s)}(x)}{-\log \mu(B(x, r; s))} \leq \min \left\{ \frac{1}{\beta^s}, \frac{s}{\alpha} \right\}.
\]

Proof. From Lemma 3.1 (ii) for any $C > 0$ and $\epsilon > 0$ we have infinitely many even i’s such that
\[
q_i^{\beta^s} \|q_i\| < C.
\]
Put $r = \|q_i\| + \|q_{i+1}\|/2$ for such i. Then
\[
\|q_{i-1}\| < \mu(B(x, r; s)) \leq 2r \leq 2\|q_i\| + \|q_{i+1}\| \leq \|q_{i-2}\|.
\]
If $\mu(B(x, r; s)) \leq \|q_{i-1}\|$, then by Lemma 3.4 (i), we have
\[
R_{B(x, r; s)}(x) = q_i < \frac{C^{1/(\beta^s)}}{\|q_i\|^{1/(\beta^s)}} < \frac{C^{1/(\beta^s)}}{r^{1/(\beta^s)}}.
\]
If \( \|q_{i-1}\theta\| < \mu(B(x, r; s)) \leq \|q_{i-2}\theta\| \), then
\[
R_{B(x, r; s)}(x) \leq q_i + q_{i-1} \leq 2q_i < \frac{2C^{1/(\beta-\epsilon)}}{\|q_i\theta\|^{1/(\beta-\epsilon)}} < \frac{2C^{1/(\beta-\epsilon)}}{r^{1/(\beta-\epsilon)}}.
\]
Hence
\[
\liminf_{r \to 0^+} \frac{-\log R_{B(x, r; s)}(x)}{-\log r} \leq \frac{1}{\beta}.
\]
(3)

Since \( \beta \geq 1 \), we only consider the case where \( 1 \leq s < \alpha \). By Lemma 3.1 (ii) there are infinitely many odd \( i \)'s such that \( q_i^{\alpha-\epsilon}\|q_i\theta\| < C \) with \( 0 < s < \alpha - \epsilon \) for any \( C > 0 \). Put \( r^x = 2\|q_i\theta\| \) for such \( i \). Then
\[
\mu(B(x, r; s)) = r + r^x \leq 4\|q_i\theta\|^{1/s} < \frac{4C^{1/s}}{q_i} < 4C^{1/s}2^{(\alpha-\epsilon)/s}\|q_{i-1}\theta\|^{{(\alpha-\epsilon)/s}}.
\]
For large \( i \) so that \( 2^{\alpha-\epsilon+2}C\|q_{i-1}\theta\|^{{\alpha-\epsilon-s}} < 1 \), we have
\[
\mu(B(x, r; s)) < \|q_{i-1}\theta\|.
\]
Thus by Lemma 3.4 (ii), we have
\[
R_{B(x, r; s)}(x) = q_i < \frac{C^{1/(\alpha-\epsilon)}}{\|q_i\theta\|^{1/(\alpha-\epsilon)}} < \frac{2^{s/(\alpha-\epsilon)}C^{1/(\alpha-\epsilon)}}{r^{s/(\alpha-\epsilon)}}
\]
for large \( i \). Hence
\[
\liminf_{r \to 0^+} \frac{-\log R_{B(x, r; s)}(x)}{-\log r} \leq \frac{s}{\alpha}.
\]
(4)

By (3) and (4), we complete the proof. \( \Box \)

**Proposition 3.8.**
\[
\limsup_{r \to 0^+} \frac{-\log R_{B(x, r; s)}(x)}{-\log \mu(B(x, r; s))} \geq \min\{\alpha, s\}.
\]

Proof. If we choose \( r \) as \( \mu(B(x, r; s)) = \|q_{i-1}\theta\| \), then
\[
R_{B(x, r; s)}(x) \geq q_i > \frac{1}{2\|q_{i-1}\theta\|} = \frac{1}{\mu(B(x, r; s))}
\]
so we have
\[
\limsup_{r \to 0^+} \frac{-\log R_{B(x, r; s)}(x)}{-\log \mu(B(x, r; s))} \geq 1.
\]
Thus we only consider the case that $s > 1$ and $\alpha > 1$:

(i) Suppose that there are only finitely many $i$’s such that

$$2^s q_{2i+1}^s \|q_{2i+1} \theta\| < 1.$$ 

In this case, $s \geq \alpha > 1$.

Choose $\epsilon$ as $0 < \epsilon < \alpha - 1$. By Lemma 3.1 (ii), there are infinitely many $i$’s such that

$$q_{2i+1}^\epsilon \|q_{2i+1} \theta\| < \frac{1}{4}.$$ 

Put $r = (1/2)\|q_{2i} \theta\|$ for such $i$. Then we have

$$\mu(B(x, r; s)) = r^s + r \leq 2r = \|q_{2i} \theta\|$$

and

$$\mu(B(x, r; s)) = r^s + r \geq \frac{1}{2} \|q_{2i} \theta\| > \frac{1}{4q_{2i+1}} > \frac{1}{4q_{2i+1}} \|q_{2i+1} \theta\|.$$ 

And for large $i$ so as to $2^s q_{2i+1}^s \|q_{2i+1} \theta\| \geq 1$, we have

$$(5) \quad r^s = \frac{1}{2} \|q_{2i} \theta\|^s < \frac{1}{2^s q_{2i+1}^s} \leq \|q_{2i+1} \theta\|.$$ 

By the definition of $K$

$$K \|q_{2i+1} \theta\| + \|q_{2i+2} \theta\| < r^s + r = \|q_{2i+1} \theta\| + \frac{1}{2} \|q_{2i} \theta\|,$$

we have

$$(K - 1) \|q_{2i+1} \theta\| + \frac{\|q_{2i+2} \theta\|}{2} < \frac{a_{2i+2}}{2} \|q_{2i+1} \theta\|$$

since $\|q_{2i} \theta\| = a_{2i+2} \|q_{2i+1} \theta\| + \|q_{2i+2} \theta\|$. Therefore $K < 1 + a_{2i+2}/2$. Since $q_{2i+2} = a_{2i+2} q_{2i+1} + q_2$, we have

$$q_{2i+2} - K q_{2i+1} > q_{2i+2} - \frac{a_{2i+2}}{2} q_{2i+1} - q_{2i+1} = \frac{1}{2} q_{2i+2} + \frac{1}{2} q_{2i} - q_{2i+1}$$

$$> \frac{1}{2} q_{2i+2} - q_{2i+1} > \frac{1}{4 \|q_{2i+1} \theta\|} - q_{2i+1}$$

$$> q_{2i+1}^{\alpha - \epsilon} - q_{2i+1} = q_{2i+1}^{\alpha - \epsilon} (1 - q_{2i+1}^{1+\epsilon - \alpha}) > \frac{1 - q_{2i+1}^{1+\epsilon - \alpha}}{\|q_{2i} \theta\|^{\alpha - \epsilon}}.$$ 

From $\alpha > 1 + \epsilon$, we have $q_{2i+1}^{\alpha - 1 - \epsilon} > 2$ for large $i$. Hence by Lemma 3.4 (ii) and (5) for large $i$, we have

$$(6) \quad R_{B(x, r; s)}(x) \geq q_{2i+2} - K q_{2i+1} > \frac{1 - q_{2i+1}^{1+\epsilon - \alpha}}{\|q_{2i} \theta\|^{\alpha - \epsilon}} > \frac{1}{2 \|q_{2i} \theta\|^{\alpha - \epsilon}} > \frac{2^{\alpha - \epsilon}}{2 r^{\alpha - \epsilon}}.$$
(ii) Suppose that there are infinitely many $i$'s such that

$$2^s q_{2i+1} \|q_{2i+1} \theta\| < 1.$$ 

In this case, $1 < s \leq \alpha$. Choose $r^s = \|q_{2i+1} \theta\|/2$ for such $i$. Then we have

$$r = \frac{\|q_{2i+1} \theta\|^{1/s}}{2^{1/s}} < \frac{1}{2^{1/s} 2q_{2i+1}} < \frac{\|q_{2i} \theta\|}{2^{1/s}}$$

and

$$\mu(B(x, r; s)) = r + r^s < \frac{\|q_{2i} \theta\|}{2^{1/s}} + \frac{\|q_{2i} \theta\|^s}{2} = \|q_{2i} \theta\| \left(2^{-1/s} + \frac{\|q_{2i} \theta\|^{-1}}{2}\right).$$

Therefore for large $i$ so as to $\|q_{2i} \theta\|^{s-1} < 2(1 - 2^{-1/s})$, we have

$$\mu(B(x, r; s)) < \|q_{2i} \theta\|.$$

Also we see

$$\mu(B(x, r; s)) = r^s + r > 2r^s = \|q_{2i+1} \theta\|.$$

Since

$$K \|q_{2i+1} \theta\| + \|q_{2i+2} \theta\| < r^s + r = \frac{\|q_{2i+1} \theta\|^{1/s}}{2} + \frac{\|q_{2i+1} \theta\|^{1/s}}{2},$$

we have

$$K \leq \frac{1}{2} + \frac{\|q_{2i+1} \theta\|^{1/s-1}}{2^{1/s}} < \frac{1}{2} + \frac{2q_{2i+2} \|q_{2i+1} \theta\|^{1/s}}{2^{1/s}} < \frac{1}{2} + \frac{2q_{2i+2}}{2^{1/s}} \frac{1}{2q_{2i+1}}.$$

Hence by Lemma 3.4 (ii)

$$R_{B(x, r; s)}(x) \geq q_{2i+2} - K q_{2i+1} > q_{2i+2} - \frac{q_{2i+2}}{2^{1/s}} - \frac{q_{2i+1}}{2},$$

$$> (1 - 2^{-1/s}) q_{2i+2} - \frac{q_{2i+1}}{2} > \frac{1 - 2^{-1/s}}{2} \|q_{2i+1} \theta\| - \frac{1}{4\|q_{2i+1} \theta\|^{1/s}}$$

$$> \frac{1 - 2^{-1/s}}{4\|q_{2i+1} \theta\|} = (1 - 2^{-1/s}) \frac{1}{8r^s}$$

for large $i$ so that

$$\|q_{2i+1} \theta\|^{1-1/s} < 1 - 2^{-1/s}.$$
Hence by (6) and (7)

\[
\limsup_{r \to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log r} \geq \min\{\alpha, \beta\},
\]

which completes the proof.

By Proposition 3.5, 3.6, 3.7 and 3.8, we have the proof of Theorem 1.3.

\[\Box\]

References


