Takeda, Y.
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ON THE K-GROUPS OF SPHERICAL VARIETIES

YUICHIRO TAKEDA

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1. Statement of results

A spherical variety is a normal variety defined over a field with a split reductive group action with a dense open orbit isomorphic to a Borel subgroup. Flag varieties, Schubert varieties and toric varieties are examples of spherical varieties. In this paper we will study the K'-groups of varieties belonging to a certain category including spherical varieties. Our main results are descriptions of K'-groups and their coniveau filtrations of such varieties by means of their equivariant K'-groups. For a smooth toric variety, they are obtained by Morelli [4, Prop. 4]. Before we state our main results explicitly, we fix some notations.

Let B be a split connected solvable group defined over a field k. Then B is isomorphic to a product of an affine space and a torus as a variety over k. In this paper we are concerned with a B-variety X with finitely many B-orbits. All B-orbits of X are indexed by a finite set Δ. For $\sigma \in \Delta$, we denote by $\mathcal{O}(\sigma)$ the corresponding B-orbit of X. Let $M = \mathrm{Hom}(B, \mathbb{G}_m)$ be the character group of B. Any orbit $\mathcal{O}(\sigma)$ is isomorphic to a quotient scheme of B by a subgroup B_σ. Hence $\mathcal{O}(\sigma)$ is also isomorphic to a product of an affine space and a torus. Let $M^\sigma = \mathrm{Hom}(B_\sigma, \mathbb{G}_m)$, then M^σ becomes a quotient module of M.

Here we introduce K-theory. We denote by $K_i'(X)$ the i-th K-group of the category of coherent sheaves on X and by $K_i'(X, B)$ the i-th K-group of the category of B-equivariant coherent sheaves on X. Moreover we denote by $K_i(X)$ the i-th K-group of the category of locally free sheaves on X and by $K_i(X, B)$ the i-th K-group of the category of B-equivariant locally free sheaves on X.

In [6] R. Thomason showed that these two equivariant K-groups are isomorphic when X is smooth over k. The equivariant K-group of the base field $K_0(k, B)$ is isomorphic to the Grothendieck group of the category of k-representations of B. Hence we have $K_0(k, B) \simeq \mathbb{Z}[M]$. From this fact we can say that the equivariant K-group $K_*'(X, B)$ admits a $\mathbb{Z}[M]$-module structure. For a $\mathbb{Z}[M]$-module R, we denote by I_R the submodule of R generated by $\{rm - r; r \in R, m \in M\}$. The quotient module R/I_R is called the group of coinvariants of R and denoted by R_M.

We need an additional assumption on the characteristic of k. When B is not a torus, we assume $\mathrm{char} k = 0$. It is needed for varieties which we treat to admit a resolution of singularities.

The main result of the present paper is the following:

Theorem 1.1. *Let X be a B-variety with finitely many orbits. Then the natural homomorphism*

$$K_0'(X,B)_M \to K_0'(X)$$

is bijective.

This theorem was proved by Morelli when X is a smooth toric variety. His proof relies on the ring structure of $K_0(X)$ and a relation between K-groups and Chow rings. So we cannot apply his method. Instead we will use K_1-group of X and group homology of M.

We assume that X is a toric variety, namely B is a split torus and X is normal. Then X is constructed by a fan and many geometrical informations about X are expressed by the combinatorial data of the fan. But its K-group $K_0'(X)$ cannot be determined by the combinatorial data by the same reason as in the case of rational homology [3]. On the other hand, the equivariant K-group $K_0'(X,B)$ is a free abelian group generated by the structure sheaf of B-invariant closed subschemes and their twists by characters of B. Hence it is determined only by orbits of X as an abelian group. But as seen in the proof of the theorem, the $\mathbb{Z}[M]$-module structure of $K_0'(X,B)$ is very complicated and Theorem 1.1 says that it cannot be determined by the combinatorial data of the fan.

Next we consider the coniveau filtration F of $K_0'(X)$. This is defined as

$$F^p K_0'(X) = \mathrm{Im}\Big(\bigoplus_{\substack{Y \subset X \\ \mathrm{codim} \geq p}} K_0'(Y) \to K_0'(X)\Big).$$

We note that the filtration F is associated with Brown Gersten spectral sequence [5].

Given a nonnegative integer i, the union of all B-orbits whose codimensions are greater than i is a closed subscheme of X. It is denoted by X^i. We set $Y^i = X^i \backslash X^{i+1}$, which is an open subscheme of X^i. Y^i becomes a disjoint union of all B-orbits of codimensions i. Let n be the dimension of X, then we have the sequence of closed subschemes of X:

$$\phi = X^{n+1} \subset X^n \subset X^{n-1} \subset \cdots \subset X^0 = X.$$

We put $E^{p,q}(X) = K'_{-p-q}(Y^p)$ and the morphism $d : E^{p,q}(X) \to E^{p+1,q}(X)$ is defined by

$$K'_{-p-q}(Y^p) \to K'_{-p-q-1}(X^{p+1}) \to K'_{-p-q-1}(Y^{p+1}),$$

where the left arrow is the connecting homomorphism of the localization exact sequence and the right arrow is the restriction of the open immersion $Y^{p+1} \hookrightarrow X^{p+1}$. Then $(E^{\cdot,q}(X), d)$ becomes a complex.

Let $R^{\cdot,q}(X)$ be the Gersten complex of X, that is, $R^{p,q}(X) = \oplus_{x \in X^{(p)}} K_{-p-q}(k(x))$ where $X^{(p)}$ is the set of all points of X whose Zariski closures are of codimension p and $k(x)$ is the residue field of x. We obtain a canonical morphism of complexes $E^{\cdot,q}(X) \to R^{\cdot,q}(X)$.

Proposition 1.2. *The morphism*

$$E^{\cdot,q}(X) \to R^{\cdot,q}(X)$$

is a quasi-isomorphism.

Since $H^p(R^{\cdot,-p}(X))$ is isomorphic to the Chow group of X of codimension p by [5, Prop. 5.14] or by [2, Cor. 7.20], this proposition gives us the representation of the Chow group of X by generators and relations, which is the same result as the one obtained by Fulton et. al. [1] and by Totaro [7].

By the above proposition we have an isomorphism

$$F^p K_0'(X) = \mathrm{Im}(K_0'(X^p) \to K_0'(X))$$

and together with Theorem 1.1 we can describe the coniveau filtration by the equivariant K'-group.

Corollary 1.3. *We define a decreasing filtration F_B^p on $K_0'(X, B)_M$ by*

$$F_B^p K_0'(X, B)_M = \mathrm{Im}(K_0'(X^p, B)_M \to K_0'(X, B)_M).$$

Then for a nonnegative integer p, we have an isomorphism

$$F_B^p K_0'(X, B)_M \simeq F^p K_0'(X).$$

2. Proof of Theorem 1.1

We will prove that

$$K_0'(X^p, B)_M \to K_0'(X^p)$$

is bijective by descending induction on p. Given a $\mathbb{Z}[M]$-module R, let

$$H_i(M; R) = \mathrm{Tor}_i^{\mathbb{Z}[M]}(\mathbb{Z}, R)$$

be the i-th homology of M with coefficient R. The 0-th homology is isomorphic to the group of coinvariants R_M. The homology is calculated by a $\mathbb{Z}[M]$-projective resolution of $\mathbb{Z}$. By choosing a basis of M we can construct a $\mathbb{Z}[M]$-free resolution of $\mathbb{Z}$. Namely for a basis $(m_1, \cdots, m_a)$ of M we set $P_q = \mathbb{Z}[M] \otimes \wedge^q M$ and define $\partial_q : P_{q+1} \to P_q$ by

$$\begin{aligned}&\partial_q(r \otimes m_{i_1} \wedge \cdots \wedge m_{i_{q+1}}) \\ &= \sum_{j=1}^{q+1} (-1)^{j+1} r([m_{i_j}] - [0]) \otimes m_{i_1} \wedge \cdots \wedge m_{i_{j-1}} \wedge m_{i_{j+1}} \wedge \cdots \wedge m_{i_{q+1}}.\end{aligned}$$

Then

$$\cdots \to P_{q+1} \xrightarrow{\partial_q} P_q \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

becomes a $\mathbb{Z}[M]$-free resolution of $\mathbb{Z}$. Hence we have

$$H_q(M; R) \simeq H_q(R \otimes_{\mathbb{Z}[M]} P.).$$

If the action of M on R is trivial, then the differentials in $R \otimes_{\mathbb{Z}[M]} P.$ are all zero. In Particular, it holds that

$$H_q(M; \mathbb{Z}) \simeq \wedge^q M.$$

Since $\mathcal{O}(\sigma) \simeq B/B_\sigma$, by [6] we have

$$\begin{aligned} K'_*(Y^p, B) &= \bigoplus_{\operatorname{codim}\mathcal{O}(\sigma)=p} K'_*(\mathcal{O}(\sigma), B) \\ &\simeq \bigoplus_{\operatorname{codim}\mathcal{O}(\sigma)=p} K'_*(B/B_\sigma, B) \\ &\simeq \bigoplus_{\operatorname{codim}\mathcal{O}(\sigma)=p} K'_*(k, B_\sigma) \\ &\simeq K_*(k) \otimes (\bigoplus_{\operatorname{codim}\mathcal{O}(\sigma)=p} \mathbb{Z}[M^\sigma]). \end{aligned}$$

In other words, $K'_*(Y^p, B)$ is isomorphic to $K_*(k) \otimes K'_0(Y^p, B)$ as a $K_*(k)$-module. Since the boundary homomorphism of the localization exact sequence

$$K'_*(Y^p, B) \to K'_{*-1}(X^{p+1}, B)$$

preserves the $K_*(k)$-module structure, it becomes zero.

Hence we have a short exact sequence of $\mathbb{Z}[M]$-modules

$$0 \to K'_0(X^{p+1}, B) \to K'_0(X^p, B) \to K'_0(Y^p, B) \to 0.$$

So we have

$$\begin{aligned} K_0'(X,B) &\simeq \bigoplus_p K_0'(Y^p,B) \\ &\simeq \bigoplus_{\sigma\in\Delta} \mathbb{Z}[M^\sigma] \end{aligned}$$

as an abelian group. Hence we can say that $K_0'(X,B)$ is determined only by orbits of X. The above short exact sequence induces the long exact sequence

$$\begin{aligned} H_1(M;K_0'(Y^p,B)) &\to K_0'(X^{p+1},B)_M \\ &\to K_0'(X^p,B)_M \to K_0'(Y^p,B)_M \to 0. \end{aligned}$$

We set $M_\sigma = \mathrm{Ker}(M \to M^\sigma)$.

Lemma 2.1.

$$H_1(M;K_0'(Y^p,B)) \simeq \bigoplus_{\mathrm{codim}\mathcal{O}(\sigma)=p} M_\sigma.$$

Proof. Since

$$K_0'(Y^p,B) \simeq \bigoplus_{\mathrm{codim}\mathcal{O}(\sigma)=p} \mathbb{Z}[M^\sigma],$$

we have only to prove $H_1(M;\mathbb{Z}[M^\sigma]) \simeq M_\sigma$. Since $M^\sigma \simeq M/M_\sigma$, the $\mathbb{Z}[M]$-module $\mathbb{Z}[M^\sigma]$ is isomorphic to the induced module of the $\mathbb{Z}[M_\sigma]$-module $\mathbb{Z}$. Hence we have

$$\begin{aligned} H_1(M;\mathbb{Z}[M^\sigma]) &\simeq H_1(M;\mathrm{Ind}_{M_\sigma}^M \mathbb{Z}) \\ &\simeq H_1(M_\sigma;\mathbb{Z}) \\ &\simeq M_\sigma, \end{aligned}$$

which completes the proof. □

Lemma 2.2. *Given an integer* $0 \le p \le r$*, there exists an exact sequence*

$$\bigoplus_{\mathrm{codim}\mathcal{O}(\sigma)=p} M_\sigma \xrightarrow{\partial} K_0'(X^{p+1}) \to K_0'(X^p) \to K_0'(Y^p) \to 0.$$

Proof. By localization exact sequence of K'-theory we have

$$K_1'(Y^p) \to K_0'(X^{p+1}) \to K_0'(X^p) \to K_0'(Y^p) \to 0.$$

By [5], we have the following isomorphism:

$$\begin{aligned} K_1'(Y^p) &= \bigoplus_{\mathrm{codim}\mathcal{O}(\sigma)=p} K_1'(\mathcal{O}(\sigma)) \\ &\simeq \bigoplus_{\mathrm{codim}\mathcal{O}(\sigma)=p} (K_1(k) \oplus (M_\sigma)). \end{aligned}$$

Since the maps in the localization exact sequence preserves the $K_*(k)$-module structures, the images of components $K_1(k)$ by $K_1'(Y^p) \to K_0'(X^{p+1})$ are all zero. This completes the proof. □

Lemma 2.3. *The diagram*

$$\begin{tikzcd} \bigoplus_{\mathrm{codim}\mathcal{O}(\sigma)=p} M_\sigma \arrow[r, "\partial"] & K_0'(X^{p+1}) \\ H_1(M; K_0'(Y^p, B)) \arrow[u] \arrow[r] & K_0'(X^{p+1}, B)_M \arrow[u] \end{tikzcd}$$

commutes, where the left vertical arrow is the isomorphism proved in Lemma 2.1.

Proof. We choose an element $m \in M_\sigma \simeq H_1(M, K_0'(\mathcal{O}(\sigma), B))$ for $\sigma \in \Delta$ and consider the image of m by the above diagram. But the support of the image is contained in the closure of $\mathcal{O}(\sigma)$ in X. So we may assume that $\mathcal{O}(\sigma)$ is the only dense open orbit. In other words, we have only to prove the result when $p = 0$ and X is irreducible.

Let $\pi : \tilde{X} \to X$ be a B-equivariant birational morphism such that $\tilde{X}$ is a smooth variety. The morphism π exists by virtue of the existence of equivariant resolution of singularities. Then horizontal arrows in the above diagram factor through K'-groups of $\tilde{X}$, namely,

$$\begin{tikzcd} M_\sigma \arrow[r, "\partial"] & K_0'(\tilde{X}^1) \arrow[r, "\pi_*"] & K_0'(X^1) \\ H_1(M, K_0'(\mathcal{O}(\sigma), B)) \arrow[u] \arrow[r, "\partial"] & K_0'(\tilde{X}^1, B)_M \arrow[u] \arrow[r, "\pi_*"] & K_0'(X^1, B)_M \arrow[u] \end{tikzcd}$$

Since the right diagram commutes, we have only to prove that the left diagram commutes. Hence we may assume that X is a smooth variety.

We first consider the image of m by the bottom horizontal map. We choose a basis $(m_1, \cdots, m_a)$ of M such that $m = \Sigma s_i m_i$ for $s_i \in \mathbb{Z}$. Then we obtain a $\mathbb{Z}[M]$-free resolution of $\mathbb{Z}$ as mentioned above and represent $m \in M_\sigma \simeq H_1(M; K_0'(Y^0, B))$ by a chain in the complex $\mathbb{Z}[M^\sigma] \otimes P.$. The chain corresponding to m by the isomorphism in Lemma 2.1 becomes $[0] \otimes m \in \mathbb{Z}[M^\sigma] \otimes M$. The bottom horizontal map is the connecting homomorphism and its image is $\Sigma s_i[\mathcal{O}_X]([0]-[m_i])$. Since its support is in X^1, we can regard it as an element of $K_0'(X^1,B)_M$.

We regard m_i as a rational function on X and let $D_{i,0}$ and $D_{i,\infty}$ be the divisors of zeros and poles of m_i respectively. Then in the same way as in [4, Prop. 4] it holds that

$$[\mathcal{O}_X]([0]-[m_i]) = [\mathcal{O}_{D_{i,0}}] - [\mathcal{O}_{D_{i,\infty}}][m_i]$$

in $K_0'(X^1,B)$. Hence the image of $\Sigma s_i[\mathcal{O}_X]([0]-[m_i])$ by the right vertical arrow is

$$\sum s_i([\mathcal{O}_{D_{i,0}}] - [\mathcal{O}_{D_{i,\infty}}]) = \partial(m). \qquad \square$$

We have the following isomorphisms for Y^p

$$\begin{aligned} K_0'(Y^p,B)_M &\simeq \bigoplus_{\operatorname{codim}\mathcal{O}(\sigma)=p} \mathbb{Z}[M^\sigma]_M \\ &\simeq \bigoplus_{\operatorname{codim}\mathcal{O}(\sigma)=p} \mathbb{Z} \\ &\simeq K_0'(Y^p). \end{aligned}$$

Then the theorem follows from the five lemma for the diagram

$$\begin{tikzcd} \bigoplus_{\operatorname{codim}\mathcal{O}(\sigma)=p} M_\sigma \arrow[r, "\partial"] & K_0'(X^{p+1}) \arrow[r] & K_0'(X^p) \arrow[r] & K_0'(Y^p) \arrow[r] & 0 \\ H_1(M, K_0'(Y^p,B)) \arrow[r] \arrow[u] & K_0'(X^{p+1},B)_M \arrow[r] \arrow[u] & K_0'(X^p,B)_M \arrow[r] \arrow[u] & K_0'(Y^p,B)_M \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

and descending induction on p.

3. Proof of Proposition 1.2

For an inclusion $X^{i+1} \hookrightarrow X^i$, we have a short exact sequence of Gersten complexes

$$0 \to R^{\cdot,q+1}(X^{i+1})[-1] \to R^{\cdot,q}(X^i) \to R^{\cdot,q}(Y^i) \to 0,$$

where $[-1]$ means the degree shift. Since $Y^i = \coprod_{\operatorname{codim}\mathcal{O}(\sigma)=i} \mathcal{O}(\sigma)$ and $\mathcal{O}(\sigma)$ is isomorphic to a product of an affine space and a torus, we have

$$H^p(R^{\cdot,q}(Y^i)) \simeq \begin{cases} \bigoplus_{\operatorname{codim}\mathcal{O}(\sigma)=i} K_{-q}(\mathcal{O}(\sigma)) & \text{if } p=0 \\ 0 & \text{if } p \neq 0. \end{cases}$$

Hence we have an isomorphism

$$H^{p-1}(R^{\cdot,q+1}(X^{i+1})) \simeq H^p(R^{\cdot,q}(X^i))$$

if $p \geq 2$ and an exact sequence

$$\begin{aligned} 0 \to H^0(R^{\cdot,q}(X^i)) &\to H^0(R^{\cdot,q}(Y^i)) \\ &\to H^0(R^{\cdot,q+1}(X^{i+1})) \to H^1(R^{\cdot,q}(X^i)) \to 0. \end{aligned}$$

Hence for $p \geq 1$ we have

$$\begin{aligned} H^p(R^{\cdot,q}(X)) &= H^p(R^{\cdot,q}(X^0)) \\ &\simeq H^{p-1}(R^{\cdot,q+1}(X^1)) \\ &\simeq \qquad \vdots \\ &\simeq H^1(R^{\cdot,p+q-1}(X^{p-1})). \end{aligned}$$

We consider the diagram

$$\begin{tikzcd} & H^0(R^{\cdot,p+q-1}(Y^{p-1})) \arrow[d] & & 0 \arrow[d] \\ 0 \arrow[r] & H^0(R^{\cdot,p+q}(X^p)) \arrow[r] \arrow[d] & H^0(R^{\cdot,p+q}(Y^p)) \arrow[r] & H^0(R^{\cdot,p+q+1}(X^{p+1})) \arrow[d] \\ & H^0(R^{\cdot,p+q-1}(X^{p-1})) \arrow[d] & & H^0(R^{\cdot,p+q+1}(Y^{p+1})). \\ & 0 & & \end{tikzcd}$$

Then this yields

$$\begin{aligned} H^1(R^{\cdot,p+q-1}(X^{p-1})) &\simeq \frac{\mathrm{Ker}(H^0(R^{\cdot,p+q}(Y^p)) \to H^0(R^{\cdot,p+q+1}(Y^{p+1})))}{\mathrm{Im}(H^0(R^{\cdot,p+q-1}(Y^{p-1})) \to H^0(R^{\cdot,p+q}(Y^p)))} \\ &\simeq \frac{\mathrm{Ker}(K_{-p-q}(Y^p) \to K_{-p-q-1}(Y^{p+1}))}{\mathrm{Im}(K_{-p-q+1}(Y^{p-1}) \to K_{-p-q}(Y^p))}. \end{aligned}$$

Hence we have

$$\begin{aligned} H^p(R^{\cdot,q}(X)) &\simeq \frac{\mathrm{Ker}(K_{-p-q}(Y^p) \to K_{-p-q-1}(Y^{p+1}))}{\mathrm{Im}(K_{-p-q+1}(Y^{p-1}) \to K_{-p-q}(Y^p))} \\ &\simeq H^p(E^{\cdot,q}(X)), \end{aligned}$$

which holds when $p = 0$ if we put $Y^{-1} = \phi$.

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Department of Mathematics
Faculty of Science
Tokyo Metropolitan University
Minami-osawa 1-1
Hachioji-shi, Tokyo 192-0364, Japan
e-mail: takeda@math.metro-u.ac.jp