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SURFACES WITH $K^2 = 8$, $p_g = 4$ AND CANONICAL INVOLUTION

INGRID C. BAUER and ROBERTO PIGNATELLI

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Abstract

In this paper we classify completely all regular minimal surfaces with $K^2 = 8$, $p_g = 4$ whose canonical map is composed with an involution. We obtain six unirational families. The last two are irreducible components of the moduli space of minimal surfaces of general type with $K^2 = 8$, $p_g = 4$. These families hit three different topological types.

Introduction

The aim of this paper is to classify regular minimal surfaces S with $K^2 = 8$ and $p_g = 4$ whose canonical map factors through an involution (short: *having a canonical involution*).

The study of surfaces with geometric genus $p_g = h^0(S, \Omega_S^2) = 4$ began with Enriques' celebrated book *Le Superficie Algebriche* ([7]), where he summarized his research of over fifty years.

By standard inequalities, minimal surfaces with geometric genus $p_g = 4$ satisfy $4 \leq K_S^2 \leq 45$. While for high values of K_S^2 it is already difficult to prove existence, the challenge for low values is to completely classify all surfaces with the given value of K_S^2. More ambitiously, one would like to understand the topology of the moduli space, i.e., the irreducible and connected components of the moduli space.

The lowest possible values $K_S^2 = 4, 5$ were already treated by Enriques and the corresponding moduli spaces were completely understood in the 70's. For $K_S^2 = 6$ the situation is far more complicated. In [12] Horikawa completely classifies all surfaces with $p_g = 4$ and $K^2 = 6$, obtaining a stratification of the moduli space in 11 strata. Moreover he shows that there are 4 irreducible components, and at most three connected components. In [2] it is shown that the number of connected components actually cannot be bigger than two. Let us point out that all these surfaces are homeomorphic.

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The complete classification of minimal surfaces with $K_S^2 = 7$ and $p_g = 4$ was achieved by the first author in [1]. Moreover, it is shown there that all these surfaces are homeomorphic, and that there are three irreducible components and at most two connected components.

The first open case $K_S^2 = 8$ is more complicated already for topological reasons. By work of Ciliberto, Francia, Mendes Lopes, Oliverio and Pardini (cf. [5], [6], [14], [16]) there are at least three different topological types, therefore at least three connected components of the moduli space.

The analysis of the cases $K^2 \leq 7$ is based on a detailed study of the behaviour of the canonical map $\varphi_{K_S}\colon S \dashrightarrow \mathbb{P}^3$, as already suggested by Enriques. For $K^2 = 8$ this approach produces too many strata and the question how they glue together becomes intractable. Therefore it is necessary to find a less fine stratification of the moduli space.

We summarize our main result in the following

Theorem. *Let S be a minimal regular surface with $p_g = 4$ and $K^2 = 8$ whose canonical map factors through an involution i on S. Then*:

1) *the number τ of isolated fixed points of i is* 0, 2, 4 *or* 20;

2) *if $\tau = 20$, S is a canonical bidouble cover and the two additional involutions have* $\tau = 0$;

3) *the surface S belongs to exactly one of six unirational families. In the table below we give, for each family, the dimension and the reference where this family is described*;

Family	dim	reference
$\mathcal{M}_0^{(\mathrm{div})}$	29	3.3
$\mathcal{M}_0$	28	3.5
$\mathcal{M}_2^{(0)}$	32	4.2
$\mathcal{M}_2^{(1)}$	33	4.3
$\mathcal{M}_4^{(\mathrm{DV})}$	38	5.1
$\mathcal{M}_4^{(2)}$	34	5.9

4) *exactly two of these families, namely $\mathcal{M}_4^{(\mathrm{DV})}$ and $\mathcal{M}_4^{(2)}$, are irreducible components of the moduli space*;

5) *the surfaces in $\mathcal{M}_0^{(\mathrm{div})}$ and the surfaces in $\mathcal{M}_4^{(2)}$ are not homeomorphic and not homeomorphic to any of the others.*

Actually, we prove more:

REMARK. The index $\tau \in \{0, 2, 4\}$ in the above families means that there is a canonical involution on S having τ isolated fixed points. In fact, the only surfaces having more than one canonical involution are canonical bidouble covers having an involution with $\tau = 20$ and two involutions with $\tau = 0$: they give a subfamily of $\mathcal{M}_0^{(\mathrm{div})}$ (when the canonical image is a quadric cone) and a subfamily of $\mathcal{M}_0$ (when the canonical image is smooth).

The surfaces in $\mathcal{M}_0^{(\mathrm{div})}$ are the only ones in the above list with 2-divisible canonical system. The surfaces in $\mathcal{M}_4^{(\mathrm{DV})}$ (so called because they are *Du Val double planes*) are the only ones in this list with nontrivial torsion subgroup of the Picard group. The surfaces in $\mathcal{M}_4^{(2)}$ are all minimal surfaces of general type with $K^2 = 8$ and $p_g = 4$ having a genus 2 pencil.

The paper is organized as follows.

In Section 1 we recall some general facts about involutions and show that the number of isolated fixed points is 0, 2, 4 or 20.

Sections 2, 3, 4 and 5 are devoted to the classification and to the detailed description of all surfaces having a canonical involution with respectively $\tau = 20$, $\tau = 0$, $\tau = 2$ and $\tau = 4$. For $\tau = 0, 2$ we use the MMP for pairs (as e.g. in [17]). The surfaces (minimal, regular with $p_g = 4$ and $K^2 = 8$) having a canonical involution with $\tau = 4$ are exactly the surfaces (with the same invariants) whose bicanonical map is not birational. Those without genus 2 pencil are classified in [6]. We classify those with a genus 2 pencil using the techniques developped in [4].

In Section 5 we calculate the dimensions of each family.

1. Canonical involutions

Let S be a regular minimal surface of general type and let i be an involution on S.

Since S is minimal i is biregular, and its fixed locus consists of τ isolated points and a nonsingular (not necessarily connected) curve R.

The quotient $T := S/i$ has τ nodes. Resolving them we get a cartesian diagram of morphisms

$$\begin{tikzcd} \hat{S} \arrow[r, "\epsilon"] \arrow[d, "\hat{\pi}"'] & S \arrow[d, "\pi"] \\ \hat{T} \arrow[r] & T \end{tikzcd} \tag{1}$$

with vertical maps finite of degree 2 and horizontal maps birational. We denote by Δ the branch curve $\pi(R)$ and by $E_1, \ldots, E_\tau$ the exceptional curves of ϵ.

The action of i on $\hat{S}$ yields a decomposition $\hat{\pi}_*\mathcal{O}_{\hat{S}} = \mathcal{O}_{\hat{T}} \oplus \mathcal{O}_{\hat{T}}(-\hat{\delta})$, with $2\hat{\delta} \equiv \Delta + \sum_1^\tau \hat{\pi}(E_i)$. Recall that $K_{\hat{S}} \equiv \hat{\pi}^*(K_{\hat{T}} + \hat{\delta})$.

Lemma 1.1.

$$\chi(\mathcal{O}_{\hat{T}}) = \frac{1}{2}\chi(\mathcal{O}_S) - \frac{1}{8}(K_S R - \tau), \tag{2}$$

$$\chi(\mathcal{O}_{\hat{T}}(-\hat{\delta})) = \frac{1}{2}\chi(\mathcal{O}_S) + \frac{1}{8}(K_S R - \tau). \tag{3}$$

Proof. By Riemann-Roch

$$\begin{aligned}\chi(\mathcal{O}_{\hat{T}}) - \chi(\mathcal{O}_{\hat{T}}(-\hat{\delta})) &= -\frac{1}{2}\hat{\delta}(K_{\hat{T}} + \hat{\delta}) \\ &= -\frac{1}{4}\left(R + \sum_1^\tau E_i\right)\left(K_S + \sum_1^\tau E_i\right) = -\frac{1}{4}(K_S R - \tau).\end{aligned}$$

The result follows then from $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{\hat{T}}) + \chi(\mathcal{O}_{\hat{T}}(-\hat{\delta}))$. □

We will also use the following (cf. e.g. [15])

$$0 \le \tau = K_S^2 + 6\chi(\mathcal{O}_{\hat{T}}) - 2\chi(\mathcal{O}_S) - 2h^0(\mathcal{O}_{\hat{T}}(2K_{\hat{T}} + \hat{\delta})). \tag{4}$$

REMARK 1.2. If the canonical map factors through the involution i, then either $p_g(\hat{T}) = p_g(S)$ (equivalently, all 2-forms are invariant) or $p_g(\hat{T}) = 0$ (i.e., all 2-forms are anti invariant).

Lemma 1.3. *Assume that i is a canonical involution and let p be an isolated fixed point of i.*
- *If $p_g(\hat{T}) = 0$, then p is a base point of $|K_S|$.*
- *If $p_g(\hat{T}) = p_g(S)$, then R is contained in the fixed part of $|K_S|$.*

Proof. There are local coordinates around p such that $i(x, y) = (-x, -y)$. In particular $i^*(x^a y^b\, dx \wedge dy) = (-1)^{a+b} x^a y^b\, dx \wedge dy$.

If $p_g(\hat{T}) = 0$, every global 2-form ω on S is anti invariant. Writing $\omega = \sum \omega_{a,b} x^a y^b\, dx \wedge dy$ it follows $\omega_{a,b} = 0$ for $a + b$ even. In particular ω vanishes in p.

The other case is similar, since there are local coordinates around any point of R such that $i(x, y) = (-x, y)$ and $R = \{x = 0\}$. □

REMARK 1.4. If $p_g(\hat{T}) = 0$, by Hurwitz' formula and Riemann-Roch (as in Lemma 1.1)

$$K_{\hat{T}}\hat{\delta} = -2 - K_{\hat{T}}^2 - \frac{1}{2}\tau,$$
$$\hat{\delta}^2 = 8 + K_{\hat{T}}^2 + \frac{1}{2}\tau.$$

From now on S will be a minimal surface of general type with $K^2 = 8,\ p_g = 4$, and $q = 0$.

REMARK 1.5. The canonical map of S is not composed with a pencil.

More generally, by results of Zucconi and Konno (cf. [21] and [13]) the canonical map of regular surfaces with $p_g \geq 3$ and $K_S^2 < 4p_g - 6$ is not composed with a pencil.

Proposition 1.6. *If the canonical map of S factors through an involution i, then either*

1) $p_g(\hat{T}) = 0,\ \tau \in \{0, 2, 4\}$, *or*
2) $p_g(\hat{T}) = 4,\ R = \emptyset,\ \tau = 20$.

Proof. If $p_g(\hat{T}) = 4$, the canonical map cannot have degree 2 (since then $\hat{T}$ is birational to the canonical image which has degree at most 4), therefore it has degree 4 and K_S is base point free, so, by Lemma 1.3, $R = \emptyset$. $\tau = 20$ follows from (2).

Otherwise $p_g(\hat{T}) = 0$. By (4) $\tau = 4 - 2h^0(\mathcal{O}_{\hat{S}/i}(2K_{\hat{S}/i} + \hat{\delta}))$. □

2. Canonical involutions with $p_g(\hat{T}) = 4$

In this section S is a minimal surface of general type with $K_S^2 = 8,\ p_g(S) = 4$ and a canonical involution such that $p_g(\hat{T}) = 4$.

Consider a Hirzebruch surface $\mathbb{F}_k,\ k \in \{0, 2\}$. Then, if $k = 0$, we denote by $|\Gamma_1|$, $|\Gamma_2|$ the two rulings of $\mathbb{F}_0$. Otherwise, we denote by $|\Gamma_2|$ the ruling of $\mathbb{F}_2$ and by $|\Gamma_1| := \Gamma_\infty + |\Gamma_2|$, Γ_∞ being the (-2)-curve.

We will show the following

Theorem 2.1. *S is a bidouble cover (i.e., a Galois cover with group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$) of $\mathbb{F}_k,\ k \in \{0, 2\}$, which is a fiber product of two double covers branched in two general divisors $B_1 \in |4\Gamma_1 + 2\Gamma_2|,\ B_2 \in |2\Gamma_1 + 4\Gamma_2|$.*

First we need the following:

Lemma 2.2. *Let C be a curve of genus 2 and let $f: D \to C$ be an étale double cover with associated involution ξ. Then the hyperelliptic involution of C lifts to an involution on D which commutes with ξ.*

Proof. The hyperelliptic involution σ' acts on $\mathrm{Pic}^0(C)$ as $L \mapsto L^*$, and therefore it fixes any 2-torsion bundle. Since (connected) étale double covers are classified by non trivial 2-torsion bundles, considering the fiber product

$$\begin{tikzcd} D' \arrow[r, "\sigma"] \arrow[d, "\sigma'^* f"'] & D \arrow[d, "f"] \\ C \arrow[r, "\sigma'"] & C \end{tikzcd}$$

it follows that $D' \cong D$ and σ is a lift of σ' to D.

Since σ' is an involution, σ^2 is either the identity or ξ, which has no fixed points. But in this last case (by Hurwitz) D/σ would have genus $3/2$, a contradiction. □

By Proposition 1.6, if $p_g(\hat{T}) = 4$, then $R = \emptyset$, so $K_T^2 = 8/2 = 4$. By [10] T is a canonical double cover of an irreducible quadric in $\mathbb{P}^3$ branched in the complete intersection with a general sextic. Moreover the canonical map of S is the composition of π with the canonical map of T.

Lemma 2.3. *S is a canonical Galois cover of a quadric in $\mathbb{P}^3$ with Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

Proof. The pull-back of a ruling of the quadric is a genus 2 pencil on T and (since $R = \emptyset$) a genus 3 pencil on S whose general element is an étale double cover of the corresponding genus 2 curve. Then by Lemma 2.2 we can lift the canonical involution of T to an involution on S commuting with i, and the canonical map is the quotient by these two commuting involutions. □

S has two more canonical involutions, and we denote them by σ and σi.

Lemma 2.4. *σ and σi do not have isolated fixed points.*

Proof. Recall that the action of i on $H^0(K_S)$ is the identity. Since $p_g(S/_{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}) = 0$, the action of σ on $H^0(K_S)$ is multiplication by -1, and $p_g(S/\sigma) = p_g(S/\sigma i) = 0$. Since $\deg(\varphi_{|K_S|}) = 4$, $|K_S|$ is base point free, and the claim follows from Lemma 1.3. □

Proof of Theorem 2.1. We have a commutative diagram of finite morphisms of degree 2:

$$\begin{tikzcd} & S \arrow[dl, "\pi_\sigma"'] \arrow[d, "\pi"] \arrow[dr, "\pi_{\sigma i}"] & \\ S/\sigma \arrow[dr, "q_{\sigma i}"'] & T \arrow[d, "q"] & S/\sigma i \arrow[dl, "q_\sigma"] \\ & S_{/\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}} & \end{tikzcd}$$

The ramification locus of π_σ is a smooth divisor R_σ, the ramification locus of $\pi_{\sigma i}$ is a smooth divisor $R_{\sigma i}$, the ramification locus of π is a set of 20 points $\mathfrak{P}$. R_σ and $R_{\sigma i}$ intersect transversally and obviously $\mathfrak{P} \supset R_\sigma \cap R_{\sigma i}$. On the other hand, since $\varphi_{|K_T|}$ factors through $\mathbb{F}_k$ ([10], Lemma 1.5), the same holds for $\varphi_{|K_S|}$ and therefore $S_{/\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$ is a blow-up of $\mathbb{F}_k$. So $S_{/\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}$ is smooth, and also the other inclusion must hold, i.e., $\mathfrak{P} = R_\sigma \cap R_{\sigma i}$.

We consider the branch divisors $B_\sigma = q \circ \pi(R_\sigma)$ of q_σ, and $B_{\sigma i} := q \circ \pi(R_{\sigma i})$ of $q_{\sigma i}$. It follows that $B_\sigma B_{\sigma i} = 20$. We denote by D_σ, $D_{\sigma i}$ the respective images on $\mathbb{F}_k$. Since B_σ, $B_{\sigma i}$ are 2-divisible, we can write $B_\sigma = D_\sigma + 2\sum_j a_j E_j$, $B_{\sigma i} = D_{\sigma i} + 2\sum_j \alpha_j E_j$ where E_j are the exceptional divisors of the first kind of the contraction to $\mathbb{F}_k$.

$\pi^* q^* E_i$ is contracted by $\varphi_{|K_S|}$ and $2K_S = \pi^* q^*(B_\sigma + B_{\sigma i} + 2K_{S_{/\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}})$. Then $a_j + \alpha_j = -1$ for all j, so $a_j\alpha_j \le 0$ is even and it follows that $D_\sigma D_{\sigma i} = 20 - 8k$ for some nonnegative integer k.

D_σ, $D_{\sigma i}$ are 2-divisible, effective and $D_\sigma + D_{\sigma i}$ is the branch curve of q, so belongs to $|6\Gamma_1 + 6\Gamma_2|$. Therefore either $D_\sigma \in |4\Gamma_1 + 2\Gamma_2|$, $D_{\sigma i} \in |2\Gamma_1 + 4\Gamma_2|$, or $D_\sigma \in |2\Gamma_j|$, $j \in \{1, 2\}$.

A smooth bidouble cover of type (in the language of [3]) $((2, 0), (4, 6), (0, 0))$ has $K^2 = 8$ and $p_g = 6$. By the formulas on page 109 of [3] there is no configuration of singularities that changes p_g without changing K^2. $\square$

Bidouble covers of a smooth quadric were already studied by Catanese [3], and later Gallego and Purnaprajna [8] and [9] classified canonical Galois covers of degree 4 of a surface of minimal degree. All these surfaces can be found in those papers. Note however that these surfaces, because of the other two canonical involutions they have, are also special cases of the surfaces studied in the next section.

3. Canonical involutions with $p_g(\hat{T}) = 0$, $\tau = 0$

In this case T is smooth. By Remark 1.4

$$K_T\delta = -2 - K_T^2, \qquad \delta^2 = 8 + K_T^2. \tag{5}$$

We inductively contract all (-1)-curves E on T contained in the image of the fundamental cycles of S, and we denote by $\alpha\colon T \to P$ the composition of all these contractions.

REMARK 3.1. We observe that every (-1)-curve E contained in the image of a fundamental cycle of S fulfills $\Delta E = 2$. It follows that equations (5) hold also for K_P, δ_P.

Let $\lambda \in \mathbb{Q} \cup \{\infty\}$ be the maximal number such that $\lambda K_P + \delta_P$ is nef. Since the pull back of $K_P + \delta_P$ to S is K_S, $\lambda \ge 1$. In fact, $\lambda > 1$, since $\lambda = 1$ implies that there is an extremal ray l such that $(K_P+\delta_P)l=0$. By a.i.t., $l^2<0$, whence l is a (-1)-curve whose pull-back to S is contained in a fundamental cycle. But these have already been contracted.

Proposition 3.2. *There are the following two possibilities*:

- $K_P^2=1$ *and* $3K_P+\delta_P$ *is trivial*;
- $K_P^2=0$ *and* $|2K_P+\delta_P|$ *is a genus* 0 *pencil without base points*.

Proof. By the algebraic index theorem: $K_P^2\delta_P^2\le(K_P\delta_P)^2$. Equations (5) imply $K_P^2\le 1$.

If $K_P^2=1$, equality holds in the a.i.t. and $3K_P+\delta_P$ is numerically trivial. By equation (4) $2K_P+\delta_P$ is effective, hence Riemann-Roch implies $h^0(3K_P+\delta_P)\ge 1$. Therefore $3K_P+\delta_P$ is trivial.

Otherwise $K_P^2\le 0$. Let l be an extremal ray with $(\lambda K_P+\delta_P)l=0$. Since P is neither $\mathbb{P}^2$ nor a $\mathbb{P}^1$-bundle, l has to be a (-1)-curve, whence $\lambda=\delta_P l\in\mathbb{Z}$. In particular, $2K_P+\delta_P$ is nef.

Since $2K_P+\delta_P$ is effective, whence $0\le(2K_P+\delta_P)^2=K_P^2\le 0$. Therefore $2K_P+\delta_P$ is a nef divisor with selfintersection 0 and negative canonical degree. This implies that $|(-2/(K_P(2K_P+\delta_P)))(2K_P+\delta_P)|$ is a base point free genus 0 pencil. Since in our case $K_P(2K_P+\delta_P)=K_P\delta_P=-2$ we are done. □

We get two families, according to the value of K_P^2.

Theorem 3.3. *If* $K_P^2=1$, *then* K_S *is* 2*-divisible and* S *is a double cover of a Del Pezzo surface of degree* 1 *branched in a general divisor in* $|-6K|$.

Proof. By Proposition 3.2 $3K_P+\delta_P$ is trivial, so $K_P+\delta_P=-2K_p$ is 2-divisible and the same holds for its pull-back $K_S=\pi^*\alpha^*(K_P+\delta_P)$. Note that since $K_P+\delta_P$ is ample, P is a Del Pezzo surface. □

REMARK 3.4. Oliverio proves in [16] that if the canonical system of a regular minimal surface with $K_S^2=8$ and $p_g=4$ is 2-divisible, either K_S has base points and the canonical map has degree 3 (so it is not our case), or the semicanonical ring $R(S,(1/2)K_S)$ embeds the canonical model of S as a complete intersection of two sextics in $\mathbb{P}(1,1,2,3,3)$.

Theorem 3.5. *If* $K_P^2=0$, *then there is a natural number* $0\le r\le 3$, *such that* S *is the minimal resolution of a double cover of a Hirzebruch-Segre surface* $\mathbb{F}_r$ *branched in a curve in* $|8\Gamma_\infty+(10+4r)f|$, *where* Γ_∞ *denotes the section at infinity and* f *a fibre, having* 8 *singular points* (*possibly infinitely near*) *of multiplicity* 4 *as only essential singularities*.

Proof. By Proposition 3.2, $|2K_P + \delta_P|$ is a genus 0 pencil without base points.

Contracting $8 - K_P^2 = 8$ (-1)-curves (contained in fibres) we get a birational morphism $\eta\colon P \to \mathbb{F}_r$. Note that there might be different choices for the 8 contractions yielding different r's.

The strict transform of the $(-r)$-section Γ_∞ of $\mathbb{F}_r$ is an irreducible rational curve B_∞ on P with $B_\infty(2K_P + \delta_P) = 1$. Let E be a (-1)-curve contained in a fibre of $|2K_P + \delta_P|$: then $B_\infty E$ is 0 or 1. If $B_\infty E = 1$, $2K_P + \delta_P - E$ is again an exceptional divisor of the first kind, so it contains an other (-1)-curve E' and $B_\infty E' = 0$. Therefore we can choose η such that for all contracted curves holds $B_\infty E = 0$.

Now, B_∞ is a smooth rational curve with $B_\infty^2 = -r$, so $K_P B_\infty = r - 2$. Therefore $0 \le (K_P + \delta_P)B_\infty = (2K_P + \delta_P)B_\infty - K_P B_\infty = 3 - r$ whence $r \le 3$.

We write $\delta_P = \eta^*(a\Gamma_\infty + bf) - \sum_1^8 c_i E_i$.

First of all, for all i, $c_i = \delta_P E_i = (2K_P + \delta_P)E_i - 2K_P E_i = 2$. Moreover, by formulae (5) $a = \delta_P(\eta^* f) = \delta_P(2K_P + \delta_P) = -4 + 8 = 4$. Finally, by $8 = \delta_P^2 = (a\Gamma_\infty + bf)^2 - 8c_i^2 = -16r + 8b - 32$ we get $40 = 8b - 16r$, whence $b = 5 + 2r$. $\square$

REMARK 3.6. At first sight the surfaces in the previous theorem fall into four distinct families, according to the different values of r. But, as follows clearly from the proof, the surface $\mathbb{F}_r$ is obtained from P by choosing 8 (-1)-curves to contract, and different choices yield different r's.

Let $P_1, \dots, P_8 \in \mathbb{F}_r$ be the (not necessarily pairwise distinct) images of the chosen exceptional curves on P. Since $2\delta_P - K_P$ is ample, $h^1(2\delta_P) = 0$. In particular, the dimension of $|\Delta_P|$ is constant.

If $r \neq 0$ and r of the points P_i do not belong to the negative section Γ_∞ of $\mathbb{F}_r$, we can modify the choice of the curves we contract in order to obtain $r = 0$. It follows that the family with $r = 0$ is open and dense in the subscheme of the moduli space of surfaces of general type given by the surfaces described in Theorem 3.5 which, in particular, is unirational.

4. Canonical involutions with $p_g(\hat{T}) = 0,\ \tau = 2$

We recall diagram (1):

$$\begin{tikzcd} \hat{S} \arrow[r, "\epsilon"] \arrow[d, "\hat{\pi}"'] & S \arrow[d, "\pi"] \\ \hat{T} \arrow[r] & T \end{tikzcd}$$

In this case ϵ is the blow up of S in two distinct points p_1 and p_2. We denote by A_i the (-2)-curve $\hat{\pi}(\epsilon^{-1}(p_i))$. Note that A_i is a component of the branch curve of $\hat{\pi}$ with $\hat{\delta} A_i = -1$.

We define the $\mathbb{Q}$-divisor $\overline{\delta} := \hat{\delta} - (1/2)(A_1 + A_2)$. We have

$$K_{\hat{T}}\overline{\delta} = -3 - K_{\hat{T}}^2, \qquad (6)$$
$$\overline{\delta}^2 = 10 + K_{\hat{T}}^2.$$

Observe that, by a.i.t., $K_{\hat{T}}^2(10+K_{\hat{T}}^2) = K_{\hat{T}}^2\overline{\delta}^2 \le (K_{\hat{T}}\overline{\delta})^2 = (K_{\hat{T}}^2+3)^2$, therefore $K_{\hat{T}}^2 \le 2$.

Let λ be the maximal (rational) number such that $\lambda K_{\hat{T}} + \overline{\delta}$ is nef. Note that $\hat{\pi}^*(K_{\hat{T}} + \overline{\delta}) = \epsilon^* K_S$, whence $K_{\hat{T}} + \overline{\delta}$ is nef, so $\lambda \ge 1$.

Assume that $\lambda = 1$ and let l be an extremal ray with $(K_{\hat{T}} + \overline{\delta})l = 0$. Since $K_{\hat{T}}^2 \le 2$, we know that $\hat{T}$ is neither $\mathbb{P}^2$ nor a $\mathbb{P}^1$-bundle. Therefore l is a (-1)-curve, and we contract it. Note that after this contraction the equations (6) remain valid (if by slight abuse of notation we denote the pushforward of $\overline{\delta}$ again by $\overline{\delta}$), since K^2, $\overline{\delta}^2$ increase by 1, while $K\overline{\delta}$ decreases by 1. In particular, by the index theorem we get $K^2 \le 2$.

Therefore, we can inductively apply the above argument and get a sequence of contractions $c_1 : \hat{T} \to P$, such that (6) holds on P (so $K_P^2 \le 2$) and there are no extremal rays in $(K_P + \overline{\delta})^\perp$.

Now, let λ be the maximal rational number such that $\lambda K_P + \overline{\delta}$ is nef. Then $\lambda > 1$. Since $K_P^2 \le 2$, an extremal ray l has to be a (-1)-curve, whence $\lambda = \overline{\delta} l \in (1/2)\mathbb{Z}$ (since $2\overline{\delta}$ is integral), i.e., $\lambda \geq 3/2$.

In particular, $(3/2)K_P + \overline{\delta}$ is nef and, since by (4) $2K_P + \overline{\delta}$ is effective, we have $0 \le ((3/2)K_P + \overline{\delta})(2K_P + \overline{\delta}) = (1/2)(K_P^2 - 1)$. Therefore, $K_P^2 \in \{1, 2\}$.

Proposition 4.1. *One of the following occurs*:

- $K_P^2 = 2$ *and* $|4K_P + 2\overline{\delta}|$ *is a genus* 0 *pencil without base points*;
- $K_P^2 = 1$, *there is a birational morphism* $c : P \to P_1$ *onto a Del Pezzo surface of degree* 5, *contracting* (-1)-*curves* l *with* $(K + \overline{\delta})l = 1/2$, *and* $2K_{P_1} + \overline{\delta} \equiv 0$.

Proof. We know that $\lambda \ge 3/2$. Assume that $\lambda = 3/2$ and let l be an extremal ray with $((3/2)K + \overline{\delta})l = 0$. By a.i.t., since $((3/2)K + \overline{\delta})^2 = 1 + K^2/4 > 0$, $l^2 < 0$. Contracting l we add 1 to K^2, 9/4 to $\overline{\delta}^2$, and we subtract 3/2 from $K\overline{\delta}$, in particular, we do not change $((3/2)K + \overline{\delta})^2$. Therefore we can repeat the argument and inductively contract all, say s, (-1)-curves l with $((3/2)K + \overline{\delta})l = 0$. We get a birational morphism $c : P \to P_1$, such that on P_1, $\lambda > 3/2$.

Since $K_{P_1} + \overline{\delta}$ is nef and $2K_{P_1} + \overline{\delta}$ is effective, we have $0 \le (K_{P_1} + \overline{\delta})(2K_{P_1} + \overline{\delta}) = 1 - s/4$, i.e., $s \le 4$. In particular, $K_{P_1}^2 \le 2 + s \le 6$, so, as above, an extremal ray l has to be a (-1)-curve, whence $\lambda = \overline{\delta} l \in (1/2)\mathbb{Z}$, so $\lambda \ge 2$. Therefore $2K_{P_1} + \overline{\delta}$ is a nef and effective divisor with selfintersection $(2K_{P_1} + \overline{\delta})^2 = (2K_P + \overline{\delta})^2 + s/4 = K_P^2 - 2 + s/4$.

If $K_P^2 = 1$, it follows that $s = 4$ and $(2K_{P_1} + \overline{\delta})^2 = (K_{P_1} + \overline{\delta})(2K_{P_1} + \overline{\delta}) = 0$. By a.i.t. $2K_{P_1} + \overline{\delta}$ is trivial.

Else $K_P^2 = 2$, and the inequality $K_{P_1}^2\overline{\delta}^2 \le (K_{P_1}\overline{\delta})^2$ gives $(2+s)(12+(9/4)s) \le (5 + (3/2)s)^2 \Leftrightarrow s \le 2/3$: we have $s = 0$. In this case, $P = P_1$ and $2K_P + \delta_P$ is nef with selfintersection 0 and canonical degree -1, so $|2(2K_P + \overline{\delta})|$ is a base point free genus 0 pencil. $\square$

Therefore we get two families, according to the value of K_P^2.

Theorem 4.2. *If $K_P^2 = 1$, then S is the minimal resolution of a double cover of a Del Pezzo surface of degree 5 branched in a divisor in $|-4K|$ having two $(3,3)$-points.*

Proof. Let $l \subset \hat{T}$ be a (-1)-curve with $(K_{\hat{T}} + \overline{\delta})l = 0$. Since the intersection form restricted to $(K_{\hat{T}} + \overline{\delta})^{\perp}$ is negative definite and since $l, A_1, A_2 \in (K_{\hat{T}} + \overline{\delta})^{\perp}$, $l(A_1 + A_2) \le 1$. Because $\overline{\delta}l$, $\hat{\delta}l \in \mathbb{Z}$, we have $l(A_1 + A_2)$ even, thus $lA_1 = lA_2 = 0$.

This shows that the images of A_1 and A_2 are still (-2)-curves in P. We show that they will be contracted by c. Recall that c is (any) sequence of 4 contractions of extremal rays in $((3/2)K_{\hat{T}} + \overline{\delta})^{\perp}$.

The first extremal ray is a (-1)-curve l with $((3/2)K_{\hat{T}} + \overline{\delta})l = 0$. By the same argument as above, $l(A_1 + A_2) \le 1$. $\overline{\delta}l \notin \mathbb{Z}$, $\hat{\delta}l \in \mathbb{Z}$, therefore w.l.o.g. $lA_1 = 1$ and $lA_2 = 0$.

After contracting l, A_1 becomes a (-1)-curve contained in $((3/2)K_{\hat{T}} + \overline{\delta})^{\perp}$, and we can choose A_1 as second extremal ray.

By the same argument the third extremal ray l' has $l'A_2 = 1$ and we can choose A_2 as last extremal ray.

Now, P_1 is a Del Pezzo of degree 5, and P is the blow up of P_1 in four points. We call the exceptional divisors $E_1, \dots, E_4$. By the above arguments, we can assume that $A_1 = E_3 - E_4$, $A_2 = E_1 - E_2$, and on P we have: $\overline{\delta} = c^*\overline{\delta} - \sum_{i=1}^4 (\overline{\delta}E_i)E_i = c^*(-2K_{P_1}) - \sum_{i=1}^4 (3/2)E_i$. The direct image of $\hat{\delta}$ on P is therefore $c^*(-2K_{P_1}) - \sum_{i=1}^4 2E_i + E_1 + E_3$. $\square$

Theorem 4.3. *If $K_P^2 = 2$, hen there is a natural number $0 \le r \le 2$, such that S is the minimal resolution of a double cover of a Hirzebruch-Segre surface $\mathbb{F}_r$ branched in a fibre $\Gamma \in |f|$ and a curve in $|8\Gamma_\infty + (9 + 4r)f|$, where Γ_∞ denotes the section at infinity and f a fibre having 6 singular points $x_1, \dots, x_6$ of multiplicity 4 as only essential singularities, with $x_5 \in \Gamma$ and x_6 infinitely near to x_5 and belonging to the strict transform of Γ.*

Proof. By Proposition 4.1, $|4K_P + 2\overline{\delta}|$ is a genus 0 pencil without base points. As in the previous proof we note that A_1 and A_2 are still (-2)-curves on P, which are contained in fibres of the pencil.

Contracting $8 - K_P^2 = 6$ (-1)-curves (contained in fibres) we get a birational morphism $\eta\colon P \to \mathbb{F}_r$. Repeating the same argument as in the proof of 3.5, we obtain that $r \le 5/2$.

Let l be one of these 6 (-1)-curves; then being l, A_1 and A_2 all contained in fibres, by Zariski's lemma $lA_i \le 1$. But it cannot be $lA_1 = 0$ for all l, since $\mathbb{F}_r$, $r \le 2$

does not contain two disjoint (-2)-curves. Therefore one of these extremal rays has $lA_1 = 1$, say E_6. $\overline{\delta}l$, $\hat{\delta}l \in \mathbb{Z}$, therefore $E_6(A_1+A_2)$ even, thus $E_6A_1 = E_6A_2 = 1$. So, after this contraction A_1 and A_2 become (-1)-curves contained in a fibre with $A_1A_2 = 1$. One will be contracted and the other will map isomorphically onto a fibre of $\mathbb{F}_r$.

We have $\eta^* f = 4K_P + 2\overline{\delta}$ and we write $\overline{\delta} = \eta^*(a\Gamma_\infty + bf) - \sum_1^6 c_i E_i$.

Then, for all i, $c_i = \overline{\delta}E_i = 2$. Moreover, by formulae (6) $a = \overline{\delta}(\eta^* f) = \overline{\delta}(4K_P + 2\overline{\delta}) = 4$. Finally, by $12 = \overline{\delta}^2 = -16r + 8b - 24$ we get $b = 9/2 + 2r$, whence $\overline{\delta} = \eta^*(4\Gamma_\infty + (9/2 + 2r)f) - 2\sum_1^6 E_i$.

Therefore the direct image of $\hat{\delta}$ on P is $\eta^*(4\Gamma_\infty + (5+2r)f) - \sum_1^5 2E_i - 3E_6$. $\square$

REMARK 4.4. 1) The same argument as in Remark 3.6 shows that the surfaces with $r = 0$ form an open and dense set in the subscheme of the moduli space of surfaces of general type given by the surfaces described in Theorem 4.3 which, in particular, is unirational.
2) We observe that the surfaces classified in this section are exactly those whose canonical map is a double cover of a cubic surface in $\mathbb{P}^3$.

5. Canonical involutions with $p_g(\hat{T}) = 0$, $\tau = 4$

This case can be treated with the same techniques as in the previous two sections, but the calculations become more demanding. We choose a different approach.

By equation (4), $h^0(\mathcal{O}_{\hat{S}/i}(2K_{\hat{S}/i} + \hat{\delta})) = 0$, in particular, the bicanonical map factors through the involution i. In [6] the authors classify all surfaces with $p_g \geq 4$, nonbirational bicanonical map having no genus 2 pencil. In particular, they obtain

Theorem 5.1 ([6], Theorem 3.1 and Remark 3.10). *If $\tau = 4$ and S has no genus 2 pencil, then S belongs to one of the following two families*
i) *S is birational to a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ with branch curve $\tilde{\Delta} = L_1 + L_1' + L_2 + L_2' + D$ where L_i, L_i' are distinct lines in $|\Gamma_i|$ and $D \in |8\Gamma_1 + 8\Gamma_2|$ has quadruple points at the intersection of the 4 lines as only essential singularities.*
ii) *S is birational to a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ with branch curve $\tilde{\Delta} = L_2 + L_2' + D$ where L_2, L_2' are distinct lines in $|\Gamma_2|$ and $D \in |8\Gamma_1 + 8\Gamma_2|$ has (4, 4) points at the intersection of the 2 lines with a line L_1 in $|\Gamma_1|$, having as tangent line L_2 resp. L_2', as only essential singularities.*
The torsion subgroup of Pic(S) *is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The second case is a specialization of the first one.*

REMARK 5.2. It is wellknown that, if a surface has a genus 2 pencil, the involution on each fibre induces an involution on S such that both the canonical and the bicanonical map of S factor through it. In particular, the induced involution is canonical and, if the surfaces is regular with $K^2 = 8$ and $p_g = 4$, it has $\tau = 4$.

It follows that none of the preceedingly studied surfaces has a genus 2 pencil.

In the following S is assumed to be a surface of general type with $K^2 = 8,\ p_g = 4$ and $q = 0$ having a genus 2 pencil $f: S \to \mathbb{P}^1$.

REMARK 5.3. Since $\tau = 4$, the canonical system has base points (cf. Lemma 1.3) and therefore the canonical map has degree two onto a cubic or a quadric.

Let $\omega_{S|\mathbb{P}^1} := \omega_S \otimes f^*\omega_{\mathbb{P}^1}^{-1}$ be the relative canonical sheaf. The sheaves $f_*\omega^n_{S|\mathbb{P}^1}$ are vector bundles and there are the relative n-canonical maps $\varphi_n : S \dashrightarrow \mathbb{P}(f_*\omega^n_{S|\mathbb{P}^1}) :=$ **Proj**(Sym $f_*\omega^n_{S|\mathbb{P}^1}$), whose restriction to each fibre is its n-canonical map. Note that for $g = 2$ the target of the relative n-canonical map is a $\mathbb{P}^1$-bundle for $n = 1$ and a $\mathbb{P}^2$-bundle for $n = 2$.

REMARK 5.4. Let $f: S \to \mathbb{P}^1$ be a genus 2 fibration with fibres $f^{-1}(t) =: F_t \in |F|$ and assume

$$\forall t \in \mathbb{P}^1 \quad \text{the restriction map} \quad H^0(\omega_S) \to H^0(\omega_{F_t}) \quad \text{is surjective.} \tag{7}$$

Then the canonical map of S factors through the relative canonical map. The resulting map $\mathbb{P}(f_*\omega_{S|\mathbb{P}^1}) \to \varphi_{|K_S|}(S)$ is a surjective morphism mapping each "line" of the ruling of $\mathbb{P}(f_*\omega_{S|\mathbb{P}^1})$ to a line of $\mathbb{P}^{p_g-1}$.

If S is regular, then the cokernels of the restriction maps in (7) are all isomorphic (to $H^1(\omega_S(-F))$). In particular, the maps are *all* surjective if and only if one of them is surjective, i.e., if and only if $|K_S|$ is not composed with $|F|$.

REMARK 5.5. The canonical map of S is a double cover of a quadric. In fact, by the above considerations the canonical image is covered by lines. On the other hand, as it is seen by the same argument as in Lemma 3.14 of [1], if the canonical image of S is a cubic, it has isolated singularities, whence cannot be covered by lines.

Proposition 5.6. *Let S be a regular surface, whose canonical map is a double cover of a quadric surface Q, and let $f: S \to \mathbb{P}^1$ be a genus 2 fibration. If Q is smooth then $f_*\omega_{S|\mathbb{P}^1} \cong 2\mathcal{O}_{\mathbb{P}^1}(3)$. If Q is a quadric cone then $f_*\omega_{S|\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4)$.*

Proof. $\mathbb{P}(f_*\omega_{S|\mathbb{P}^1})$ is a Hirzebruch surface $\mathbb{F}_k$ having, by Remark 5.4, a birational morphism onto Q. If Q is smooth, then $k = 0$, and if the quadric is a cone, then $k = 2$. We conclude, since by standard computations (e.g., [4], Remark 2.11) deg $f_*\omega_{S|\mathbb{P}^1} = \chi(\mathcal{O}_S) + 1 = 6$. □

Lemma 5.7. *With the same hypotheses as in* Proposition 5.6, *if Q is a quadric cone, then the branch curve of the relative canonical map $\varphi_1 : S \dashrightarrow \mathbb{F}_2$ cannot contain Γ_∞.*

Proof. Assume by contradiction that Γ_∞ is contained in the branch locus of φ_1. Then the preimage of the vertex of the cone under the canonical map is a point $p \in S$. Since the genus two pencil maps onto the ruling of Q, it has a base point, contradicting Kodaira's lemma ([11] or [19], Proposition 5.1). □

We will use some of the techniques developped in [4], which for sake of simplicity will only be briefly reported in the case of genus 2 fibrations $f\colon S \to \mathbb{P}^1$ with $p_g(S) = 4$.

We consider the exact sequence

$$0 \to \mathrm{Sym}^2\, f_*\omega_{S|\mathbb{P}^1} \xrightarrow{\sigma_2} f_*\omega^2_{S|\mathbb{P}^1} \to \mathcal{O}_{\mathfrak{t}} \to 0, \tag{8}$$

where σ_2 is the natural map induced by the tensor product of canonical sections of the fibers of f, and $\mathfrak{t}$ is an effective divisor on $\mathbb{P}^1$ of degree $K_S^2 - 4$ (cf. Lemma 4.1 of [4]). The map σ_2 yields a rational map $\nu\colon \mathbb{P}(f_*\omega_{S|\mathbb{P}^1}) \dashrightarrow \mathbb{P}(f_*\omega^2_{S|\mathbb{P}^1})$ (relative version of 2-Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$) birational onto a conic bundle $\mathcal{C}$.

The following exact sequence defines the vector bundle $\mathcal{A}_6$ as quotient of $\mathrm{Sym}^3\, f_*\omega^2_{S|\mathbb{P}^1}$, the vector bundle of relative cubics on $\mathbb{P}(f_*\omega^2_{S|\mathbb{P}^1})$, by the subbundle of cubics vanishing on $\mathcal{C}$ (cf. Lemma 4.4 of [4]):

$$0 \to f_*\omega^2_{S|\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(12) \xrightarrow{i_3} \mathrm{Sym}^3\, f_*\omega^2_{S|\mathbb{P}^1} \to \mathcal{A}_6 \to 0. \tag{9}$$

The branch curve Δ of the map $S \to \mathcal{C}$ is given (cf. Theorem 4.7 and Proposition 4.8 of [4]) by a map

$$\delta\colon \mathcal{O}_{\mathbb{P}^1}(2K_S^2 + 4) \hookrightarrow \mathcal{A}_6. \tag{10}$$

Lemma 5.8. *Under the assumptions of* Proposition 5.6, *if moreover* $K_S^2 \geq 6$, *then each direct summand of* $f_*\omega^2_{S|\mathbb{P}^1}$ *has degree at least* 6.

Proof. Being σ_2 an injective morphism between two vector bundles of the same rank, if each summand of the source has degree at least 6, the same holds for the target. Therefore by Proposition 5.6 we can assume $f_*\omega_{S|\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4)$.

Assume by contradiction that (writing coordinates on $f_*\omega_{S|\mathbb{P}^1}$, $f_*\omega^2_{S|\mathbb{P}^1}$)

$$f_*\omega_{S|\mathbb{P}^1} = x_0\mathcal{O}_{\mathbb{P}^1}(2) \oplus x_1\mathcal{O}_{\mathbb{P}^1}(4)$$
$$f_*\omega^2_{S|\mathbb{P}^1} = y_0\mathcal{O}_{\mathbb{P}^1}(a) \oplus y_1\mathcal{O}_{\mathbb{P}^1}(b) \oplus y_2\mathcal{O}_{\mathbb{P}^1}(c)$$

with $a \leq 5$.

In these coordinates we have that Γ_∞ has equation $x_1 = 0$. From $a \leq 5$ it follows that $\sigma_2(x_0x_1)$, $\sigma_2(x_1^2)$ belong to $\mathrm{Span}(y_1, y_2)$, whence $\nu(\Gamma_\infty) = \{y_1 = y_2 = 0\}$.

Since $\nu(\Gamma_\infty) \subset \mathcal{C}$, y_0^2 does not appear in the equation of $\mathcal{C}$ and therefore y_0^3 does not appear in the equation of any relative cubic vanishing in $\mathcal{C}$. This means that the row of the matrix of i_3 corresponding to the direct summand $y_0^3\mathcal{O}_{\mathbb{P}^1}(3a)$ of $\mathrm{Sym}^3\, f_*\omega^2_{S|\mathbb{P}^1}$ is a line of zeroes. Therefore this summand maps isomorphically onto a direct summand of $\mathcal{A}_6$.

$K_S^2 \geq 6$ implies $2K_S^2 + 4 > 15 \geq 3a$ and therefore the composition of δ with the projection on this summand is zero. But this implies $\Delta \supset \nu(\Gamma_\infty)$, contradicting Lemma 5.7. $\square$

Let now S be a minimal surface of general type with $K^2 = 8$, $p_g = 4$ and $q = 0$ having a genus 2 pencil $f: S \to \mathbb{P}^1$.

By the above arguments we know:

- $\mathbb{P}(f_*\omega_{S|\mathbb{P}^1}) \cong \mathbb{F}_k$ for $k \in \{0, 2\}$;
- $f_*\omega^2_{S|\mathbb{P}^1} \cong r\mathcal{O}_{\mathbb{P}^1}(6) \oplus V$ for $r \in \{0, 1, 2\}$, where V is a sum of line bundles of degree at least 7.

Note that $r \neq 3$, since $\deg f_*\omega^2_{S|\mathbb{P}^1} = 18 + \deg \mathfrak{t} = 22$.

Theorem 5.9. *The moduli space of surfaces with $K^2 = 8$, $p_g = 4$ and $q = 0$ having a genus 2 pencil $f: S \to \mathbb{P}^1$ is unirational of dimension 34.*

Proof. We use the structure theorem for genus 2 fibrations (cf. Theorem 4.13 in [4]). For each case we have to describe the associated 5-tuple $(B, V_1, \mathfrak{t}, \xi, w)$. We treat separately the cases $k = 0$ and $k = 2$.

***k* = 0.** The first three elements are easy: $B = \mathbb{P}^1$, $V_1 = f_*\omega_{S|\mathbb{P}^1} = 2\mathcal{O}_{\mathbb{P}^1}(3)$ and $\mathfrak{t}$ is an effective divisor on $\mathbb{P}^1$ of degree 4.

ξ is an element of $\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_\mathfrak{t}, \mathrm{Sym}^2 V_1)_{/\mathrm{Aut}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_\mathfrak{t})}$, giving the short exact sequence (8). In order to give explicitly these extension classes we fix a section $f_\mathfrak{t} \in H^0(\mathcal{O}_{\mathbb{P}^1}(\mathfrak{t}))$ and, applying to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(3) \xrightarrow{f_\mathfrak{t}} \mathcal{O}_{\mathbb{P}^1}(7) \to \mathcal{O}_\mathfrak{t} \to 0 \tag{11}$$

the functor $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\,\cdot\,, 3\mathcal{O}_{\mathbb{P}^1}(6))$, we get

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_\mathfrak{t}, \mathrm{Sym}^2 V_1) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}(3), 3\mathcal{O}_{\mathbb{P}^1}(6)) \cong H^0(3\mathcal{O}_{\mathbb{P}^1}(3)) \cong \mathbb{C}^{12}.$$

This isomorphism is explicitly given as follows: for any triple of cubics (c_0, c_1, c_2), the resulting $f_*\omega^2_{S|\mathbb{P}^1}$ is given by the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(3) \xrightarrow{c} \mathcal{O}_{\mathbb{P}^1}(7) \oplus 3\mathcal{O}_{\mathbb{P}^1}(6) \to f_*\omega^2_{S|\mathbb{P}^1} \to 0 \tag{12}$$

for c being the transpose of $(-f_\mathfrak{t}, c_1, c_2, c_3)$; σ_2 is then the restriction to the last three summands $(3\mathcal{O}_{\mathbb{P}^1}(6))$ of the projection on $f_*\omega^2_{S|\mathbb{P}^1}$.

These 4 data give us the exact sequence (8) and therefore the conic bundle $\mathcal{C}$. To complete the 5-tuple we have to give an element $w \in (\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(20), \mathcal{A}_6) \setminus \{0\})_{/\mathbb{C}^*}$ corresponding to the map δ in (10), and then to the branch curve $\Delta \subset \mathcal{C}$.

From the exact sequence (9), $\dim(\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(20), \mathcal{A}_6)) = \chi(\mathcal{A}_6(-20)) + h^1(\mathcal{A}_6(-20)) = 29 + h^1(\mathcal{A}_6(-20))$. Moreover, $H^1(\mathcal{A}_6(-20))$ is isomorphic to the cokernel of the map $H^1(i_3(-20))$.

By Lemma 5.8, all summands of the source and of the target of the map $i_3(-20)$ have degree at least -2. More precisely, the source has $r \leq 2$ summands of degree -2, the target r^2, and $H^1(i_3(-20))$ is a map $\mathbb{C}^r \to \mathbb{C}^{r^2}$. In particular,

$$r^2 - r \leq h^1(\mathcal{A}_6(-20)) \leq r^2. \tag{13}$$

In fact, the map $H^1(i_3(-20))$ is easily obtained by the matrix of i_3 by taking the $r^2 \times r$ submatrix A given by the rows and the columns of the summands of degree 18 (both in the source and in the target).

We have three cases, according to the value of $h^1(\mathcal{A}_6(-20))$.

$\underline{h^1(\mathcal{A}_6(-20)) = 0}$. This happens for a general choice of ξ, since dualizing the exact sequence (12) one sees that, if the three cubics c_1, c_2, c_3 are linearly independent, $r = 0$.

We have 4 parameters for $\mathfrak{t}$, $12 - 4 = 8$ for ξ and $29 - 1 = 28$ for w: 40 parameters. Since we must take the quotient by the action of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$, this family is unirational of dimension 34.

$\underline{h^1(\mathcal{A}_6(-20)) = 1}$. By (13), then $r = 1$, i.e., there is a nontrivial relation $\alpha c_1 + \beta c_2 + \gamma c_3 = 0$ between the three cubics: these are two conditions for ξ. Moreover, the row of the matrix of σ_2 corresponding to the degree 6 summand of the target is (α, β, γ), and $A = (\alpha\gamma - \beta^2)$. In order to get $r = 1$ we need to further assume $\alpha\gamma = \beta^2$; we have three conditions on ξ, and therefore this gives a family of dimension $34 - 3 + h^1(\mathcal{A}_6(-20)) = 32$.

$\underline{h^1(\mathcal{A}_6(-20)) \geq 2}$. By (13), then $r = 2$, i.e., the three cubics span a space of dimension 1: these are six conditions. Moreover, if the submatrix of σ_2 corresponding to the degree 6 summands of the target is

$$\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{pmatrix}, \tag{14}$$

the matrix A is

$$\begin{pmatrix} \alpha_1\gamma_1 - \beta_1^2 & 0 \\ \alpha_1\gamma_2 + \alpha_2\gamma_1 - 2\beta_1\beta_2 & \alpha_1\gamma_1 - \beta_1^2 \\ \alpha_2\gamma_2 - \beta_2^2 & \alpha_1\gamma_2 + \alpha_2\gamma_1 - 2\beta_1\beta_2 \\ 0 & \alpha_2\gamma_2 - \beta_2^2 \end{pmatrix}.$$

It follows: rank $A \neq 2 \Leftrightarrow A = 0$. If $A = 0$, then $(\alpha_1 y_1 + \alpha_2 y_2)(\gamma_1 y_1 + \gamma_2 y_2) - (\beta_1 y_1 + \beta_2 y_2)^2 = 0$, and this implies that the matrix (14) has not rank 2, contradicting the injectivity of σ_2. Therefore $h^1(\mathcal{A}_6(-20)) = 2$ and this gives a family of dimension $34 - 6 + h^1(\mathcal{A}_6(-20)) = 30$.

***k* = 2.** Here $V_1 = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4)$. The main difference to the first case is that here to describe the extension class we need to apply the functor $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\cdot\,, \mathrm{Sym}^2 V_1)$ to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(1) \xrightarrow{f_{\mathfrak{t}}} \mathcal{O}_{\mathbb{P}^1}(5) \to \mathcal{O}_{\mathfrak{t}} \to 0 \tag{15}$$

getting only a short exact sequence

$$\begin{aligned}0 \to{}& \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(5), \mathrm{Sym}^2 V_1)\\ &\to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(1), \mathrm{Sym}^2 V_1) \to \mathrm{Ext}^1(\mathcal{O}_{\mathfrak{t}}, \mathrm{Sym}^2 V_1) \to 0.\end{aligned}$$

To induce any extension as described in the first case we need maps $\mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(8)$ (but not in a unique way): the dimension of the Ext^1 is in fact $18 - 6 = 12$ as in the first case. We distinguish two cases.

$\underline{h^1(\mathcal{A}_6(-20)) = 0}$. This happens for general choice of ξ, since also in this case, if ξ is general, then $r = 0$. The analysis of this case is identical to the analogous case for $k = 0$, so we find again 40 parameters. Since $\dim \mathrm{Aut}(\mathbb{F}_2) = 7$, awe get an unirational family of dimension $40 - 7 = 33$.

$\underline{h^1(\mathcal{A}_6(-20)) \geq 1}$. By (13) in this case $r \geq 1$. Let us first assume $r = 1$: then the row of the matrix of σ_2 corresponding to the degree 6 summand of the target is $(\alpha, \beta, 0)$ (where $\deg \alpha = 2$, $\beta \in \mathbb{C}$), and therefore the matrix A is $(-\beta^2)$. It follows that $h^1(\mathcal{A}_6(-20)) = 1$ forces $\beta = 0$.

We are now in the same situation as in the proof of Lemma 5.8: $\sigma_2(x_0 x_1)$, $\sigma_2(x_1^2)$ belong to $\mathrm{Span}(y_1, y_2)$. Arguing as there, we conclude that $\Delta \supset \nu(\Gamma_\infty)$ contradicting Lemma 5.7.

The case $r = 2$ is similar and even easier, since in this case we can always assume (up to a change of coordinates in the target) that the submatrix of σ_2 corresponding to the degree 6 summands has the form

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \beta_2 & 0 \end{pmatrix}.$$

Summing up we have found 4 families, one generically smooth unirational of dimension 34, say the “main” family, and three more of respective dimensions 32, 30 and 33. To conclude, we have to show that the general surface in each of those last three families admits a small deformation to a surface belonging to the “main” family.

This is easy for surfaces in the family with $k = 2$. In fact, we first deform $\mathbb{F}_2$ to $\mathbb{F}_0$ (i.e., the vector bundle V_1). Then, leaving $\mathfrak{t}$ fixed, we can deform the extension class ξ, since all the Ext^1 groups have the same dimension 12: geometrically this corresponds to deform $\mathcal{C}$ to a family of conic bundles. Finally, we can deform the last datum, w, since we have seen that (for $k = 2$) $h^1(\mathcal{A}_6(-20)) = 0$, so by semicontinuity it must be zero also on nearby fibres, and therefore $h^0(\mathcal{A}_6(-20))$ remains constant for a small deformation: this geometrically corresponds to deform Δ.

This argument does not work for the other two families, since in these cases $h^1(\mathcal{A}_6(-20)) \neq 0$ and therefore, once we have fixed a 1-parameter deformation of $\mathcal{C}$, we will not be able to deform all possible curves Δ.

We use a different argument. Each of the two families is contained in a irreducible component of the subscheme of the moduli space given by the surfaces having a canonical involution. We claim that it has dimension at least 34.

For the general surface in each of our two families, $\mathcal{C}$ has $\deg \mathfrak{t} = 4$ nodes (the vertices of the singular conics), none of them in Δ, which is smooth. Let $\tilde{\mathcal{C}}$ be a minimal desingularization of $\mathcal{C}$; the 4 (-2)-curves on $\tilde{\mathcal{C}}$ give rise to 4 (-1) curves on the associated double cover $\tilde{S}$, the exceptional locus of the birational morphism $\tilde{S} \to S$. The finite double cover $\varphi\colon \tilde{S} \to \tilde{\mathcal{C}}$ branches in $\tilde{\Delta}$, union of the pull-back of Δ with the (-2) curves.

The invariant part of $\varphi_*(\Omega^1_{\tilde{S}} \otimes \Omega^2_{\tilde{S}})$ is isomorphic to $\Omega^1_{\tilde{\mathcal{C}}}(\log \tilde{\Delta}) \otimes \Omega^2_{\tilde{\mathcal{C}}}$.

The morphism $\tilde{\mathcal{C}} \to \mathbb{P}(V_1)$ is the contraction of the strict transforms of each component of the singular conics, so of $2 \deg \mathfrak{t} = 8$ exceptional curves of the first kind. If $\mathfrak{T}_\cdot$ denotes the tangent sheaf, $\chi(\mathfrak{T}_{\tilde{\mathcal{C}}}) = \chi(\mathfrak{T}_{\mathbb{F}_0}) - 4 \deg \mathfrak{t} = 6 - 16 = -10$. Then our claim follows from

$$\begin{aligned} & h^1(\Omega^1_{\tilde{\mathcal{C}}}(\log \tilde{\Delta}) \otimes \Omega^2_{\tilde{\mathcal{C}}}) - h^2(\Omega^1_{\tilde{\mathcal{C}}}(\log \tilde{\Delta}) \otimes \Omega^2_{\tilde{\mathcal{C}}}) \\ & \geq -\chi(\Omega^1_{\tilde{\mathcal{C}}}(\log \tilde{\Delta}) \otimes \Omega^2_{\tilde{\mathcal{C}}}) = -\chi(\Omega^1_{\tilde{\mathcal{C}}} \otimes \Omega^2_{\tilde{\mathcal{C}}}) - \chi(\mathcal{O}_{\tilde{\Delta}}(\Omega^2_{\tilde{\mathcal{C}}})) \\ & = -\chi(\mathfrak{T}_{\tilde{\mathcal{C}}}) - \chi(\Omega^2_{\tilde{\mathcal{C}}}) + \chi(\Omega^2_{\tilde{\mathcal{C}}}(-\tilde{\Delta})) = 10 + \frac{1}{2}\tilde{\Delta}(\tilde{\Delta} - K_{\tilde{\mathcal{C}}}) \\ & = 6 + \frac{1}{2}\Delta(\Delta - K_{\mathcal{C}}) = 34 \end{aligned}$$

where $\Delta(\Delta - K_{\mathcal{C}}) = 56$ is a standard intersection computation (note that $\mathcal{C} \in |\mathcal{O}_{\mathbb{P}(V_2)}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(-12)|$, Δ is a divisor in the linear system induced on $\mathcal{C}$ by $|\mathcal{O}_{\mathbb{P}(V_2)}(3) \otimes \mathcal{O}_{\mathbb{P}^1}(-20)|$).

Then, since for a small deformation preserving the involution also the bicanonical map factors through it, either the two families are in the closure of the "main" family or these surface can be deformed to surfaces as in Theorem 5.1. But this is impossible for topological reasons, since the surfaces in Theorem 5.1 have non trivial 2-torsion in $\mathrm{Pic}(S)$ whereas every surface with a linear pencil of genus 2 curves and slope < 3 (in our case $8/3$) is simply connected by [20], Theorem 3. $\square$

6. Moduli

In the previous sections we classified all pairs (S, i) where S is a minimal regular surfaces with $K_S^2 = 8,\ p_g = 4,$ and i is a canonical involution on $S,$ finding 8 families.

Family	Theorem	*short description*
$\mathcal{M}_+^{(0)}$	2.1	bidouble covers of $\mathbb{F}_0$ branched in $(4, 2),\ (2, 4)$
$\mathcal{M}_+^{(2)}$	2.1	bidouble covers of $\mathbb{F}_2$ branched in $(4, 2),\ (2, 4)$
$\mathcal{M}_0^{(\mathrm{div})}$	3.3	double covers of a Del Pezzo of degree 1 branched in $-6K$
$\mathcal{M}_0$	3.5	the general surface is a double cover of $\mathbb{F}_0$ branched in $(8, 10) - 4\sum_1^8 E_i$
$\mathcal{M}_2^{(0)}$	4.2	double covers of a Del Pezzo of degree 5 branched in $-4K$ with two $(3, 3)$
$\mathcal{M}_2^{(1)}$	4.3	the general surface is a double cover of $\mathbb{F}_0$ branched in $(8, 10)$ with certain singularities
$\mathcal{M}_4^{(2)}$	5.9	the surfaces having a genus 2 pencil
$\mathcal{M}_4^{(\mathrm{DV})}$	5.1	$2K$ non birational, but no genus 2 pencil

REMARK 6.1. The first two are the families for which $H^0(K_S)$ is invariant. These surfaces have in fact (Lemmas 2.3 and 2.4) two more involutions for which $H^0(K_S)$ is antiinvariant and $\tau = 0$. In fact, for the family $\mathcal{M}_+^{(0)}$ the two further involutions are in $\mathcal{M}_0$, for the family $\mathcal{M}_+^{(2)}$ the two further involutions are in $\mathcal{M}_0^{(\mathrm{div})}$.

On the other hand, since the canonical map has maximal degree 4, if one of these surface has more than one canonical involution, it must have one involution for which $H^0(K_S)$ is invariant: so these two families give all surfaces having more than one canonical involution.

Our results yield then a stratification of the corresponding subscheme of the moduli space of minimal regular surfaces of general type with $K_S^2 = 8,\ p_g = 4$ in six families, image of the last 6 families of the above table.

The aim of this section is to prove the following

Theorem 6.2. *$\mathcal{M}_4^{(\mathrm{DV})}$ and $\mathcal{M}_4^{(2)}$ give unirational irreducible components of the moduli space of minimal regular surfaces of general type with $K_S^2 = 8,\ p_g = 4$ of respective dimensions* 38 *and* 34.

The remaining 4 *families $\mathcal{M}_0^{(\mathrm{div})}$, $\mathcal{M}_0$, $\mathcal{M}_2^{(0)}$, $\mathcal{M}_2^{(1)}$ give unirational strata of respective dimensions* 29, 28, 32, 33.

REMARK 6.3. By Kuranishi's theorem each irreducible component of the moduli space of minimal surfaces of general type with $K^2 = 8$, $p_g = 4$ has dimension at least $10\chi - 2K^2 = 34$. It follows that the last four families are not irreducible components of the moduli space.

Observe that the general point of the irreducible component in which each of these families is contained is a surface without a canonical involution. In fact, it cannot be in $\mathcal{M}_4^{(DV)}$ or in $\mathcal{M}_4^{(2)}$ because τ is invariant under deformations preserving the involution.

REMARK 6.4. $\mathcal{M}_4^{(DV)}$ and $\mathcal{M}_4^{(2)}$ are generically smooth. This is proved in [18] for $\mathcal{M}_4^{(DV)}$. The same calculation as in [1], Theorem 5.32, shows it for $\mathcal{M}_4^{(2)}$.

REMARK 6.5. Minimal surfaces of general type with $K^2 = 8$, $p_g = 4$ belong to at least three different topological types (in particular, the moduli space has at least three connected components). The surfaces in $\mathcal{M}_0^{(div)}$ are the only ones in our list with 2-divisible canonical class, the surfaces in $\mathcal{M}_4^{(DV)}$ are the only ones in our list with non trivial torsion in the Picard group.

Proof of Theorem 6.2. The statement about $\mathcal{M}_4^{(DV)}$ is Theorem 1.3 of [18].

By Theorem 5.9 $\mathcal{M}_4^{(2)}$ is unirational of dimension 34. To prove that it is an irreducible component of the moduli space we need to show that for a general surface in this family the antiinvariant part (with respect to the involution) of $H^1(\Omega_S^1 \otimes \Omega_S^2)$ is trivial.

This computation works almost identically as the analogous one in [1], Section 5.3. We sketch it.

Using the same notation as in the proof of Theorem 5.9, recall that for a general surface S in $\mathcal{M}_4^{(2)}$, we have a finite double cover $S \to \mathcal{C} = S/i$ branched in the $\deg \mathfrak{t} = 4$ nodes of $\mathcal{C}$, and in the smooth divisor Δ. Resolving the singular points of $\mathcal{C}$ and blowing up their preimages in S we get a finite double cover $\varphi\colon \tilde{S} \to \tilde{\mathcal{C}}$ whose branch locus is a smooth divisor $\tilde{\Delta}$, union of the pull-back of Δ with the (-2) curves.

Now we can compute the dimension of the antiinvariant part of $H^1(\Omega_{\tilde{S}}^1 \otimes \Omega_{\tilde{S}}^2)$ with respect to the lifting of the involution i to $\tilde{S}$ exactly as in the proof of Theorem 5.32 of [1]: the result is 8. Since $b\colon \tilde{S} \to S$ is a sequence of 4 blow ups, by Lemma 5.34 of [1] the dimension of the antiinvariant part of $H^1(\Omega_S^1 \otimes \Omega_S^2)$ is $8 - 2 \cdot 4 = 0$.

We prove now the second part of the statement. In all 4 cases S is a double cover of a surface P such that the movable part of the branch curve is 2δ where δ is a $\mathbb{Q}$-divisor such that $\lambda K_P + \delta$ is ample for $\lambda \leq 1$. In particular, $2\delta_P - K_P$ is ample, therefore $h^1(2\delta_P) = 0$, and the dimension of the linear system $|2\delta|$ can be computed by Riemann-Roch.

$\underline{\mathcal{M}_0^{(div)}}$. Del Pezzo surfaces of degree 1 are obtained by choosing 8 points in $\mathbb{P}^2$, therefore, modulo $\mathrm{Aut}(\mathbb{P}^2)$, they depend on 8 (unirational) parameters. Curves in $|-6K|$ depend on $1 + (1/2)(42K^2) - 1 = 21$ parameters.

$\underline{\mathcal{M}_0}$. By Remark 3.6, we know that $\mathcal{M}_0$ is unirational and that for a general surface in $\mathcal{M}_0$ we can assume that P is the blow up of $\mathbb{F}_0$ in 8 general points, branched in a curve in $\left|8\Gamma_1 + 10\Gamma_2 - 4\sum_1^8 E_i\right|$. Since 8 points in $\mathbb{F}_0$ depend on 16 parameters and $\dim \mathrm{Aut}(\mathbb{F}_0) = 6$, P depends on 10 parameters. The branch curve depends on 18 parameters.

$\underline{\mathcal{M}_2^{(0)}}$. P is the blow up of a Del Pezzo of degree 5 in 4 points, 2 of which are infinitely near to the other two. Therefore P depends on 6 parameters. The branch curve depends on 26 parameters.

$\underline{\mathcal{M}_2^{(1)}}$. We know already (cf. the remark after Theorem 4.3) that $\mathcal{M}_2^{(1)}$ is irreducible and for a general surface we can assume that P is the blow up of $\mathbb{F}_0$ in 6 points, the last determined by the previous one. Therefore P depends on $10 - 6 = 4$ parameters. The branch curve depends on 29 parameters. $\square$

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References

[1] I.C. Bauer: Surfaces with $K^2 = 7$ and $p_g = 4$, Mem. Amer. Math. Soc. **152**, 2001.

[2] I.C. Bauer, F. Catanese and R. Pignatelli: *The moduli space of surfaces with $K^2 = 6$ and $p_g = 4$*, Math. Ann. **336** (2006), 421–438.

[3] F. Catanese: *Singular bidouble covers and the construction of interesting algebraic surfaces*; in Algebraic Geometry: Hirzebruch 70 (Warsaw, 1998), Contemp. Math. **241**, Amer. Math. Soc., Providence, RI, 1999, 97–120.

[4] F. Catanese and R. Pignatelli: *Fibrations of low genus*, I, Ann. Sci. École Norm. Sup. (4) **39** (2006), 1011–1049.

[5] C. Ciliberto: *Canonical surfaces with $p_g = p_a = 4$ and $K^2 = 5, \dots, 10$*, Duke Math. J. **48** (1981), 121–157.

[6] C. Ciliberto, P. Francia and M. Mendes Lopes: *Remarks on the bicanonical map for surfaces of general type*, Math. Z. **224** (1997), 137–166.

[7] F. Enriques: Le Superficie Algebriche, Nicola Zanichelli, Bologna, 1949.

[8] F.J. Gallego and B.P. Purnaprajna: *Classification of quadruple Galois canonical covers*. I, preprint, `http://arxiv.org/abs/math/0302045`.

[9] F.J. Gallego and B.P. Purnaprajna: *Classification of quadruple Galois canonical covers*. II, J. Algebra **312** (2007), 798–828.

[10] E. Horikawa: *Algebraic surfaces of general type with small c_1^2*. I, Ann. of Math. (2) **104** (1976), 357–387.

[11] E. Horikawa: *On algebraic surfaces with pencils of curves of genus* 2; in Complex Analysis and Algebraic Geometry, Iwanami Shoten, Tokyo, 1977, 79–90.

[12] E. Horikawa: *Algebraic surfaces of general type with small c_1^2*. III, Invent. Math. **47** (1978), 209–248.

[13] K. Konno: *On the irregularity of special non-canonical surfaces*, Publ. Res. Inst. Math. Sci. **30** (1994), 671–688.

[14] M. Mendes Lopes and R. Pardini: *Triple canonical surfaces of minimal degree*, Internat. J. Math. **11** (2000), 553–578.
[15] M. Mendes Lopes and R. Pardini: *The bicanonical map of surfaces with* $p_g = 0$ *and* $K^2 \geq 7$. II, Bull. London Math. Soc. **35** (2003), 337–343.
[16] P.A. Oliverio: *On even surfaces of general type with* $K^2 = 8$, $p_g = 4$, $q = 0$, Rend. Sem. Mat. Univ. Padova **113** (2005), 1–14.
[17] M. Reid: *Surfaces of small degree*, Math. Ann. **275** (1986), 71–80.
[18] P. Supino: *On moduli of regular surfaces with* $K^2 = 8$ *and* $p_g = 4$, Port. Math. (N.S.) **60** (2003), 353–358.
[19] G. Xiao: Surfaces Fibrées en Courbes de Genre Deux, Lecture Notes in Math. **1137**, Springer, Berlin, 1985.
[20] G. Xiao: π_1 *of elliptic and hyperelliptic surfaces*, Internat. J. Math. **2** (1991), 599–615.
[21] F. Zucconi: *Numerical inequalities for surfaces with canonical map composed with a pencil*, Indag. Math. (N.S.) **9** (1998), 459–476.

Ingrid C. Bauer
Mathematisches Institut
Universitätstraße 30
D-95447 Bayreuth
Germany

Roberto Pignatelli
Dipartimento di Matematica
Università di Trento
via Sommarive 14 loc. Povo
I-38050 Trento
Italy