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<th>On the K-groups of spherical varieties</th>
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1. Statement of results

A spherical variety is a normal variety defined over a field with a split reductive group action with a dense open orbit isomorphic to a Borel subgroup. Flag varieties, Schubert varieties and toric varieties are examples of spherical varieties. In this paper we will study the $K'$-groups of varieties belonging to a certain category including spherical varieties. Our main results are descriptions of $K'$-groups and their coniveau filtrations of such varieties by means of their equivariant $K'$-groups. For a smooth toric variety, they are obtained by Morelli [4, Prop. 4]. Before we state our main results explicitly, we fix some notations.

Let $B$ be a split connected solvable group defined over a field $k$. Then $B$ is isomorphic to a product of an affine space and a torus as a variety over $k$. In this paper we are concerned with a $B$-variety $X$ with finitely many $B$-orbits. All $B$-orbits of $X$ are indexed by a finite set $\Delta$. For $\sigma \in \Delta$, we denote by $O(\sigma)$ the corresponding $B$-orbit of $X$. Let $M = \text{Hom}(B, \mathbb{G}_m)$ be the character group of $B$. Any orbit $O(\sigma)$ is isomorphic to a quotient scheme of $B$ by a subgroup $B_\sigma$. Hence $O(\sigma)$ is also isomorphic to a product of an affine space and a torus. Let $M^\sigma = \text{Hom}(B_\sigma, \mathbb{G}_m)$, then $M^\sigma$ becomes a quotient module of $M$.

Here we introduce $K$-theory. We denote by $K_i^e(X)$ the $i$-th $K$-group of the category of coherent sheaves on $X$ and by $K_i^e(X, B)$ the $i$-th $K$-group of the category of $B$-equivariant coherent sheaves on $X$. Moreover we denote by $K_i(X)$ the $i$-th $K$-group of the category of locally free sheaves on $X$ and by $K_i(X, B)$ the $i$-th $K$-group of the category of $B$-equivariant locally free sheaves on $X$.

In [6] R. Thomason showed that these two equivariant $K$-groups are isomorphic when $X$ is smooth over $k$. The equivariant $K$-group of the base field $K_0(k, B)$ is isomorphic to the Grothendieck group of the category of $k$-representations of $B$. Hence we have $K_0(k, B) \simeq \mathbb{Z}[M]$. From this fact we can say that the equivariant $K$-group $K_i^e(X, B)$ admits a $\mathbb{Z}[M]$-module structure. For a $\mathbb{Z}[M]$-module $R$, we denote by $I_R$ the submodule of $R$ generated by $\{ rm - r; r \in R, m \in M \}$. The quotient module $R/I_R$ is called the group of coinvariants of $R$ and denoted by $R_M$.

We need an additional assumption on the characteristic of $k$. When $B$ is not a torus, we assume $\text{char} k = 0$. It is needed for varieties which we treat to admit a resolution of singularities.
The main result of the present paper is the following:

**Theorem 1.1.** Let $X$ be a $B$-variety with finitely many orbits. Then the natural homomorphism

$$K'_0(X, B)_M \rightarrow K'_0(X)$$

is bijective.

This theorem was proved by Morelli when $X$ is a smooth toric variety. His proof relies on the ring structure of $K_0(X)$ and a relation between $K$-groups and Chow rings. So we cannot apply his method. Instead we will use $K$-group of $X$ and group homology of $M$.

We assume that $X$ is a toric variety, namely $B$ is a split torus and $X$ is normal. Then $X$ is constructed by a fan and many geometrical informations about $X$ are expressed by the combinatorial data of the fan. But its $K$-group $K'_0(X)$ cannot be determined by the combinatorial data by the same reason as in the case of rational homology [3]. On the other hand, the equivariant $K$-group $K'_0(X, B)$ is a free abelian group generated by the structure sheaf of $B$-invariant closed subschemes and their twists by characters of $B$. Hence it is determined only by orbits of $X$ as an abelian group. But as seen in the proof of the theorem, the $\mathbb{Z}[M]$-module structure of $K'_0(X, B)$ is very complicated and Theorem 1.1 says that it cannot be determined by the combinatorial data of the fan.

Next we consider the coniveau filtration $F$ of $K'_0(X)$. This is defined as

$$F^pK'_0(X) = \text{Im}( \bigoplus_{Y \subset X \text{ codim } \geq p} K'_0(Y) \rightarrow K'_0(X)).$$

We note that the filtration $F$ is associated with Brown Gersten spectral sequence [5].

Given a nonnegative integer $i$, the union of all $B$-orbits whose codimensions are greater than $i$ is a closed subscheme of $X$. It is denoted by $X^i$. We set $Y^i = X^i \setminus X^{i+1}$, which is an open subscheme of $X^i$. $Y^i$ becomes a disjoint union of all $B$-orbits of codimensions $i$. Let $n$ be the dimension of $X$, then we have the sequence of closed subschemes of $X$:

$$\phi = X^{n+1} \subset X^n \subset X^{n-1} \subset \cdots \subset X^0 = X.$$  

We put $E^{p,q}(X) = K'_{p-q}(Y^p)$ and the morphism $d : E^{p,q}(X) \rightarrow E^{p+1,q}(X)$ is defined by

$$K'_{p-q}(Y^p) \rightarrow K'_{p-q-1}(X^{p+1}) \rightarrow K'_{p-q-1}(Y^{p+1}),$$
where the left arrow is the connecting homomorphism of the localization exact sequence and the right arrow is the restriction of the open immersion $Y^{p+1} \hookrightarrow X^{p+1}$. Then $(E^{\cdot,q}(X), d)$ becomes a complex.

Let $R^{\cdot,q}(X)$ be the Gersten complex of $X$, that is, $R^{p,q}(X) = \oplus_{x \in X^{(p)}} K_{p-q}(k(x))$ where $X^{(p)}$ is the set of all points of $X$ whose Zariski closures are of codimension $p$ and $k(x)$ is the residue field of $x$. We obtain a canonical morphism of complexes $E^{\cdot,q}(X) \to R^{\cdot,q}(X)$.

**Proposition 1.2.** The morphism

$$E^{\cdot,q}(X) \to R^{\cdot,q}(X)$$

is a quasi-isomorphism.

Since $H^p(R^{\cdot,-p}(X))$ is isomorphic to the Chow group of $X$ of codimension $p$ by [5, Prop. 5.14] or by [2, Cor. 7.20], this proposition gives us the representation of the Chow group of $X$ by generators and relations, which is the same result as the one obtained by Fulton et. al. [1] and by Totaro [7].

By the above proposition we have an isomorphism

$$F^pK^0_0(X) = \text{Im}(K^0_0(X^p) \to K^0_0(X))$$

and together with Theorem 1.1 we can describe the coniveau filtration by the equivariant $K'$-group.

**Corollary 1.3.** We define a decreasing filtration $F^p_B$ on $K^0_0(X,B)_M$ by

$$F^p_B K^0_0(X,B)_M = \text{Im}(K^0_0(X^p,B)_M \to K^0_0(X,B)_M).$$

Then for a nonnegative integer $p$, we have an isomorphism

$$F^p_B K^0_0(X,B)_M \simeq F^p K^0_0(X).$$

2. **Proof of Theorem 1.1**

We will prove that

$$K^0_0(X^p,B)_M \to K^0_0(X^p)$$

is bijective by descending induction on $p$. Given a $\mathbb{Z}[M]$-module $R$, let

$$H_i(M; R) = \text{Tor}^\mathbb{Z}[M]_i(\mathbb{Z}, R)$$
be the $i$-th homology of $M$ with coefficient $R$. The 0-th homology is isomorphic to the group of coinvariants $R_M$. The homology is calculated by a $\mathbb{Z}[M]$-projective resolution of $\mathbb{Z}$. By choosing a basis of $M$ we can construct a $\mathbb{Z}[M]$-free resolution of $\mathbb{Z}$. Namely for a basis $(m_1, \cdots, m_n)$ of $M$ we set $P_q = \mathbb{Z}[M] \otimes \wedge^q M$ and define $\partial_q : P_{q+1} \to P_q$ by

$$
\partial_q(r \otimes m_{i_1} \wedge \cdots \wedge m_{i_{q+1}})
= \sum_{j=1}^{q+1} (-1)^{j+1} r([m_{i_j}] - [0]) \otimes m_{i_1} \wedge \cdots \wedge m_{i_{j-1}} \wedge m_{i_{j+1}} \wedge \cdots \wedge m_{i_{q+1}}.
$$

Then

$$
\cdots \to P_{q+1} \xrightarrow{\partial_q} P_q \to \cdots \to P_0 \to \mathbb{Z} \to 0
$$

becomes a $\mathbb{Z}[M]$-free resolution of $\mathbb{Z}$. Hence we have

$$
H_q(M; R) \simeq H_q(R \otimes_{\mathbb{Z}[M]} P).
$$

If the action of $M$ on $R$ is trivial, then the differentials in $R \otimes_{\mathbb{Z}[M]} P$ are all zero. In particular, it holds that

$$
H_q(M; \mathbb{Z}) \simeq \wedge^q M.
$$

Since $O(\sigma) \simeq B/B_\sigma$, by [6] we have

$$
K'_*(Y^p, B) = \bigoplus_{\text{codim} O(\sigma) = p} K'_*(O(\sigma), B)
\simeq \bigoplus_{\text{codim} O(\sigma) = p} K'_*(B/B_\sigma, B)
\simeq \bigoplus_{\text{codim} O(\sigma) = p} K'_*(k, B_\sigma)
\simeq K_*(k) \otimes (\bigoplus_{\text{codim} O(\sigma) = p} \mathbb{Z}[M^\sigma]).
$$

In other words, $K'_*(Y^p, B)$ is isomorphic to $K_*(k) \otimes K'_0(Y^p, B)$ as a $K_*(k)$-module. Since the boundary homomorphism of the localization exact sequence

$$
K'_*(Y^p, B) \to K'_{*-1}(X^{p+1}, B)
$$

preserves the $K_*(k)$-module structure, it becomes zero.

Hence we have a short exact sequence of $\mathbb{Z}[M]$-modules

$$
0 \to K'_0(X^{p+1}, B) \to K'_0(X^p, B) \to K'_0(Y^p, B) \to 0.
$$
So we have

\[ K'_0(X, B) \cong \bigoplus_{p} K'_0(Y^p, B) \cong \bigoplus_{\sigma \in \Delta} \mathbb{Z}[M^\sigma] \]

as an abelian group. Hence we can say that \( K'_0(X, B) \) is determined only by orbits of \( X \). The above short exact sequence induces the long exact sequence

\[ H_1(M; K'_0(Y^p, B)) \to K'_0(X^{p+1}, B)_M \to K'_0(X^p, B)_M \to K'_0(Y^p, B)_M \to 0. \]

We set \( M_\sigma = \text{Ker}(M \to M^\sigma) \).

**Lemma 2.1.**

\[ H_1(M; K'_0(Y^p, B)) \cong \bigoplus_{\text{codim}(\sigma) = p} M_\sigma. \]

**Proof.** Since

\[ \text{codim}(\sigma) = p \]

we have only to prove \( H_1(M; \mathbb{Z}[M^\sigma]) \cong M_\sigma \). Since \( M^\sigma \cong M/M_\sigma \), the \( \mathbb{Z}[M^\sigma] \)-module \( \mathbb{Z}[M^\sigma] \) is isomorphic to the induced module of the \( \mathbb{Z}[M_\sigma] \)-module \( \mathbb{Z} \). Hence we have

\[ H_1(M; \mathbb{Z}[M^\sigma]) \cong H_1(M; \text{Ind}_{M_\sigma}^M \mathbb{Z}) \cong H_1(M_\sigma; \mathbb{Z}) \cong M_\sigma, \]

which completes the proof. \( \square \)

**Lemma 2.2.** Given an integer \( 0 \leq p \leq r \), there exists an exact sequence

\[ \bigoplus_{\text{codim}(\sigma) = p} M_\sigma \to K'_0(X^{p+1}) \to K'_0(X^p) \to K'_0(Y^p) \to 0. \]

**Proof.** By localization exact sequence of \( K' \)-theory we have

\[ K'_1(Y^p) \to K'_0(X^{p+1}) \to K'_0(X^p) \to K'_0(Y^p) \to 0. \]
By [5], we have the following isomorphism:

\[
K_1'(Y^p) = \bigoplus_{\text{codim} \mathcal{O}(\sigma) = p} K_1'(\mathcal{O}(\sigma)) \cong \bigoplus_{\text{codim} \mathcal{O}(\sigma) = p} (K_1(k) \oplus (M_\sigma)).
\]

Since the maps in the localization exact sequence preserves the \(K_*(k^*)\)-module structures, the images of components \(K_1(k)\) by \(K_1'(Y^p) \to K_0'(X^{p+1})\) are all zero. This completes the proof.

\[\square\]

**Lemma 2.3.** The diagram

\[
\begin{array}{ccc}
\bigoplus_{\text{codim} \mathcal{O}(\sigma) = p} M_\sigma & \xrightarrow{\partial} & K_0'(X^{p+1}) \\
\uparrow & & \uparrow \\
H_1(M; K_0'(Y^p, B)) & \longrightarrow & K_0'(X^{p+1}, B)_M
\end{array}
\]

commutes, where the left vertical arrow is the isomorphism proved in Lemma 2.1.

**Proof.** We choose an element \(m \in M_\sigma \cong H_1(M, K_0'(\mathcal{O}(\sigma), B))\) for \(\sigma \in \Delta\) and consider the image of \(m\) by the above diagram. But the support of the image is contained in the closure of \(\mathcal{O}(\sigma)\) in \(X\). So we may assume that \(\mathcal{O}(\sigma)\) is the only dense open orbit. In other words, we have only to prove the result when \(p = 0\) and \(X\) is irreducible.

Let \(\pi : \tilde{X} \to X\) be a \(B\)-equivariant birational morphism such that \(\tilde{X}\) is a smooth variety. The morphism \(\pi\) exists by virtue of the existence of equivariant resolution of singularities. Then horizontal arrows in the above diagram factor through \(K'\)-groups of \(\tilde{X}\), namely,

\[
\begin{array}{ccc}
M_\sigma & \xrightarrow{\partial} & K_0'(\tilde{X}^1) \\
H_1(M, K_0'(\mathcal{O}(\sigma), B)) & \longrightarrow & K_0'(\tilde{X}^1, B)_M
\end{array}
\]

Since the right diagram commutes, we have only to prove that the left diagram commutes. Hence we may assume that \(X\) is a smooth variety.

We first consider the image of \(m\) by the bottom horizontal map. We choose a basis \((m_1, \ldots, m_\alpha)\) of \(M\) such that \(m = \Sigma s_i m_i\) for \(s_i \in \mathbb{Z}\). Then we obtain a \(\mathbb{Z}[M]\)-free resolution of \(\mathbb{Z}\) as mentioned above and represent \(m \in M_\sigma \cong H_1(M; K_0'(Y^0, B))\) by a chain in the complex \(\mathbb{Z}[M^\sigma] \otimes P\). The chain corresponding to \(m\) by the isomorphism in Lemma 2.1 becomes \([0] \otimes m \in \mathbb{Z}[M^\sigma] \otimes M\). The bottom horizontal
map is the connecting homomorphism and its image is $\sum s_i[O_X](0 - [m_i])$. Since its support is in $X^1$, we can regard it as an element of $K'_0(X^1, B)_M$.

We regard $m_i$ as a rational function on $X$ and let $D_{i,0}$ and $D_{i,\infty}$ be the divisors of zeros and poles of $m_i$ respectively. Then in the same way as in [4, Prop. 4] it holds that

$$[O_X](0 - [m_i]) = [O_{D_{i,0}}] - [O_{D_{i,\infty}}][m_i]$$

in $K'_0(X^1, B)$. Hence the image of $\sum s_i[O_X](0 - [m_i])$ by the right vertical arrow is

$$\sum s_i([O_{D_{i,0}}] - [O_{D_{i,\infty}}]) = \partial(m).$$

We have the following isomorphisms for $Y^p$

$$K'_0(Y^p, B)_M \simeq \bigoplus_{\text{codim} O(\sigma) = p} \mathbb{Z}[M^p]_M$$

$$\simeq \bigoplus_{\text{codim} O(\sigma) = p} \mathbb{Z} \simeq K'_0(Y^p).$$

Then the theorem follows from the five lemma for the diagram

$$\bigoplus_{\text{codim} O(\sigma) = p} M_\sigma \xrightarrow{\partial} K'_0(X^{p+1}) \rightarrow K'_0(X^p) \rightarrow K'_0(Y^p) \rightarrow 0$$

and descending induction on $p$.

3. Proof of Proposition 1.2

For an inclusion $X^{i+1} \hookrightarrow X^i$, we have a short exact sequence of Gersten complexes

$$0 \rightarrow R \cdot q + 1(X^{i+1})[-1] \rightarrow R \cdot q(X^i) \rightarrow R \cdot q(Y^i) \rightarrow 0,$$

where $[-1]$ means the degree shift. Since $Y^i = \coprod_{\text{codim} O(\sigma) = i} O(\sigma)$ and $O(\sigma)$ is isomorphic to a product of an affine space and a torus, we have

$$H^p(R \cdot q(Y^i)) \simeq \begin{cases} 
\bigoplus_{\text{codim} O(\sigma) = i} K_{-q}(O(\sigma)) & \text{if } p = 0 \\
0 & \text{if } p \neq 0.
\end{cases}$$
Hence we have an isomorphism
\[ H^{p-1}(R'^q(X^{i+1})) \simeq H^p(R'^q(X^i)) \]
if \( p \geq 2 \) and an exact sequence
\[
0 \to H^0(R'^q(X^i)) \to H^0(R'^q(Y^i)) \\
\quad \to H^0(R'^{q+1}(X^{i+1})) \to H^1(R'^q(X^i)) \to 0.
\]
Hence for \( p \geq 1 \) we have
\[
H^p(R'^q(X)) = H^p(R'^q(X^0)) \\
\quad \simeq H^{p-1}(R'^{q+1}(X^1)) \\
\quad \simeq \ldots \\
\quad \simeq H^1(R'^{p+q-1}(X^{p-1})).
\]

We consider the diagram
\[
\begin{array}{ccc}
H^0(R'^{p+q-1}(Y^{p-1})) & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & H^0(R'^{p+q}(X^p)) \\
\downarrow & & \downarrow \\
& H^0(R'^{p+q-1}(X^{p-1})) & \to \quad H^0(R'^{p+q+1}(Y^{p+1})) \\
\downarrow & \downarrow & \\
0 & & H^0(R'^{p+q+1}(Y^{p+1})).
\end{array}
\]
Then this yields
\[
H^1(R'^{p+q-1}(X^{p-1})) \simeq \frac{\text{Ker}(H^0(R'^{p+q}(Y^p)) \to H^0(R'^{p+q+1}(Y^{p+1})))}{\text{Im}(H^0(R'^{p+q-1}(Y^{p-1}) \to H^0(R'^{p+q+1}(Y^p)))}
\]
\[
\simeq \frac{\text{Ker}(K_{-p-q}(Y^p) \to K_{-p-q-1}(Y^{p+1}))}{\text{Im}(K_{-p-q+1}(Y^{p-1}) \to K_{-p-q}(Y^p))}.
\]
Hence we have
\[
H^p(R'^q(X)) \simeq \frac{\text{Ker}(K_{-p-q}(Y^p) \to K_{-p-q-1}(Y^{p+1}))}{\text{Im}(K_{-p-q+1}(Y^{p-1}) \to K_{-p-q}(Y^p))}
\]
\[
\simeq H^p(E'^q(X)),
\]
which holds when \( p = 0 \) if we put \( Y^{-1} = \phi \).
References


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