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ON INVARIANT MEASURES OF CRITICAL MULTITYPE GALTON-WATSON PROCESSES

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1. Introduction

A multitype Galton-Watson process with discrete time (abbreviated to MGWP) is a mathematical description of a population growth involving several types of individuals, where each individual produces offspring by a certain stochastic law independently of others. Suppose that one i-type individual produces x^1 individuals of 1-type, x^2 ones of 2-type, $\cdots$, x^N ones of N-type during a unit time with probability $P^i(x)$ $(x=(x^1, \cdots, x^N))$. Then an MGWP is defined as a Markov chain on the space S of all N-tuples of nonnegative integers, with the one step transition probability

$$(1.1)\qquad P(x,y)=\begin{cases} \underbrace{P^1 * \cdots * P^1}_{x^1} * \cdots * \underbrace{P^N * \cdots * P^N}_{x^N}(y)\,, & \\ \qquad\qquad \text{if } 0 \neq x=(x^1, \cdots, x^N) \in S,\, y \in S\,, & \\ \delta_0(y), \qquad \text{if } x=0 \in S,\, y \in S\,, & \end{cases}$$

where $*$ means the convolution of distributions. Since the state $0 \in S$ is a trap for our process, invariant measures on the whole state space S are trivial in most cases. But invariant measures of the MGWP restricted onto $S-\{0\}$ are not trivial in general, and it is important to study them. For the case of $N=1$, many authors have investigated this subject (cf. Harris [4] pp. 22–31, Athreya and Ney [2] pp. 67–73, 87–93]. Especially, Kesten, Ney and Spitzer [7] gave the definitive results on the existence and uniqueness of invariant measures of critical simple GW processes.

In this paper, we shall prove the existence and uniqueness of invariant measures on $S-\{0\}$ of a critical, positively regular and nonsingular MGWP, under the hypothesis of finite $x^2\log x$-moments (cf. (H.1)$\sim$(H.4) in §2). The statement of the theorem is given in section 2. In section 3 we shall prove a basic lemma which was proved in [7] ((2.16), p. 517) in the case of critical simple GW processes. It will be proved by elaborating those results in [5] since the proof in [7] does not seem applicable to the case of multitype GW processes. Finally, in section 4, we shall prove the theorem with the aid of the basic lemma.

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2. Definitions and the statement of Theorem

For $s=(s^1, \cdots, s^N)\in R^N$ and $y=(y^1, \cdots, y^N)\in S$, we define $s^y=(s^1)^{y^1}\cdots(s^N)^{y^N}$, $||s||=\max_{1\leqslant i\leqslant N}|s^i|$ and $|y|=y^1+\cdots+y^N$. Special vectors $(0, 0, \cdots, 0)$ and $(1, 1, \cdots, 1)$ are abbreviated simply as 0 and 1 respectively.

We denote the MGWP defined in section 1 by $X=(Z(n), P_x)$, $n=0, 1, 2, \cdots$, $x\in S$. The probability generating function (p.g.f.) $F^i(s)$ of the distribution $\{P^i(y)\}$ is given by

$$F^i(s)=\sum_{y\in S}P^i(y)s^y,\ ||s||\leqslant 1\,.$$

Then, by (1.1), we have

$$\sum_{y\in S}P(x, y)s^y = F(s)^x,\ x\in S\,,$$

where $F(s)=(F^1(s), \cdots, F^N(s))$. Therefore it follows from the Markov property that

$$\sum_{y\in S}P_n(x, y)s^y = F(n; s)^x,\ x\in S\,, \tag{2.1}$$

where $P_n(x, y)=P_x(Z(n)=y)$ and $F(n; s)$ is the n-th iteration of $F(s)$:

$$F(0; s) = s,\ F(n+1; s)=F(F(n; s)),\ n = 0, 1, 2, \cdots .$$

The mean matrix $M=[m^i_j]^N_{i,j=1}$ is defined by

$$m^i_j = F^i_j(1-) = \sum_{y\in S}y^jP^i(y),\ 1\leqslant i, j\leqslant N\,,$$

where $F^i_j(s)=\partial F^i(s)/\partial s^j$. It is well known that the (i, j)-component $m^i_j(n)$ of the n-fold product M^n of the matrix M is equal to

$$m^i_j(n) = F^i_j(n; 1-) = \sum_{y\in S}y^jP_n(e_i, y),\ 1\leqslant i, j\leqslant N\,,$$

where e_i is the unit vector with i-th component equal to 1. Since every $m^i_j\geqslant 0$, M has a nonnegative characteristic root ρ with the greatest absolute value. This root is called Perron-Frobenius root ($P-F$ root) of the matrix M.

In this paper, we shall deal with those MGWP's which satisfy the following hypotheses:

(H.1) (Positive regularity). *There exists a positive integer* n *such that* $m^i_j(n)>0$ *for all* $1\leqslant i, j\leqslant N$.

(H.2) (Critical property). *The* $P-F$ *root* ρ *of the matrix* M *is equal to* 1.

(H.3) (Nonsingularity). *Every* p.g.f. $F^i(s)$ *is not of homogeneous linear form.*

(H.4) (Moment assumption). *For each* $1 \leqslant i, j \leqslant N$, *it holds that*

$$\sum_{y \in S} P^i(y)\,(y^j)^2 \log y^j < \infty \,.$$

In this paper a nonnegative measure $\{\nu(x)\}$ on $S - \{0\}$ is said to be an invariant measure of the MGWP X if

$$\sum_{x \in S - \{0\}} \nu(x) P(x, y) = \nu(y),\ y \in S - \{0\} \,. \tag{2.2}$$

By hypothesis (H.1), the $P-F$ root ρ of M is simple and there correspond the unique right eigenvector $u = (u^1, \cdots, u^N)$[1] and left eigenvector $v = (v_1, \cdots, v_N)$ whose components are positive with the normalization

$$|u| = \sum_{i=1}^N u^i = 1,\ vu = \sum_{i=1}^N v_i u^i = 1 \,. \tag{2.3}$$

It also holds for some $0 < \rho_1 < \rho$ and $K_1 > 0$ that

$$\|M^n - M^*\| \leqslant K_1 \rho_1^n, \qquad n = 0, 1, 2, \cdots, \tag{2.4}$$

where $M^* = [m_j^{*i}]_{i,j=1}^N = [u^i v_j]_{i,j=1}^N$ and $\|M^n - M^*\| = \max\limits_{1 \leqslant i, j \leqslant N} |m_j^i(n) - m_j^{*i}|$ (cf. Gantmacher [3], Joffe and Spitzer [5], and Harris [4]).

We set

$$B = \frac{1}{2} \sum_{i,j,k=1}^N v_i F_{jk}^i(1-) u^j u^k \,,$$

where $F_{jk}^i(s) = \partial^2 F^i(s) / \partial s^j \partial s^k$. Then it follows that $B > 0$ holds by the hypotheses (H.1)$\sim$(H.3) (cf. [5] p. 429).

It will be seen in the sequel that

$$G(x, y) = \sum_{n=0}^\infty P_n(x, y) < \infty,\ x, y \in S - \{0\} \,. \tag{2.5}$$

The purpose of this paper is to prove the following theorem.

Theorem. *Under the hypotheses* (H.1)$\sim$(H.4), *there exists a unique invariant measure* $\{\mu(x)\}$ *of the MGWP X up to a constant multiple. Further it is given by*

$$\mu(y) = \frac{B}{xu} \lim_{n \to \infty} n^2 P_n(x, y) = \lim_{|z| \to \infty} G(z, y),\ x, y, z \in S - \{0\} \,. \tag{2.6}$$

Throughout the following sections, we assume (H.1)$\sim$(H.4).

3. Basic lemma

Here we shall introduce some order relations between two vectors $s_1 = (s_1^1, \cdots, s_1^N)$ and $s_2 = (s_2^1, \cdots, s_2^N)$: $s_1 \leqslant s_2$ means $s_1^i \leqslant s_2^i$ for all $1 \leqslant i \leqslant N$. Similarly $s_1 < s_2$

1) We shall not distinct row vectors and column vectors in this note.

[resp. $s_1 \ll s_2$] stands for $s_1 \leqslant s_2$ and $s_1 \neq s_2$ [resp. $s_1^i < s_2^i$ for all $1 \leqslant i \leqslant N$]. These notations are extended for matrices in a natural way.

The purpose of this section is to prove the following

Basic Lemma. *There exists the finite limit*

$$U(s) = \lim_{n\to\infty} B\, n^2 v\{F(n; s) - F(n; 0)\},\ 0 \leqslant s \ll 1. \tag{3.1}$$

Furthermore, $U(s) \not\equiv 0$ *and*

$$\lim_{n\to\infty} n^2\{F(n; s) - F(n; 0)\} = \frac{U(s)}{B} u, \qquad 0 \leqslant s \ll 1. \tag{3.2}$$

We start with some preliminary remarks. We set $R(n; s) = 1 - F(n; s)$, $R^i(n; s) = 1 - F^i(n; s)$ (i.e. i-th component of the vector $R(n; s)$), $r_n(s) = vR(n; s)$, $\tilde{R}(n; s) = R(n; s)/r_n(s)$, $T(n; s) = F(n; s) - F(n; 0) = R(n; 0) - R(n; s)$, $T^i(n; s) = R^i(n; 0) - R^i(n; s)$ and $t_n(s) = vT(n; s) = v\{F(n; s) - F(n; 0)\}$. We often omit the variable s like $R(n; s) = R(n)$, etc.

The Taylor expansion of $F^i(s)$ from the left at $s=1$ gives

$$1 - F(s) = (M - E(s))(1 - s), \qquad 0 \leqslant s \leqslant 1, \tag{3.3}$$

$$E_j^i(s) = \int_0^1 \{F_j^i(1-) - F_j^i(1 - (1-s)\theta)\} d\theta = \sum_{k=1}^N B_{jk}^i(s)(1 - s^k), \tag{3.4}$$

$$\begin{aligned} B_{jk}^i(s) &= \int_0^1 (1-\theta) F_{jk}^i(1 - (1-s)\theta) d\theta \\ &= \sum_{y\in S} P^i(y)(y^j y^k - y^j \delta_k^j) \int_0^1 (1-\theta)\{1 - (1-s)\theta\}^{y - e_j - e_k} d\theta, \end{aligned} \tag{3.5}$$

where δ_k^j is the Kronecker's delta. (cf. Joffe and Spitzer [5] pp. 426–427 or Ogura [8] (4.15)). Obviously, (3.4) and (3.5) imply that

$$0 \leqslant E(s) \leqslant M, \tag{3.6}$$

$$\begin{cases} 0 \leqslant B_{jk}^i(s) \leqslant B_{jk}^i(s') \leqslant \frac{1}{2} F_{jk}^i(1-),\ 0 \leqslant s \leqslant s' \leqslant 1, \\ B_{jk}^i(s) \nearrow \frac{1}{2} F_{jk}^i(1-) \ \text{ as } \ s \nearrow 1,\ 1 \leqslant i, j, k \leqslant N. \end{cases} \tag{3.7}$$

Replacing s in (3.3) by $F(n-1; s)$, we obtain

$$R(n; s) = (M - E(n-1; s)) R(n-1; s), \tag{3.8}$$

where

$$E_j^i(l; s) = E_j^i(F(l; s)) = \sum_{k=1}^N B_{jk}^i(F(l; s)) R^k(l; s). \tag{3.9}$$

Before the proof of Basic Lemma, we need to prepare several lemmas. We note that Lemma 1~4 are valid under the hypothesis of finite second moments i.e. $F^{i}_{jk}(1-)<\infty$ instead of (H.4).

Lemma 1. (Joffe and Spitzer [5])[2] *If $n\to\infty$, then*

$$nR(n;\,s)=\frac{u}{B}+o(1),\ 0\leqslant s<1\,,$$

so that $\lim_{n\to\infty} n\, r_n(s)=1/B$ *and* $\lim_{n\to\infty}\tilde{R}(n;\,s)=u$ *for* $0\leqslant s<1$.

Lemma 1 implies that $R(n;\,s)=r_n(s)u+o(1/n)$.
The next lemma is a refinement of this fact.

Lemma 2. *If $n\to\infty$, then*

$$R(n;\,s)=r_n(s)u+O\left(\frac{\log n}{n^2}\right),\ 0\leqslant s<1\,,$$

so that $\tilde{R}(n;\,s)=u+O((\log n)/n)$.

Proof. We fix an s in $0\leqslant s<1$ and abbreviate it in the descriptions. Using (3.8) inductively, we have

$$R(n)=(M-E(n-1))(M-E(n-2))\cdots(M-E(n-m))R(n-m)\,.$$

Further, applying Lemma 1 to (3.9) and noting that $m^{i}_{j}=0$ yields $E^{i}_{j}(l)=0$ by (3.6), we have

$$O\leqslant E(l)\leqslant\frac{K}{l}M,\ l=1,\,2,\,\cdots$$

for some constant $K>0$. Hence it follows that

$$\prod_{l=n-m}^{n-1}\left(1-\frac{K}{l}\right)M^{m}R(n-m)\leqslant R(n)\leqslant M^{m}R(n-m)\,,$$

if $n-m$ is large enough. But $(1-K_1\rho_1^m)M^*\leqslant M^m\leqslant(1+K_1\rho_1^m)M^*$ by (2.4), and $M^*R(n-m)=r_{n-m}u$ by the definition. Hence we have

$$\begin{aligned}(3.10)\qquad &\left(1-\sum_{l=n-m}^{n-1}\frac{K}{l}-K_1\rho_1^m\right)r_{n-m}u\\ &\leqslant\prod_{l=n-m}^{n-1}\left(1-\frac{K}{l}\right)(1-K_1\rho_1^m)r_{n-m}u\\ &\leqslant R(n)\leqslant(1+K_1\rho_1^m)r_{n-m}u\,,\end{aligned}$$

and simultaneously, by taking the inner product with the left eigenvector v,

2) In their paper there is an error: the inequality (4.42) should be read as

$$-\frac{\alpha(s)^2v(1-s)}{1-v(1-s)(\alpha(s)-\varepsilon(s))}\leqslant\delta(s)\leqslant\varepsilon(s),\ s\in C-\{1\}\,.$$

But their assertion is valid.

$$(3.11) \qquad \Big(1-\sum_{l=n-m}^{n-1}\frac{K}{l}-K_1\rho_1^m\Big)r_{n-m} \leqslant r_n \leqslant (1+K_1\rho_1^m)r_{n-m}\,.$$

Multiplying (3.11) by the right eigenvector u, and subtracting it from (3.10), we obtain

$$|R^i(n)-r_nu^i| \leqslant \Big(2K_1\rho_1^m+\sum_{l=n-m}^{n-1}\frac{K}{l}\Big)r_{n-m}u^i,\ 1\leqslant i\leqslant N\,.$$

Now we take a constant $c>1/(-\log\rho_1)$ and set $m=m(n)=[c\log n]$. Then it is clear by Lemma 1 that

$$\frac{n^2}{\log n}\rho_1^{m(n)}r_{n-m(n)}\leqslant\frac{K'n^2e^{m(n)\log\rho_1}}{(n-m(n))\log n}\to 0,\ \text{ as }\ n\to\infty\,,$$

$$\frac{n^2}{\log n}\sum_{l=n-m(n)}^{n-1}\frac{r_{n-m(n)}}{l}\leqslant\frac{K''n^2[c\log n]}{(n-m(n))^2\log n}\,.$$

Hence we have shown the boundedness of the sequence $|R^i(n)-r_nu^i|n^2/\log n$, which proves the first formula of the lemma. The second formula follows from the first one and Lemma 1.

Lemma 3. *If* $n\to\infty$, *then*

$$t_n(s)=r_n(0)-r_n(s)=O\Big(\Big(\frac{\log n}{n}\Big)^2\Big),\ 0\leqslant s\ll 1\,.$$

Proof. Taking the inner product of (3.8) and the left eigenvector v with the aid of (3.9), we have

$$(3.12) \qquad r_n(s)=r_{n-1}(s)-b_{n-1}(s)r_{n-1}(s)^2\,,$$

where

$$(3.13) \qquad b_l(s)=\sum_{i,j,k=1}^{N}v_iB^i_{jk}(F(l;s))\tilde{R}^j(l;s)\tilde{R}^k(l;s)$$

(note that $r_l(s)>0$ for all $0\leqslant s<1$ and $l=0, 1, 2, \cdots$). From (3.12) it follows that $1-b_l(s)r_l(s)>0$ and

$$\frac{1}{r_n(s)}-\frac{1}{r_0(s)}=\sum_{l=0}^{n-1}\frac{b_l(s)}{1-b_l(s)r_l(s)}\,.$$

Hence

$$(3.14) \qquad \Big(\frac{1}{r_n(s)}-\frac{1}{r_n(0)}\Big)-\Big(\frac{1}{r_0(s)}-\frac{1}{r_0(0)}\Big) =\sum_{l=0}^{n-1}\frac{b_l(s)-b_l(0)}{(1-b_l(s)r_l(s))(1-b_l(0)r_l(0))}$$

$$-\sum_{l=0}^{n-1}\frac{b_l(0)b_l(s)(r_l(0)-r_l(s))}{(1-b_l(s)r_l(s))(1-b_l(0)r_l(0))}$$

$$=I_n-II_n \quad \text{(say)}.$$

Since $0\leqslant s\ll 1$ and $F(n;0)\nearrow 1$ as $n\to\infty$, one can find a positive integer l_0 with $s\leqslant F(l_0;0)\leqslant 1$. Then, since $b_l(s)$ and $b_l(0)$ are bounded in l and $r_l(s)\to 0$, $r_l(0)\to 0$ as $l\to\infty$, we have

$$0\leqslant\frac{b_l(0)b_l(s)(r_l(0)-r_l(s))}{(1-b_l(s)r_l(s))(1-b_l(0)r_l(0))}\leqslant Kv\{F(l;s)-F(l;0)\}$$
$$\leqslant Kv\{F(l+l_0;0)-F(l;0)\}$$

for some constant $K>0$. Hence

$$\lim_{n\to\infty} II_n=\sum_{l=0}^{\infty}\frac{b_l(0)b_l(s)(r_l(0)-r_l(s))}{(1-b_l(s)r_l(s))(1-b_l(0)r_l(0))} \tag{3.15}$$
$$\leqslant K\sum_{l=0}^{l_0-1}v\{1-F(l;0)\}<\infty\,.$$

To estimate I_n, we use the equation

$$\begin{aligned}b_l(s)-b_l(0)=&\sum_{i,j,k=1}^{N}v_i\{B^i_{jk}(F(l;s))-B^i_{jk}(F(l;0))\}\tilde{R}^j(l;s)\tilde{R}^k(l;s)\\&+\sum_{i,j,k=1}^{N}v_iB^i_{jk}(F(l;0))\{\tilde{R}^j(l;s)-\tilde{R}^j(l;0)\}\tilde{R}^k(l;s)\\&+\sum_{i,j,k=1}^{N}v_iB^i_{jk}(F(l;0))\tilde{R}^j(l;0)\{\tilde{R}^k(l;s)-\tilde{R}^k(l;0)\}\,.\end{aligned}\tag{3.16}$$

Then Lemma 1 and 2 yield the estimate

$$|I_n|<K\Big(\sum_{i,j,k=1}^{N}\sum_{l=0}^{n-1}\{B^i_{jk}(F(l;s))-B^i_{jk}(F(l;0))\}+\sum_{l=1}^{n-1}\frac{\log l}{l}\Big)$$

for some constant $K>0$. But since

$$0\leqslant\sum_{l=0}^{\infty}\{B^i_{jk}(F(l;s))-B^i_{jk}(F(l;0))\} \tag{3.17}$$
$$\leqslant\sum_{l=0}^{\infty}\{B^i_{jk}(F(l+l_0;0))-B^i_{jk}(F(l;0))\}$$
$$=\sum_{l=0}^{l_0-1}\Big\{\frac{1}{2}F^i_{jk}(1-)-B^i_{jk}(F(l;0))\Big\}$$

by (3.7), and since $\sum_{l=1}^{n-1}(\log l)/l=O((\log n)^2)$ as $n\to\infty$, it follows that $I_n=O((\log n)^2)$. Now combining this fact with (3.14) and (3.15), we obtain

$$\frac{1}{r_n(s)}-\frac{1}{r_n(0)}=O((\log n)^2), \text{ as } n\to\infty .$$

Hence by virtue of Lemma 1 we have

$$\begin{aligned} r_n(0)-r_n(s) &= O((\log n)^2 r_n(s) r_n(0)) \\ &= O\Big(\Big(\frac{\log n}{n}\Big)^2\Big), \text{ as } n\to\infty . \end{aligned}$$

Lemma 4. *If* $n\to\infty$, *then*

$$T(n; s)=t_n(s)u+O\Big(\Big(\frac{\log n}{n}\Big)^3\Big),\ 0\leqslant s\ll 1 .$$

Proof. Noting that $B^i_{jk}=B^i_{kj}$, we have from (3.8) that

$$T(n)=(M-G(n-1))T(n-1)+D(n-1), \tag{3.18}$$

where $G^i_j(l)=\sum_{k=1}^{N} B^i_{jk}(F(l; 0))\{R^k(l; 0)+R^k(l; s)\}$,

and $D^i(l)=\sum_{j,k=1}^{N}\{B^i_{jk}(F(l; s))-B^i_{jk}(F(l; 0))\}R^j(l; s)R^k(l; s)$.

Since $m^i_j=0$ implies $F^i_j(s)\equiv 0$ and hence $B^i_{jk}(s)=0$, $1\leqslant k\leqslant N$ by (3.5), it follows from Lemma 1 that

$$0\leqslant G(l)\leqslant\frac{K}{l}M,\ l=1, 2, \cdots . \tag{3.19}$$

Further it holds that

$$0\leqslant D(l)\leqslant d_l u,\ d_l=O\Big(\frac{(\log l)^2}{l^3}\Big) \text{ as } l\to\infty . \tag{3.20}$$

Indeed, by (3.5), Lemma 1 and 3, we have

$$\begin{aligned} 0 &\leqslant B^i_{jk}(F(l; s))-B^i_{jk}(F(l; 0)) \\ &\leqslant \sum_{y\in S} P^i(y)y^j y^k\|T(l; s)\|\int_0^1(|y|-2)\{1-\theta(1-\|F(l; s)\|)\}^{|y|-3}d\theta \\ &\leqslant K\sum_{y\in S} P^i(y)y^j y^k\frac{(\log l)^2}{l} \end{aligned} \tag{3.21}$$

for some constant $K>0$, where we have used the relation

$$\prod_{i=1}^{k}\alpha_i-\prod_{i=1}^{k}\beta_i=\sum_{i=1}^{k}\alpha_1\cdots\alpha_{i-1}(\alpha_i-\beta_i)\beta_{i+1}\cdots\beta_k$$

in the second inequality. We obtain (3.20) from the definition of $D(l)$ and Lemma 1. Now using (3.18) repeatedly, we have

$$\begin{aligned} T(n) &= (M-G(n-1))\cdots(M-G(n-m))T(n-m) \\ &\quad+\sum_{l=n-m}^{n-2}(M-G(n-1))\cdots(M-G(l+1))D(l)+D(n) . \end{aligned}$$

Hence, by the similar arguments as in the proof of Lemma 2, we can use (3.19) and (3.20) to obtain

$$|T^i(n)-t_nu^i| \leqslant \left\{\left(2K_1\rho_1^m+\sum_{l=n-m}^{n-1}\frac{K}{l}\right)t_{n-m}+\sum_{l=n-m}^{n-1}d_l\right\}u^i,\ 1\leqslant i\leqslant N,$$

if $n-m$ is large enough. Take the same $m=m(n)=[c\log n]$ as in the proof of Lemma 2. Then by virtue of Lemma 3 and (3.20), we have

$$\left(\frac{n}{\log n}\right)^3\rho_1^{m(n)}t_{n-m(n)}\leqslant\frac{K'n^3e^{m(n)\log\rho_1}}{(n-m(n))^2\log n}\to 0,\ \text{ as } n\to\infty,$$

$$\left(\frac{n}{\log n}\right)^3 t_{n-m(n)}\sum_{l=n-m(n)}^{n}\frac{1}{l}\leqslant\frac{K''n^3(\log(n-m(n)))^2[c\log n]}{(n-m(n))^3(\log n)^3},$$

$$\left(\frac{n}{\log n}\right)^3\sum_{l=n-m(n)}^{n}d_l\leqslant\frac{K'''n^3(\log(n-m(n)))^2[c\log n]}{(n-m(n))^3(\log n)^3}$$

Hence the sequence $|T^i(n)-t_nu^i|(n/\log n)^3$ is bounded in n, which completes the proof of Lemma 4.

Lemma 5. *If* $n\to\infty$, *then*

$$r_n(s)=\frac{1}{nB}+o\left(\frac{1}{n}\right),\ 0\leqslant s<1. \tag{3.22}$$

Moreover

$$\sum_{n=1}^{\infty}\left|r_n(s)-\frac{1}{nB}\right|<\infty,\ 0\leqslant s<1. \tag{3.23}$$

Proof. First we note that (3.12) yields

$$\frac{1}{r_n}=nB+\frac{1}{r_0}+\sum_{l=0}^{n-1}c_l \tag{3.24}$$

where $c_l=\dfrac{b_l-B+Bb_lr_l}{1-b_lr_l}$.

We shall show that $\sum_{l=1}^{\infty}|c_l|/l<\infty$ by proving $\sum_{l=1}^{\infty}|b_l-B|/l<\infty$. Using a decomposition of $B-b_l$ similar to (3.16), we have an estimate

$$0\leqslant|B-b_l|\leqslant K\left(\sum_{i,j,k=1}^{N}\left\{\frac{1}{2}F^i_{jk}(1-)-B^i_{jk}(F(l))\right\}+\frac{\log l}{l}\right).$$

On the other hand

$$0\leqslant\frac{1}{2}F^i_{jk}(1-)-B^i_{jk}(F(l))$$

$$=\sum_{y\in S}P^i(y)(y^jy^k-y^j\delta^j_k)\int_0^1(1-\theta)\{1-(1-\theta R(l))^{y-e_j-e_k}\}d\theta$$

$$\leqslant \sum_{y\in S} P^i(y)y^jy^k\int_0^1(1-\theta)\{[(|y|-2)\|R(l)\|]\wedge 1\}d\theta$$

$$\leqslant \sum_{y\in S} P^i(y)y^jy^k\left(\frac{|y|K}{l}\wedge 1\right),$$

where $a\wedge b=\min\{a, b\}$. Hence it follows from the hypothesis (H.4) that

$$0\leqslant \sum_{l=1}^{\infty}\frac{1}{l}\left\{\frac{1}{2}F^i_{jk}(1-)-B^i_{jk}(F(l))\right\} \tag{3.25}$$

$$\leqslant \sum_{y\in S} P^i(y)y^jy^k\left\{\sum_{l=1}^{[|y|K]}\frac{1}{l}+\sum_{l=[|y|K]}^{\infty}\frac{|y|K}{l^2}\right\}$$

$$\leqslant \sum_{y\in S} P^i(y)y^jy^k(\log|y|K')<\infty,$$

which proves that $\sum_{l=1}^{\infty}|b_l-B|/l<\infty$ and therefore $\sum_{l=1}^{\infty}|c_l|/l<\infty$. Since $c_l\to 0$ as $l\to\infty$, (3.24) yields

$$\left|r_n-\frac{1}{nB}\right|=\frac{1}{nB}\frac{\left|\frac{1}{nBr_0}+\frac{1}{nB}\sum_{l=0}^{n-1}c_l\right|}{\left|1+\frac{1}{nBr_0}+\frac{1}{nB}\sum_{l=0}^{n-1}c_l\right|} \tag{3.26}$$

$$\leqslant\frac{K_2}{n^2}+\frac{K_3}{n^2}\sum_{l=0}^{n-1}|c_l|$$

for some constants $K_2, K_3>0$. But

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{l=0}^{n-1}|c_l|=\sum_{l=0}^{\infty}|c_l|\sum_{n=l+1}^{\infty}\frac{1}{n^2}$$

$$\leqslant K_4+\sum_{l=1}^{\infty}\frac{|c_l|}{l}<\infty,$$

which proves (3.23). The first relation (3.22) is a direct consequence of (3.26).

Lemma 6. *Suppose that* $\varepsilon_n=o\left(\frac{1}{n}\right)$ (as $n\to\infty$) *and* $\sum_{n=0}^{\infty}|\varepsilon_n|<\infty$. *If we set*

$$\beta_n=\frac{2}{n}+\varepsilon_n, \tag{3.27}$$

then for each $l=1, 2, \cdots$, *there exists the limit*

$$\beta^*(l)=\lim_{n\to\infty}n^2\prod_{m=l}^{n}(1-\beta_m). \tag{3.28}$$

Moreover there exists a constant $K>0$ *such that*

$$n^2\prod_{m=l}^{n}(1-\beta_m)\leqslant Kl^2 \text{ for every } n \text{ and } l\geqslant 1. \tag{3.29}$$

Proof. We take an m_0 satisfying $0\leqslant\beta_m<1$ for all $m\geqslant m_0$. Since

$$\log(1-\beta_m) = -\beta_m - \eta_m,\ \eta_m = O\left(\frac{1}{m^2}\right),\ m \to \infty\ ,$$

it follows from (3.27) that

$$\log\{n^2 \prod_{m=l}^{n}(1-\beta_m)\} = 2\left(\log n - \sum_{m=1}^{n}\frac{1}{m}\right) + 2\sum_{m=1}^{l-1}\frac{1}{m} - \sum_{m=l}^{n}(\varepsilon_m + \eta_m)$$

for $l \geqslant m_0$. The right side converges as $n \to \infty$ and hence the $\beta^*(l)$ exists for $l \geqslant m_0$. For $l < m_0$, we have only to note that $\beta^*(l) = \beta^*(m_0)\prod_{m=l}^{m_0-1}(1-\beta_m)$. Inequality (3.29) is clear since

$$\begin{aligned}\log\{n^2 \prod_{m=l}^{n}(1-\beta_m)\} &= 2\log n - \sum_{m=l}^{n}\frac{2}{m} - \sum_{m=l}^{n}(\varepsilon_m + \eta_m)\\ &\leqslant 2\log n - \int_l^n \frac{2}{x}\,dx + \log K\\ &= \log(Kl^2),\ l \geqslant m_0\,.\end{aligned}$$

Now we are ready to prove Basic Lemma.

Proof of Basic Lemma. Multiplying the left eigenvector v to the both sides of (3.18), we have

$$t_{n+1} = t_n - vG(n)T(n) + vD(n)\,,$$

which is rewritten as

$$t_{n+1} = (1-\beta_n)t_n + \gamma_n \tag{3.30}$$

where

$$\beta_n = vG(n)u$$

and

$$\gamma_n = vG(n)(t_n u - T(n)) + vD(n)\,.$$

Since

$$\begin{aligned}\beta_n &= \sum_{i,j,k=1}^{N} v_i B^i_{jk}(F(n;0))\{R^k(n;0) + R^k(n;s)\}u^j\\ &= \frac{2}{n} + \sum_{i,j,k=1}^{N} v_i \frac{1}{2}F^i_{jk}(1-)\Big\{(R^k(n;0) - r_n(0)u^k)\\ &\quad + (R^k(n;s) - r_n(s)u^k) + \left(r_n(s) + r_n(0) - \frac{2}{nB}\right)u^k\Big\}u^j\\ &\quad + \sum_{i,j,k=1}^{N} v_i\left\{B^i_{jk}(F(n;0)) - \frac{1}{2}F^i_{jk}(1-)\right\}\{R^k(n;0) + R^k(n;s)\}u^j\,,\end{aligned}$$

it follows from Lemma 2 and 5 that $\{\beta_n\}$ satisfies (3.27). Next we observe that

(3.31) $$\sum_{n=0}^{\infty} n^2 |\gamma_n| < \infty .$$

Indeed, Lemma 4 and (3.19) imply that

$$vG(n)(t_n u - T(n)) = O\Big(\frac{(\log n)^3}{n^4}\Big) .$$

The finiteness of $\sum_{n=0}^{\infty} n^2 vD(n)$ is a direct consequence of Lemma 1 and (3.17). Now using (3.30) repeatedly, we have

$$\begin{aligned} n^2 t_{n+1} &= n^2 \prod_{m=0}^{n} (1-\beta_m) t_0 + \sum_{l=0}^{n-1} n^2 \prod_{m=l+1}^{n} (1-\beta_m) \gamma_l + n^2 \gamma_n \\ &= I_n + II_n + III_n \text{ (say).} \end{aligned}$$

It is obvious that $\lim_{n\to\infty} I_n = \beta^*(0)$ and $\lim_{n\to\infty} n^2 \gamma_n = 0$. Each term of II_n converges as $n \to \infty$ and it is dominated by $Kl^2 |\gamma_l|$. But since (3.31) is valid, the convergence of II_n follows from the Lebesgue dominated convergence theorem. Since $t_n = v\{F(n; s) - F(n; 0)\}$ we have proved (3.1).

To show the nontriviality of the function $U(s)$, we note that

$$\lim_{n\to\infty} n^2 (r_{n-1}(s) - r_n(0)) = \frac{1}{B}$$

by (3.12), (3.13) and Lemma 1. Then it holds from (3.1) that

(3.32) $$U(F(0)) = \lim_{n\to\infty} Bn^2 (r_n(0) - r_{n+1}(0)) = 1$$

Finally, it is clear that (3.2) follows from (3.1) and Lemma 4.

4. Proof of Theorem

In this section, we shall prove the theorem stated in §2. The next lemma is a direct consequence of Basic Lemma by the standard argument on a convergent sequence of analytic functions with nonnegative coefficients (eg. [7] p. 518 for a complete proof).

Lemma 7. *The function $U(s)$ in* (3.1) *is analytic in $||s|| < 1$ and it is expressed by the power series with nonnegative coefficients $\mu(x)$, $x \in S - \{0\}$* :

(4.1) $$U(s) = \sum_{x \in S - \{0\}} \mu(x) s^x, \quad ||s|| < 1 ,$$

and

(4.2) $$\mu(x) = \lim_{n\to\infty} Bn^2 \sum_{i=1}^{N} v_i P_n(e_i, x) .$$

Now we shall restate the "existence" part of the theorem and prove it.

Proposition 1. *The measure $\{\mu(x)\}$ on $S - \{0\}$ is an invariant measure of the MGWP X. Further it holds that*

(4.3) $$\lim_{n\to\infty} n^2 P_n(x, y) = \frac{xu}{B}\mu(y),\ x, y\in S-\{0\}\ ,$$

and

(4.4) $$\lim_{|x|\to\infty} G(x, y) = \mu(y),\ y\in S-\{0\}\ .$$

Proof. From (3.1) and (3.32), we have

$$U(F(s)) = U(s)+1,\ 0\leqslant s\ll 1\ .$$

Hence, comparing those coefficients of s^x of the above equation, we see that $\{\mu(x)\}$ is an invariant measure of X.

For (4.3), it is enough to show that

$$\lim_{n\to\infty} n^2\{F(n; s)^x - F(n; 0)^x\} = \frac{xu}{B}U(s),\ 0\leqslant s\ll 1\ .$$

But this is clear from (3.2) since

(4.5) $$F(n; s)^x - F(n; 0)^x = \sum_{i=1}^{N}\sum_{l=1}^{x^i} F^1(n; x)^{x^1}\cdots F^{i-1}(n; s)^{x^{i-1}}$$
$$F^i(n; s)^{x^i-l}\{F^i(n; s)-F^i(n; 0)\}F^i(n; 0)^{l-1}F^{i+1}(n; 0)^{x^{i+1}}\cdots F^N(n; 0)^{x^N}\ .$$

We note that (4.3) yields $G(x, y)<\infty$.

To show (4.4), we fix an $0\leqslant s\ll 1$ and take any $\varepsilon>0$. Then, by means of Lemma 1 and (3.2), there exists a positive integer n_0 such that

$$\exp\left\{-\frac{u^i}{nB}(1+\varepsilon)^2\right\}\leqslant F^i(n; s)\leqslant \exp\left\{-\frac{u^i}{nB}(1-\varepsilon)\right\},$$

$$\exp\left\{-\frac{u^i}{nB}(1+\varepsilon)^2\right\}\leqslant F^i(n; 0)\leqslant \exp\left\{-\frac{u^i}{nB}(1-\varepsilon)\right\},$$

$$(1-\varepsilon)U(s)\frac{u^i}{n^2B}\leqslant F^i(n; s)-F^i(n; 0)\leqslant (1+\varepsilon)U(s)\frac{u^i}{n^2B}$$

$$\text{and}\quad \exp\left\{\frac{u^i}{nB}\right\}\leqslant 1+\varepsilon \quad\text{for any}\quad 1\leqslant i\leqslant N,\ n\geqslant n_0\ .$$

Hence it follows from (4.5) that

$$(1-\varepsilon)U(s)\sum_{n=n_0}^{\infty}\frac{xu}{n^2B}\exp\left\{-\frac{xu}{nB}(1+\varepsilon)^2\right\}\leqslant \sum_{y\in S-\{0\}} G(x, y)s^y$$
$$\leqslant \sum_{n=1}^{n_0-1}\sum_{y\in S-\{0\}} P_n(x, y)s^y + (1+\varepsilon)^2 U(s)\sum_{n=n_0}^{\infty}\frac{xu}{n^2B}\exp\left\{-\frac{xu}{nB}(1-\varepsilon)\right\}.$$

But since

$$\lim_{|x|\to\infty}\sum_{n=n_0}^{\infty}\frac{xu}{n^2B}\exp\left\{-\frac{xu}{nB}c\right\} = \int_0^{\infty}\frac{1}{t^2}e^{-c/t}dt = \frac{1}{c},\ c>0\ ,$$

$$\lim_{|x|\to\infty}\sum_{y\in S-\{0\}} P_n(x, y)s^y = \lim_{|x|\to\infty}\{F(n; s)^x - F(n; 0)^x\} = 0\,,$$

we have

$$\frac{1-\varepsilon}{(1+\varepsilon)^2}U(s) \leqslant \liminf_{|x|\to\infty}\sum_{y\in S-\{0\}} G(x, y)s^y$$
$$\leqslant \limsup_{|x|\to\infty}\sum_{y\in S-\{0\}} G(x, y)s^y \leqslant \frac{(1+\varepsilon)^2}{1-\varepsilon}U(s)\,.$$

Because $\varepsilon>0$ is arbitrary, it follows that

$$\lim_{|x|\to\infty}\sum_{y\in S-\{0\}} G(x, y)s^y = U(s)\,,$$

so that we have (4.4) by the standard arguments.

To prove the uniqueness of invariant measures, we define the positive integer n_1 and the distributions $\{\tilde{P}^i(y)\}$ on S by

$$n_1 = \min\{n\geqslant 1;\, F(n; 0)\gg 0\}\,,$$

$$\tilde{F}^i(s) = \sum_{y\in S}\tilde{P}^i(y)s^y = F^i(n_1; s),\ 1\leqslant i\leqslant N,\ 0\leqslant s\leqslant 1\,. \tag{4.6}$$

We denote the MGWP corresponding to $\{\tilde{P}^i(y)\}$ by $\tilde{X}=(\tilde{Z}(n), \tilde{P}_x)$. It is easily seen that the MGWP $\tilde{X}$ also satisfies the hypotheses (H.1)$\sim$(H.4) as well as

$$\tilde{P}_n(x, 0)>0 \quad\text{for}\quad n\geqslant 1,\ x\in S\,, \tag{4.7}$$

where $\tilde{P}_n(x, y)=\tilde{P}_x(\tilde{Z}(n)=y)$ (especially for (H.4) cf. Athreya [1] or Sevastyanov [9] Chapter III, §3). Let $\tilde{U}(s)$ be the function associated with $\tilde{X}$ in Basic Lemma, and $\tilde{\mu}(y)$ the coefficient of s^y of $\tilde{U}(s)$. By Proposition 1, $\{\tilde{\mu}(y)\}$ is an invariant measure of $\tilde{X}$ and

$$\lim_{|x|\to\infty}\tilde{G}(x, y) = \tilde{\mu}(y),\ y\in S-\{0\}\,, \tag{4.8}$$

where $\tilde{G}(x, y)=\sum_{n=0}^{\infty}\tilde{P}_n(x, y)$. By the branching property (2.1), it also holds that

$$\tilde{P}_n(x, y) = \underbrace{\tilde{P}_n(e_1, \cdot)*\cdots*\tilde{P}_n(e_1, \cdot)}_{x^1}*\cdots*\underbrace{\tilde{P}_n(e_N, \cdot)*\cdots*\tilde{P}_n(e_N, \cdot)}_{x^N}(y) \tag{4.9}$$
$$n\geqslant 0,\ x, y\in S,\ x\neq 0\,.$$

As in [7], we set

$$T(x) = \{y\in S-\{0\};\, \tilde{G}(x, y)>0\},\ x\in S-\{0\}\,,$$
$$T = \bigcup_{x\in S-\{0\}} T(x)\,.$$

Lemma 8. $T=T(x)$ *for all* $x\in S-\{0\}$.

Proof. Since (4.9) implies $\tilde{P}_n(y, z)\geqslant\tilde{P}_n(y-x, 0)\tilde{P}_n(x, z)$ for all $x, y, z\in S$,

$n \geqslant 1$ with $y \geqslant x \neq 0$, it follows from (4.7) that

$$(4.10) \qquad T(x) \subset T(y),\ 0 \neq x \leqslant y\,.$$

Next we shall show

$$(4.11) \qquad T(e_i) = T(e_j),\ 1 \leqslant i, j \leqslant N\,.$$

By the hypothesis (H.1), there is a positive integer n_0 such that

$$(4.12) \qquad \tilde{P}_{e_i}(\tilde{Z}^j(n_0) \geqslant 1) > 0,\ 1 \leqslant i, j \leqslant N\,.$$

Hence there exists an $y_{ij} \geqslant e_j (y_{ij} \in S)$ such that $\tilde{P}_{n_0}(e_i, y_{ij}) > 0$. It follows from this and the Markov property that $T(e_i) \supset T(y_{ij})$. But $T(y_{ij}) \supset T(e_j)$ by (4.10), so that $T(e_i) \supset T(e_j)$, which proves (4.11).

Finally, take an $x \in S - \{0\}$ and fix it. Because $B > 0$ (cf. section 2), it is clear that $\tilde{P}_{e_{i_0}}(|\tilde{Z}(1)| \geqslant 2) > 0$ for some $1 \leqslant i_0 \leqslant N$. Hence, from the property of iteration of p.g.f.s', it holds that $\tilde{P}_{e_{i_0}}(|\tilde{Z}(n_2)| > |x|) > 0$ for some integer n_2; i.e. $P_{n_2}(e_{i_0}, z) > 0$ for some $|z| > |x|$. We partition the set

$$I \equiv \{\underbrace{e_1, \cdots, e_1}_{z^1}, \cdots, \underbrace{e_N, \cdots, e_N}_{z^N}\} = I_1 + \cdots + I_N$$

such that $I_i \cap I_j = \phi$, $i \neq j$, and $\# I_i \geqslant x^i$. (4.12) assures that for each $e \in I$, there exists $y_e \in S$ such that $y_e \geq e_j$ and $\tilde{P}_{n_0}(e, y_e) > 0$ if $e \in I_j$. Then, setting $y = \sum_{e \in I} y_e$, we have $y \geqslant x$ and $\tilde{P}_{n_0}(z, y) > 0$ by (4.9).
Hence by the Markov property it follows that $\tilde{P}_{n_0+n_2}(e_{i_0}, y) > 0$ and $T(e_{i_0}) \supset T(y)$. Since $T(y) \supset T(x)$ by (4.10), it holds that $T(e_{i_0}) \supset T(x)$.
Combining this fact with (4.10) and (4.11), we obtain the conclusion.

Corollary. *Every non-trivial invariant measure for the MGWP $\tilde{X}$ is positive on T and zero off T.*

The proof is not difficult and will be omitted (cf. [7] p. 521).

Applying the Martin entrance boundary theory to the MGWP $\tilde{X}$ restricted on $T \cup \{0\}$, we have the following proposition from (4.8) and Lemma 8 (cf. [7] pp. 521–522 or [6] pp. 366–368 for complete proofs).

Proposition 2. *Each invariant measure of the MGWP $\tilde{X}$ on $S - \{0\}$ is a constant multiple of the measure $\{\tilde{\mu}(x)\}$.*

Now the "uniqueness" part of the theorem is proved since every invariant measure of the MGWP X is also an invariant measure of the MGWP $\tilde{X}$.

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Added in proof.

After this paper was submitted, we received a preprint from Dr. Fred Hope, entitled "The critical Bienaymé-Galton-Watson process." (to appear in "Stochastic Processes and their Applications"), where the same problem is treated without moment assumptions.