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FAMILIES OF FINITE COVERINGS OF THE RIEMANN SPHERE

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Abstract

For a finite group G and its subgroup H which does not contain any normal subgroup of G except the identity, criterions for the existence or non-existence of Hurwitz families of special type of (G, H)-coverings of the Riemann sphere are given.

1. Introduction

Let G be a finite group and H its subgroup which does not contain any normal subgroup of G except the identity $\{1\}$. We call a finite covering $f\colon X \to Y$, a (G, H)-*covering* if, roughly speaking, its permutation monodromy representation is equivalent to the representation of G on the set G/H of left cosets (see §2 for a rigorous definition). In particular, a $(G, \{1\})$-covering is a Galois covering with its Galois group isomorphic to G. We simply call it a G-*covering*.

In this paper, we discuss non-degenerate families of (G, H)-coverings of the Riemann sphere $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$, for a fixed (G, H). We call a non-degenerate family of (G, H)-coverings of $\mathbb{P}^1$, a *Hurwitz family* if

(i) it contains all (G, H)-coverings, up to isomorphisms, which are topologically equivalent to a given $f_0\colon X_0 \to \mathbb{P}^1$ and

(ii) any different members of it are not isomorphic.

Its parameter space is called a *Hurwitz parameter space*.

A Hurwitz parameter space M for a given f_0: $X_0 \to \mathbb{P}^1$ always and uniquely exists. (M is a connected complex manifold of dimension s, the number of branch points of f_0, and is a finite unramified covering $M \to \mathbb{P}^s - \Delta$, where Δ is the discriminant locus.) On the other hand, a Hurwitz family may not exist. Fried [7] asked and discussed the problem of existence of Hurwitz families, and gave applications of his results to arithmetic problems. The same problem was discussed and developed in Biggers–Fried [2], Fried–Völklein [8], Völklein [14], Dèbes–Douai [5], Dèbes–Douai–Emsalem [6], Dèbes [4], Bailey–Fried [1] etc., with applications to arithmetic problems.

In this paper, we discuss the same problem, using a little different principle, explained below, from Fried [7] or Biggers–Fried [2], and then we define a *Hurwitz fam-*

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ily of special type (see §5). We give a criterion for the existence of a Hurwitz family of special type. In particular, we prove

Theorem (cf., Theorem 5.6). *Let Z_G be the center of G and $N_G(H)$ be the normalizer of H in G. If $N_G(H) = H \cdot Z_G$ and if the exact sequence $1 \to Z_G \to G \to \mathrm{Inn}(G) \to 1$ does not split, then there does not exist a Hurwitz family of special type of (G, H)-coverings of $\mathbb{P}^1$.*

Our principle in this paper is as follows: We make use of the Galois correspondence between subgroups of a fundamental group and *branched* coverings, using the extension theorem of Grauert–Remmert (see §2). We assume that the locus

$$q_1^0, \ldots, q_s^0$$

of a given $f_0 \colon X_0 \to \mathbb{P}^1$ does not contain ∞, the point at infinity. The divisor

$$D^0 = (q_1^0) + \cdots + (q_s^0)$$

on $\mathbb{P}^1$ is then a point in $\mathbb{C}^s = \mathbb{P}^s - H_\infty$, where $\mathbb{P}^s$ is the s-dimensional complex projective space and H_∞ is the hyperplane at infinity. The fundamental group $\pi_1(\mathbb{C}^s - \Delta, D_0)$ is isomorphic to the Artin braid group B_s of s-strings and acts on the fundamental group $\pi_1(\mathbb{P}^1 - \{q_1^0, \ldots, q_s^0\}, \infty)$. Using this action, the Hurwitz parameter space $M \to \mathbb{P}^s - \Delta$ corresponds to a suitable subgroup $\hat{\Gamma}$ in $\pi_1(\mathbb{C}^s - \Delta, D_0)$, under the Galois correspondence. Now the existence problem of the Hurwitz families reduces to the problem of a kind of group extensions with respect to $\hat{\Gamma}$

Next, using the Galois correspondence, we see that there exists a family

$$g \colon Y \to \mathbb{P}^1 \times M_1,$$

which has a lift of the ∞-section and is 'near' from a Hurwitz family, whose parameter space M_1 is a finite covering $\pi_1 \colon M_1 \to M$ of M (see §4).

Now a Hurwitz family of special type is a Hurwitz family whose pull-back over the map π_1 is isomorphic to the family g.

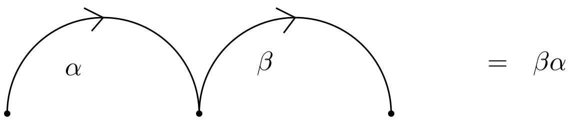
In this paper, we discuss the problem only from the geometric point of view. If our result is combined to those in Fried [7] or Biggers–Fried [2], then arithmetic applications will be obtained.

The author would like to thank the referee for his/her many valuable comments. Following his/her suggestions, many parts of the original version of the paper are revised.

2. Preliminary remarks and notations

In this paper, we use the following notations:

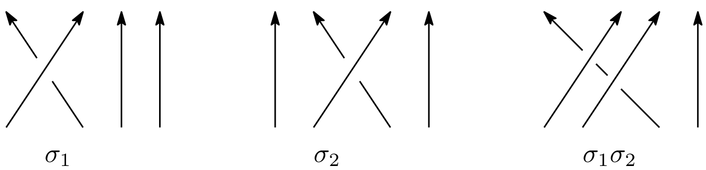
(i) For paths α and β such that the end point of α coincides with the initial point of β, the composition of paths connecting α and β is denoted by $\beta\alpha$.



The universal covering space $\hat{X}$ of a connected space X is given by the set of all homotopy equivalence classes of paths with the end point o (a reference point). The fundamental group $\pi_1(X, o)$ acts on $\hat{X}$ from the left as compositions of paths.
(ii) The products of permutations are defined, e.g., as follows:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

(iii) Braids and their products are denoted and defined, e.g., as follows:



In the following, we mainly follow terminologies in Namba [11] and Mizuta–Namba [10].
A *finite covering* of a connected complex manifold Y is, by definition, a finite proper holomorphic map $f: X \to Y$ of an irreducible normal complex space X onto Y. Finite coverings $f: X \to Y$ and $f': X' \to Y$ of Y are said to be *isomorphic*, denoted by $f \simeq f'$, if there is a biholomorphic map $\psi: X \to X'$ such that $f' \cdot \psi = f$. The set $\mathrm{Aut}(f)$ of all automorphisms of f forms a group under compositions, called the *automorphism group* of f. This is a finite subgroup of the automorphism group $\mathrm{Aut}(X)$ of X. Each $\psi \in \mathrm{Aut}(f)$ acts on every fiber of f. f is called a *Galois covering* if $\mathrm{Aut}(f)$ acts trasitively on every fiber of f. In this case, $\mathrm{Aut}(f)$ is sometimes called the *Galois group* of f. Y is, in this case, canonically biholomorphic to $X/\mathrm{Aut}(f)$. A Galois covering is called a *cyclic* (resp. *abelian*) *covering* if its Galois group is cyclic (resp. abelian).

Finite coverings $f: X \to Y$ and $f': X' \to Y'$ are said to be *holomorphically equivalent* (resp. *topologically equivalent*, resp. *meromorphically equivalent*) if there are biholomorphic maps (resp. orientation preserving homeomorphisms, resp. bimeromorphic maps) $\psi: X \to X'$ and $\varphi: Y \to Y'$ such that $f' \cdot \psi = \varphi \cdot f$. We denote this relation by $f \sim_{\mathrm{hol}} f'$ (resp. $f \sim_{\mathrm{top}} f'$, resp. $f \sim_{\mathrm{mer}} f'$).

For a finite covering $f\colon X \to Y$, put

$$R_f = \{p \in X \mid f \text{ is not biholomorphic around } p\},$$
$$B_f = f(R_f).$$

Then they are hypersurfaces of X and Y, respectively, because of the non-singularity of Y. They are called the *ramification locus* and the *branch locus* of f, respectively. If B is a hypersurface of Y such that $B_f \subset B$, then f is said to *branch at most at* B. In this case, the restriction

$$f'\colon X' = f^{-1}(Y - B) \to Y - B$$

is an unbranched (i.e., ordinary topological) covering. We define the *degree* of f, $\deg f$, to be the mapping degree of f'. Note that the singular loci relate as follows (see, e.g., Namba [11]);

$$\operatorname{Sing} X \subset f^{-1}(\operatorname{Sing} B),$$
$$\operatorname{Sing} R_f = \operatorname{Sing} f^{-1}(B_f) \subset f^{-1}(\operatorname{Sing} B).$$

The following theorem is fundamental in our discussion.

Theorem 2.1 (Grauert–Remmert [9]). *Let Y and B be a connected complex manifold and its hypersurface. Then any finite unbranched covering $f'\colon X' \to Y - B$ can be uniquely* (*up to isomorphisms*) *extended to a finite covering $f\colon X \to Y$ which branches at most at B.*

This theorem implies in particular that there exists a one-to-one correspondence (Galois correspondence) between isomorphism classes of finite coverings $f\colon X \to Y$ of Y which branches at most at B, and conjugacy classes of subgroups $\mathscr{H}$ of finite index of the fundamental group $\pi_1(Y - B, q_0)$, where q_0 is a reference point. The *Galois closure* $\hat{f}\colon \hat{X} \to Y$ of f is the Galois covering of Y, which corresponds to the intersection $\mathscr{K}$ of all subgroups which are conjugate to $\mathscr{H}$ in $\pi_1(Y - B, q_0)$. If $\mathscr{H}'$ is a subgroup of $\pi_1(Y - B, q_0)$ such that $\mathscr{H} \subset \mathscr{H}'$, and if $f'\colon X' \to Y$ is the finite covering which corresponds to $\mathscr{H}'$, then there exists a finite proper surjective holomorphic map $h\colon X \to X'$ such that $f' \cdot h = f$. This follows from the following proposition.

Proposition 2.2. *Let $f\colon X \to Y$ be a finite covering which branches at most at a hypersurface B of Y and corresponds to a subgroup $\mathscr{H}$ of $\pi_1(Y - B, q_0)$. Take a point p_0 in $f^{-1}(q_0)$. Let $g\colon Z \to Y$ be a holomorphic map of an irreducible normal complex space Z into Y such that $g^{-1}(B)$ is a hypersurface of Z with* $\operatorname{Sing} Z \subset g^{-1}(B)$. *Suppose that, for a point $o \in g^{-1}(q_0)$, the homomorphism $g_*\colon \pi_1(Z - g^{-1}(B), o) \to \pi_1(Y - B, q_0)$ induced by g satisfies $g_*(\pi_1(Z - g^{-1}(B), o)) \subset \mathscr{H}$. Then there exists a* *unique holomorphic map* $h: Z \to X$ *such that* $h(o) = p_0$ *and* $f \cdot h = g$. (h *is called a lift of* g). *Conversely, if there is such a holomorphic map* h, *then* g *satisfies* $g_*(\pi_1(Z - g^{-1}(B), o)) \subset \mathscr{H}$.

Proof. For a point $z \in Z - g^{-1}(B)$, let γ be a path in $Z - g^{-1}(B)$ with the initial point o and the end point z. We define $h(z)$ to be the end point of the lift $\hat{\gamma}$ in $X - f^{-1}(B)$ of the path $g(\gamma)$ with the initial point p_0. This is well defined. In fact, if γ' is another such path in $Z - g^{-1}(B)$, then $g_*(\gamma'^{-1}\gamma)$ is in $\mathscr{H}$ by the assumption. This implies that $\hat{\gamma}'^{-1}\hat{\gamma}$ is a loop in $X - f^{-1}(B)$, so the end point of $\hat{\gamma}'$ is equal to the end point of $\hat{\gamma}$.

The map $h: Z - g^{-1}(B) \to X - f^{-1}(B)$, thus defined, satisfies $f \cdot h = g$. The map h is holomorphic, for $f: X - f^{-1}(B) \to Y - B$ is locally biholomorphic.

For a point $z \in g^{-1}(B)$, put $f^{-1}(g(z)) = \{p_1, \ldots, p_k\}$, $(k \leq \deg f)$. Let W be a connected open neighborhood of $g(z)$ such that $f^{-1}(W) = \bigcup_{j=1}^{k} V_j$ be the decomposition into the connected components such that $p_j \in V_j$ and f maps V_j onto W. Let U be a connected open neighborhood of z in Z such that $g(U) \subset W$. Then $U \cap (Z - g^{-1}(B)) = U - g^{-1}(B)$ is also connected. Hence $h(U \cap (Z - g^{-1}(B)))$ is contained in V_j for some j. We put $h(z) = p_j$ in this case. Thus h is extended continuously to $g^{-1}(B)$. The map $h: Z \to X$ is holomorphic, for Z is normal. h satisfies $f \cdot h = g$. The map h is uniquely determined, by the principle of analytic continuation.

The converse is obvious. □

Now, as before, let $f: X \to Y$ be a finite covering which branches at most at a hypersurface B of Y and corresponds to a subgroup $\mathscr{H}$ of $\pi_1(Y - B, q_0)$. Put

$$f^{-1}(q_0) = \{p_1, \ldots, p_d\}, \quad (d = \deg f).$$

The homotopy class of a loop γ in $\pi_1(Y - B, q_0)$ defines a permutation $\Phi_f(\gamma)$ on the set $\{p_1, \ldots, p_d\}$, where $\Phi_f(\gamma)(p_j)$ is the end point of the lift of γ with the initial point p_j.

$$\Phi_f: \pi_1(Y - B, q_0) \to S_d$$

is a homomorphism whose image is a transitive subgroup of S_d. Φ_f is called the *permutation monodromy representation* of f. The image $\Phi_f(\pi_1(Y - B, q_0))$ is called the *permutation monodromy group* of f. The image $\Phi_f(\mathscr{H})$ of $\mathscr{H}$ is the isotropy subgroup of $\Phi_f(\pi_1(Y - B, q_0))$ for a point in $\{p_1, \ldots, p_d\}$.

Now let (G, H) be a pair of a finite group G and its subgroup H which contains no normal subgroup of G except the identity $\{1\}$. G acts effectively on the set of left cosets $\{Ha\}$ as follows:

$$(g, Ha) \in G \times G/H \mapsto Hag^{-1} \in G/H.$$

This gives a permutation representation of G:

$$R\colon g \in G \mapsto \begin{pmatrix} Ha \\ Hag^{-1} \end{pmatrix} \in S_d$$

($d = [G : H]$). By the condition on H, the representation R is faithful, that is, the homomorphism R is injective. Hence we may regard G as a transitive subgroup of S_d through $R\colon G \subset S_d$. In this identification, H is written as

$$H = G \cap S_{d-1},$$

where S_{d-1} is the isotropy subgroup of S_d of the letter 1, say.

Now we call a finite covering $f\colon X \to Y$ of Y which branches at most at B, a (G, H)-*covering* if there is a surjective homomorphism

$$\xi : \pi_1(Y - B, q_0) \to G$$

such that
(i) $R \cdot \xi$ is equivalent to the monodromy representation Φ_f and
(ii) $\mathscr{H} = \xi^{-1}(H)$ corresponds to f.

A $(G, \{1\})$-covering is simply called a G-*covering*. This is a Galois covering with the Galois group isomorphic to G.

A (G, H)-covering on the Riemann sphere $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ is given as follows: Consider a presentation

$$G = \langle g_1, \ldots, g_s \mid g_1 \cdots g_s = 1, g_1^{e_1} = 1, \ldots, g_s^{e_s} = 1, *, \ldots, * \rangle \tag{2.1}$$

of G, where $e_1, \ldots, e_s$ are integers ≥ 2 and $*, \ldots, *$ are other relations. Let $q_1, \ldots, q_s$ be distinct points in $\mathbb{P}^1$. We identify the set $\{q_1, \ldots, q_s\}$ with the divisor $D = (q_1) + \cdots + (q_s)$, ($(q_j)$: the point divisor). Take a reference point q_0 in $\mathbb{P}^1 - D$. Then $\pi_1(\mathbb{P}^1 - D, q_0)$ is presented as follows:

$$\pi_1(\mathbb{P}^1 - D, q_0) = \langle \gamma_1, \ldots, \gamma_s \mid \gamma_1 \cdots \gamma_s = 1 \rangle.$$

Here γ_j are (the homotopy classes of) the meridians around q_j as in Fig. 1:

We define a surjective homomorphism

$$\xi : \pi_1(\mathbb{P}^1 - D, q_0) \to G \tag{2.2}$$

by $\xi(\gamma_j) = g_j$ $(j = 1, \ldots, s)$.

The finite covering $f\colon X \to \mathbb{P}^1$ which corresponds to the subgroup $\mathscr{H} = \xi^{-1}(H)$ of $\pi_1(\mathbb{P}^1 - D, q_0)$ is a (G, H)-covering. Conversely, any (G, H)-covering of $\mathbb{P}^1$ can be obtained in this way. The G-covering $\hat{f}\colon \hat{X} \to \mathbb{P}^1$ which corresponds to $\mathscr{K} = \mathrm{Ker}(\xi)$ is the Galois closure of f.

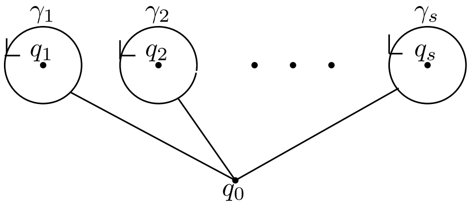


Fig. 1.

3. Non-degenerate family of (G, H)-coverings

Let T be a connected complex manifold. A *non-degenerate family* of (G, H)-coverings of $\mathbb{P}^1$ *with the parameter space* T means a finite covering $f: X \to \mathbb{P}^1 \times T$ such that
(i) the branch locus B_f is a non-singular hypersurface of $\mathbb{P}^1 \times T$ such that the restriction of the projection $\mathbb{P}^1 \times T \to T$ to each connected component of B_f is a finite unramified covering of T and
(ii) the restriction

$$f_t: X_t = f^{-1}(\mathbb{P}^1 \times t) \to \mathbb{P}^1 \times t \quad (= \mathbb{P}^1)$$

of f is a (G, H)-covering such that $B_{f_t} = B_f \cap (\mathbb{P}^1 \times t)$, for each $t \in T$ and
(iii) the number s of branch points of f_t is constant for $t \in T$.

In this case, X is a connected complex manifold and R_f is a non-singular hypersurface of X.

Two finite coverings $f_1: X_1 \to \mathbb{P}^1$ and $f_2: X_2 \to \mathbb{P}^1$ of $\mathbb{P}^1$ are said to be *deformation equivalent* if there is a non-degenerate family $f: X \to \mathbb{P}^1 \times T$ of (G, H)-coverings of $\mathbb{P}^1$ and points t_1 and t_2 in T such that f_1 (resp. f_2) is isomorphic to f_{t_1} (resp. f_{t_2}).

Theorem 3.1. *Finite coverings $f_1: X_1 \to \mathbb{P}^1$ and $f_2: X_2 \to \mathbb{P}^1$ of $\mathbb{P}^1$ are deformation equivalent if and only if they are topologically equivalent.*

The 'only if' part of the theorem can be proved in a similar way to the proof of Theorem 4.1 in Mizuta–Namba [10] on G-coverings of $\mathbb{P}^1$. The 'if' part will be shown later (see Remark 4.11).

This theorem clearly implies

Corollary 3.2. *For a non-degenerate family $f: X \to \mathbb{P}^1 \times T$ of (G, H)-coverings of $\mathbb{P}^1$, the monodromy representations Φ_{f_t} of f_t and $\Phi_{f_{t'}}$ of $f_{t'}$ are equivalent for all $t, t' \in T$.*

Now, for a non-degenerate family $f: X \to \mathbb{P}^1 \times T$ of (G, H)-coverings of $\mathbb{P}^1$, take a reference point $o \in T$. We assume that the branch locus

$$B_{f_o} = \{q_1^o, \dots, q_s^o\}$$

of f_o does not contain the point ∞ at infinity:

$$\infty \notin B_{f_o}. \tag{3.1}$$

REMARK 3.3. If B_{f_o} contains ∞, then we take an automorphism φ (i.e., a coordinate change) of $\mathbb{P}^1$ such that the branch locus of $\varphi \cdot f_o$ does not contain ∞. Then, instead of f, we consider the non-degenerate family $(\varphi, id) \cdot f: X \to \mathbb{P}^1 \times T$ ($id =$ the identity map on T), which is holomorphically equivalent to f.

Consider the closed complex subspace

$$T_\infty = \{t \in T \mid \infty \in B_{f_t}\}$$

of T and its complement

$$T_{\text{fin}} = T - T_\infty, \tag{3.2}$$

which is a Zariski open set of T. The map

$$\rho: t \in T_{\text{fin}} \mapsto (\infty, t) \in (\mathbb{P}^1 \times T_{\text{fin}}) - B_f \tag{3.3}$$

is a holomorphic section of the projection

$$(\mathbb{P}^1 \times T_{\text{fin}}) - B_f \to T_{\text{fin}}.$$

We call it the ∞-section. Take a point $p_\infty \in X_{t_0}$ such that

$$f_o(p_\infty) = \infty.$$

Then the maps

$$f_o: (X_o, p_\infty) \to (\mathbb{P}^1, \infty),$$
$$f: (X_{|T_{\text{fin}}}, p_\infty) \to (\mathbb{P}^1 \times T_{\text{fin}}, (\infty, o))$$

($X_{|T_{\text{fin}}} = f^{-1}(\mathbb{P}^1 \times T_{\text{fin}})$) give injective homomorphisms

$$(f_o)_*: \pi_1(X_o - f_o^{-1}(D_o), p_\infty) \to \pi_1(\mathbb{P}^1 - D_o, \infty),$$
$$f_*: \pi_1(X_{|T_{\text{fin}}} - f^{-1}(B_f), p_\infty) \to \pi_1((\mathbb{P}^1 \times T_{\text{fin}}) - B_f, (\infty, o)).$$

By Lemma 4.2 of Mizuta–Namba [10], the projection $(\mathbb{P}^1 \times T_{\mathrm{fin}}) - B_f \to T_{\mathrm{fin}}$ (resp. the projection $X_{|T_{\mathrm{fin}}} - f^{-1}(B_f) \to (\mathbb{P}^1 \times T_{\mathrm{fin}}) - B_f \to T_{\mathrm{fin}}$) is a topological fiber bundle with the standard fiber $\mathbb{P}^1 - D_o$ ($D_o = (q_1^o) + \cdots + (q_s^o)$), (resp. the standard fiber $X_o - f_o^{-1}(D_o)$). Hence, by Steenrod [12], there exists a commutative diagram of long exact sequences:

$$
\begin{array}{cccccc}
\longrightarrow \pi_2(T_{\mathrm{fin}}, o) & \longrightarrow & \pi_1(X_o - f_o^{-1}(D_o), p_\infty) & \longrightarrow & \pi_1(X_{|T_{\mathrm{fin}}} - f^{-1}(B_f), p_\infty) & \longrightarrow \\
\| & & \downarrow{\scriptstyle (f_o)_*} & & \downarrow{\scriptstyle f_*} & \\
\underset{\rho_*}{\rightleftarrows} \pi_2(T_{\mathrm{fin}}, o) & \longrightarrow & \pi_1(\mathbb{P}^1 - D_o, \infty) & \longrightarrow & \pi_1((\mathbb{P}^1 \times T_{\mathrm{fin}}) - B_f, (\infty, o)) & \longrightarrow
\end{array}
$$

$$
\begin{array}{ccccc}
\longrightarrow \pi_1(T_{\mathrm{fin}}, o) & \longrightarrow & \pi_0(X_o - f_o^{-1}(D_o), p_\infty) & \longrightarrow & \pi_0(X_{|T_{\mathrm{fin}}} - f^{-1}(B_f), p_\infty) \\
\| & & \downarrow & & \downarrow \\
\underset{\rho_*}{\rightleftarrows} \pi_1(T_{\mathrm{fin}}, o) & \longrightarrow & \pi_0(\mathbb{P}^1 - D_o, \infty) & \longrightarrow & \pi_0((\mathbb{P}^1 \times T_{\mathrm{fin}}) - B_f, (\infty, o)).
\end{array}
$$

Here ρ_* is the homomorphism induced by ∞-section ρ in (3.3). By the existence of ρ_*, we have the following commutative diagram of short exact sequences:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_o - f_o^{-1}(D_o), p_\infty) & \longrightarrow & \pi_1(X_{|T_{\mathrm{fin}}} - f^{-1}(B_f), p_\infty) & \longrightarrow & \pi_1(T_{\mathrm{fin}}, o) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle (f_o)_*} & & \downarrow{\scriptstyle f_*} & & \| & & \\
1 & \longrightarrow & \pi_1(\mathbb{P}^1 - D_o, \infty) & \longrightarrow & \pi_1((\mathbb{P}^1 \times T_{\mathrm{fin}}) - B_f, (\infty, o)) & \underset{\rho_*}{\rightleftarrows} & \pi_1(T_{\mathrm{fin}}, o) & \longrightarrow & 1.
\end{array}
$$

(The surjectivity of $\pi_1(X_{|T_{\mathrm{fin}}} - f^{-1}(B_f), p_\infty) \to \pi_1(T_{\mathrm{fin}}, o)$ follows from the connectedness of $X_{|T_{\mathrm{fin}}} - f^{-1}(B_f)$ and $\mathbb{P}^1 - D_o$).

We put

$$
\begin{aligned}
\mathscr{H}_o &= (f_o)_*(\pi_1(X_o - f_o^{-1}(D_o), p_\infty)), \\
\mathscr{L} &= f_*(\pi_1(X_{|T_{\mathrm{fin}}} - f^{-1}(B_f), p_\infty)).
\end{aligned}
\tag{3.4}
$$

Then we have the following commutative diagram of short exact sequences:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathscr{H}_o & \longrightarrow & \mathscr{L} & \longrightarrow & \pi_1(T_{\mathrm{fin}}, o) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \| & & \\
1 & \longrightarrow & \pi_1(\mathbb{P}^1 - D_o, \infty) & \longrightarrow & \pi_1((\mathbb{P}^1 \times T_{\mathrm{fin}}) - B_f, (\infty, o)) & \underset{\rho_*}{\rightleftarrows} & \pi_1(T_{\mathrm{fin}}, o) & \longrightarrow & 1,
\end{array}
\tag{3.5}
$$

where i and j are inclusion maps.

REMARK 3.4. In the diagram (3.5), the upper exact sequence may not split. By Proposition 2.2, this splits and the diagram with the splitting is still commutative if and only if there is a lift

$$\hat{\rho}\colon T_{\mathrm{fin}} \to X_{|T_{\mathrm{fin}}} - f^{-1}(B_f)$$

of the ∞-section ρ in (3.3) such that $\hat{\rho}(o) = p_\infty$.

Next, for a non-degenerate family $f\colon X \to \mathbb{P}^1 \times T$ of (G, H)-coverings, we consider a holomorphic map

$$\Theta\colon t \in T \mapsto D_t = (q_1^t) + \cdots + (q_s^t) \in \mathbb{P}^s - \Delta,$$

where $\{q_1^t, \ldots, q_s^t\} = B_{f_t}$, $\mathbb{P}^s$ is regarded as the s-th symmetric product of $\mathbb{P}^1$ and Δ is the discriminant locus. Let T_{fin} be the Zariski open set of T in (3.2). The homomorphism

$$\Theta_*\colon \pi_1(T_{\mathrm{fin}}, o) \to B_s = \pi_1(\mathbb{C}^s - \Delta, D_o) \tag{3.6}$$

induced by Θ is called the *braid monodromy*. Here $\pi_1(\mathbb{C}^s - \Delta, D_o)$ is identified with the Artin braid group B_s of s-strings. Henceforth, we assume that

$$s \geq 3. \tag{3.7}$$

The Artin braid group B_s acts on $\pi_1(\mathbb{P}^1 - D_o, \infty)$ as follows:

$$\begin{aligned} \sigma_i(\gamma_i) &= \gamma_{i+1}, \\ \sigma_i(\gamma_{i+1}) &= \gamma_{i+1}^{-1}\gamma_i\gamma_{i+1}, \\ \sigma(\gamma_j) &= \gamma_j \quad (j \neq i, i+1), \end{aligned} \tag{3.8}$$

where γ_j are the meridians in Fig. 1 with $q_0 = \infty$. Note that the action is not effective. The action of $B_s/Z(B_s)$ is effective, where $Z(B_s)$ is the center of B_s, which is the cyclic group of infinite order generated by $(\sigma_1 \cdots \sigma_{s-1})^s$ (see Birman [3]).

Lemma 3.5. *For* $\delta \in \pi_1(T_{\mathrm{fin}}, o)$ *and* $\gamma \in \pi_1(\mathbb{P}^1 - D_o, \infty)$,

$$\Theta_*(\delta)(\gamma) = \rho_*(\delta)\gamma\rho_*(\delta)^{-1},$$

where the product of the right hand side is that of $\pi_1(\mathbb{P}^1 \times T_{\mathrm{fin}} - B_f, (\infty, o))$, *in which* $\pi_1(\mathbb{P}^1 - D_o, \infty)$ *is a normal subgroup.*

Proof. The action of B_s on $\pi_1(\mathbb{P}^1 - D_o, \infty)$ is defined to be that of the mapping classes of $(\mathbb{P}^1, D_o)$. Hence there is a map of the cylinder $S^1 \times [0, 1]$ (S^1: the unit circle) into $\mathbb{P}^1 \times T_{\mathrm{fin}} - B_f$ whose image is as in Fig. 2. The image of the map in Fig. 2 shows that $\Theta_*(\delta)(\gamma)$ is homotopic to $\rho_*(\delta)\gamma\rho_*(\delta)^{-1}$. □

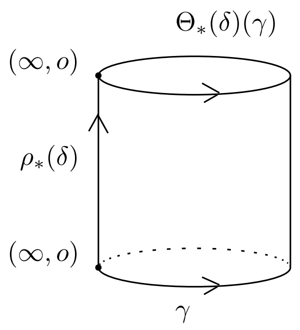


Fig. 2.

Let

(3.9) $$\xi_o\colon \pi_1(\mathbb{P}^1 - D, \infty) \to G$$

be the surjective homomorphism defined, as in §2, by

$$\xi_o(\gamma_j) = g_j \quad (j = 1, \dots, s).$$

Put

(3.10) $$\mathscr{K}_o = \mathrm{Ker}(\xi_o),\ \mathscr{H}_o = \xi_o^{-1}(H).$$

Then $\mathscr{H}_o$ is the subgroup which appeared in (3.4):

$$\mathscr{H}_o = (f_o)_*(\pi_1(X_o - f_o^{-1}(D_o),\ p_\infty)).$$

Consider the subgroup $\hat{\Gamma}_0$ of B_s defined by

(3.11) $$\hat{\Gamma}_0 = \{\sigma \in B_s \mid \sigma(\mathscr{K}_o) = \mathscr{K}_o\}.$$

Then $\hat{\Gamma}_0$ can be rewritten as follows:

(3.12) $$\hat{\Gamma}_0 = \{\sigma \in B_s \mid \text{there is an automorphism } \Lambda(\sigma) \in \mathrm{Aut}(G) \text{ such that } \Lambda(\sigma)\cdot\xi_o = \xi_o\cdot\sigma\}.$$

The map

(3.13) $$\Lambda\colon \hat{\Gamma}_0 \to \mathrm{Aut}(G)$$

is a homomorphism. We denote its image by Γ_0:

(3.14) $$\Gamma_0 = \Lambda(\hat{\Gamma}_0).$$

Note that Γ_0 depends on the presentation (2.1) of G.

Consider moreover the subgroup $\hat{\Gamma}$ of B_s defined by

(3.15)
$$\hat{\Gamma} = \{\sigma \in B_s \mid \sigma \text{ maps } \mathscr{H}_o \text{ to one of its conjugate subgroups in } \pi_1(\mathbb{P}^1 - D_o, \infty)\}.$$

Each element $\sigma \in \hat{\Gamma}$ maps $\mathscr{H}_o$ to its conjugate. Hence σ maps its conjugate to another conjugate. Hence σ maps their intersection $\mathscr{K}_o$ to itself. Thus we have

$$\operatorname{Ker}(\Lambda) \subset \hat{\Gamma} \subset \hat{\Gamma}_0. \tag{3.16}$$

Let Γ be the image of $\hat{\Gamma}$ by Λ:

$$\Gamma = \Lambda(\hat{\Gamma}). \tag{3.17}$$

Then we have easily

$$\Gamma = \{\psi \in \Gamma_0 \mid \psi \text{ maps } H \text{ to one of its conjugate subgroups in } G\}.$$

Note that each $\psi \in \Gamma$ induces a permutation on the set of all subgroups in G which are conjugate to H.

Lemma 3.6. (i) $\hat{\Gamma}$ *contains the inner automorphism group* $\operatorname{Inn}(\pi_1(\mathbb{P}^1 - D_o, \infty))$ *of* $\pi_1(\mathbb{P}^1 - D_o, \infty)$.
(ii) Γ *contains the inner automorphism group* $\operatorname{Inn}(G)$ *of* G.

Proof. (i) Put

$$\tau = \sigma_1 \cdots \sigma_{s-1}\sigma_{s-1} \cdots \sigma_1,$$
$$\eta = \sigma_1 \cdots \sigma_{s-1}.$$

Then we have

$$\tau(\gamma_j) = \gamma_1^{-1}\gamma_j\gamma_1 \quad (j = 1, \ldots, s),$$
$$\eta^k\tau\eta^{-k}(\gamma_j) = \gamma_{k+1}^{-1}\gamma_j\gamma_{k+1} \quad (j = 1, \ldots, s;\ k = 1, \ldots, s-1).$$

Hence τ and $\eta^k\tau\eta^{-k}$ generate the inner automorphism group of $\pi_1(\mathbb{P}^1 - D_o, \infty)$. By the definition of $\hat{\Gamma}$, τ and $\eta^k\tau\eta^{-k}$ are contained in $\hat{\Gamma}$. Hence $\operatorname{Inn}(\pi_1(\mathbb{P}^1 - D_o, \infty)) \subset \hat{\Gamma}$.
(ii) follows from (i). □

Let

$$M_1 \to M \to M_0 \to \mathbb{P}^s - \Delta \tag{3.18}$$

be the extensions (by Theorem 2.1) of the unbranched coverings of $\mathbb{C}^s - \Delta$ corresponding to the subgroups

$$\operatorname{Ker}(\Lambda) \subset \hat{\Gamma} \subset \hat{\Gamma}_0 \subset B_s = \pi_1(\mathbb{C}^s - \Delta, D_0) \tag{3.19}$$

with respect to the Galois correspondence.

They are connected complex manifolds, because the hyperplane H_∞ at infinity is non-singular.

Lemma 3.7. (i) *The coverings* $M \to \mathbb{P}^s - \Delta$ *and* $M_0 \to \mathbb{P}^s - \Delta$ *in* (3.18) *are unramified coverings.*
(ii) $M_1 \to \mathbb{P}^s - \Delta$ *is unbranched at* $H_\infty - \Delta$ *if and only if* G *is abelian and* $H = \{1\}$.

Proof. (i) A meridian of $H_\infty - \Delta$ is written as $\sigma\tau\sigma^{-1}$, where $\tau = \sigma_1 \cdots \sigma_{s-1}\sigma_{s-1}\cdots\sigma_1$ in the proof of Lemma 3.6 and $\sigma \in B_s$. Then

$$\sigma\tau\sigma^{-1}(\gamma) = \sigma(\gamma_1^{-1}\sigma^{-1}(\gamma)\gamma_1) = \sigma(\gamma_1)^{-1}\gamma\sigma(\gamma_1),$$

for any γ in $\pi_1(\mathbb{P}^1 - D_o, \infty)$. Hence $\sigma\tau\sigma^{-1}$ belongs to $\hat{\Gamma}$. Hence the extensions $M \to \mathbb{P}^s - \Delta$ and $M_0 \to \mathbb{P}^s - \Delta$ are still unbranched at $H_\infty - \Delta$ (see Lemma 3.2 of Namba [11]).

(ii) Using the notations in the proof of (i), $\sigma\tau\sigma^{-1}$ belongs to $\operatorname{Ker}(\Lambda)$ if and only if

$$\xi(\sigma(\gamma_1))^{-1}\xi(\gamma)\xi(\sigma(\gamma_1)) = \xi(\gamma),$$

for any γ in $\pi_1(\mathbb{P}^1 - D_o, \infty)$. This holds if and only if $\xi(\sigma(\gamma_1))$ belongs to the center of G. Note that σ can be taken any element of B_s and $\sigma(\gamma_1)$ is a conjugate of some γ_j in $\pi_1(\mathbb{P}^1 - D_o, \infty)$. This proves (ii). □

Proposition 3.8. (i) $\Theta_*(\pi_1(T_{\mathrm{fin}}, o)) \subset \hat{\Gamma}$.
(ii) *There is a lift* $\hat{\Theta}\colon T \to M$ *of* Θ.

Proof. (i) From the diagram (3.5), for any $\delta \in \pi_1(T_{\mathrm{fin}}, o)$, there is $\gamma \in \pi_1(\mathbb{P}^1 - D_o, \infty)$ such that $\gamma \cdot \rho_*(\delta) \in \mathscr{L}$. Since $\mathscr{H}_o$ is normal in $\mathscr{L}$, we have

$$(\gamma \cdot \rho_*(\delta))\mathscr{H}_o(\gamma \cdot \rho_*(\delta))^{-1} = \mathscr{H}_o.$$

Hence

$$\rho_*(\delta)\mathscr{H}_o\rho_*(\delta)^{-1} = \gamma^{-1}\mathscr{H}_o\gamma.$$

Hence, by Lemma 3.5,

$$\Theta_*(\delta)(\mathscr{H}_o) = \gamma^{-1}\mathscr{H}_o\gamma.$$

This shows that $\Theta_*(\delta) \in \hat{\Gamma}$.

(ii) follows from (i) and Proposition 2.2. □

For the later use, we mention two more lemmas. Consider the hypersurface

$$B = \{(p, m) \in \mathbb{P}^1 \times M \mid p \in D = \pi(m)\} \tag{3.20}$$

of $\mathbb{P}^1 \times M$, where

$$\pi : M \to \mathbb{P}^s - \Delta$$

is the projection in (3.18). Put $M_\infty = \pi^{-1}(H_\infty)$. Then

$$\rho : m \in M - M_\infty \mapsto (\infty, m) \in \mathbb{P}^1 \times (M - M_\infty) - B \tag{3.21}$$

is a holomorphic section of the projection

$$\mathbb{P}^1 \times M - B \to M,$$

which we call the ∞-*section* again. This projection is a topological fiber bundle with the standard fiber $\mathbb{P}^1 - D_o$ (see Lemma 4.2 of Mizuta–Namba[10]). Hence, as in (3.5), we get the following lemma:

Lemma 3.9. *There is the following splitting exact sequence*:

$$1 \to \pi_1(\mathbb{P}^1 - D_o, \infty) \to \mathscr{S} \underset{\rho_*}{\rightleftarrows} \hat{\Gamma} \to 1,$$

where $\mathscr{S} = \pi_1(\mathbb{P}^1 \times (M - M_\infty) - B, (\infty, o))$.

The following lemma can be proved in a similar way to that of Lemma 3.5, so we omit its proof.

Lemma 3.10. *For* $\sigma \in \hat{\Gamma}$ *and* $\gamma \in \pi_1(\mathbb{P}^1 - D_o, \infty)$, *the following equality holds*:

$$\sigma(\gamma) = \rho_*(\sigma)\gamma\rho_*(\sigma)^{-1},$$

where ρ *is the* ∞-*section in* (3.21) *and the product of the right hand side is that in*

$$\mathscr{S} = \pi_1(\mathbb{P}^1 \times (M - M_\infty) - B, (\infty, o)) = \pi_1(\mathbb{P}^1 - D_o, \infty) \cdot \rho_*(\hat{\Gamma}),$$

in which $\pi_1(\mathbb{P}^1 - D_o, \infty)$ *is a normal subgroup.*

4. Hurwitz families

A non-degenerate family $f: X \to \mathbb{P}^1 \times T$ of (G, H)-coverings of $\mathbb{P}^1$ is called a *Hurwitz family* if
(i) for any (G, H)-covering $g_1: X_1 \to \mathbb{P}^1$ of $\mathbb{P}^1$ which is topologically equivalent to a member of the family, there is a point $t \in T$ such that g_1 is isomorphic to f_t and
(ii) for any distinct points t and t' in T, f_t and $f_{t'}$ are not isomorphic.

Lemma 4.1. *The parameter space T of a Hurwitz family is biholomorphic to M in* (3.17) *through a lift $\hat{\Theta}$ of Θ in* Proposition 3.8.

Proof. A path in $\mathbb{P}^s - \Delta$ with the initial point D_o and the terminal point D_1 defines an isotopy of $(\mathbb{P}^1, \{s$ points of $\mathbb{P}^1\})$, which give an orientation preserving homeomorphism $\varphi: (\mathbb{P}^1, D_o) \to (\mathbb{P}^1, D_1)$. We introduce another complex structure X_1 on X_0 so that $f_1 = \varphi \cdot f_o$ is holomorphic. Then the identity map $\psi: X_o \to X_1$ and $\varphi: \mathbb{P}^1 \to \mathbb{P}^1$ give a topological equivalence of f_o to f_1.

Conversely, any $g_1: X_1 \to \mathbb{P}^1$ which is topologically equivalent to f_o, can be obtained in this way up to isomorphisms.

Hence, by the definition of $M \to \mathbb{P}^s - \Delta$ in (3.18), M can be regarded as the set of all isomorphism classes of (G, H)-coverings which are topologically equivalent to f_o.

Hence, by the properties (i) and (ii) of a Hurwitz family, $\hat{\Theta}$ is a bijective holomorphic map of T to M. Since a bijective holomorphic map between connected complex manifolds is a biholomorphic map, $\hat{\Theta}$ is a biholomorphic map. □

Henceforth we identify T of a Hurwitz family $f: X \to \mathbb{P}^1 \times T$ with M in (3.18) through $\hat{\Theta}$. We call M the *Hurwitz parameter space*.

REMARK 4.2. The connected complex manifold M is nothing but the connected component of the (*absolute*) *Hurwitz space*, containing the isomorphism class of a given $f_0: X_0 \to \mathbb{P}^1$, of the *Nielsen class* whose representative is given by ξ in (2.2) (Biggers–Fried [2], p. 88). A Hurwitz family is a *total representing family* with M as its parameter space.

The Hurwitz parameter space M always exists, while a Hurwitz family may not exist as Fried [7] pointed out. The existence problem of Hurwitz families is known to be a delicate problem. Some of the known results which are easy to state on the existence problem of Hurwitz families are as follows:

Theorem 4.3 (Fried [7]). *If $N_G(H) = H$, then there exists a universal Hurwitz family $f: X \to \mathbb{P}^1 \times M$ of (G, H)-coverings of $\mathbb{P}^1$, having a given $f_0: X_0 \to \mathbb{P}^1$ as a member.*

Here $N_G(H)$ is the normalizer of H in G. Also, a Hurwitz family $f: X \to \mathbb{P}^1 \times M$ is said to be *universal* if, for any non-degenerate family $g: Y \to \mathbb{P}^1 \times T$ with a point t_0 such that g_{t_0} is isomorphic to f_0, there is a unique holomorphic map $\Phi: T \to M$ with $\Phi(t_0) = 0$, such that the family g is isomorphic to the family induced by f over Φ.

Theorem 4.4 (Völklein [14]). *For any G-covering $f_0: X_0 \to \mathbb{P}^1$, there exists a Hurwitz family of $\mathbb{P}^1$ having f_0 as a member.*

This theorem is a by-product of his main arithmetic theorem in Völklein [14].

In the next section, we discuss the problem of existence of Hurwitz families *of special type*. For the rest of this section, we discuss, from our point of view, some properties of (general) Hurwitz families, which are essentially contained in Fried [7] and Biggers–Fried [2], for the preparation of the next section.

If there exists a Hurwitz family

$$f: X \to \mathbb{P}^1 \times M$$

the argument in §3 shows that f is the extension by Theorem 2.1 of an unramified covering

$$f: X' \to \mathbb{P}^1 \times (M - M_\infty) - B,$$

where B is a non-singular hypersurface of $\mathbb{P}^1 \times M$ defined by

$$B = \{(p, m) \in \mathbb{P}^1 \times M \mid p \in D = \pi(m)\}$$

($\pi: M \to \mathbb{P}^s - \Delta$ is the projection) and $M_\infty = \pi^{-1}(H_\infty)$, (see (3.18) and (3.20)).

Recall that there is the splitting exact sequence in Lemma 3.9. Now, a similar argument to that in §3 shows that, under the Galois correspondence, f corresponds to a subgroup $\mathscr{L}$ of $\pi_1(\mathbb{P}^1 \times (M - M_\infty) - B, (\infty, o))$ such that there is the following commutative diagram of exact sequences:

(4.1)
$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathscr{H}_o & \longrightarrow & \mathscr{L} & \longrightarrow & \hat{\Gamma} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \| & & \\
1 & \longrightarrow & \pi_1(\mathbb{P}^1 - D_o, \infty) & \longrightarrow & \mathscr{S} & \underset{\rho_*}{\rightleftarrows} & \hat{\Gamma} & \longrightarrow & 1.
\end{array}$$

(ρ is the ∞-section in (3.20), $\mathscr{S} = \pi_1(\mathbb{P}^1 \times (M - M_\infty) - B, (\infty, o))$ and i and j are the inclusion maps.)

REMARK 4.5. Note that f is still unramified on $\mathbb{P}^1 \times M - B$, because an element in the inverse image of $\tau = \sigma_1 \cdots \sigma_{s-1}\sigma_{s-1} \cdots \sigma_1$ in the proof of Lemma 3.7 by the surjective homomorphism $\mathscr{L} \to \hat{\Gamma}$ can be regarded as a meridian of $\mathbb{P}^1 \times M_\infty$ in $\mathbb{P}^1 \times M$.

Conversely, if there is such a subgroup $\mathscr{L}$ of $\pi_1(\mathbb{P}^1 \times (M - M_\infty) - B, (\infty, o))$, then the family $f\colon X \to \mathbb{P}^1 \times M$ corresponding to $\mathscr{L}$ is a Hurwitz family. In fact, the composition

$$X' \to \mathbb{P}^1 \times (M - M_\infty) - B \to M - M_\infty$$

(X' is the unramified covering corresponding to $\mathscr{L}$) is a topological fiber bundle with the standard fiber $X'_o = f^{-1}((\mathbb{P}^1 - D_o) \times o)$. Hence $f_m\colon X_m \to \mathbb{P}^1 \times m$ is topologically equivalent to $f_o\colon X_o \to \mathbb{P}^1 \times o$ for every $m \in M$, so is a (G, H)-covering with $B_{f_m} = B \cap (\mathbb{P}^1 \times m)$. Hence $f\colon X \to \mathbb{P}^1 \times M$ is a non-degenerate family of (G, H)-coverings of $\mathbb{P}^1$. By the definition of f, the lift $\hat{\Theta}$ with $\hat{\Theta}(o) = o$ of the braid monodromy Θ of the family is the identity map $id\colon M \to M$. The argument in the proof of Lemma 4.1 implies that the family $f\colon X \to \mathbb{P}^1 \times M$ is a Hurwitz family.

Thus we have proved the following proposition:

Proposition 4.6. *A Hurwitz family exists if and only if there is a subgroup $\mathscr{L}$ of*

$$\mathscr{S} = \pi_1(\mathbb{P}^1 \times (M - M_\infty) - B, (\infty, o)) = \pi_1(\mathbb{P}^1 - D_o, \infty) \cdot \rho_*(\hat{\Gamma}),$$

such that the commutative diagram of exact sequences in (4.1) *exists.*

Lemma 4.7. *There is the following exact sequence*:

$$1 \to N_{\pi_1(\mathbb{P}^1 - D_o, \infty)}(\mathscr{H}_o) \to N_{\mathscr{S}}(\mathscr{H}_o) \to \hat{\Gamma} \to 1,$$

where $N_{\pi_1(\mathbb{P}^1 - D_o, \infty)}(\mathscr{H}_o)$ (resp. $N_{\mathscr{S}}(\mathscr{H}_o)$) is the normalizer of $\mathscr{H}_o$ in $\pi_1(\mathbb{P}^1 - D_o, \infty)$ (resp. in $\mathscr{S}$).

Proof. By the definition of $\hat{\Gamma}$, for any element $\sigma \in \hat{\Gamma}$, there is $\gamma \in \pi_1(\mathbb{P}^1 - D_o, \infty)$ such that

$$\sigma(\mathscr{H}_o) = \gamma^{-1} \mathscr{H}_o \gamma.$$

Hence, by Lemma 3.10,

$$(\gamma \cdot \rho_*(\sigma))\mathscr{H}_o(\gamma \cdot \rho_*(\sigma))^{-1} = \gamma \rho_*(\sigma)\mathscr{H}_o \rho_*(\sigma)^{-1}\gamma^{-1} = \gamma\sigma(\mathscr{H}_o)\gamma^{-1} = \mathscr{H}_o.$$

This shows that $\gamma \cdot \rho_*(\sigma) \in N_{\mathscr{S}}(\mathscr{H}_o)$. Hence the homomorphism

$$N_{\mathscr{S}}(\mathscr{H}_o) \to \hat{\Gamma}, \quad \gamma \cdot \rho_*(\sigma) \mapsto \sigma$$

is surjective. The kernel of the homomorphism is clearly $N_{\pi_1(\mathbb{P}^1 - D_o, \infty)}(\mathscr{H}_o)$. □

Lemma 4.8. *There exists a commutative diagram of exact sequences in* (4.1) *if and only if there exists the following commutative diagram*:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathscr{H}_o & \longrightarrow & \mathscr{L} & \longrightarrow & \hat{\Gamma} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \| & & \\ 1 & \longrightarrow & N_{\pi_1(\mathbb{P}^1 - D_o, \infty)}(\mathscr{H}_o) & \longrightarrow & N_{\mathscr{S}}(\mathscr{H}_o) & \longrightarrow & \hat{\Gamma} & \longrightarrow & 1 \end{array} \tag{4.2}$$

(i, j: *the inclusion maps*).

Proof. In the diagram in (4.1), $\mathscr{H}_o$ is a normal subgroup of $\mathscr{L}$. Hence $\mathscr{L} \subset N_{\mathscr{S}}(\mathscr{H}_o)$, so the diagram in (4.2) is obtained. The converse is obvious. □

Lemma 4.9.

$$\rho_*(\mathrm{Ker}\,\Lambda) \subset N_{\mathscr{S}}(\mathscr{H}_o).$$

Proof. Take any $\sigma \in \mathrm{Ker}\,\Lambda$. Then

$$\xi_o(\sigma(\gamma)) = \xi_o(\gamma) \quad \text{for all} \quad \gamma \in \pi_1(\mathbb{P}^1 - D_o, \infty).$$

Hence, by Lemma 3.10,

$$\rho_*(\sigma)\gamma\rho_*(\sigma)^{-1}\gamma^{-1} \in \mathscr{K}_o \quad \text{for all} \quad \gamma \in \pi_1(\mathbb{P}^1 - D_o, \infty).$$

Since $\mathscr{K}_o \subset \mathscr{H}_o$, we have

$$\rho_*(\sigma)\mathscr{H}_o\rho_*(\sigma)^{-1} = \mathscr{H}_o.$$

□

Lemma 4.10. (i) $\mathscr{H}_o \cdot \rho_*(\mathrm{Ker}\,\Lambda)$ *is a subgroup of* $N_{\mathscr{S}}(\mathscr{H}_o)$.
(ii) *There is the following exact sequences*:

$$1 \to \mathscr{H}_o \to \mathscr{H}_o \cdot \rho_*(\mathrm{Ker}\,\Lambda) \to \mathrm{Ker}\,\Lambda \to 1. \tag{4.3}$$

Proof. (i) follows from Lemma 4.9. (ii) is obvious. □

From this lemma, there exists a non-degenerate family

$$g\colon Y \to \mathbb{P}^1 \times M_1 \tag{4.4}$$

of (G, H)-coverings of $\mathbb{P}^1$ with the parameter space M_1 in (3.17), corresponding to $\mathscr{H}_o \cdot \rho_*(\mathrm{Ker}\,\Lambda)$ in (4.3). We will make use of this family in the sequel.

REMARK 4.11. This family has the following properties:

(i) By Proposition 2.2, this family has a lift $\hat{\rho}$ of the ∞-section

$$\rho\colon m_1 \in M_1 - M_{1\infty} \mapsto (\infty, m_1) \in \mathbb{P}^1 \times (M_1 - M_{1\infty}) - B_1,$$

where $B_1 = \{(p, m_1) \in \mathbb{P}^1 \times M_1 \mid p \in D = \pi \cdot \pi_1(m_1)\}$, $\pi_1\colon M_1 \to M$, $\pi\colon M \to \mathbb{P}^s - \Delta$ are coverings in (3.18) and $M_{1\infty} = (\pi \cdot \pi_1)^{-1}(H_\infty)$.

(ii) For any (G, H)-covering $h_0\colon Z_0 \to \mathbb{P}^1$ of $\mathbb{P}^1$ which is topologically equivalent to the given f_0, there is a point $m_1 \in M_1$ such that h_0 is isomorphic to g_{m_1}. Hence the existence of this family implies the 'if' part of Theorem 3.1.

(iii) For any two points m_1 and m_1' in M_1, g_{m_1} and $g_{m_1'}$ are isomorphic if and only if m_1 and m_1' are in a same fiber of the covering $\pi_1\colon M_1 \to M$ in (3.18).

5. Hurwitz families of special type

Now we consider the following condition on $\mathscr{L}$ in (4.2):

$$\text{Condition}\colon \rho_*(\operatorname{Ker} \Lambda) \subset \mathscr{L}, \tag{5.1}$$

where ρ is the ∞-section in (3.21).

We say that a Hurwitz family $f\colon X \to \mathbb{P}^1 \times M$ is *of special type* if the group $\mathscr{L}$ corresponding to $f\colon X \to \mathbb{P}^1 \times M$ satisfies the condition 5.1.

REMARK 5.1. If the condition 5.1 is satisfied, then we have

$$\mathscr{H}_0 \cdot \rho_*(\operatorname{Ker} \Lambda) \subset \mathscr{L} \tag{5.2}$$

(see Lemma 4.10). Hence, by Proposition 2.2, a Hurwitz family $f\colon X \to \mathbb{P}^1 \times M$ is of special type if and only if the family induced by f over the covering $\pi_1\colon M_1 \to M$ in (3.18) is isomorphic to the family $g\colon Y \to \mathbb{P}^1 \times M$ in (4.4).

Suppose that the group $\mathscr{L}$ in (4.1) or (4.2) satisfies the condition (5.1). Then we have the following commutative diagrams of exact sequence of finite groups:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & H & \longrightarrow & L & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \| & & \\
1 & \longrightarrow & G & \longrightarrow & S & \rightleftarrows & \Gamma & \longrightarrow & 1,
\end{array} \tag{4.1$'$}$$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & H & \longrightarrow & L & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \| & & \\
1 & \longrightarrow & N_G(H) & \longrightarrow & N_S(H) & \longrightarrow & \Gamma & \longrightarrow & 1.
\end{array} \tag{4.2$'$}$$

(i, j: the inclusion maps.) Here $\Gamma = \Lambda(\hat{\Gamma})$, $L = (\xi_o, \Lambda)(\mathscr{L})$ and

$$S = G \times \Gamma$$

as a set, whose group structure is defined by

$$(g_1, \psi_1)(g_2, \psi_2) = (g_1\psi_1(g_2), \psi_1\psi_2).$$

($\Gamma \subset \Gamma_0 \subset \mathrm{Aut}(G)$. See (3.14) and (3.17).) We identify $(g, 1) \in S$ with $g \in G$ and $(1, \psi) \in S$ with $\psi \in \Gamma$. Then

$$(g, \psi) = g \cdot \psi,$$
$$\psi(g) = \psi \cdot g \cdot \psi^{-1}.$$

Hence S is the semi-direct product

$$S = G \cdot \Gamma$$

of G and Γ such that the surjective homomorphism $S \to T$ splits:

(5.3) $$1 \to G \to S \rightleftharpoons \Gamma \to 1.$$

In fact, (4.1)′ (resp. (4.2)′) is obtained from (4.1) (resp. (4.2)) by operating on (4.1) (resp. (4.2)) the surjective homomorphism

$$(\xi_o, \Lambda)\colon \gamma \cdot \rho_*(\sigma) \in \mathscr{S} \mapsto \xi_o(\gamma)\Lambda(\sigma) \in S$$

($\gamma \in \pi_1(\mathbb{P}^1 - D_o, \infty)$, $\sigma \in \hat{\Gamma}$), whose kernel is

$$\mathrm{Ker}(\xi_o, \Lambda) = \mathscr{K}_o \cdot \rho_*(\mathrm{Ker}\,\Lambda)$$

and is contained in $\mathscr{L}$.

Conversely, by taking the inverse image by (ξ_o, Λ) of the commutative diagram of exact sequences in (4.1)′ (resp. (4.2)′), we obtain the commutative diagram of exact sequences (4.1) (resp. (4.2)) in which $\mathscr{L} = (\xi_o, \Lambda)^{-1}(L)$ satisfies the condition (5.1).

Thus the problem of the existence of Hurwitz families of special type can be reduced to the existence of L in (4.1)′ or (4.2)′.

Furthermore, we can easily show that there exists a commutative diagram of exact sequences in (4.2)′ if and only if the surjective homomorphism

$$N_S(H)/H \to \Gamma$$

of the exact sequence:

$$1 \to N_G(H)/H \to N_S(H)/H \to \Gamma \to 1$$

splits. Thus

Theorem 5.2. *There exists a Hurwitz family of special type if and only if the surjective homomorphism* $N_S(H)/H \to \Gamma$ *in the exact sequence*

$$1 \to N_G(H)/H \to N_S(H)/H \to \Gamma \to 1 \tag{5.4}$$

splits.

In particular, if $N_G(H) = H$, then the exact sequence (5.4) clearly splits. Hence a Hurwitz family of special type exists by Theorem 5.2. This shows the existence part of Theorem 4.3.

As for G-coverings $f_0\colon X_0 \to \mathbb{P}^1$, we have $H = \{1\}$ and $\Gamma = \Gamma_0$ in (3.14). In this case, $\Gamma_G(H)/H = G$ and $N_S(H)/H = S$. Hence the exact sequence (5.4) is reduced to the exact sequence (5.3) (with $\Gamma = \Gamma_0$), which splits. Thus a Hurwitz family of special type exists by Theorem 5.2. This shows Theorem 4.4.

REMARK 5.3. Note that the exact sequence (5.4) and the split condition appeared in Fried [7], (4.12) and Proposition 5.

Now, we further discuss the exact sequence (5.4). Let G/H be the set of left cosets. Put

$$J = \{(Hb, \psi) \in (G/H) \times \Gamma \mid \psi(H) = b^{-1}Hb\}.$$

We introduce a group structure in J by

$$(Hb_1, \psi_1)(Hb_2, \psi_2) = (Hb_1\psi_1(b_2), \psi_1\psi_2).$$

This is well defined by the definition of J. Moreover, the map

$$b\psi \in N_S(H) \mapsto (Hb, \psi) \in J$$

is a surjective homomorphism, whose kernel is H. Hence

$$N_S(H)/H \simeq J.$$

Under this isomorphism, the surjective homomorphism $N_S(H)/H \to \Gamma$ corresponds to the surjective homomorphism

$$(Hb, \psi) \in J \mapsto \psi \in \Gamma,$$

whose kernel is

$$\{(Hb, 1) \mid b \in N_G(H)\} \simeq N_G(H)/H.$$

Hence we have the exact sequence

$$1 \to N_G(H)/H \to J \to \Gamma \to 1 \tag{5.5}$$

which corresponds to the exact sequence (5.4) under the isomorphism $N_S(H)/H \simeq J$.

Thus the problem of the existence of Hurwitz families of special type is reduced to the problem of existence of splittings of the surjective homomorphism $J \to \Gamma$ in (5.5).

Proposition 5.4. *There is a one-to-one correspondence between splittings of $J \to \Gamma$ and maps*

$$\lambda\colon \Gamma \to G/H$$

with the following 3 *properties*:
(i) $\psi(H) = \lambda(\psi)^{-1}H\lambda(\psi)$ *for all* $\psi \in \Gamma$,
(ii) $\lambda(1) = H$,
(iii) $\lambda(\psi_1\psi_2) = \lambda(\psi_1)\psi_1(\lambda(\psi_2))$ *for all* $\psi_1, \psi_2 \in \Gamma$.

Proof. A splitting $\hat{\lambda}\colon \Gamma \to J$ of $J \to \Gamma$ is given by

$$\hat{\lambda}\colon \psi \in \Gamma \mapsto (Hb, \psi) \in J.$$

We define a map

$$\lambda\colon \Gamma \to G/H$$

by $\lambda(\psi) = Hb$. Then, since $\hat{\lambda}$ is an injective homomorphism, λ must satisfy the above properties (i), (ii) and (iii).

Conversely, if a map $\lambda\colon \Gamma \to G/H$ satisfies the properties (i), (ii) and (iii), then

$$\hat{\lambda}\colon \psi \mapsto (\lambda(\psi), \psi)$$

is an injective homomorphism of Γ into J and gives a splitting of $J \to \Gamma$. □

Let $\lambda\colon \Gamma \to G/H$ be a map which satisfies the properties (i), (ii), and (iii) in Proposition 5.4.

By Lemma 3.6, the inner automorphism group $\mathrm{Inn}(G)$ of G is contained in Γ. Let

$$I : g \in G \mapsto \psi^g \in \mathrm{Inn}(G)$$

be the surjective homomorphism with the kernel

$$\mathrm{Ker}(I) = Z_G \quad \text{(the center of } G\text{)},$$

where

$$\psi^g(x) = gxg^{-1} \quad \text{for} \quad x \in G.$$

We define a map

$$\mu\colon G \to G/H$$

by

$$\mu(g) = (\lambda \cdot I)(g) \cdot g = \lambda(\psi^g)g \quad \text{for} \quad g \in G$$

Lemma 5.5. (i) *$\mu(G) \subset N_G(H)/H$.*
(ii) *$\mu\colon G \to N_G(H)/H$ is a homomorphism.*
(iii) *$\mu(z) = Hz$ for $z \in Z_G$.*

Proof. (i) Put $\lambda(\psi^g) = Hb$. Then

$$gHg^{-1} = \psi^g(H) = b^{-1}Hb.$$

Hence

$$(bg)H(bg)^{-1} = H$$

Hence

$$\mu(g) = Hbg \in N_G(H)/H.$$

(ii) Put $\lambda(\psi^{g_1}) = Hb_1$ and $\lambda(\psi^{g_2}) = Hb_2$. Then

$$\begin{aligned} \mu(g_1g_2) &= \lambda(\psi^{g_1}\psi^{g_2})g_1g_2 \\ &= \lambda(\psi^{g_1})\psi^{g_1}(\lambda(\psi^{g_2}))g_1g_2 \\ &= Hb_1g_1b_2g_1^{-1} \cdot g_1g_2 \\ &= Hb_1g_1b_2g_2 = (Hb_1g_1)(Hb_2g_2) \\ &= \mu(g_1)\mu(g_2). \end{aligned}$$

(iii) For $z \in Z_G$, $I(z) = \psi^z = 1$. Hence $\mu(z) = \lambda(1)z = Hz$. □

Note that the subgroup $H \cdot Z_G$ of G is contained in $N_G(H)$. Since H contains no normal subgroup of G except $\{1\}$, we have

$$H \cap Z_G = \{1\}.$$

Hence

$$(H \cdot Z_G)/H \simeq Z_G.$$

Hence we may regard Z_G as a subgroup of $N_G(H)/H$:

$$Z_G \subset N_G(H)/H.$$

Now, consider the following condition on (G, H):

$$\text{Condition}: N_G(H) = H \cdot Z_G. \tag{5.6}$$

Under the condition in (5.6), $N_G(H)/H$ can be identified with Z_G:

$$N_G(H)/H = Z_G.$$

Hence the homomorphism μ in Lemma 5.5 can be regarded as a homomorphism

$$\mu: G \to Z_G$$

such that $\mu|Z_G = id$ (the identity map). We then define a map

$$\nu: G \to G$$

by

$$\nu(g) = \mu(g)^{-1}g \quad \text{for} \quad g \in G.$$

We can easily check that ν is a homomorphism with the kernel

$$\mathrm{Ker}(\nu) = Z_G.$$

Moreover, we can easily check that
(i) $\nu(G) \simeq \mathrm{Inn}(G)$,
(ii) $Z_G \cap \nu(G) = \{1\}$,
(iii) $G = Z_G \cdot \nu(G) = \nu(G) \cdot Z_G$.
Hence G is the direct product of Z_G and $\nu(G)$. Hence $\nu(G)$ gives the splitting of the exact sequence $1 \to Z_G \to G \to \mathrm{Inn}(G) \to 1$.

Thus we get the following theorem:

Theorem 5.6. *Assume that*
(i) $N_G(H) = H \cdot Z_G$ *and*
(ii) *the exact sequence* $1 \to Z_G \to G \to \mathrm{Inn}(G) \to 1$ *does not split.*
Then there does not exist a Hurwitz family of special type of (G, H)*-covering of* $\mathbb{P}^1$.

We give one of the simplest examples of (G, H) with the conditions (i) and (ii) in Theorem 5.6.

EXAMPLE 5.7. Let A_4 be the alternating group of 4 letters and A_3 its isotropy subgroup of the letter 1. The Schur multiplier $M(A_4)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (see, e.g., §3.2 of Suzuki [13]). Hence there is a central extension

$$1 \to Z \to G \to^{\alpha} A_4 \to 1$$

with $Z \simeq \mathbb{Z}/2\mathbb{Z}$, which does not split. In this case, Z is the center of G. From the exact sequence, we have the following exact sequence:

$$1 \to Z \to \alpha^{-1}(A_3) \to A_3 \to 1,$$

which clearly splits. Hence there is a subgroup H of G such that

$$Z \cap H = \{1\}, \quad \alpha^{-1}(A_3) = Z \cdot H \quad \text{and} \quad \alpha \colon H \simeq A_3.$$

Then we have easily

$$N_G(H) = Z \cdot H.$$

Hence (G, H) satisfies the conditions (i) and (ii) in Theorem 5.6.

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