Osaka Mathematical Journal,
Vol. 2, No. 1, March, 1950

Mixing Up Property of Brownian Motion.

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1. Let X be the space of Brownian motions. Any element x of X is to be considered here as a set function defined for intervals on the infinite line. We denote the value of x for the interval (a, b) by $x(b)-x(a)$, but it is to be noticed that $x(a)$ or $x(b)$ alone has no meaning in our case. Let $\varphi_t\ (-\infty < t < \infty)$ be the flow of translations on X defined by $\varphi_t x(a, b) = x(a+t, b+t)$, that is, $x' = \varphi_t x$ means that $x'(b)-x'(a) = x(b+t)-x(a+t)$. It is well known that φ_t is strongly mixing.[1]

THEOREM 1. *Let X be the space of Brownian motions as set functions and let φ_t be the flow of translations on X above defined. Then for any measurable ergodic flow ψ_t on a measure space Y, the skew product flow T_t, which is defined in the following way on the direct product measure space Ω of X and Y, is strongly mixing.*

$$T_t(x, y) = (\varphi_t x, \psi_{x(t)-x(0)}\, y), \quad \textit{where } (x, y) \in \Omega = X \times Y.$$

At first, it is easily verified that for any fixed t, T_t is a measure preserving transformation on Ω, from the fact that for any null set N in the usual Lebesgue measure space of the infinite line, the set $\{x \mid x(t)-x(0) \in N\}$ is a null set in X.[2] And further we may show in the same way that T_t is a measurable flow on Ω from the fact that the (t, x)-set $\{(t, x) \mid x(t)-x(0) \in N\}$ is a set of (t, x)-measure zero.[2] In order to prove that T_t is strongly mixing, it is sufficient to show that

$$(1) \qquad \lim_{t\to\infty} \iint f(\varphi_t x)\, f_1(\psi_{x(t)-x(0)} y)\, g(x)\, g_1(y)\, dxdy = \iint f(x)\, f_1(y)\, dxdy \iint g(x)\, g_1(y)\, dxdy$$

holds for any $f(x),\ g(x) \in L^2(X)$ and $f_1(y),\ g_1(y) \in L^2(Y)$.

Let E_λ be the spectral resolution of the one-parameter group of unitary transformations of $L^2(Y)$, which corresponds to the flow ψ_t on Y, and put $v(\lambda) = (E_\lambda f_1,\ g_1)$. Then the left hand side of (1) is equal to

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1) E. HOPF, Ergodentheorie, Berlin, 1937, p. 59 §16. Masstheorie im Raum der additiven Mengenfunktionen. Das Spektrum der Translationen.

2) See (7) of **2** of this note.

$$(2)\qquad \lim_{t\to\infty}\int f(\varphi_t x)\, g(x)\left\{\int f_1(\psi_{x(t)-x(0)}\, y)\, g_1(y)\, dy\right\} dx$$

$$= \lim_{t\to\infty}\int f(\varphi_t x)\, g(x)\left\{\int_{-\infty}^{\infty} \exp\left(i(x(t)-x(0))\,\lambda\right) d(E_\lambda f_1, g_1)\right\} dx$$

$$= \lim_{t\to\infty}\int f(\varphi_t x)\, g(x)\left\{\int_{-\infty}^{\infty} \exp\left(i(x(t)-x(0))\,\lambda\right) dv(\lambda)\right\} dx$$

$$= \lim_{t\to\infty}\int f(\varphi_t x)\, g(x)\left\{\int_{-\infty}^{\infty} \exp\left(i(x(t)-x(0))\,\lambda\right) du(\lambda)\right\} dx$$

$$+ \int \left\{v(+0)-v(-0)\right\} \lim_{t\to\infty}\int f(\varphi_t x)\, g(x)\, dx,$$

where

$$(3)\qquad u(\lambda)=\begin{cases} v(\lambda)-\left\{v(+0)-v(-0)\right\} & \text{if } \lambda \geq 0 \\ v(\lambda) & \text{if } \lambda < 0. \end{cases}$$

Since the ergodicity of ψ_t implies

$$(4)\qquad v(+0)-v(-0)=\int f_1(y)\, dy \int g_1(y)\, dy,$$

and since the strong mixing property of φ_t implies

$$(5)\qquad \lim_{t\to\infty}\int f(\varphi_t x)\, g(x)\, dx = \int f(x)\, dx \int g(x)\, dx,$$

so in order to prove (1) it suffices to prove that

$$(6)\qquad \lim_{t\to\infty}\int f(\varphi_t x)\, g(x)\left\{\int_{-\infty}^{\infty} \exp\left(i(x(t)-x(0))\,\lambda\right) du(\lambda)\right\} dx$$

$$= \lim_{t\to\infty}\int_{-\infty}^{\infty}\left\{\int f(\varphi_t x)\, g(x)\, \exp\left(i(x(t)-x(0))\,\lambda\right) dx\right\} du(\lambda) = 0$$

holds for any $f(x),\ g(x) \in L^2(X)$. We may restrict ourselves to the case in which $f(x)$ and $g(x)$ are functions of the following form:

$$(7)\qquad f(x)=\exp\left\{ i\sum_{k=1}^{n} \sigma_k \left(x(a_k)-x(a_{k-1}\right) \right\}$$

$$a_0 < a_1 < \dots < a_p = 0 < a_{p+1} \dots < a_n$$

$$g(x)=\exp\left\{ i\sum_{j=1}^{m} \tau_j \left(x(b_j)-x(b_{j-1})\right)\right\}$$

$$b_0 < b_1 < \dots < b_q = 0 < b_{q+1} \dots < b_m$$

where $\sigma_k\,(1 \leqq k \leqq n)$ and $\tau_j\,(1 \leqq j \leqq m)$ are arbitrary real numbers. This is because linear combinations of such functions are dense in $L^2(X)$. Further it is sufficient to show that

$$(8)\quad \lim_{t\to\infty} \int f(\varphi_t x)\, g(x) \exp\left\{i\big(x(t)-x(0)\big)\lambda\right\} dx = 0$$

holds for any $\lambda \neq 0$. This is because $u(\lambda)$ is continuous at $\lambda=0$ and the integral in (8), namely the integrand in (6) is uniformly bounded in t. Let t be greater than $b_m - a_0$, then the integral in (8) is equal to

$$(9)\quad \int dx \prod_{k=p+1}^{n} \exp\left\{i\sigma_k\big(x(a_k+t)-x(a_{k-1}+t)\big)\right\}$$

$$\times \prod_{k=1}^{p} \exp\left\{i(\sigma_k+\lambda)\big(x(a_k+t)-x(a_{k-1}+t)\big)\right\}$$

$$\times \exp\left\{i\lambda\big(x(a_0+t)-x(b_m)\big)\right\}$$

$$\times \prod_{j=q+1}^{m} \exp\left\{i(\tau_j+\lambda)\big(x_j(b)-x(b_{j-1})\big)\right\}$$

$$\times \prod_{j=1}^{q} \exp\left\{i\tau_j\big(x(b_j)-x(b_{j-1})\big)\right\}$$

$$= \prod_{k=p+1}^{n} \int dx \exp\left\{i\sigma_k\big(x(a_k+t)-x(a_{k-1}+t)\big)\right\}$$

$$\times \prod_{k=1}^{p} \int dx \exp\left\{i(\sigma_k+\lambda)\big(x(a_k+t)-x(a_{k-1}+t)\big)\right\}$$

$$\times \int dx \exp\left\{i\lambda\big(x(a_0+t)-x(b_m)\big)\right\}$$

$$\times \prod_{j=q+1}^{m} \int dx \exp\left\{i(\tau_j+\lambda)\big((x(b_j)-x(b_{j-1})\big)\right\}$$

$$\times \prod_{j=1}^{q} \int dx \exp\left\{i\tau_j\big(x(b_j)-x(b_{j-1})\big)\right\}$$

$$= \prod_{k=p+1}^{n} \exp\left\{-\frac{1}{2}\sigma_k^2(a_k-a_{k-1})\right\} \prod_{k=1}^{n} \exp\left\{-\frac{1}{2}(\sigma_k+\lambda)^2(a_k-a_{k-1})\right\}$$

$$\times \exp\left\{-\frac{1}{2}\lambda^2(a_0+t-b_m)\right\} \prod_{j=q+1}^{m} \exp\left\{-\frac{1}{2}(\tau_j+\lambda)^2(b_j-b_{j-1})\right\}$$

$$\times \prod_{j=1}^{q} \exp\left\{-\frac{1}{2}\tau_j^2(b_j-b_{j-1})\right\},$$

which tends to zero as $t \to \infty$. The proof of the theorem is completed.

This is the phenomenon which cannot be observed in the direct product flow S_t of φ_t and ψ_t on Ω, where the direct product flow S_t means $S_t(x, y) = (\varphi_t x, \psi_t y)$. It is true that S_t is ergodic when ψ_t is ergodic, since φ_t has no point spectrum except the trivial eigen-frequency zero. But, in this case, if ψ_t has an eigen-frequency λ, λ is again an eigen-frequency for S_t. Generally speaking, any spectral measure which is a convolution of spectral measures of φ_t and ψ_t appears in the spectrum of S_t.

COROLLARY. *Let ψ_t be any ergodic flow on a measure space Y. Let Ξ be the space of Brownian motions $\xi(t)$ starting from the origin 0 at the instant $t = 0$: $\xi(0) = 0$. [Here we consider $\xi(t)$ as a usual point function and not as a set function.]*

Then for any $f(y) \in L^1(Y)$, for almost all $\xi \in \Xi$, there exists a null set N in Y, which depends on f and ξ, such that

$$\lim_{A\to\infty} \frac{1}{A} \int_0^A f(\psi_{\xi(t)}\, y)\, dt = \int f(y)\, dy$$

holds for any $y \notin N$.

This is an immediate consequence of the ergodicity of the skew product flow T_t. This may be considered as an example of the random ergodic theorems of S. ULAM and J. V. NEUMANN[3] in continuous parameter cases, where the ergodic limit is independent both of the continuous path $\xi(t)$ (corresponding to the infinite sequences in their theorem) and of the space point y.

2. In this section we are going to show that for some flows, contrary to the case of **1**, essentially skew product compositions are impossible, that is, every skew product flow is equivalent with a direct product flow.

We begin with general definitions. Let $(E_1, \mathfrak{M}_{E_1}, \mu_1)$ and $(E_2, \mathfrak{M}_{E_2}, \mu_2)$ be measure spaces, $\mathfrak{M}_{E_1} \times \mathfrak{M}_{E_2}$ means the least Borel field of the sets of the product space $E_1 \times E_2$, which contains the sets of the form $A_1 \times A_2$, where $A_1 \in \mathfrak{M}_{E_1}$, $A_2 \in \mathfrak{M}_{E_2}$, and $\mathfrak{M}_{E_1} \otimes \mathfrak{M}_{E_2}$ means the completion of $\mathfrak{M}_{E_1} \times \mathfrak{M}_{E_2}$ with respect to the direct product measure $\mu_1 \times \mu_2$ defined on $\mathfrak{M}_{E_1} \times \mathfrak{M}_{E_2}$. Denote by $(R, \mathfrak{M}_R, m)$ the usual Lebesgue measure space of the set of all real numbers, and by $\mathfrak{B}_R$ the family of usual Borel sets on R. By *Borel-measurable flow* ψ_t $(-\infty < t < \infty)$ on

3) S. M. ULAM and J. v. NEUMANN, 165. Random ergodic theorems, Bull. A. M. S. Vol. 51, No. 9, 1945, p. 660.

a measure space $(E, \mathfrak{M}_E, \mu)$, we understand a measurable flow on E, such that $\{(t, p) \mid \psi_t p \in A\} \in \mathfrak{B}_R \times \mathfrak{M}_E$ for $A \in \mathfrak{M}_E$, where $p \in E$, while simply a *measurable flow* ψ_t means that

$$\{(t, p) \mid \psi_t p \in A\} \in \mathfrak{B}_R \otimes \mathfrak{M}_E = \mathfrak{M}_R \otimes \mathfrak{M}_E, \text{ for } A \in \mathfrak{M}_E .$$

Let $(\Omega, \mathfrak{M}_\Omega, \lambda)$ be the completed product measure space of the given measure spaces $(X, \mathfrak{M}_X, \mu)$ and $(Y, \mathfrak{M}_Y, \nu)$.

$$(\Omega, \mathfrak{M}_\Omega, \lambda) = (X, \mathfrak{M}_X, \mu) \otimes (Y, \mathfrak{M}_Y, \nu) .$$

Suppose there are given measurable flows φ_t and ψ_t respectively on X and Y. We consider a real-valued function $\alpha(t, x)$ defined on $R \times X$, which satisfies the following conditions, and such a function is called an *α-function.*

(1) For every $t \in R$, $\alpha(t, x)$ is a measurable function on X.

(2) $\alpha(t, x)$ is as a function of two variables t and x, $\mathfrak{M}_R \otimes \mathfrak{M}_X$-measurable.

(3) For every s and t of R

$$\alpha(s, x) + \alpha(t, \varphi_s x) = \alpha(s + t, x) .$$

By making use of an α-function $\alpha(t, x)$, we define one-to-one transformations T_t on Ω by

$$T_t(x, y) = (\varphi_t x, \psi_{\alpha(t, x)} y), \quad -\infty < t < \infty .$$

By the condition (3), T_t is a one-parameter group:

$$T_{s+t} = T_s T_t .$$

In order that T_t be a measurable flow, it is necessary and sufficient that $\alpha(t, x)$ satisfies the following conditions, and then T_t is called a *skew product flow with the α-function* $\alpha(t, x)$:

(4) $\{(x, y) \mid \psi_{\alpha(t, x)} y \in B\} \in \mathfrak{M}_X \otimes \mathfrak{M}_Y$, for any fixed $t \in R$, and $B \in \mathfrak{M}_Y$.

(5) $\{(t, x, y) \mid \psi_{\alpha(t, x)} y \in B\} \in \mathfrak{M}_R \otimes \mathfrak{M}_X \otimes \mathfrak{M}_Y$, for $B \in \mathfrak{M}_Y$.

It is still an open question to the author whether the conditions (4) and (5) are consequences of the conditions (1), (2), and (3) or not. But, for instance, in each of the following cases, it is easily verified that the conditions (4), (5) are satisfied:

(6) ψ_t is a Borel-measurable flow on Y.

(7) $\mu\{x \mid \alpha(t, x) \in N\} = 0$, for $N \in \mathfrak{B}_R$ such that $m(N) = 0$, for every fixed $t \in R$.

(8) $\alpha(t, x)$ takes only countably many values.

Let $\beta(x)$ be a real-valued measurable function on X. Then $V(x, y) = (x, \psi_{\beta(x)} y)$ is a measure preserving transformation on Ω, if and only if

(9) $\{(x, y) \mid \psi_{\beta(x)} y \in B\} \in \mathfrak{M}_X \otimes \mathfrak{M}_Y$, for $B \in \mathfrak{M}_Y$.

The condition (9) is satisfied, for instance, in each of the following cases:

(6) ψ_t is a Borel-measurable flow on Y.

(7′) $\mu\{x \mid \beta(x) \in N\} = 0$ for $N \in \mathfrak{B}^R$ such that $m(N) = 0$.

(8′) $\beta(x)$ takes only countably many values.

If for two α-functions $\alpha(t, x)$ and $\gamma(t, x)$ fulfilling (4), (5), there exists a measurable $\beta(x)$ satisfying (9), such that $\alpha(t, x) - \gamma(t, x) = \beta(\varphi_t x) - \beta(x)$, then between the corresponding skew product flows T_t and S_t, there exists the following relation:

$$T_t = VS_tV^{-1}, \text{ where } T_t(x, y) = (\varphi_t x, \psi_{\alpha(t, x)} y),$$
$$S_t(x, y) = (\varphi_t x, \psi_{\gamma(t, x)} y), \ V(x, y) = (x, \psi_{\beta(x)} y).$$

In this case, $\alpha(t, x)$ and $\gamma(t, x)$, T_t and S_t, are called *equivalent to each other*. And if $\gamma(t, x) = 0$, that is, if $S_t(x, y) = (\varphi_t x, y)$, $\alpha(t, x)$ $(= \beta(\varphi_t x) - \beta(x))$ is called a *trivial α-function* and T_t is called a *trivial flow*.

THEOREM 2. *Let X be the infinite line, and let φ_t be the translation by t on X: $\varphi_t x = t + x$. Then for any α-function $\alpha(t, x)$, there exists a measurable function $\beta(x)$ on X such that $\alpha(t, x) = \beta(\varphi_t x) - \beta(x)$.* [This is the only case in this note in which the total measure of the space is not finite.]

PROOF: Put $x = 0$ in the condition (3), then we get

(10) $\alpha(t, s) = \alpha(s + t, 0) - \alpha(s, 0)$.

Considering $s \in R$ as an element of X, we may regard $\beta(s) = \alpha(s, 0)$ as a function defined on X. If we replace s in (10) by x, we get

(11) $\alpha(t, x) = \beta(t + x) - \beta(x) = \beta(\varphi_t x) - \beta(x)$.

By applying Fubini's theorem, the condition (2) implies the existence of an $x_0 \in X$, such that $\alpha(t, x_0)$ is a measurable function of t. Hence $\beta(t + x_0) - \beta(x_0)$ is a measurable function of t. By replacing again the variable t by x, we may conclude the measurability of $\beta(x)$.

THEOREM 3. *Let X be the circle of unit length, and let φ_t be the* *rotation of X by the angle $2\pi t$. Then for any α-function $\alpha(t, x)$, there exists a constant c and a measurable function $\beta(x)$ on X, such that $\alpha(t, x) - ct = \beta(\varphi_t x) - \beta(x)$.*

PROOF: X is R (mod 1). Hence from the equality (10), $\alpha(s+t, 0) - \alpha(s, 0)$ is a periodic function of s with the period 1;

(12) $$\alpha(s+t, 0) - \alpha(s, 0) = \alpha(s+t+1, 0) - \alpha(s+1, 0).$$

(12′) $$\alpha(s+t+1, 0) - \alpha(s+t, 0) = \alpha(s+1, 0) - \alpha(s, 0).$$

(12′) shows that $\alpha(s+1, 0) - \alpha(s, 0)$ is a constant c.

Put $\gamma(s) = \alpha(s, 0)$ for $0 \leqq s \leqq 1$, then for $0 \leqq s \leqq 1$,

$$\alpha(s+1, 0) = c + \gamma(s),\ \alpha(s+2, 0) = 2c + \gamma(s), \dots .$$

Therefore we get for $-\infty < s < \infty$,

(13) $$\alpha(s, 0) = c\,[s] + \gamma(\{s\}),$$

where $[s]$ is the integral part of s, and $\{s\} = s - [s]$.

(13′) $$\alpha(s, 0) = cs + \gamma(\{s\}) - c\,\{s\}.$$

Putting $\beta(\{s\}) = \gamma(\{s\}) - c\,\{s\}$, from (10) and (13′) we get

(14) $$\alpha(t, s) = ct + \beta(\{s+t\}) - \beta(\{s\}).$$

Considering s as an element of X, replace it by x, then we get

$$\alpha(t, x) = ct + \beta(\varphi_t x) - \beta(x) .$$

The same argument as in the proof of Theorem 2 assures the measurability of $\beta(x)$.

A skew product flow T_t of the rotations φ_t on the circle X and of a Borel-measurable flow ψ_t on a space Y is always equivalent with a direct product flow S_t;

$$S_t(x, y) = (\varphi_t x,\ \psi_{ct}\, y), \text{ where } c \text{ is a constant.}$$

It was shown by S. KAKUTANI that if φ_t is an ergodic flow with pure point spectrum on a measure space X, the statement of Theorem 3 is valid under the following supplementary conditions (15) and (22):

(15) $$\int |\alpha(t, x)|^2\, dx < \infty \text{ for every } -\infty < t < \infty.$$

Since φ_t is an ergodic flow with pure point spectrum, we may expand $\alpha(t, x)$, as a function of $L^2(X)$, by the system of characteristic functions $\theta_p(x)$, $p = 0, 1, 2, \dots$, where $\theta_p(\varphi_t x) = \exp(i\lambda_p t)\,\theta_p(x)$, $[\lambda_0 = 0,\ \theta_0(x) \equiv 1.]$ and $\int |\theta_p(x)|^2\, dx = 1$.

(16) $$\alpha(t, x) = \sum_{p=0}^{\infty} a_p(t)\, \theta_p(x)$$

where

(17) $$\sum_p |a_p(t)|^2 < \infty .$$

From (3) and (16), and from the fact that $\theta_p(\varphi_s x) = \exp(i\lambda_p s)\ \theta_p(x)$,

(18) $$\sum_p a_p(s)\, \theta_p(x) + \sum_p a_p(t) \exp(i\lambda_p s)\, \theta_p(x) = \sum a_p(s+t)\ \theta_p(x).$$

By comparing the coefficients of the functions $\theta_p(x)$, we get

(19) $$a_p(s) + \exp(i\lambda_p s)\, a_p(t) = a_p(s+t). \quad p = 0, 1, 2, \dots .$$

It is easily seen that the measurable solutions of (19) are

(20) $a_p(t) = c_p(\exp(i\lambda_p t) - 1)$, where c_p is a constant, $p \neq 0$,

(21) $a_0(t) = ct$, where c is a constant.

If

(22) $$\sum_{p=1}^{\infty} |c_p|^2 < \infty$$

holds, then

(23) $$\beta(x) = \sum_{p=1}^{\infty} c_p \theta_p(x)$$

is a function belonging to $L^2(X)$.
From (20), (21) and (16), we have

$$\alpha(t, x) = ct + \beta(\varphi_t x) - \beta(x) .$$

(Received December 16, 1949)