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THE RECURRENCE TIME FOR IRRATIONAL ROTATIONS

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Abstract

Let T be a measure preserving transformation on $X \subset \mathbb{R}^d$ with a Borel measure μ and R_E be the first return time to a subset E. If (X, μ) has positive pointwise dimension for almost every x, then for almost every x

$$\limsup_{r \to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log \mu(B(x, r))} \leq 1,$$

where $B(x, r)$ the the ball centered at x with radius r. But the above property does not hold for the neighborhood of the 'skewed' ball. Let $B(x, r; s) = (x - r^s, x + r)$ be an interval for $s > 0$. For arbitrary $\alpha \geq 1$ and $\beta \geq 1$, there are uncountably many irrational numbers whose rotation satisfy that

$$\limsup_{r \to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log \mu(B(x, r; s))} = \alpha \quad \text{and} \quad \liminf_{r \to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log \mu(B(x, r; s))} = \frac{1}{\beta}$$

for some s.

1. Introduction

Let μ be a probability measure on X and $T : X \to X$ be a μ-preserving transformation. For a measurable subset $E \subset X$ with $\mu(E) > 0$ and a point $x \in E$ which returns to E under iteration by T, we define the first return time R_E on E by

$$R_E(x) = \min \left\{ j \geq 1 : T^j x \in E \right\}.$$

Kac's lemma [5] states that

$$\int_E R_E(x)\, d\mu \leq 1.$$

If T is ergodic, then the equality holds.

For a decreasing sequence of subsets $\{E_n\}$ containing x, R_{E_n} is an increasing sequence. The asymptotic behavior between R_{E_n} and the measure of E_n has been studied after Wyner and Ziv's work [13] for ergodic processes. Let $\mathcal{P}$ be a partition of X and $\{\mathcal{P}_n\}$ be a sequence of partitions of X obtained by $\mathcal{P}_n = \mathcal{P} \vee T^{-1}\mathcal{P} \vee \cdots \vee T^{-n+1}\mathcal{P}$,

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where $\mathcal{P} \vee \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, \ Q \in \mathcal{Q}\}$. Ornstein and Weiss [9] showed that if T is ergodic, then

$$\lim_{n\to\infty} \frac{\log R_{P_n(x)}(x)}{n} = h(T, \mathcal{P}) \quad \text{a.e.},$$

where $P_n(x)$ is the element in $\mathcal{P}_n$ containing x. Therefore, by the Shannon-McMillan-Brieman theorem, if the entropy with respect to a partition $\mathcal{P}$, $h(T, \mathcal{P})$ is positive, then we have

$$\lim_{n\to\infty} \frac{\log R_{P_n(x)}(x)}{-\log \mu(P_n(x))} = 1 \quad \text{a.e.}$$

Let (X, d) be a metric space and $B(x, r) = \{y : d(x, y) < r\}$. Define the upper and lower pointwise dimension of μ at x by

$$\overline{d}_\mu(x) = \limsup_{r\to 0^+} \frac{\log \mu(B(x, r))}{\log r}, \quad \underline{d}_\mu(x) = \liminf_{r\to 0^+} \frac{\log \mu(B(x, r))}{\log r}.$$

Now we have another recurrence theorem for the decreasing sequence of balls.

Theorem 1.1. *Let $T : X \to X$ be a Borel measurable transformation on a measurable set $X \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$ and μ be a T-invariant probability measure on X. If $\underline{d}_\mu(x) > 0$ for μ-almost every x, then we have*

$$\limsup_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log \mu(B(x, r))} \leq 1$$

for μ-almost every x.

This theorem is a modified version of Barreira and Saussol's result [1] which states that

$$\limsup_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log r} \leq \overline{d}_\mu(x), \quad \liminf_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log r} \leq \underline{d}_\mu(x).$$

See also [2], [3], [7], and [11] for the transformations which satisfy that

$$\lim_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log r} = \text{dimension of } \mu.$$

Note that for some irrational rotations the limit does not exist [4].

So one might expect that if we choose a decreasing sequence of sets E_n as 'good' neighborhoods of x

$$\limsup_n \frac{\log R_{E_n}(x)}{-\log \mu(E_n)} \leq 1.$$

However, we show that even for interval E_n's on X the limsup can be larger than 1 for some irrational rotations.

For $t \in \mathbb{R}$ we define $\|\cdot\|$ and $\{\cdot\}$ by

$$\|t\| = \min_{n\in\mathbb{Z}} |t-n|, \qquad \{t\} = t - \lfloor t \rfloor,$$

i.e., the distance to the nearest integer and the nearest integer which is less than or equal to t, respectively.

An irrational number θ, $0<\theta<1$, is said to be of type η if

$$\eta = \sup\left\{ t > 0 \colon \liminf_{j\to\infty} j^t \|j\theta\| = 0 \right\}.$$

Every irrational number is of type $\eta \geq 1$. The set of irrational numbers of type 1 has measure 1 and includes the set of irrational numbers with bounded partial quotients, which is of measure 0. There exist numbers of type ∞, called Liouville numbers. Here we introduce a new definition on type of irrational numbers:

DEFINITION 1.2. An irrational number θ, $0<\theta<1$, is said to be of *type* (α, β) if

$$\alpha = \sup\left\{ t > 0 \colon \liminf_{j\to\infty} j^t \{-j\theta\} = 0 \right\},$$
$$\beta = \sup\left\{ t > 0 \colon \liminf_{j\to\infty} j^t \{j\theta\} = 0 \right\}.$$

For example, if the partial quotients of an irrational number θ is $a_{2k} = 2^{2^k}$ for $k \geq 1$ and $a_{2k+1} = 1$ for $k \geq 0$, then θ is of type $(2,1)$. Note that $\alpha, \beta \geq 1$ and $\eta = \max\{\alpha, \beta\}$. For each $\alpha, \beta > 1$ there are uncountably many (but measure zero) θ's which are of type (α, β).

Let $0<\theta<1$ be an irrational number and $T\colon [0,1) \to [0,1)$ an irrational rotation, i.e.,

$$Tx = x + \theta \pmod 1.$$

Then T preserves the Lebesgue measure μ on $X = [0,1)$.

Let $B(x,r;s)$ be an interval $(x-r^s, x+r)$, $s>0$ and put $B(x,r;\infty) = [x, x+r)$.

Theorem 1.3. *If θ is of type (α,β), then for $1 \leq s \leq \infty$ and any $x \in [0,1)$, we have*

$$\limsup_{r\to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log \mu(B(x,r;s))} = \min\{\alpha, s\}, \qquad \liminf_{r\to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log \mu(B(x,r;s))} = \min\left\{\frac{1}{\beta}, \frac{s}{\alpha}\right\}$$

and for $0 < s < 1$ *and any* $x \in [0,1)$, *we have*

$$\limsup_{r\to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log \mu(B(x,r;s))} = \min\left\{\beta, \frac{1}{s}\right\}, \quad \liminf_{r\to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log \mu(B(x,r;s))} = \min\left\{\frac{1}{\alpha}, \frac{1}{s\beta}\right\}.$$

By the symmetry, we have

$$\limsup_{r\to 0^+} \frac{\log R_{(x-r,x]}(x)}{-\log r} = \beta, \quad \liminf_{r\to 0^+} \frac{\log R_{(x-r,x]}(x)}{-\log r} = \frac{1}{\alpha}.$$

Note that if $s = 1$ the theorem is reduced to

$$\limsup_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log \mu(B(x,r))} = 1, \quad \liminf_{r\to 0^+} \frac{\log R_{B(x,r)}(x)}{-\log \mu(B(x,r))} = \frac{1}{\eta},$$

which was shown in [4].

2. Return time for measure space

In this section we prove Theorem 1.1. Let $X \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$. Define

$$\overline{\mathcal{Q}_n} = \left\{\left[i_1 2^{-n}, (i_1+1)2^{-n}\right) \times \cdots \times \left[i_d 2^{-n}, (i_d+1)2^{-n}\right) : (i_1, \ldots, i_d) \in \mathbb{Z}^d\right\}$$

to be the dyadic partition of $\mathbb{R}^d$ and $\mathcal{Q}_n = \{X \cap A : A \in \overline{\mathcal{Q}_n}\}$. Let $Q_n(x)$ as the element of $\mathcal{Q}_n$ containing x.

In order to prove Theorem 1.1 we need a lemma, which is a slight modification of the weakly diametrically regularity in [1].

Lemma 2.1. *Let* μ *be a Borel probability measure on* $\mathbb{R}^d$. *For* μ*-almost every* x *we have*

$$\mu\left(B\left(x, 2^{-n}\right)\right) \le n^2 \mu(Q_n(x))$$

for sufficiently large n.

Proof. Let

$$E_n = \left\{x : \mu\left(B\left(x, 2^{-n}\right)\right) > n^2 \mu(Q_n(x))\right\}.$$

For each $A \in \mathcal{Q}_n$ with $A \cap E_n \neq \emptyset$ choose one $x \in A \cap E_n$ and let F be a set of such x's. Then we have

$$E_n \subset \bigcup_{x \in F} Q_n(x)$$

and

$$\mu(E_n) \leq \sum_{x \in F} \mu(Q_n(x)) < \sum_{x \in F} n^{-2} \mu\left(B\left(x, 2^{-n}\right)\right).$$

There is a constant D depending on d such that for each $y \in \mathbb{R}^d$, there are at most D x's in F such that $x \in B(y, 2^{-n})$. Therefore, we have

$$\sum_{x \in F} \mu\left(B\left(x, 2^{-n}\right)\right) \leq D \cdot \mu\left(\mathbb{R}^d\right) = D$$

and

$$\mu(E_n) < \sum_{x \in F} n^{-2} \mu\left(B\left(x, 2^{-n}\right)\right) \leq D n^{-2}.$$

Since

$$\sum_n \mu(E_n) < D \sum_n n^{-2} < \infty,$$

the first Borel-Cantelli lemma completes the proof. □

Proposition 2.2. *Let $T: X \to X$ be a Borel measurable transformation on a measurable set $X \subset \mathbb{R}^d$ and μ be a T-invariant probability measure on X. If $\underline{d}_\mu(x) > 0$ for μ-almost every x, then*

$$\limsup_{n \to \infty} \frac{\log R_{Q_n(x)}(x)}{-\log \mu(Q_n(x))} \leq 1$$

for μ-almost every x.

Proof. Choose an arbitrary $\epsilon > 0$. For an $A \in \mathcal{Q}_n$, we have by Markov's inequality

$$\mu\left(\left\{x \in A : R_A(x) \geq \frac{2^{n\epsilon}}{\mu(A)}\right\}\right) \leq \mu(A) 2^{-n\epsilon} \int_A R_A(x)\, d\mu.$$

By Kac's lemma we have

$$\mu\left(\left\{x \in A : R_A(x) \geq \frac{2^{n\epsilon}}{\mu(A)}\right\}\right) \leq \mu(A) 2^{-n\epsilon}.$$

Hence we have

$$\mu\left(\left\{x \in X\colon R_{Q_n(x)}(x) \geq \frac{2^{n\epsilon}}{\mu(Q_n(x))}\right\}\right) \leq \sum_{A\in\mathcal{Q}_n} \mu(A)2^{-n\epsilon} \leq 2^{-n\epsilon}$$

and

$$\sum_{n=1}^{\infty} \mu\left(\left\{x \in X\colon R_{Q_n(x)}(x) \geq \mu(Q_n(x))^{-1}2^{-n\epsilon}\right\}\right) < \infty.$$

By the first Borel-Cantelli lemma, for almost every x we have

$$R_{Q_n(x)}(x) < \frac{2^{n\epsilon}}{\mu(Q_n(x))}$$

eventually. Thus for almost every x

$$\begin{aligned}\limsup_{n\to\infty} \frac{\log R_{Q_n(x)}(x)}{-\log\mu(Q_n(x))} &\leq 1 + \epsilon \cdot \limsup_{n\to\infty} \frac{-n\log 2}{\log\mu(Q_n(x))} \\ &\leq 1 + \epsilon \cdot \limsup_{n\to\infty} \frac{-n\log 2}{\log\mu(B(x, 2^{-n}))} \\ &\leq 1 + \epsilon \cdot \limsup_{r\to 0} \frac{\log r}{\log\mu(B(x, r))}\end{aligned}$$

since $Q_n(x) \subset B(x, 2^{-n})$. Hence we have

$$\limsup_{n\to\infty} \frac{\log R_{Q_n(x)}(x)}{-\log\mu(Q_n(x))} \leq 1 + \frac{\epsilon}{\underline{d}_\mu(x)}.$$

By the assumption of $\underline{d}_\mu(x) > 0$ for almost every x, we have

$$\limsup_{n\to\infty} \frac{\log R_{Q_n(x)}(x)}{-\log\mu(Q_n(x))} \leq 1$$

for almost every x. □

Proof of Theorem 1.1. By Lemma 2.1 we have $\log\mu(B(x, 2^{-n})) \leq \log\mu(Q_n(x)) + 2\log n$ and $\log R_{B(x,2^{-n})}(x) \leq \log R_{Q_n(x)}(x)$ from $Q_n(x) \subset B(x, 2^{-n})$. Therefore,

$$\frac{\log R_{B(x,2^{-n})}(x)}{-\log\mu(B(x, 2^{-n}))} \leq \frac{\log R_{Q_n(x)}(x)}{-\log\mu(Q_n(x)) - 2\log n}$$

for sufficiently large n. By Proposition 2.2

$$\limsup_{n\to\infty} \frac{\log R_{B(x,2^{-n})}(x)}{-\log \mu(B(x,2^{-n}))} \le \limsup_{n\to\infty} \left(\frac{\log R_{Q_n(x)}(x)}{-\log \mu(Q_n(x))} \cdot \frac{\log \mu(Q_n(x))}{\log \mu(Q_n(x)) + 2\log n} \right)$$
$$\le \limsup_{n\to\infty} \frac{1}{1 + 2\log n / \log \mu(Q_n(x))}.$$

Since

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \le \liminf_{n\to\infty} \frac{\log \mu(B(x,2^{-n}))}{-n\log 2} \le \liminf_{n\to\infty} \frac{\log \mu(Q_n(x))}{-n\log 2},$$

for large n we see

$$\log \mu(Q_n(x)) < -\frac{n}{2}\underline{d}_\mu(x)\log 2.$$

Hence we have

$$\limsup_{n\to\infty} \frac{\log R_{B(x,2^{-n})}(x)}{-\log \mu(B(x,2^{-n}))} \le \limsup_{n\to\infty} \left(1 - \frac{4\log n}{n\underline{d}_\mu(x)\log 2}\right)^{-1} = 1. \qquad \square$$

3. Return time for irrational rotations

In this section we prove Theorem 1.3.

We need some properties on diophantine approximations. For more details, consult [6] and [10]. For an irrational number $0 < \theta < 1$, we have a unique continued fraction expansion;

$$\theta = [a_1, a_2, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

if $a_i \ge 1$ for all $i \ge 1$. Put $p_0 = 0$ and $q_0 = 1$. Choose p_i and q_i for $i \ge 1$ such that $(p_i, q_i) = 1$ and

$$\frac{p_i}{q_i} = [a_1, a_2, \dots, a_i] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_i}}}}.$$

We call each a_i the i-th partial quotient and p_i/q_i the i-th convergent. Then the denominator q_i and the numerator p_i of the i-th convergent satisfy the following properties: $q_{i+2} = a_{i+2}q_{i+1} + q_i$, $p_{i+2} = a_{i+2}p_{i+1} + p_i$ and

$$\frac{1}{2q_{i+1}} < \frac{1}{q_{i+1} + q_i} < \|q_i\theta\| < \frac{1}{q_{i+1}}$$

for $i \ge 1$.

It is well known [6] that $\|j\theta\| \geq \|q_i\theta\|$ for $0 < j < q_{i+1}$ and $\theta - p_i/q_i$ is positive if and only if i is even. Thus, by the definition of type (α, β) in Definition 1.2, we have

$$\eta = \sup\left\{t > 0\colon \liminf_{i\to\infty} q_i^t \|q_i\theta\| = 0\right\},$$

$$\alpha = \sup\left\{t > 0\colon \liminf_{i\to\infty} q_{2i+1}^t \|q_{2i+1}\theta\| = 0\right\},$$

$$\beta = \sup\left\{t > 0\colon \liminf_{i\to\infty} q_{2i}^t \|q_{2i}\theta\| = 0\right\}.$$

And we have the following lemma:

Lemma 3.1. *For any $\epsilon > 0$ and $C > 0$, we have* (i)

$$q_{2i+1}^{\alpha+\epsilon}\|q_{2i+1}\theta\| > C \quad \textit{and} \quad q_{2i}^{\beta+\epsilon}\|q_{2i}\theta\| > C.$$

for sufficiently large integer i, and (ii) *there are infinitely many odd i's such that $q_i^{\alpha-\epsilon}\|q_i\theta\| < C$ and even i's such that $q_i^{\beta-\epsilon}\|q_i\theta\| < C$.*

It is known that the first return time R_E of an irrational rotation T has at most three values if E is an interval [12]. For the proof consult [8].

FACT 3.2. Let T be an irrational rotation and $b \in (0, \|\theta\|]$ a fixed real number. Moreover let $i \geq 0$ be an integer such that $\|q_i\theta\| < b \leq \|q_{i-1}\theta\|$ and put

$$K = \max\{k \geq 0\colon k\|q_i\theta\| + \|q_{i+1}\theta\| < b\}.$$

If i is even, then

$$R_{(0,b)}(x) = \begin{cases} q_i, & 0 < x < b - \|q_i\theta\|, \\ q_{i+1} - (K-1)q_i, & b - \|q_i\theta\| \leq x \leq K\|q_i\theta\| + \|q_{i+1}\theta\|, \\ q_{i+1} - Kq_i, & K\|q_i\theta\| + \|q_{i+1}\theta\| < x < b. \end{cases}$$

If i is odd, then

$$R_{(0,b)}(x) = \begin{cases} q_{i+1} - Kq_i, & 0 < x < b - K\|q_i\theta\| - \|q_{i+1}\theta\|, \\ q_{i+1} - (K-1)q_i, & b - K\|q_i\theta\| - \|q_{i+1}\theta\| \leq x \leq \|q_i\theta\|, \\ q_i, & \|q_i\theta\| < x < b. \end{cases}$$

And we have $R_{[0,b)}(0) = q_i$ for even i and $R_{[0,b)}(0) = q_{i+1} - Kq_i$ for odd i.

Note that the value at the middle interval is the sum of the other two values and $0 \leq K \leq a_{i+1} - 1$ since $\|q_{i-1}\theta\| = a_{i+1}\|q_i\theta\| + \|q_{i+1}\theta\|$.

REMARK 3.3. (i) For all i, $q_{i+1} - Kq_i > q_i$. (ii) By Kac's lemma $q_{i+1} - (K-1)q_i > 1/b$.

Lemma 3.4. *Let i be an integer such that $\|q_i\theta\| < \mu(B(x,r;s)) \leq \|q_{i-1}\theta\|$. Put $K = \max\{k \geq 0 : k\|q_i\theta\| + \|q_{i+1}\theta\| < \mu(B(x,r;s))\}$ as in* Fact 3.2. *Then*
(i) *if i is even, then $R_{B(x,r;s)}(x) = q_i$ for $r > \|q_i\theta\|$ and $R_{B(x,r;s)}(x) \geq q_{i+1} - Kq_i$ for $r \leq \|q_i\theta\|$,*
(ii) *if i is odd, then $R_{B(x,r;s)}(x) = q_i$ for $r^s > \|q_i\theta\|$ and $R_{B(x,r;s)}(x) \geq q_{i+1} - Kq_i$ for $r^s \leq \|q_i\theta\|$.*

Proof. Put $b = \mu(B(x,r;s)) = r^s + r$ and apply Fact 3.2. Then $R_{\mu(B(x,r;s))}(x) = R_{(0,b)}(r^s)$ for $s < \infty$ and $R_{\mu(B(x,r;s))}(x) = R_{[0,b)}(0)$ for $s = \infty$. □

By the symmetry, we only consider the case $s \geq 1$.

Proposition 3.5.

$$\liminf_{r\to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log \mu(B(x,r;s))} \geq \min\left\{\frac{1}{\beta}, \frac{s}{\alpha}\right\}.$$

Proof. If $\|q_{2i}\theta\| < \mu(B(x,r;s)) \leq \|q_{2i-1}\theta\|$, then for any $C > 0$ and $\epsilon > 0$ by Lemma 3.4 (i) and Lemma 3.1 (i) we have

$$R_{B(x,r;s)}(x) \geq q_{2i} > \frac{C^{1/(\beta+\epsilon)}}{\|q_{2i}\theta\|^{1/(\beta+\epsilon)}} > \frac{C^{1/(\beta+\epsilon)}}{\mu(B(x,r;s))^{1/(\beta+\epsilon)}}.$$

If $\|q_{2i+1}\theta\| < \mu(B(x,r;s)) \leq \|q_{2i}\theta\|$ and $r^s > \|q_{2i+1}\theta\|$, then

$$R_{B(x,r;s)}(x) = q_{2i+1} > \frac{C^{1/(\alpha+\epsilon)}}{\|q_{2i+1}\theta\|^{1/(\alpha+\epsilon)}} > \frac{C^{1/(\alpha+\epsilon)}}{\mu(B(x,r;s))^{s/(\alpha+\epsilon)}}.$$

If $\|q_{2i+1}\theta\| < \mu(B(x,r;s)) \leq \|q_{2i}\theta\|$ and $r^s \leq \|q_{2i+1}\theta\|$, then by Remark 3.3

$$R_{B(x,r;s)}(x) \geq q_{2i+2} - Kq_{2i+1} > \frac{1}{2}(q_{2i+2} - (K-1)q_{2i+1}) > \frac{1}{2\mu(B(x,r;s))}. \qquad \square$$

Proposition 3.6.

$$\limsup_{r\to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log \mu(B(x,r;s))} \leq \min\{\alpha, s\}.$$

Proof. Suppose $\|q_{2i+1}\theta\| < \mu(B(x,r;s)) \leq \|q_{2i}\theta\|$. If $r^s > \|q_{2i+1}\theta\|$, then

$$R_{B(x,r;s)}(x) = q_{2i+1} < \frac{1}{\|q_{2i}\theta\|} \leq \frac{1}{\mu(B(x,r;s))}.$$

If $r^s \le \|q_{2i+1}\theta\|$, then

$$\mu(B(x,r;s)) \le \|q_{2i+1}\theta\| + \|q_{2i+1}\theta\|^{1/s} \le 2\|q_{2i+1}\theta\|^{1/s},$$

so we have

$$R_{B(x,r;s)}(x) \le q_{2i+2} + q_{2i+1} < 2q_{2i+2} < \frac{2}{\|q_{2i+1}\theta\|} \le \frac{2\cdot 2^s}{\mu(B(x,r;s))^s}. \tag{1}$$

Also by Lemma 3.1 (i) for any $C > 0$ and $\epsilon > 0$ we have

$$R_{B(x,r;s)}(x) < \frac{2}{\|q_{2i+1}\theta\|} < \frac{2q_{2i+1}^{\alpha+\epsilon}}{C} < \frac{2}{C\|q_{2i}\theta\|^{\alpha+\epsilon}} \le \frac{2}{C\mu(B(x,r;s))^{\alpha+\epsilon}}. \tag{2}$$

Suppose $\|q_{2i}\theta\| < \mu(B(x,r;s)) \le \|q_{2i-1}\theta\|$. If $r > \|q_{2i}\theta\|$, then

$$R_{B(x,r;s)}(x) = q_{2i} < \frac{1}{\|q_{2i-1}\theta\|} \le \frac{1}{\mu(B(x,r;s))}.$$

If $r \le \|q_{2i}\theta\|$, then

$$R_{B(x,r;s)}(x) \le q_{2i+1} + q_{2i} < 2q_{2i+1} < \frac{2}{\|q_{2i}\theta\|} \le \frac{2}{r} \le \frac{4}{\mu(B(x,r;s))}.$$

Since $\alpha \ge 1$ and $s \ge 1$, by (1) and (2), we have

$$\limsup_{r\to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log \mu(B(x,r;s))} \le \min\{\alpha, s\}. \qquad \square$$

Proposition 3.7.

$$\liminf_{r\to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log \mu(B(x,r;s))} \le \min\left\{\frac{1}{\beta}, \frac{s}{\alpha}\right\}.$$

Proof. From Lemma 3.1 (ii) for any $C > 0$ and $\epsilon > 0$ we have infinitely many even i's such that

$$q_i^{\beta-\epsilon}\|q_i\theta\| < C.$$

Put $r = \|q_i\theta\| + \|q_{i+1}\theta\|/2$ for such i. Then

$$\|q_{i-1}\theta\| < \mu(B(x,r;s)) \le 2r \le 2\|q_i\theta\| + \|q_{i+1}\theta\| \le \|q_{i-2}\theta\|.$$

If $\mu(B(x,r;s)) \le \|q_{i-1}\theta\|$, then by Lemma 3.4 (i), we have

$$R_{B(x,r;s)}(x) = q_i < \frac{C^{1/(\beta-\epsilon)}}{\|q_i\theta\|^{1/(\beta-\epsilon)}} < \frac{C^{1/(\beta-\epsilon)}}{r^{1/(\beta-\epsilon)}}.$$

If $\|q_{i-1}\theta\| < \mu(B(x,r;s)) \le \|q_{i-2}\theta\|$, then

$$R_{B(x,r;s)}(x) \le q_i + q_{i-1} \le 2q_i < \frac{2C^{1/(\beta-\epsilon)}}{\|q_i\theta\|^{1/(\beta-\epsilon)}} < \frac{2C^{1/(\beta-\epsilon)}}{r^{1/(\beta-\epsilon)}}.$$

Hence

$$\liminf_{r\to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log r} \le \frac{1}{\beta}. \tag{3}$$

Since $\beta \ge 1$, we only consider the case where $1 \le s < \alpha$. By Lemma 3.1 (ii) there are infinitely many odd i's such that $q_i^{\alpha-\epsilon}\|q_i\theta\| < C$ with $0 < s < \alpha - \epsilon$ for any $C > 0$. Put $r^s = 2\|q_i\theta\|$ for such i. Then

$$\mu(B(x,r;s)) = r + r^s \le 4\|q_i\theta\|^{1/s} < \frac{4C^{1/s}}{q_i^{(\alpha-\epsilon)/s}} < 4C^{1/s}2^{(\alpha-\epsilon)/s}\|q_{i-1}\theta\|^{(\alpha-\epsilon)/s}.$$

For large i so that $2^{\alpha-\epsilon+2}C\|q_{i-1}\theta\|^{\alpha-\epsilon-s} < 1$, we have

$$\mu(B(x,r;s)) < \|q_{i-1}\theta\|.$$

Thus by Lemma 3.4 (ii), we have

$$R_{B(x,r;s)}(x) = q_i < \frac{C^{1/(\alpha-\epsilon)}}{\|q_i\theta\|^{1/(\alpha-\epsilon)}} < \frac{2^{s/(\alpha-\epsilon)}C^{1/(\alpha-\epsilon)}}{r^{s/(\alpha-\epsilon)}}$$

for large i. Hence

$$\liminf_{r\to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log r} \le \frac{s}{\alpha}. \tag{4}$$

By (3) and (4), we complete the proof. □

Proposition 3.8.

$$\limsup_{r\to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log \mu(B(x,r;s))} \ge \min\{\alpha, s\}.$$

Proof. If we choose r as $\mu(B(x,r;s)) = \|q_{i-1}\theta\|$, then

$$R_{B(x,r;s)}(x) \ge q_i > \frac{1}{2\|q_{i-1}\theta\|} = \frac{1}{\mu(B(x,r;s))}$$

so we have

$$\limsup_{r\to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log \mu(B(x,r;s))} \ge 1.$$

Thus we only consider the case that $s > 1$ and $\alpha > 1$:

(i) Suppose that there are only finitely many i's such that

$$2^s q_{2i+1}^s \|q_{2i+1}\theta\| < 1.$$

In this case, $s \geq \alpha > 1$.

Choose ϵ as $0 < \epsilon < \alpha - 1$. By Lemma 3.1 (ii), there are infinitely many i's such that

$$q_{2i+1}^{\alpha-\epsilon}\|q_{2i+1}\theta\| < \frac{1}{4}.$$

Put $r = (1/2)\|q_{2i}\theta\|$ for such i. Then we have

$$\mu(B(x,r;s)) = r^s + r \leq 2r = \|q_{2i}\theta\|$$

and

$$\mu(B(x,r;s)) = r^s + r \geq \frac{1}{2}\|q_{2i}\theta\| > \frac{1}{4q_{2i+1}} > \frac{1}{4q_{2i+1}^{\alpha-\epsilon}} > \|q_{2i+1}\theta\|.$$

And for large i so as to $2^s q_{2i+1}^s\|q_{2i+1}\theta\| \geq 1$, we have

$$r^s = \frac{1}{2^s}\|q_{2i}\theta\|^s < \frac{1}{2^s q_{2i+1}^s} \leq \|q_{2i+1}\theta\|. \tag{5}$$

By the definition of K

$$K\|q_{2i+1}\theta\| + \|q_{2i+2}\theta\| < r^s + r = \|q_{2i+1}\theta\| + \frac{1}{2}\|q_{2i}\theta\|,$$

we have

$$(K-1)\|q_{2i+1}\theta\| + \frac{\|q_{2i+2}\theta\|}{2} < \frac{a_{2i+2}}{2}\|q_{2i+1}\theta\|$$

since $\|q_{2i}\theta\| = a_{2i+2}\|q_{2i+1}\theta\| + \|q_{2i+2}\theta\|$. Therefore $K < 1 + a_{2i+2}/2$. Since $q_{2i+2} = a_{2i+2}q_{2i+1} + q_{2i}$, we have

$$\begin{aligned}
q_{2i+2} - Kq_{2i+1} &> q_{2i+2} - \frac{a_{2i+2}}{2}q_{2i+1} - q_{2i+1} = \frac{1}{2}q_{2i+2} + \frac{1}{2}q_{2i} - q_{2i+1} \\
&> \frac{1}{2}q_{2i+2} - q_{2i+1} > \frac{1}{4\|q_{2i+1}\theta\|} - q_{2i+1} \\
&> q_{2i+1}^{\alpha-\epsilon} - q_{2i+1} = q_{2i+1}^{\alpha-\epsilon}\left(1 - q_{2i+1}^{1+\epsilon-\alpha}\right) > \frac{1 - q_{2i+1}^{1+\epsilon-\alpha}}{\|q_{2i}\theta\|^{\alpha-\epsilon}}.
\end{aligned}$$

From $\alpha > 1 + \epsilon$, we have $q_{2i+1}^{\alpha-1-\epsilon} > 2$ for large i. Hence by Lemma 3.4 (ii) and (5) for large i, we have

$$R_{B(x,r;s)}(x) \geq q_{2i+2} - Kq_{2i+1} > \frac{1 - q_{2i+1}^{1+\epsilon-\alpha}}{\|q_{2i}\theta\|^{\alpha-\epsilon}} > \frac{1}{2\|q_{2i}\theta\|^{\alpha-\epsilon}} > \frac{2^{\alpha-\epsilon}}{2r^{\alpha-\epsilon}}. \tag{6}$$

(ii) Suppose that there are infinitely many i's such that

$$2^s q_{2i+1}^s \|q_{2i+1}\theta\| < 1.$$

In this case, $1 < s \leq \alpha$.

Choose $r^s = \|q_{2i+1}\theta\|/2$ for such i. Then we have

$$r = \frac{\|q_{2i+1}\theta\|^{1/s}}{2^{1/s}} < \frac{1}{2^{1/s}2q_{2i+1}} < \frac{\|q_{2i}\theta\|}{2^{1/s}}$$

and

$$\mu(B(x,r;s)) = r + r^s < \frac{\|q_{2i}\theta\|}{2^{1/s}} + \frac{\|q_{2i}\theta\|^s}{2} = \|q_{2i}\theta\| \left(2^{-1/s} + \frac{\|q_{2i}\theta\|^{s-1}}{2}\right).$$

Therefore for large i so as to $\|q_{2i}\theta\|^{s-1} < 2(1-2^{-1/s})$, we have

$$\mu(B(x,r;s)) < \|q_{2i}\theta\|.$$

Also we see

$$\mu(B(x,r;s)) = r^s + r > 2r^s = \|q_{2i+1}\theta\|.$$

Since

$$K\|q_{2i+1}\theta\| + \|q_{2i+2}\theta\| < r^s + r = \frac{\|q_{2i+1}\theta\|}{2} + \frac{\|q_{2i+1}\theta\|^{1/s}}{2^{1/s}},$$

we have

$$K \leq \frac{1}{2} + \frac{\|q_{2i+1}\theta\|^{1/s-1}}{2^{1/s}} < \frac{1}{2} + \frac{2q_{2i+2}\|q_{2i+1}\theta\|^{1/s}}{2^{1/s}} < \frac{1}{2} + \frac{2q_{2i+2}}{2^{1/s}}\frac{1}{2q_{2i+1}}.$$

Hence by Lemma 3.4 (ii)

$$\begin{aligned} R_{B(x,r;s)}(x) &\geq q_{2i+2} - Kq_{2i+1} > q_{2i+2} - \frac{q_{2i+2}}{2^{1/s}} - \frac{q_{2i+1}}{2} \\ &> \left(1-2^{-1/s}\right) q_{2i+2} - \frac{q_{2i+1}}{2} > \frac{1-2^{-1/s}}{2\|q_{2i+1}\theta\|} - \frac{1}{4\|q_{2i+1}\theta\|^{1/s}} \\ &> \frac{1-2^{-1/s}}{4\|q_{2i+1}\theta\|} = \left(1-2^{-1/s}\right)\frac{1}{8r^s} \end{aligned} \tag{7}$$

for large i so that

$$\|q_{2i+1}\theta\|^{1-1/s} < 1 - 2^{-1/s}.$$

Hence by (6) and (7)

$$\limsup_{r\to 0^+} \frac{\log R_{B(x,r;s)}(x)}{-\log r} \geq \min\{\alpha, s\},$$

which completes the proof. □

By Proposition 3.5, 3.6, 3.7 and 3.8, we have the proof of Theorem 1.3.

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