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A FUNCTIONAL CALCULUS AND FRACTIONAL POWERS FOR MULTIVALUED LINEAR OPERATORS

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1. Introduction

The class of univalent linear operators is unstable under the operations closure, inverse and adjoint. This is not the case if we consider the more general class of multivalued linear operators.

On the other hand, Favini and Yagi [7] and Yagi [15] have proved existence and uniqueness theorems of the strict solutions of degenerate evolution equations by means of this class of operators.

For these reasons, it is interesting to extend some results of functional calculus to the multivalued linear case as well as obtaining a theory of fractional powers for multivalued linear operators.

This problem has already been studied by Alaarabiou [1, 2]. He extended the well-known Hirsch functional calculus (see [8, 9]) to the set $\mathcal{M}$ of multivalued non-negative linear operators in a Banach space. His main idea was to endow $\mathcal{M}$ with an appropriate topology so that if $f \in \mathcal{T}_+$ (that is, $f(1/z)$ is a Stieltjes transform of a non-negative Radon measure), then $f : \mathcal{M} \to \mathcal{M}$ is continuous.

The basic properties of a functional calculus were proved in [1] by the above mentioned continuity. Nevertheless, this kind of reasoning does not allow us to obtain two fundamental properties: the product formula and the spectral mapping theorem. Moreover, this functional calculus does not generate interesting operators such as the fractional powers of complex exponent, or the semigroup generated by the fractional powers either, because the functions z^α, $0 < \operatorname{Re}\alpha < 1$, and $e^{-t z^\alpha}$, $0 < \alpha < 1/2$ and $t > 0$, do not belong to the class $\mathcal{T}_+$.

Sections 3 and 4 of this paper are devoted to improving a functional calculus valid for a wider class of functions $\mathcal{T}$ which contains the earlier mentioned functions. This process is not trivial because we have neither $f(\mathcal{M}) \subset \mathcal{M}$ nor continuity of f. So we have developed an original method to obtain the main properties of a functional calculus.

First of all, in Theorem 3.2 we study the inverse operator of $f(A)$. Then, in Proposition 3.5, we relate $f(A)$ and $f(A + \varepsilon)$ for $\varepsilon > 0$. These results enable us to

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extend the functional calculus on $\mathcal{L}(X) \cap \mathcal{M}$ to $\mathcal{M}$.

Afterwards, in Theorem 3.10 we show a result that will be essential from now on: for a subclass of functions that includes the fractional powers, $f(A)$ is given by

$$f(A) = (1 + A)f(A_D)(1 + A_D)^{-1}, \tag{1.1}$$

where A_D denotes the restriction of A to the Banach space $\overline{D(A)}$. By means of (1.1) we prove the product formula, the stability under composition and the spectral mapping theorem.

In the third part of [1], Alaarabiou constructed a theory of fractional powers valid for exponents α, $0 < \alpha < 1$, that verifies the following fundamental properties:
(i) $(A^{-1})^\alpha = (A^\alpha)^{-1}$.
(ii) $(A^\beta)^\alpha = A^{\beta\alpha}$ for $0 < \beta < 1$.
(iii) $A^\alpha A^\beta = A^{\alpha+\beta}$ for $\alpha, \beta > 0$ with $\alpha + \beta \leq 1$.
(iv) $A = \liminf_{\alpha \to 1} A^\alpha$.
Nevertheless, the following basic ones remain unanswered:
(v) $\sigma(A^\alpha) = \{z^\alpha : z \in \sigma(A)\}$.
(vi) If A is w-sectorial and $0 < \beta w \leq \pi$, then $A^\beta \in \mathcal{M}$ and $(A^\beta)^\alpha = A^{\beta\alpha}$.
(vii) $(A^*)^\alpha = (A^\alpha)^*$.

In section 5 we give a new definition of fractional powers A^α, $\operatorname{Re}\alpha > 0$, based on formula (1.1). By applying our functional calculus to the function z^α, $0 < \operatorname{Re}\alpha < 1$, and by means of the theory of fractional powers of densely defined operators (see [5, 10]), we get the properties (i) to (vii). Finally, we extend this theory to exponents $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha < 0$.

2. Multivalued linear operators and Stieltjes transforms.

Throughout this paper $(X, \|\ \|)$ will be a complex Banach space.

DEFINITION 2.1. A linear subspace A of $X \times X$ is said to be a multivalued linear operator in X. From now on, we use the following notation about A:

$$\begin{aligned} D(A) &= \{u \in X : \exists v \in X \text{ such that } (u, v) \in A\}, \\ Au &= \{v \in X : (u, v) \in A\}, \quad u \in D(A), \\ R(A) &= \{v \in Au : u \in D(A)\} \quad \text{and} \quad \ker A = \{u \in D(A) : 0 \in Au\}. \end{aligned}$$

When $A0 = \{0\}$, we say that A is a univalent linear operator.

By $\mathcal{L}(X)$ we denote the Banach algebra of bounded univalent linear operators defined on all X.

If A and B denote multivalued linear operators and $a \in \mathbb{C}$, we can also consider the multivalued operators:

$$A + B = \{(u, v + w) \in X \times X : (u, v) \in A,\ (u, w) \in B\},$$
$$AB = \{(u, v) \in X \times X : \exists w \in X \text{ such that } (u, w) \in B \text{ and } (w, v) \in A\},$$

and

$$aA = \{(u, av) \in X \times X : (u, v) \in A\}.$$

In short, the operator $\{(u, au) \in X \times X : u \in X\}$ will be denoted by a.

Moreover, we can always consider the inverse of A

$$A^{-1} = \{(u, v) \in X \times X : (v, u) \in A\},$$

and the adjoint of A

$$A^* = \{(u^*, v^*) \in X^* \times X^* : \langle v^*, u\rangle = \langle u^*, v\rangle, \quad \forall (u, v) \in A\}.$$

Note that A^{-1} is univalent if and only if A is one-to-one, and, by the Hahn-Banach theorem, A^* is univalent if and only if $\overline{D(A)} = X$.

By $\rho(A)$ we denote the resolvent set of A, that is,

$$\rho(A) = \{z \in \mathbb{C} : (z - A)^{-1} \in \mathcal{L}(X)\},$$

and by $\sigma(A) = \mathbb{C} \setminus \rho(A)$ the spectrum of A. As in the univalent case, $\rho(A)$ is open, the resolvent $(z-A)^{-1}$ is a holomorphic function from $\rho(A)$ to $\mathcal{L}(X)$ and the resolvent identity

$$(z - A)^{-1} - (y - A)^{-1} = (y - z)(z - A)^{-1}(y - A)^{-1}, \quad \forall z, y \in \rho(A), \tag{2.1}$$

holds (see [7, Theorem 2.6]).

DEFINITION 2.2. Given $\omega \in]0, \pi]$, we say that a multivalued linear operator A is ω–sectorial if

$$\sigma(A) \subseteq S_\omega := \{z \in \mathbb{C} : |\arg z| < \omega\} \cup \{0\},$$

and there is a constant $K \geq 0$ such that

$$\|z(z - A)^{-1}\| \leq K, \quad \forall z \in \mathbb{C} \setminus S_\omega. \tag{2.2}$$

If $\omega = \pi$, we say that A is non-negative. We denote by $\mathcal{M}(X)$ ($\mathcal{M}$ for short) the set of non-negative multivalued linear operators in X.

REMARK 2.1. By (2.1) it is not hard to show that the condition (2.2) is equivalent to that

$$M(\theta) = \sup_{\lambda>0} \lambda \|(\lambda e^{i\theta} - A)^{-1}\| < \infty, \quad \text{for } \omega \leq |\theta| \leq \pi.$$

Moreover, if A is ω–sectorial, then A^{-1} and A^* also are.

We associate to $A \in \mathcal{M}$ the family of bounded operators

$$A_\lambda = \frac{1}{\lambda}(1 - J_\lambda^A), \quad \forall \lambda > 0,$$

usually called Yosida's regularization of A, where $J_\lambda^A = (1 + \lambda A)^{-1}$, and the constant $M(A) = \sup_{\lambda>0} \|J_\lambda^A\| < \infty$. Note that if $A = \{0\} \times X$, then $M(A) = 0$. Otherwise $M(A) \geq 1$. It is easy to check that

$$AJ_\lambda^A u = A_\lambda u + A0, \; \forall u \in X, \; \text{ and } A_\lambda u = J_\lambda^A Au, \; \forall u \in D(A).$$

By (2.1), $\{A_\lambda\}_{\lambda>0}$ is a resolvent family.

EXAMPLE 2.1. We can obtain multivalued operators by considering the inverse of a univalent non-negative linear operator which is not one-to-one. For example, let us consider the Banach space $L^\infty(]0, +\infty[; \mathbb{C})$. The integral operator $\mathcal{I}$ given by:

$$\begin{cases} D(\mathcal{I}) = \left\{ f \in L^\infty : \displaystyle\int_0^x f(s)\,ds \in L^\infty \right\}, \\ \mathcal{I}f = \left\{ -\displaystyle\int_0^x f(s)\,ds + K : K \in \mathbb{C} \right\}, \quad \forall f \in D(\mathcal{I}), \end{cases}$$

belongs to $\mathcal{M}(L^\infty)$ with

$$(J_\lambda^{\mathcal{I}} g)(x) = \lambda \int_x^\infty (1 - g(s))e^{-\lambda(s-x)}\,ds, \quad \forall x \geq 0 \text{ and } \forall g \in L^\infty.$$

It is evident that $-\mathcal{I}$ is the inverse of the derivative operator.

EXAMPLE 2.2. Let $\Omega \subset \mathbb{R}^n$ be an open set with a smooth boundary $\partial\Omega$, and $X = (L^2(\Omega))^n = X_\sigma \oplus X_\nabla$, where

$$X_\sigma = \overline{\{u \in (C_0^\infty(\Omega))^n : \operatorname{div} u = 0 \text{ in } \Omega\}} \quad \text{and} \quad X_\nabla = \{\nabla p : p \in H^1(\Omega)\}.$$

In [7, Example 6.2], in order to formulate the Stokes equation in multivalued form, the multivalued linear operator

$$\begin{cases} D(A) = (H^2(\Omega))^n \cap (H_0^1(\Omega))^n \cap X_\sigma \\ Au = -\Delta u + X_\nabla \end{cases}$$

was introduced. It is proved that $A \in \mathcal{M}(X)$. Note that A is not one-to-one.

DEFINITION 2.3. Let $\{A_i\}_{i\in\mathcal{I}}$ be a net of multivalued linear operators in X. As in [1], we call $\liminf_{\mathcal{I}} A_i$ to the multivalued linear operator in X given by

$$\begin{aligned} \liminf_{\mathcal{I}} A_i = \{&(u, v) \in X \times X : \exists\{(u_i, v_i)\}_{i\in\mathcal{I}} \text{ convergent to } (u, v) \\ &\text{in } X \times X, \text{ where } (u_i, v_i) \in A_i, \forall i \in \mathcal{I}\}. \end{aligned}$$

REMARK 2.2. Note that, if $A_i \in \mathcal{L}(X)$, $\forall i \in \mathcal{I}$, and $\limsup_{\mathcal{I}} \|A_i\| < +\infty$, then $\liminf_{\mathcal{I}} A_i$ is a univalent linear operator.

In the following proposition we have collected the main properties of operators in the class $\mathcal{M}$. The proof can be found in [2, Proposition 1.I]. However, we give a constructive proof of (iii) that will be very useful in next section.

Proposition 2.1. *Let $A \in \mathcal{M}$. The following assertions hold:*

(i) *A is closed.*

(ii) *If $\lambda > 0$, then $A_\lambda \in \mathcal{L}(X) \cap \mathcal{M}$ with*

$$J_\mu^{A_\lambda} = \frac{1}{\mu + \lambda}(\lambda + \mu J_{\mu+\lambda}^A), \quad \forall \mu > 0,$$

and $M(A_\lambda) \leq \max\{M(A), 1\}$. Moreover,

$$(A_\lambda)_\mu = A_{\lambda+\mu}, \quad \forall \mu > 0.$$

(iii) $A = \liminf_{\lambda\to 0} A_\lambda$.

(iv) $\overline{D(A)} = \{u \in X : \lim_{\lambda\to 0} J_\lambda^A u = u\} = \overline{D(A^n)}, \quad \forall n \in \mathbb{N}$. *Hence,*

$$A0 \cap \overline{D(A)} = \{0\}. \tag{2.3}$$

(v) *The univalent linear operator $A_D = A \cap \left(\overline{D(A)} \times \overline{D(A)}\right)$ is non-negative and densely defined on the Banach space $\overline{D(A)}$.*

(vi) *If X is reflexive, then $X = A0 \oplus \overline{D(A)}$.*

Proof. (iii) First of all,

$$\lim_{\lambda\to 0} J_\mu^{A_\lambda} u = J_\mu^A u, \quad \forall \mu > 0 \text{ and } \forall u \in X,$$

since

$$\|J_\mu^{A_\lambda} u - J_\mu^A u\| \leq \lambda(M(A) + 1)\|A_\mu u\| + \|J_{\lambda+\mu}^A u - J_\mu^A u\|.$$

Let $(u, v) \in A$. We have

$$\lim_{\lambda\to 0} \left(J_1^{A_\lambda}(u + v), A_\lambda J_1^{A_\lambda}(u + v)\right) = (u, v),$$

and therefore $(u, v) \in \liminf_{\lambda\to 0} A_\lambda$. Let us now suppose that $(u, v) \in \liminf_{\lambda\to 0} A_\lambda$, that is, there is a net $\{u_\lambda\}_{\lambda>0}$ that satisfies

$$\lim_{\lambda\to 0}(u_\lambda, A_\lambda u_\lambda) = (u, v).$$

It is easily seen that

$$\lim_{\lambda\to 0} J_1^{A_\lambda}(1+A_\lambda)u_\lambda = J_1^A(u+v),$$

and then $u = J_1^A(u+v)$. Therefore $(u, v) \in A$. □

REMARK 2.3. From (2.3) it follows that if $A \in \mathcal{M}$ is a densely defined operator, then A is univalent, and if A has dense range, then A is one-to-one. Moreover, from (vi) of the previous theorem it is deduced that in reflexive Banach spaces the converses are also true.

In the following definition we introduce the class of functions we shall use to construct our functional calculus. We will denote by $\mathbb{R}_+ = [0, +\infty[$ and by $\mathbb{R}_- =]-\infty, 0]$.

DEFINITION 2.4. Let μ be a complex Radon measure on $\mathbb{R}_+$ satisfying that there is $z_0 \in \mathbb{C} \setminus \mathbb{R}_-$ such that

$$\int_{\mathbb{R}_+} \frac{1}{|z_0+t|}\, d\,|\mu|\,(t) < \infty, \tag{2.4}$$

where $|\mu|$ denotes the total variation of μ. Let $a \in \mathbb{C}$ be a complex number. We call Stieltjes transform of the measure μ with value a at infinity the function $f : \mathbb{C}\setminus\mathbb{R}_- \to \mathbb{C}$ given by:

$$f(z) = a + \int_{\mathbb{R}_+} \frac{1}{z+t}\, d\mu(t).$$

Condition (2.4) states that f is well-defined. Moreover, f is holomorphic in $\mathbb{C} \setminus \mathbb{R}_-$.

The Stieltjes transforms are determined by the measure μ and its value a at $+\infty$. For this reason, we will use the notation $f(z) = (a, \mu)(z)$. By $\mathcal{S}$ we denote the set of Stieltjes transforms and by $\mathcal{S}_+$ the subset of $\mathcal{S}$ of the functions $f(z) = (a, \mu)(z)$ with value $a \geq 0$ and measure $\mu \geq 0$. Finally,

$$\mathcal{S}_0 = \left\{ f(z) = (a, \mu)(z) \in \mathcal{S} : \mu(\{0\}) = 0 \text{ and } \int_{]0,\infty[} \frac{1}{t}\, d|\mu|(t) < \infty \right\}.$$

In [13, Theorem 1.2] the following result is proved:

Theorem 2.2. *Let $f \in \mathcal{S}_+$. The functions $f_\lambda(z) = f(z)/(1+\lambda f(z))$, $\lambda > 0$, and $\tilde{f}(z) = 1/f(1/z)$, if f does not vanish, belong to $\mathcal{S}_+$.*

REMARK 2.4. By monotone convergence, $f_\lambda \in \mathcal{S}_0$, too.

Now, we introduce the following classes of functions:

$$\mathcal{T} = \{f : f(1/z) \in \mathcal{S}\}, \ \mathcal{T}_+ = \{f : f(1/z) \in \mathcal{S}_+\},$$
$$\tilde{\mathcal{T}} = \{f \in \mathcal{T} : \tilde{f} \in \mathcal{T}\} \quad \text{and} \quad \mathcal{T}_0 = \{f \in \tilde{\mathcal{T}} : \tilde{f}(0) = 0\}.$$

It is not hard to see that $\mathcal{S}_0 \subset \mathcal{T}$ and that $\mathcal{T}_+ \subset \mathcal{T}_0 \cup \mathcal{S}_0$.

From Theorem 2.2 we have $\mathcal{T}_+ \setminus \{0\} \subset \tilde{\mathcal{T}}$, but, in general, it is not true that if $f \in \mathcal{T}$ does not vanish, then $\tilde{f} \in \mathcal{T}$. In fact, the function $f(z) = 1/(1+z) \in \mathcal{T}$ and $\tilde{f}(z) \notin \mathcal{T}$ since $\tilde{f}(0)$ does not exist.

The product of two functions of $\mathcal{T}$ does not belong, in general, to $\mathcal{T}$. For example, $z \in \mathcal{T}_+$ but $z^2 \notin \mathcal{T}$ (see the remark next to [13, Lemma 2.1]).

As in [8, Theorem 3.3] the following result holds:

Theorem 2.3. *If $f \in \mathcal{T}$ ($\mathcal{T}_+$) and $g \in \mathcal{T}_+$, then $(f \circ g) \in \mathcal{T}$ ($\mathcal{T}_+$).*

EXAMPLE 2.3. Let $f(z)$ be an absolutely concave function (also named non-negative operator monotone function in the literature). That is, f is continuous and non-negative on $]0, \infty[$, $\lim_{z \to 0^+} f(z) < \infty$ and f has an analytic continuation into the upper half-plane $\operatorname{Im} z > 0$ such that $\operatorname{Im} f(z) \geq 0$. By the Nevanlinna integral representation for functions preserving the upper half-plane (see [6, Theorem. I, p. 20] or [14, p. 84]), it is not hard to check that $f \in \mathcal{T}_+$. The converse is also true.

Consequently, z^α, $0 < \alpha < 1$, $\ln(1+z)$ and $\sqrt{z}\arctan(\beta\sqrt{z})$, $\beta > 0$, belong to $\mathcal{T}_+ \cap \mathcal{T}_0$. We can also prove this fact by means of the integral representations:

$$z^\alpha = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \frac{z}{1+tz} t^{-\alpha} dt,$$
$$\ln(1+z) = \int_0^1 \frac{z}{1+tz} dt,$$
$$\sqrt{z}\arctan(\beta\sqrt{z}) = \int_0^\beta \frac{z}{1+tz} \frac{1}{2\sqrt{z}} dt.$$

The first one is due to Cauchy's integral formula. It is also true for $\alpha \in \mathbb{C}$, $0 < \operatorname{Re}\alpha < 1$. Therefore, we have that $z^\alpha \in \mathcal{T}_0$ for $0 < \operatorname{Re}\alpha < 1$.

EXAMPLE 2.4. Let $f : \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$ be holomorphic such that $f(e^z)$ have an analytic continuation to $|\operatorname{Im} z| < \pi + \delta$ for some $\delta > 0$. We will say that f belongs to the class $\mathcal{F}_0$ if

$$\lim_{z\to 0} f(z) \text{ exists and } \lim_{|z|\to\infty} f(z)/z = 0.$$

We will say that f belongs to the class $\mathcal{F}_1$ if

$$\lim_{z\to 0} z f(z) = 0 \quad \text{and} \quad \lim_{|z|\to\infty} f(z) \text{ exists.}$$

Let $f \in \mathcal{F}_0 \cup \mathcal{F}_1$. For all $\lambda > 0$ there exist the limits

$$\phi_1(\lambda) = \lim_{\substack{z \to -\lambda \\ \operatorname{Im} z > 0}} f(z) \quad \text{and} \quad \phi_2(\lambda) = \lim_{\substack{z \to -\lambda \\ \operatorname{Im} z < 0}} f(z).$$

We will write $\phi(\lambda) = (\phi_1(\lambda) - \phi_2(\lambda))/(2\pi i)$. We define the classes

$$\begin{aligned} \mathcal{H}_0 &= \left\{ f \in \mathcal{F}_0 : \lambda^{-1}\phi \,|_{]0,1]} \in L^1(]0,1]) \quad \text{and} \quad \lambda^{-2}\phi \in L^1(]1,\infty[) \right\}, \\ \mathcal{H}_1 &= \left\{ f \in \mathcal{F}_1 : \lambda^{-1}\phi \in L^1(]0,\infty[) \right\}. \end{aligned}$$

We have $\mathcal{H}_0 \subset \mathcal{T}$ and $\mathcal{H}_1 \subset \mathcal{S}_0$ (see [12, Proposition 2.1]).

By this result, it is a simple matter to show that $z^\alpha \in \mathcal{T}_0$, $0 < \operatorname{Re}\alpha < 1$, and that $e^{-v z^a} \in \mathcal{S}_0$, $0 < \alpha < 1/2$ and $v \in S_{(\pi/2)-\alpha\pi}$. Nevertheless, the reader can directly prove that

$$e^{-v z^a} = \frac{1}{\pi} \int_0^\infty \frac{1}{z+t}\, e^{-v t^\alpha \cos\alpha\pi} \sin(v t^\alpha \sin\alpha\pi)\, dt,$$

by Cauchy's integral formula.

Note that $(z+\varepsilon)^\alpha \in \mathcal{T}$ and $(z+\varepsilon)^{-\alpha} \in \mathcal{S}_0$, but these functions do not belong to $\mathcal{H}_0 \cup \mathcal{H}_1$ because $(e^z+\varepsilon)^\alpha$ and $(e^z+\varepsilon)^{-\alpha}$ have not an analytic continuation to $|\operatorname{Im} z| < \pi + \delta$ for any $\delta > 0$.

3. Construction of the functional calculus.

In the sequel, $A \in \mathcal{M}$. If we write $f \in \mathcal{T}$ we understand that $f(z) = (a, \mu)(1/z)$, and if moreover $f \in \tilde{\mathcal{T}}$, then $\tilde{f}(z) = (\tilde{a}, \tilde{\mu})(1/z)$.

Following [13], we associate to $f \in \mathcal{T}$ and $A \in \mathcal{L}(X) \cap \mathcal{M}$ the operator

$$f(A) = a + \int_{\mathbb{R}_+} A_t \, d\mu(t), \tag{3.1}$$

understanding that the integral, convergent in $\mathcal{L}(X)$, takes the value A for $t = 0$. From [13] it follows:

(i) If $f \in \tilde{\mathcal{T}}$ and $0 \in \rho(A)$, then $f(A)^{-1} = \tilde{f}(A^{-1})$.
(ii) If $g \in \mathcal{T}$ and $h = f g \in \mathcal{T}$, then $h(A) = f(A)\, g(A)$.
(iii) If $g \in \mathcal{T}_+$, then $f(g(A)) = (f \circ g)(A)$.
(iv) $\sigma(f(A)) = \{f(z) : z \in \sigma(A)\}$.

The reader can easily check that $f(A) = \liminf_{\lambda\to 0} f(A_\lambda)$. So the following definition makes sense.

DEFINITION 3.1. For $f \in \mathcal{T}$ and $A \in \mathcal{M}$ we define the multivalued linear operator $f(A)$ by

$$f(A) = \liminf_{\lambda\to 0} f(A_\lambda) = a + \int_{]1,\infty[} A_t\, d\mu(t) + \liminf_{\lambda\to 0} \int_{[0,1]} A_{\lambda+t}\, d\mu(t).$$

REMARK 3.1. In [1] the functional calculus is constructed by applying the Dunford-Riesz functional calculus to the functions of $\mathcal{T}_+$ and to the operators $A_\lambda + \lambda$, and dealt the general case by means of the definition $f(A) = \liminf_{\lambda\to 0} f(A_\lambda + \lambda)$. It is not hard to see that this concept agrees with the above definition. In [13] the previous properties (i) to (iii) were proved without techniques of Banach algebras. So, this standpoint will allow us to extend our functional calculus to any sequentially complete locally convex space.

REMARK 3.2. If $f(z) = (a, \mu)(z) \in \mathcal{S}_0$, then

$$f(A) = a + \int_{]0,\infty[} (t + A)^{-1}\, d\mu(t) \in \mathcal{L}(X),$$

as is easy to check. Moreover, if $\{A_i\}_{i\in\mathcal{I}}$ is a net in $\mathcal{M}$ satisfying

$$A = \liminf_{\mathcal{I}} A_i \quad \text{and} \quad \limsup_{\mathcal{I}} M(A_i) < \infty,$$

then $f(A) = \liminf_{\mathcal{I}} f(A_i)$. This "continuity" property remains valid in $\mathcal{T}_+$.

The following operators will be very useful in the sequel.

Given $f \in \mathcal{T}$ and $A \in \mathcal{M}$ we define the multivalued linear operator $W_f(A)$ by $D(W_f(A)) = D(A)$ and

$$W_f(A)u = au + \int_{]0,\infty[} A_t u\, d\mu(t) + \mu(\{0\})\, Au, \quad \forall u \in D(A).$$

If $S, T \in \mathcal{L}(X)$ are given by

$$S = a + \int_{]1,\infty[} A_t\, d\mu(t) \quad \text{and} \quad T = \int_{[0,1]} J_t^A\, d\mu(t),$$

then, the multivalued linear operator $S + AT$ is closed and extends to $W_f(A)$.

Proposition 3.1. *Let $f \in \mathcal{T}$. The following assertions hold*:

(i) $f(A)0 \subseteq A0$.

(ii) *If $z \in \rho(A)$, then*

$$f(A)(z - A)^{-1}u = (z - A)^{-1} f(A)u + f(A)0, \quad \forall u \in D(f(A)).$$

(iii) $W_f(A) \subseteq f(A) \subseteq (1 + A)W_f(A)(1 + A)^{-1}$.

Proof. (i) Let $v \in f(A)0$. There is a net $\{u_\lambda\}_{\lambda>0}$ in X such that

$$\lim_{\lambda\to 0}\left(u_\lambda, \int_{\mathbb{R}_+} (A_\lambda)_t\, u_\lambda\, d\mu(t)\right) = (0, v).$$

Since

$$\left\| \int_{\mathbb{R}_+} J_1^A(A_\lambda)_t u_\lambda d\mu(t) \right\| \leq M(A)(M(A)+1)\, \|u_\lambda\| \times \left[|\mu|([0,1]) + \int_{]1,\infty[} \frac{1}{t}\, d|\mu|(t) \right],$$

then, by letting $\lambda \to 0$, $J_1^A v = 0$, that is, $v \in A0$.

(ii) Given $(u, v) \in f(A)$, there is $\{u_\lambda\}_{\lambda>0}$ in X such that

$$\lim_{\lambda \to 0} (u_\lambda, f(A_\lambda)u_\lambda) = (u, v).$$

Since $f(A_\lambda)$ and $(z - A)^{-1}$ commute, then

$$((z - A)^{-1}u, (z - A)^{-1}v) \in \liminf_{\lambda \to 0} f(A_\lambda) = f(A).$$

(iii) Let $(u, w) \in W_f(A)$ and $(u, v) \in A$ such that

$$w = au + \int_{]0,\infty[} J_t^A v\, d\mu(t) + \mu(\{0\})v.$$

In part (iii) of Proposition 2.1 we have proved that

$$\lim_{\lambda \to 0} (J_1^{A_\lambda}(u + v), A_{\lambda+1}(u + v)) = (u, v).$$

By dominated convergence

$$\lim_{\lambda \to 0} \int_{]0,1]} A_{\lambda+t} J_1^{A_\lambda}(u + v)\, d\mu(t) = \int_{]0,1]} J_t^A v\, d\mu(t),$$

and therefore

$$\left(u, \int_{]0,1]} J_t^A v\, d\mu(t) + \mu(\{0\})v \right) \in \liminf_{\lambda \to 0} \int_{[0,1]} A_{\lambda+t}\, d\mu(t).$$

Consequently, $(u, w) \in f(A)$.

Finally, the second inclusion follows from the previous one and part (ii). □

REMARK 3.3. Note that if $\mu(\{0\}) \neq 0$, then $A0 = f(A)0$, and that if $f(0) = 0$, then $R(f(A)) \subseteq \overline{R(A)}$.

Theorem 3.2. *If $f \in \tilde{\mathcal{T}}$, then*

$$f(A)^{-1} = \tilde{f}(A^{-1}).$$

Proof. If $\lambda > 0$, then $A_\lambda + \lambda \in \mathcal{L}(X) \cap \mathcal{M}$ and $0 \in \rho(A_\lambda + \lambda)$. Hence

$$f(A_\lambda + \lambda)^{-1} = \tilde{f}((A_\lambda + \lambda)^{-1}).$$

Since $\liminf_{\lambda \to 0} f(A_\lambda + \lambda)^{-1} = f(A)^{-1}$, the proof is completed by showing that

$$\liminf_{\lambda \to 0} f((A_\lambda + \lambda)^{-1}) = f(A^{-1}), \qquad \forall f \in \mathcal{T}.$$

For $t \geq 0$ we have

$$((A_\lambda + \lambda)^{-1})_t = \frac{1}{t+\lambda}\left(1 - \frac{1}{t+\lambda} A_{\lambda + 1/(t+\lambda)}\right),$$

and then, formula (2.1) yields

$$((A_\lambda + \lambda)^{-1})_t - (A^{-1})_{\lambda + t} = \frac{\lambda}{(t+\lambda)^2} A_{\lambda + 1/(t+\lambda)} A_{1/(t+\lambda)}.$$

Therefore

$$\begin{aligned} &\int_{\mathbb{R}_+} \|((A_\lambda + \lambda)^{-1})_t - (A^{-1})_{\lambda+t}\| d|\mu|(t) \\ &\leq (M(A)+1)^2 \int_{\mathbb{R}_+} \frac{\lambda}{1 + \lambda(t+\lambda)} d|\mu|(t), \end{aligned}$$

where, by dominated convergence, the last integral converges to zero as $\lambda \to 0$. From this, it is concluded that

$$\liminf_{\lambda \to 0} f((A_\lambda + \lambda)^{-1}) = \liminf_{\lambda \to 0} f((A^{-1})_\lambda) = f(A^{-1}). \qquad \square$$

Corollary 3.3. *If $f \in \tilde{\mathcal{T}}$, then*:

(i) $\ker f(A) \subseteq \ker A$. *Moreover, if* $\lim_{z \to 0,\, z>0} z/f(z) \neq 0$, *then the identity* $\ker f(A) = \ker A$ *holds.*

(ii) $R(A) \subseteq R(f(A))$.

Proof. The proof easily follows by applying Proposition 3.1 to $\tilde{f}$ and A^{-1}, and by Theorem 3.2. $\square$

Corollary 3.4. *If $f \in \mathcal{T}_0$, then $D(f(A)) \subseteq \overline{D(A)}$.*

Proof. By Theorem 3.2 we know that $D(f(A)) = R(\tilde{f}(A^{-1}))$, and as $\tilde{f}(0) = 0$, then $R(\tilde{f}(A^{-1})) \subseteq \overline{D(A)}$. This completes the proof. $\square$

Proposition 3.5. *Let* $f \in \mathcal{T}$ *with* $\mu(\{0\}) = 0$, *or let* $f \in \tilde{\mathcal{T}}$. *For all* $\varepsilon > 0$ *there exists an operator* $F_\varepsilon \in \mathcal{L}(X)$ *such that*

$$f(A+\varepsilon) = f(A) + F_\varepsilon,$$

and $\lim_{\varepsilon\to 0} \|F_\varepsilon\| = 0$. *Moreover,*

$$\overline{f(A)} = \liminf_{\lambda\to 0} f(A+\varepsilon).$$

Proof. Let ε, λ be two positive real numbers, and let us consider the bounded operator

$$\begin{aligned} S_{\lambda,\varepsilon} &= \int_{]0,1]} ((A+\varepsilon)_{\lambda+t} - A_{\lambda+t})\, d\mu(t) \\ &= \int_{]0,1]} \frac{\varepsilon}{1+\varepsilon(\lambda+t)}\, J^A_{\lambda+t}\, J^A_{(\lambda+t)/(1+\varepsilon(\lambda+t))}\, d\mu(t). \end{aligned}$$

It is clear that

$$S_\varepsilon := \liminf_{\lambda\to 0} S_{\lambda,\varepsilon} = \int_{]0,1]} \frac{\varepsilon}{1+\varepsilon t}\, J^A_t\, J^A_{t/(1+\varepsilon t)}\, d\mu(t) \in \mathcal{L}(X).$$

So, to prove the first assertion it is sufficient to show that

$$\liminf_{\lambda\to 0} \int_{[0,1]} (A+\varepsilon)_{\lambda+t}\, d\mu(t) = \varepsilon\, \mu(\{0\}) + S_\varepsilon + \liminf_{\lambda\to 0} \int_{[0,1]} A_{\lambda+t}\, d\mu(t). \tag{3.2}$$

Obviously this is the case if $\mu(\{0\}) = 0$. Otherwise, we have $f \in \mathcal{T}_0$, and therefore, by Corollary 3.4, $D(f(A+\varepsilon))$, $D(f(A)) \subseteq \overline{D(A)}$. Moreover, for every net $\{u_\lambda\}_{\lambda>0}$ such that $\lim_{\lambda\to 0} u_\lambda = u \in \overline{D(A)}$, we have

$$\begin{aligned} \left\| J^A_\lambda\, J^A_{\lambda/(1+\epsilon\lambda)} u_\lambda - u \right\| &\le M(A)^2 \|u_\lambda - u\| + M(A) \left\| J^A_{\lambda/(1+\epsilon\lambda)} u - u \right\| \\ &\quad + \left\| J^A_\lambda\, u - u \right\|, \end{aligned}$$

and so

$$\lim_{\lambda\to 0} J^A_\lambda\, J^A_{\lambda/(1+\epsilon\lambda)} u_\lambda = u.$$

This enables us to obtain (3.2).

On account of the above, we have $f(A+\varepsilon) = f(A) + F_\varepsilon$ where

$$F_\varepsilon = \int_{[0,\infty[} \frac{\varepsilon}{1+\varepsilon\, t}\, J^A_t\, J^A_{t/(1+\varepsilon t)}\, d\mu(t) \in \mathcal{L}(X).$$

Obviously, $\lim_{\varepsilon\to 0} \|F_\varepsilon\| = 0$.

Now, the last assertion follows from the definition of $f(A)$. □

Corollary 3.6. *If $f \in \tilde{\mathcal{T}}$, then*

$$f(A^*) = f(A)^*.$$

Proof. The result easily follows if $A \in \mathcal{L}(X) \cap \mathcal{M}$. Therefore,

$$\tilde{f}((1 + A^*)^{-1}) = \left[\tilde{f}((1 + A)^{-1})\right]^*,$$

and taking inverses,

$$f(1 + A^*) = [f(1 + A)]^*.$$

By Proposition 3.5 we obtain

$$[f(1 + A)]^* = [f(A) + F_1]^* = f(A)^* + \int_{\mathbb{R}_+} \frac{1}{1 + t} J_t^{A^*} J_{t/(1+t)}^{A^*} \, d\mu(t)$$

and

$$f(1 + A^*) = f(A^*) + \int_{\mathbb{R}_+} \frac{1}{1 + t} J_t^{A^*} J_{t/(1+t)}^{A^*} \, d\mu(t).$$

Thus, as the common addend belongs to $\mathcal{L}(X^*)$, the desired relation is proved. □

Corollary 3.7. *If $f \in \mathcal{T}_0$, then $f(A) \in \mathcal{L}(X)$ if and only if $A \in \mathcal{L}(X)$.*

Proof. If $A \in \mathcal{L}(X)$ we know that $f(A) \in \mathcal{L}(X)$. Conversely, let us suppose that $A \notin \mathcal{L}(X)$. By the spectral mapping theorem for operators of $\mathcal{L}(X) \cap \mathcal{M}$ and by Theorem 3.2 we have

$$0 = \tilde{f}(0) \in \{\tilde{f}(z) : z \in \sigma((1 + A)^{-1})\} = \sigma(f(1 + A)^{-1}),$$

and hence $f(1 + A) \notin \mathcal{L}(X)$. By Proposition 3.5 we now conclude that $f(A) \notin \mathcal{L}(X)$. □

Theorem 3.8. *Let $f \in \tilde{\mathcal{T}}$. The following assertions hold:*

(i) *$f(A)$ is closed. In particular*

$$f(A) = \liminf_{\varepsilon \to 0} f(A + \varepsilon).$$

(ii) *If A is univalent, then*

$$f(A) = (1 + A) W_f(A) (1 + A)^{-1} = S + AT.$$

Moreover, if A is densely defined, then

$$f(A) = \overline{W_f(A)}.$$

Proof. (i) By Theorem 3.2 we know that $f(1+A)$ has closed inverse and therefore it is closed. Consequently, by Proposition 3.5, $f(A)$ is closed.

(ii) Firstly, as

$$W_f(A)(1+A)^{-1} = (1+A)^{-1}[S-T]+T,$$

the second identity holds. By part (iii) of Proposition 3.1 we only need to prove that if $u \in X$ and $v = W_f(A)(1+A)^{-1}u \in D(A)$, then $u \in D(f(A))$.

If $0 \in \rho(A)$, then, being $w = (1+A)v$, we have

$$\begin{aligned}(1+A)^{-1}u &= \tilde{f}(A^{-1})(1+A)^{-1}w \\ &= \tilde{a}\,(1+A)^{-1}w + \int_{[0,\infty[} (A^{-1})_t(1+A)^{-1}w\, d\tilde{\mu}(t).\end{aligned}$$

As $(1+A)^{-1}$ commutes with $\tilde{f}(A^{-1})$, then

$$(1+A)^{-1}(\tilde{f}(A^{-1})w-u)=0, \tag{3.3}$$

and therefore $w = f(A)u$.

Let $A \in \mathcal{M}$ be arbitrary. By the previous case,

$$f(A+1) = (2+A)W_f(A+1)(2+A)^{-1}.$$

Moreover, as $v \in D(A)$, it is very easy to check that $W_f(A)(2+A)^{-1}u \in D(A)$. Now, by Proposition 3.5

$$W_f(A+1)(2+A)^{-1}u = W_f(A)(2+A)^{-1}u + F_1(2+A)^{-1}u,$$

and, as F_1 and $(2+A)^{-1}$ commute, we conclude that $u \in D(f(A+1)) = D(f(A))$.

To prove the second assertion, as $f(A)$ is closed, it is sufficient to prove that $S + AT \subseteq \overline{W_f(A)}$. Let $u \in X$ such that $Tu \in D(A)$. As $\overline{D(A)} = X$, then

$$\lim_{\lambda\to 0} J_\lambda^A u = u \quad \text{and} \quad \lim_{\lambda\to 0} W_f(A)J_\lambda^A u = \lim_{\lambda\to 0} J_\lambda^A(S+AT)u = (S+AT)u,$$

and therefore $\overline{W_f(A)}$ is an extension of $AT+S$. □

REMARK 3.4. Part (ii) of previous theorem states that our functional calculus coincides on the class $\tilde{\mathcal{T}} \cup \mathcal{S}_0$ with the one given in [13] for univalent non-negative linear operators.

Corollary 3.9. *If $f \in \mathcal{T}_0$, then*

$$f(A)_D = f(A_D).$$

Moreover, if $\mu(\{0\}) = 0$, *then* $W_f(A)$ *is closable and*

$$\overline{W_f(A)} = f(A)_D.$$

Proof. By Corollary 3.4 we know that $\overline{D(f(A))} = \overline{D(A)}$ and by the previous theorem

$$f(A_D) = \overline{W_f(A_D)} \subseteq f(A)_D.$$

On the other hand, if $(u, v) \in f(A)_D$, then, as $J_\lambda^A v = W_f(A_D)J_\lambda^A u$, we have $(u, v) \in \overline{W_f(A_D)}$. This proves the first identity.

Finally, if $\mu(\{0\}) = 0$, then the inclusion $W_f(A) \subseteq f(A)_D$ holds. So, we have the second assertion. □

Theorem 3.10. *If* $f \in \mathcal{T}_0$, *then* $A0 = f(A)0$ *and*

$$f(A) = (1 + A)W_f(A_D)(1 + A_D)^{-1} = S + AT \mid_{\overline{D(A)}}, \tag{3.4}$$

where $T \mid_{\overline{D(A)}}$ *is the restriction of* T *to* $\overline{D(A)}$.

Proof. In part (i) of Proposition 3.1 we have proved that $f(A)0 \subseteq A0$, $\forall f \in \mathcal{T}$. So, we only need to prove that if $(0, v) \in A$, then $(v, 0) \in f(A)^{-1}$. As

$$\lim_{\lambda\to 0} \left(A_{\lambda+1}v, \int_{[0,1]} (A^{-1})_{\lambda+t} A_{\lambda+1} v \, d\tilde{\mu}(t) \right) = (v, 0),$$

then

$$(v, 0) \in \liminf_{\lambda\to 0} \int_{[0,1]} (A^{-1})_{\lambda+t} \, d\tilde{\mu}(t),$$

and therefore, as $\tilde{f}(0) = 0$, we conclude that $(v, 0) \in \tilde{f}(A^{-1})$.

The proof of the first identity in (3.4) runs as in Theorem 3.8, part (ii), since $f(A)0 = (1 + A)W_f(A_D)(1 + A)^{-1}0$ and $f(A) \subseteq (1 + A)W_f(A_D)(1 + A_D)^{-1}$. Only the case $0 \in \rho(A)$ is slightly different: if $u \in \overline{D(A)}$, $v = W_f(A_D)(1 + A_D)^{-1}u \in D(A)$ and $w \in (1 + A)v$, then (3.3) holds, which, by (2.3) and the fact that $\tilde{f}(A^{-1})w \in \overline{D(A)}$, implies that $u = \tilde{f}(A^{-1})w$.

Finally, as $\forall u \in \overline{D(A)}$,

$$W_f(A_D)(1 + A_D)^{-1}u = (1 + A)^{-1}[Su - Tu] + Tu,$$

we obtain the second identity of (3.4). □

REMARK 3.5. If $f \in \mathcal{T}_0$ we also have

$$f(A) = a + \int_{]b,\infty[} A_t d\,\mu(t) + A \int_{[0,b]} J_t^A d\,\mu(t) \mid \overline{D(A)}, \quad \forall b > 0.$$

Note that for this class of functions we do not have, as in the univalent case (see Theorem 3.8), $f(A) = (1+A)W_f(A_D)(1+A)^{-1}$, since this identity implies that A is univalent. Indeed, if this identity holds, then $A0 \subseteq \overline{D(A)}$, and therefore $A0 = \{0\}$. However, if moreover $\mu(\{0\}) \neq 0$, then $f(A) = S + AT$.

Corollary 3.11. *If $f \in \tilde{\mathcal{T}}$ and $f(0) = 0$, then*

$$\ker A = \ker f(A).$$

4. Main properties of the functional calculus.

Theorem 4.1 (Product formula). *The following assertions hold:*

(i) *If $f \in \mathcal{T}_0$, $g \in \tilde{\mathcal{T}}$ and $h = f\,g \in \mathcal{T}_0$, then*

$$h(A) = f(A)g(A). \tag{4.1}$$

(ii) *If $f \in \mathcal{T}$, $g \in \mathcal{S}_0$ and $h = f\,g \in \mathcal{T}$, then*

$$g(A)f(A) \subseteq h(A) \subseteq f(A)g(A).$$

(iii) *If $f, g \in \mathcal{T}_+$, do not vanish, and $h = f\,g \in \mathcal{T}_+$, then* (4.1) *holds.*

Proof. (i) Let ε be a positive real number. By the product formula for operators of $\mathcal{L}(X) \cap \mathcal{M}$, we have

$$\tilde{h}((A+\varepsilon)^{-1}) = \tilde{g}((A+\varepsilon)^{-1})\tilde{f}((A+\varepsilon)^{-1}),$$

and taking inverses

$$h(A+\varepsilon) = f(A+\varepsilon)g(A+\varepsilon).$$

Let $(u, v) \in h(A)$ and H_ε be the operator associated to h in Proposition 3.5. There is $w_\varepsilon \in g(A+\varepsilon)u \cap D(f(A+\varepsilon)) \subseteq \overline{D(A)}$ so that $(w_\varepsilon, v + H_\varepsilon u) \in f(A+\varepsilon)$. Again by Proposition 3.5 it is readily verified that $\lim_{\varepsilon\to 0} w_\varepsilon = g(A)u \cap \overline{D(A)}$, and hence

$$(g(A)u \cap \overline{D(A)}, v) \in \liminf_{\varepsilon\to 0} f(A+\varepsilon) = f(A),$$

so that $(u, v) \in f(A)g(A)$. Therefore, $h(A) \subseteq f(A)\,g(A)$ and, as $h(A)0 = A0 = f(A)g(A)0$, we conclude the proof by showing that $D(f(A)g(A)) \subseteq D(h(A))$. To prove this, let $(u, w) \in g(A)$ and $(w, v) \in f(A)$. As

$$(1 + A)^{-1}v \in f(A)(1 + A)^{-1}w = h(A)(1 + A)^{-1}u,$$

then,

$$h(A)(1 + A)^{-1}u \cap D(A) \neq \emptyset,$$

which, by Theorem 3.10, implies that $u \in D(h(A))$.

(ii) It is a straightforward consequence of the product formula for operators of the class $\mathcal{L}(X) \cap \mathcal{M}$.

(iii) By (i) and (ii), we only need to consider the case $f \in \mathcal{S}_0$ and $g \in \mathcal{T}_0$. In this case we have

$$f(A)g(A) \subseteq h(A) = g(A)f(A).$$

Moreover, as $f \in \mathcal{S}_0$, then $f(A)g(A)0 = A0$. So, to prove that $f(A)$ and $g(A)$ commute we only need to show that if $u \in X$ and $f(A)u \in D(g(A))$, then $u \in D(g(A))$. By Proposition 3.5

$$f(1 + A)u \in D(g(A)) = D(g(1 + A)),$$

and hence, as $f(1 + A)$ and $g(1 + A)$ commute, we have $u \in D(g(A))$. □

REMARK 4.1. In general, the product formula (4.1) is not true. To see this it is sufficient to consider the functions $f(z) = (1 + z) \in \mathcal{T}_0$, $g(z) = 1/(1 + z) \in \mathcal{S}_0$ and $h(z) = 1 \in \mathcal{T}_+$. If $\{0\} \subsetneq A0$, then both inclusions of part (ii) are strict.

As a consequence of (ii) we have if $\alpha \in \mathbb{R}$, $0 < \alpha < 1/2$, then the family of bounded operators $\{S(v)\}_{v \in S_{(\pi/2)-\alpha\pi}}$ determined by

$$S(v) = (e^{-vz^\alpha})(A) = \frac{1}{\pi}\int_0^\infty e^{-v s^\alpha \cos\alpha\pi} \sin(vs^\alpha \sin\alpha\pi)(s + A)^{-1} ds,$$

satisfies the semigroup property:

$$S(v_1)S(v_2) = S(v_1 + v_2), \quad \forall v_1, v_2 \in S_{(\pi/2)-\alpha\pi}.$$

This semigroup, associated to the fractional power $-A^\alpha$, will be studied in a later paper.

The following result is proved in [2, Proposition 3.II] as a consequence of the stability under composition. It can also be proved by means of the product formula in $\mathcal{S}_0$.

Corollary 4.2. *If $f \in \mathcal{T}_+$, then $f(A) \in \mathcal{M}$ with $M(f(A)) \leq M(A)$ and $f(A)_\lambda = f_\lambda(A)$, $\forall \lambda > 0$.*

Theorem 4.3 (Stability under composition). *If $f \in \mathcal{T}_0 \cup \mathcal{S}_0$ and $g \in \mathcal{T}_+$, then*

$$f(g(A)) = (f \circ g)(A).$$

Proof. The operators $f(g(A))$ and $(f \circ g)(A)$ are well-defined, respectively, by Corollary 4.2 and Theorem 2.3.

Firstly, let us suppose that $f \in \mathcal{T}_0$. We may assume that $f \circ g \in \tilde{\mathcal{T}}$, since if g vanishes the result is evident. Consequently, from the bounded case it is deduced that

$$f(g(A+\varepsilon)) = (f \circ g)(A+\varepsilon), \quad \forall \varepsilon > 0.$$

By Theorem 3.8, part (i), we only need to prove that

$$\liminf_{\varepsilon \to 0} f(g(A+\varepsilon)) = f(g(A)). \tag{4.2}$$

Let ε be a positive real number and G_ε be the operator associated to g in Proposition 3.5. Then

$$\int_{[0,1]} (J_t^{g(A)} - J_t^{g(A+\varepsilon)}) d\,\mu(t) = \int_{[0,1]} t\, J_t^{g(A)} G_\varepsilon\, J_t^{g(A+\varepsilon)} d\,\mu(t),$$

and hence, as $g(A)$ is closed, $\forall u \in X$ we have

$$\left(\int_{[0,1]} (J_t^{g(A)} - J_t^{g(A+\varepsilon)}) u\, d\mu(t), \int_{[0,1]} (1 - J_t^{g(A)}) G_\varepsilon\, J_t^{g(A+\varepsilon)} u\, d\mu(t) \right) \in g(A).$$

Therefore, by (3.4) we conclude that

$$\begin{aligned} f(g(A+\varepsilon)) &= f(g(A)) + \int_{]1,\infty[} (g(A+\varepsilon)_t - g(A)_t)\, d\mu(t) \\ &\quad + \int_{[0,1]} J_t^{g(A)} G_\varepsilon\, J_t^{g(A+\varepsilon)}\, d\mu(t). \end{aligned}$$

If we denote by $\mathcal{F}_\varepsilon$ the last two terms in the above expression, then $\mathcal{F}_\varepsilon \in \mathcal{L}(X)$ with

$$\|\mathcal{F}_\varepsilon\| \le M(A)^2 \left[\|G_\varepsilon\|\, |\mu|([0,1]) + \int_{]1,\infty[} \left(\int_{]0,\infty[} \frac{\varepsilon}{1+\varepsilon s}\, d\nu_t(s) \right) d|\mu|(t) \right],$$

where ν_t denotes the measure associated to the function g_t. By dominated convergence the last term tends to zero as $\varepsilon \to 0$ since

$$\int_{]0,\infty[} \frac{\varepsilon}{1+\varepsilon s}\, d\nu_t(s) \le g_t(1) - g_t(0), \quad \forall \varepsilon \le 1,$$

and $g_t(1) - g_t(0)$ is $|\mu|(t)$–integrable in $]1,\infty[$. Thus, $\lim_{\varepsilon \to 0} \|\mathcal{F}_\varepsilon\| = 0$, and from this (4.2) follows easily.

Finally, if $f \in \mathcal{S}_0$, by continuity (see Remark 3.2), it is sufficient to take limits as $\lambda \to 0$ in the expression

$$f(g(A_\lambda)) = (f \circ g)(A_\lambda), \quad \forall \lambda > 0. \qquad \square$$

In the following theorem we exclude the well-known case $A \in \mathcal{L}(X)$.

Theorem 4.4 (Spectral mapping theorem). *Let $A \in \mathcal{M}$ such that $A \notin \mathcal{L}(X)$. The following assertions hold:*

(i) *If $f \in \mathcal{T}_0$, then*

$$\{f(s) : s \in \sigma(A)\} \subseteq \sigma(f(A)).$$

(ii) *If $f \in \mathcal{S}_0$, then*

$$\sigma(f(A)) = \{f(s) : s \in \sigma(A)\} \cup \{f(\infty)\}.$$

(iii) *If $f \in \mathcal{T}_+ \setminus \mathcal{S}_0$, then*

$$\sigma(f(A)) = \{f(s) : s \in \sigma(A)\}.$$

Proof. (i) If $s \in \sigma(A)$ is not zero, then

$$f(z) - f(s) = (z - s)h(z),$$

where $h(z) \in \mathcal{S}_0$. Therefore, by Theorem 4.1, part (ii), the following inclusions hold:

$$h(A)(A - s) \subseteq f(A) - f(s) \subseteq (A - s)h(A).$$

If $f(s) \in \rho(f(A))$, from the first inclusion it is deduced that $A - s$ is a one-to-one operator, and from the second one that it is surjective. Consequently, $(A-s)^{-1} \in \mathcal{L}(X)$, which is a contradiction. So, we have $f(s) \in \sigma(f(A))$.

Let us now consider the case $0 \in \sigma(A)$. The proof is completed by showing that $a = f(0) \in \sigma(f(A))$. If we suppose that $a \in \rho(f(A))$, then, by Remark 3.5, we have

$$\begin{aligned} B &:= A \int_{[0,b]} J_t^A \, d\mu(t)|_{\overline{D(A)}} \\ &= \left[1 - \int_{]b,\infty[} A_t \, (f(A) - a)^{-1} d\mu(t)\right] (f(A) - a). \end{aligned}$$

So, by choosing $b > 0$ large enough, the operator $B^{-1} \in \mathcal{L}(X)$. This gives $0 \in \rho(A)$, which contradicts our assumption.

(ii) This part runs as in [8, Theorem 3.1] by the Gelfand theory.

(iii) If $f \in \mathcal{T}_+ \setminus \mathcal{S}_0$, then we know that $f(A) \in \mathcal{M}$ and $f(A)_\lambda = f_\lambda(A)$, $\forall \lambda > 0$ (see Corollary 4.2). Moreover, by Corollary 3.7, $f(A) \notin \mathcal{L}(X)$, and as a consequence of this it is not hard to show that

$$\sigma(f(A)_\lambda) = \left\{ \frac{s}{1+\lambda s} : s \in \sigma(f(A)) \right\} \cup \left\{ \frac{1}{\lambda} \right\}. \tag{4.3}$$

On the other hand, by applying part (ii) to the function f_λ we have

$$\sigma(f_\lambda(A)) = \{ f_\lambda(s) : s \in \sigma(A) \} \cup \left\{ \frac{1}{\lambda} \right\}, \tag{4.4}$$

since $f(\infty) = \infty$. Now, the proof follows from equalizing (4.3) and (4.4). □

5. Fractional powers of multivalued non-negative linear operators.

By applying the functional calculus developed in the previous sections to the function z^α, $0 < \operatorname{Re} \alpha < 1$ (see Example 2.3), we can obtain a theory of fractional powers for multivalued non negative linear operators and for this kind of exponents.

In this section we extend this theory of fractional powers to exponents with $\operatorname{Re} \alpha \neq 0$.

The concept of fractional power that we introduce is based on formula (3.4), which points out a relationship between the theory of fractional powers of non-negative densely defined operators and the multivalued case. However, the proof of some of the main results for the multivalued case is based on the functional calculus.

Throughout this section $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$, and $A \in \mathcal{M}$.

DEFINITION 5.1. We define the fractional power A^α with base A and exponent α to be the multivalued linear operator given by

$$A^\alpha = (1 + A)[A_D]^\alpha (1 + A_D)^{-1}.$$

REMARK 5.1. If $0 < \operatorname{Re} \alpha < 1$, then $A^\alpha = z^\alpha(A) = \liminf_{\lambda \to 0} z^\alpha(A_\lambda)$ (see Theorem 3.10), and therefore, our concept extends the one given in [1]. On the other hand, this definition is also an extension of the well-known univalent case (see [11]).

Theorem 5.1. *A^α is closed.*

Proof. The proof is a straightforward consequence of the fact that $[A_D]^\alpha$ is a closed operator. □

Lemma 5.2. *If $u \in D([A_D]^\alpha)$ and there is $z \in \mathbb{C}$ such that $zu - [A_D]^\alpha u \in D(A)$, then $u \in D(A_D)$.*

Proof. Let us suppose that $0 < \operatorname{Re}\alpha < 1$. Let $n \in \mathbb{N}$ such that $\beta = (1-\alpha)/n$ satisfies $\operatorname{Re}\beta < \operatorname{Re}\alpha$. By the additivity of fractional powers of A_D (see [10, Theorem 7.1]), $D([A_D]^\alpha) \subseteq D([A_D]^\beta)$. Moreover, by Corollary 3.9, we have

$$zu - [A_D]^\alpha u \in D(A) \subseteq D([A^\beta]_D) = D([A_D]^\beta).$$

Therefore, $[A_D]^\alpha u \in D([A_D]^\beta)$ and, again by additivity, it follows that

$$u \in D([A_D]^{\alpha+\beta}) \subseteq D([A_D]^{2\beta}).$$

Reiterating this argument it is concluded that $u \in D([A_D]^{\alpha+n\beta}) = D(A_D)$.

If $\operatorname{Re}\alpha \geq 1$ we choose β with $0 < \operatorname{Re}\beta < 1$, and reasoning as in the previous case we find that $u \in D([A_D]^{\alpha+\beta}) \subseteq D(A_D)$. □

Theorem 5.3 (Additivity). *If* $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, *then*

$$A^\alpha A^\beta = A^{\alpha+\beta}.$$

Proof. It is evident that $A^\alpha A^\beta \subseteq A^{\alpha+\beta}$. Moreover, as $A^\alpha A^\beta 0 = A0 = A^{\alpha+\beta}0$, it is sufficient to show that $D(A^{\alpha+\beta}) \subseteq D(A^\alpha A^\beta)$. Let $u \in D(A^{\alpha+\beta})$. By definition,

$$\begin{aligned}&(1 + A_D)^{-1}u \in D([A_D]^{\alpha+\beta}) = D([A_D]^\alpha [A_D]^\beta) \text{ and}\\ &[A_D]^\alpha [A_D]^\beta (1 + A_D)^{-1}u \in D(A).\end{aligned}$$

By Lemma 5.2 we have $u \in D(A^\beta)$ and there is $w \in A^\beta u \cap \overline{D(A)}$. It is easy to show that $w \in D(A^\alpha)$ and, consequently, $u \in D(A^\alpha A^\beta)$. □

Theorem 5.4. *The following assertions hold*:

(i) $A^n = A^{(n \overset{times}{\ldots})} A$, $\forall n \in \mathbb{N}$.

(ii) $(A^\alpha)^{-1} = (A^{-1})^\alpha$.

(iii) $\overline{D(A^\alpha)} = \overline{D(A)}$, $\overline{R(A^\alpha)} = \overline{R(A)}$, $A^\alpha 0 = A0$ *and* $\ker A^\alpha = \ker A$.

(iv) *If* $0 < \beta < 1$, *then* $A^\beta \in \mathcal{M}$ *and* $(A^\beta)^\alpha = A^{\beta\alpha}$ *(Multiplicativity).*

Proof. Part (i) is trivial. (ii) is deduced from additivity and the fact that the result is true for $0 < \operatorname{Re}\alpha < 1$ (see Theorem 3.2). (iii) easily follows from the definition of A^α and (ii). Regarding (iv), Corollary 4.2 assures the non-negativity of A^β, and multiplicativity follows from additivity and Theorem 4.3. □

Proposition 5.5. *The following properties hold*:

$$[A_D]^\alpha = [A^\alpha]_D, \quad \rho(A^\alpha) = \rho([A_D]^\alpha) \quad \textit{and}$$

(5.1) $$(z - A^\alpha)^{-1} = (1 + A_D)(z - [A_D]^\alpha)^{-1}(1 + A)^{-1}, \quad \forall z \in \rho(A^\alpha).$$

Proof. It is a simple matter to check the validity of first identity.

Let us now prove that $\rho(A^\alpha) = \rho([A_D]^\alpha)$. The inclusion $\rho(A^\alpha) \subseteq \rho([A_D]^\alpha)$ is trivial. Let now $z \in \rho([A_D]^\alpha)$. Then $z - A^\alpha$ is a one-to-one operator, since if $(u, v) \in A^\alpha$ and $0 = zu - v$, then $(u, v) \in [A_D]^\alpha$ and therefore $u = 0$. To prove that $z - A^\alpha$ is surjective let $u \in X$. If

$$v = (z - [A_D]^\alpha)^{-1}(1 + A)^{-1}u \in D([A_D]^\alpha),$$

then $zv - [A_D]^\alpha\, v \in D(A)$, and so, by Lemma 5.2, we have $v \in D(A_D)$. The element $w = (1 + A_D)v$ belongs to $D(A^\alpha)$ since

$$[A_D]^\alpha\,(1 + A)^{-1}w = zv - (1 + A)^{-1}u \in D(A),$$

and moreover $u \in (z - A^\alpha)w$. Consequently, $z - A^\alpha$ is surjective and as it is closed, then $z \in \rho(A^\alpha)$.

Now, it is already evident that (5.1) holds. □

Corollary 5.6 (Spectral mapping theorem). *If $\sigma(A)$ is empty, then $\sigma(A^\alpha)$ also is. Otherwise,*

$$\sigma(A^\alpha) = \{z^\alpha : z \in \sigma(A)\}.$$

Proof. The proof follows from Proposition 5.5 and the spectral mapping theorem to the dense case (see [12, Theorem 3.5]) □

REMARK 5.2. Thanks to part (ii) of Theorems 4.1 and 4.4, the last Theorem can also be proved by means of the Balakrishnan technique (see [5, Theorem 3.1]). This proof is only valid in Banach spaces. However, the proof in [12] is based on integral representations of the resolvent of the operator A^α. For this reason, our proof is valid in any sequentially complete locally convex space. Through (5.1), we extend these integral representations (see [12, Theorem 3.2]) to the multivalued case in the following Corollary.

Corollary 5.7. *Let $\alpha \in \mathbb{C}$ such that $|\alpha|^2 < \operatorname{Re}\alpha$. The following properties hold:*

(i) *If $z \in \mathbb{C} \setminus \{\lambda^\alpha e^{i\theta\alpha} : \lambda \geq 0$ and $-\pi \leq \theta \leq \pi\}$, then*

$$(z - A^\alpha)^{-1} = -\frac{\sin\alpha\pi}{\pi}\int_0^\infty \frac{t^{\alpha-1}}{z^2 - 2zt^\alpha\cos\alpha\pi + t^{2\alpha}}\, J_{1/t}^A\, dt.$$

(ii) *If* $z = s^\alpha$ *with* $s \in (\mathbb{C} \setminus \mathbb{R}_-) \cap \rho(A)$, *then*

$$(z - A^\alpha)^{-1} = \frac{1}{\alpha} s^{1-\alpha}(s - A)^{-1} - \frac{\sin \alpha\pi}{\pi} \int_0^\infty \frac{t^{\alpha-1}}{s^{2\alpha} - 2s^\alpha t^\alpha \cos \alpha\pi + t^{2\alpha}} J_{1/t}^A \, dt.$$

(iii) *If* $r > 0$, *then*

$$(r^\alpha e^{\pm i\alpha\pi} - A^\alpha)^{-1} = r^{-\alpha} e^{\mp i\alpha\pi} J_{1/r}^A + \frac{\sin \alpha\pi}{\pi} \int_0^\infty \frac{t^{\alpha-2}(r-t)}{(t^\alpha - r^\alpha)(t^\alpha - e^{\pm 2i\alpha\pi} r^\alpha)} (1 - J_{1/r}^A) J_{1/t}^A \, dt.$$

In the next two results we study the sectorial property of fractional powers and we extend the multiplicativity (see part (iv) of Theorem 5.4) to exponents, depending on A, greater than unit.

Proposition 5.8. *If* A *is* ω*–sectorial,* $\omega \in]0, \pi]$, *and* $\beta > 0$ *such that* $\beta\omega \leq \pi$, *then* A^β *is* $\beta\omega$*–sectorial. In particular,* $A^\beta \in \mathcal{M}$.

Proof. By the spectral mapping theorem we have $\sigma(A^\beta) \subseteq S_{\beta\omega}$. It remains to prove that for $\beta\omega \leq |\theta| \leq \pi$, the operators $\lambda(\lambda e^{i\theta} - A^\beta)^{-1}$, $\lambda > 0$, are uniformly bounded (see Remark 2.1). For $\beta < 1$, the result follows from the integral formulas given in Corollary 5.7. For the general case, let n be a positive integer such that $n > \beta$. Let $z \notin S_{\beta\omega}$ and let $\{z_k\}_{k=1}^n$ its n-roots. The operator $A^{\beta/n}$ verifies (2.3) and from this relation is not hard to check that

$$A^\beta - z = \prod_{k=1}^n (A^{\beta/n} - z_k). \tag{5.2}$$

So, we have

$$\|z(z - A^\beta)^{-1}\| \leq \prod_{k=1}^n \|z_k(z_k - A^{\beta/n})^{-1}\|,$$

and, as $z_k \notin S_{\beta\omega/n}$ and $A^{\beta/n}$ is $\beta\omega/n$–sectorial, we conclude the proof. □

REMARK 5.3. Regarding the property (5.2), remember that the product formula is valid for multivalued linear operators and for polynomials with coefficients in $\mathbb{C}$ (see [4, Theorem 2.3]).

Theorem 5.9 (Multiplicativity). *If* A *is* ω*–sectorial,* $\omega \in]0, \pi]$, *and* $\beta > 0$ *such that* $\beta\omega \leq \pi$, *then*

$$(A^\beta)^\alpha = A^{\beta\alpha}.$$

Proof. This result easily follows from the dense case (see [10, Theorem 10.6]). □

The identity $(A^*)^\alpha = (A^\alpha)^*$, $\operatorname{Re}\alpha > 0$, has already been proved in the densely defined operators case (see [11, Theorem 4.2]), where the statement made sense. In the following theorem we extend this result to the multivalued case. By Corollary 3.6, if $0 < \operatorname{Re}\alpha < 1$, then the mentioned property holds.

Theorem 5.10. *Suppose either that* $A0 + \overline{D(A)} = X$, *or* $\ker A + \overline{R(A)} = X$. *Then*

$$(A^*)^\alpha = (A^\alpha)^*.$$

Proof. Let $n \in \mathbb{N}$, $n > \operatorname{Re}\alpha$. By additivity and by the properties of adjoint operator we have

$$(A^*)^\alpha = ((A^*)^{\alpha/n})^n = ((A^{\alpha/n})^*)^n \subseteq (A^\alpha)^*.$$

As $(A^*)^\alpha 0 = (A^\alpha)^* 0$, then the proof is completed by showing that $D((A^\alpha)^*) \subseteq D((A^*)^\alpha)$.

Let us suppose that $A0 + \overline{D(A)} = X$. We will show that the operator $(A^\alpha)^*$ satisfies (2.3). Let $u^* \in (A^\alpha)^* 0 \cap \overline{D((A^\alpha)^*)}$ and $\{(u_n^*, v_n^*)\}_{n\in\mathbb{N}}$ be a sequence in $(A^\alpha)^*$ such that $\lim_{n\to\infty} u_n^* = u^*$. As $\langle u_n^*, w\rangle = 0$, $\forall w \in A^\alpha 0$, then u^* also vanishes on $A0$, and therefore $u^* = 0$.

Let now $u^* \in D((A^\alpha)^*)$. Due to the fact that $J_1^A A^\alpha \subseteq A^\alpha J_1^A$ and that $J_1^A \in \mathcal{L}(X)$ we have

$$J_1^{A^*}(A^\alpha)^* \subseteq (A^\alpha J_1^A)^* \subseteq (J_1^A A^\alpha)^* = (A^\alpha)^* J_1^{A^*}.$$

Hence,

$$\begin{aligned}(A_1^*)^n u^* &\in (A^*)^n (J_1^{A^*})^n u^* = (A^*)^{n-\alpha}(A^*)^\alpha (J_1^{A^*})^n u^* \\ &= (A^*)^{n-\alpha}(J_1^{A^*})^n (A^\alpha)^* u^*,\end{aligned}$$

and then, there is $w^* \in A^*0$ satisfying $(A_1^*)^n u^* + w^* \in D((A^*)^\alpha)$. Since $(A_1^*)^n u^* = u^* + v^*$, with $v^* \in D(A^*) \subseteq \overline{D((A^*)^\alpha)} \subseteq \overline{D((A^\alpha)^*)}$, we can state that $w^* = \{0\}$. This gives $(A_1^*)^n u^* \in D((A^*)^\alpha)$, which implies that $u^* \in D((A^*)^\alpha)$, as is easy to check.

If $\ker A + \overline{R(A)} = X$ holds, then A^{-1} satisfies the above condition and therefore

$$((A^{-1})^*)^\alpha = ((A^{-1})^\alpha)^*.$$

The result now follows by taking inverses in this expression. □

REMARK 5.4. Remember that the hypothesis on operator A in this theorem is satisfied if X is a reflexive Banach space (see (vi) of Proposition 2.1). Moreover, from the proof of the above theorem it is evident that, in general, the identity

$$(A^*)^\alpha = (A^\alpha)^* \cap \left(\overline{D(A^*)} \times X^*\right)$$

holds.

REMARK 5.5. For exponents $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha < 0$, we define $A^\alpha = (A^{-1})^{-\alpha}$. From the case $\operatorname{Re}\alpha > 0$ we easily obtain the fundamental properties for this kind of fractional powers.

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