Mixing up property of Brownian motion

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Mixing Up Property of Brownian Motion.

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1. Let $X$ be the space of Brownian motions. Any element $x$ of $X$ is to be considered here as a set function defined for intervals on the infinite line. We denote the value of $x$ for the interval $(a, b)$ by $x(b) - x(a)$, but it is to be noticed that $x(a)$ or $x(b)$ alone has no meaning in our case. Let $\varphi_t (-\infty < t < \infty)$ be the flow of translations on $X$ defined by $\varphi_t x(a, b) = x(a + t, b + t)$, that is, $x' = \varphi_t x$ means that $x'(b) - x'(a) = x(b + t) - x(a + t)$. It is well known that $\varphi_t$ is strongly mixing. 1)

**Theorem 1.** Let $X$ be the space of Brownian motions as set functions and let $\varphi_t$ be the flow of translations on $X$ above defined. Then for any measurable ergodic flow $\varphi_t$ on a measure space $Y$, the skew product flow $T_t$, which is defined in the following way on the direct product measure space $\Omega$ of $X$ and $Y$, is strongly mixing.

At first, it is easily verified that for any fixed $t$, $T_t$ is a measure preserving transformation on $\Omega$, from the fact that for any null set $N$ in the usual Lebesgue measure space of the infinite line, the set \{ $x(x(t) - x(0) \in N$ \} is a null set in $X$. 2) And further we may show in the same way that $T_t$ is a measurable flow on $\Omega$ from the fact that the $(t, x)$-set \{ $(t, x) \mid x(t) - x(0) \in N$ \} is a set of $(t, x)$-measure zero. 2) In order to prove that $T_t$ is strongly mixing, it is sufficient to show that

$$\lim_{t \to \infty} \int \int f(\varphi_t x) f_1(\varphi_t x(t) - x(0), y) g(x) g_1(y) \, dx \, dy$$

holds for any $f(x), g(x) \in L^2(X)$ and $f_1(y), g_1(y) \in L^2(Y)$.

Let $E_\lambda$ be the spectral resolution of the one-parameter group of unitary transformations of $L^2(Y)$, which corresponds to the flow $\varphi_t$ on $Y$, and put $v(\lambda) = (E_\lambda f_1, g_1)$. Then the left hand side of (1) is equal to


2) See (7) of 2 of this note.
\begin{align*}
(2) \quad & \lim_{t \to \infty} \int f(\varphi_t x) g(x) \left\{ \int f_1(\varphi_{t(\xi)} - x(0), y) g_1(y) \, dy \right\} \, dx \\
& = \lim_{t \to \infty} \int f(\varphi_t x) g(x) \left\{ \int \exp \left( i(x(\xi) - x(0)) \lambda \right) d(E, f_1, g_1) \right\} \, dx \\
& = \lim_{t \to \infty} \int f(\varphi_t x) g(x) \left\{ \int \exp \left( i(x(\xi) - x(0)) \lambda \right) dv_1(\lambda) \right\} \, dx \\
& = \lim_{t \to \infty} \int f(\varphi_t x) g(x) \left\{ \int \exp \left( i(x(\xi) - x(0)) \lambda \right) du(\lambda) \right\} \, dx \\
& \quad + \int \{v(0) - v(-0)\} \lim_{t \to \infty} \int f(\varphi_t x) g(x) \, dx, \\
\end{align*}

where

\begin{align*}
(3) \quad & u(\lambda) = \begin{cases} 
v(\lambda) - \{v(0) - v(-0)\} & \text{if } \lambda \geq 0 \\
v(\lambda) & \text{if } \lambda < 0. 
\end{cases}
\end{align*}

Since the ergodicity of \( \varphi_t \) implies

\begin{align*}
(4) \quad & v(0) - v(-0) = \int f_1(y) \, dy \int g_1(y) \, dy,
\end{align*}

and since the strong mixing property of \( \varphi_t \) implies

\begin{align*}
(5) \quad & \lim_{t \to \infty} \int f(\varphi_t x) g(x) \, dx = \int f(x) \, dx \int g(x) \, dx,
\end{align*}

so in order to prove (1) it suffices to prove that

\begin{align*}
(6) \quad & \lim_{t \to \infty} \int f(\varphi_t x) g(x) \{ \int \exp \left( i(x(\xi) - x(0)) \lambda \right) du(\lambda) \} \, dx \\
& = \lim_{t \to \infty} \int \{ \int f(\varphi_t x) g(x) \exp \left( i(x(\xi) - x(0)) \lambda \right) dx \} du(\lambda) = 0
\end{align*}

holds for any \( f(x), g(x) \in L^2(X) \). We may restrict ourselves to the case in which \( f(x) \) and \( g(x) \) are functions of the following form:

\begin{align*}
(7) \quad & f(x) = \exp \left\{ \sum_{k=1}^{n} \sigma_k \left( x(a_k) - x(a_{k-1}) \right) \right\} \\
& a_0 < a_1 < \ldots < a_p = 0 < a_{p+1} < \ldots < a_n \\
& g(x) = \exp \left\{ \sum_{j=1}^{m} \tau_j \left( x(b_j) - x(b_{j-1}) \right) \right\} \\
& b_0 < b_1 < \ldots < b_q = 0 < b_{q+1} < \ldots < b_m
\end{align*}
where \( \sigma_k (1 \leq k \leq n) \) and \( \tau_j (1 \leq j \leq m) \) are arbitrary real numbers. This is because linear combinations of such functions are dense in \( L^2(X) \). Further it is sufficient to show that

\[
(8) \quad \lim_{t \to \infty} \int f(\varphi, x) g(x) \exp \left\{ i \left[ x(t) - x(0) \right] \lambda \right\} \, dx = 0
\]

holds for any \( \lambda \neq 0 \). This is because \( u(\lambda) \) is continuous at \( \lambda = 0 \) and the integral in (8), namely the integrand in (6) is uniformly bounded in \( t \). Let \( t \) be greater than \( b_m - a_0 \), then the integral in (8) is equal to

\[
(9) \quad \int dx \prod_{k=p+1}^{n} \exp \left\{ i \sigma_k (x(a_k + t) - x(a_{k-1} + t)) \right\} \\
\times \prod_{k=1}^{p} \exp \left\{ i (\sigma_k + \lambda) (x(a_k + t) - x(a_{k-1} + t)) \right\} \\
\times \exp \left\{ i \lambda (x(a_0 + t) - x(b_m)) \right\} \\
\times \prod_{j=q+1}^{m} \exp \left\{ i \tau_j (x(b_j) - x(b_{j-1})) \right\} \\
= \prod_{k=p+1}^{n} \int dx \exp \left\{ i \sigma_k (x(a_k + t) - x(a_{k-1} + t)) \right\} \\
\times \prod_{k=1}^{p} \int dx \exp \left\{ i (\sigma_k + \lambda) (x(a_k + t) - x(a_{k-1} + t)) \right\} \\
\times \int dx \exp \left\{ i \lambda (x(a_0 + t) - x(b_m)) \right\} \\
\times \prod_{j=q+1}^{m} \int dx \exp \left\{ i \tau_j (x(b_j) - x(b_{j-1})) \right\} \\
= \prod_{k=p+1}^{n} \exp \left\{ -\frac{1}{2} \sigma_k^2 (a_k - a_{k-1}) \right\} \prod_{k=1}^{n} \exp \left\{ -\frac{1}{2} (\sigma_k + \lambda)^2 (a_k - a_{k-1}) \right\} \\
\times \exp \left\{ -\frac{1}{2} \lambda^2 (a_0 + t - b_m) \right\} \prod_{j=q+1}^{m} \exp \left\{ -\frac{1}{2} (\tau_j + \lambda)^2 (b_j - b_{j-1}) \right\} \\
\times \prod_{j=1}^{q} \exp \left\{ -\frac{1}{2} \tau_j^2 (b_j - b_{j-1}) \right\},
\]
which tends to zero as \( t \to \infty \). The proof of the theorem is completed.

This is the phenomenon which cannot be observed in the direct product flow \( S_t \) of \( \varphi_t \) and \( \psi_t \) on \( \Omega \), where the direct product flow \( S_t \) means \( S_t(x, y) = (\varphi_t x, \psi_t y) \). It is true that \( S_t \) is ergodic when \( \psi_t \) is ergodic, since \( \varphi_t \) has no point spectrum except the trivial eigen-frequency zero. But, in this case, if \( \psi_t \) has an eigen-frequency \( \lambda \), \( \lambda \) is again an eigen-frequency for \( S_t \). Generally speaking, any spectral measure which is a convolution of spectral measures of \( \varphi_t \) and \( \psi_t \) appears in the spectrum of \( S_t \).

**Corollary.** Let \( \psi_t \) be any ergodic flow on a measure space \( Y \). Let \( \Xi \) be the space of Brownian motions \( \xi(t) \) starting from the origin \( 0 \) at the instant \( t = 0 : \xi(0) = 0 \). [Here we consider \( \xi(t) \) as a usual point function and not as a set function.]

Then for any \( f(y) \in L^1(Y) \), for almost all \( \xi \in \Xi \), there exists a null set \( N \) in \( Y \), which depends on \( f \) and \( \xi \), such that

\[
\lim_{A \to \infty} \frac{1}{A} \int_0^A f(\psi_{it}(y)) dt = \int f(y) dy
\]

holds for any \( y \notin N \).

This is an immediate consequence of the ergodicity of the skew product flow \( T_t \). This may be considered as an example of the random ergodic theorems of S. Ulam and J. V. Neumann \(^3\) in continuous parameter cases, where the ergodic limit is independent both of the continuous path \( \xi(t) \) (corresponding to the infinite sequences in their theorem) and of the space point \( y \).

2. In this section we are going to show that for some flows, contrary to the case of 1, essentially skew product compositions are impossible, that is, every skew product flow is equivalent with a direct product flow.

We begin with general definitions. Let \( (E_1, \mathcal{M}_{\sigma_1}, \mu_1) \) and \( (E_2, \mathcal{M}_{\sigma_2}, \mu_2) \) be measure spaces, \( \mathcal{M}_{\sigma_1} \times \mathcal{M}_{\sigma_2} \) means the least Borel field of the sets of the product space \( E_1 \times E_2 \), which contains the sets of the form \( A_1 \times A_2 \), where \( A_1 \in \mathcal{M}_{\sigma_1}, A_2 \in \mathcal{M}_{\sigma_2} \), and \( \mathcal{M}_{\sigma_1} \otimes \mathcal{M}_{\sigma_2} \) means the completion of \( \mathcal{M}_{\sigma_1} \times \mathcal{M}_{\sigma_2} \) with respect to the direct product measure \( \mu_1 \times \mu_2 \) defined on \( \mathcal{M}_{\sigma_1} \times \mathcal{M}_{\sigma_2} \). Denote by \( (R, \mathcal{B}_R, m) \) the usual Lebesgue measure space of the set of all real numbers, and by \( \mathcal{B}_R \) the family of usual Borel sets on \( R \). By Borel-measurable flow \( \psi_t (-\infty < t < \infty) \) on

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a measure space \((E, \mathcal{M}_E, \mu)\), we understand a measurable flow on \(E\), such that \(\{(t, p) \mid \psi_t p \in A\} \in \mathcal{B}_E \times \mathcal{M}_E\) for \(A \in \mathcal{M}_E\), where \(p \in E\), while simply a measurable flow \(\psi_t\) means that 
\[
\{(t, p) \mid \psi_t p \in A\} \in \mathcal{B}_E \otimes \mathcal{M}_E = \mathcal{M}_E \otimes \mathcal{M}_E, \quad \text{for } A \in \mathcal{M}_E.
\]

Let \((\Omega, \mathcal{M}_\Omega, \lambda)\) be the completed product measure space of the given measure spaces \((X, \mathcal{M}_X, \mu)\) and \((Y, \mathcal{M}_Y, \nu)\).

\[
(\Omega, \mathcal{M}_\Omega, \lambda) = (X, \mathcal{M}_X, \mu) \otimes (Y, \mathcal{M}_Y, \nu).
\]

Suppose there are given measurable flows \(\varphi_t\) and \(\psi_t\) respectively on \(X\) and \(Y\). We consider a real-valued function \(\alpha(t, x)\) defined on \(R \times X\), which satisfies the following conditions, and such a function is called an \(\alpha\)-function.

1. For every \(t \in R\), \(\alpha(t, x)\) is a measurable function on \(X\).
2. \(\alpha(t, x)\) is as a function of two variables \(t\) and \(x\), \(\mathcal{M}_E \otimes \mathcal{M}_X\)-measurable.
3. For every \(s\) and \(t\) of \(R\)
   \[
   \alpha(s, x) + \alpha(t, \varphi_s x) = \alpha(s + t, x).
   \]

By making use of an \(\alpha\)-function \(\alpha(t, x)\), we define one-to-one transformations \(T_t\) on \(\Omega\) by

\[
T_t(x, y) = (\varphi_t x, \psi_{\alpha(t, x)} y), \quad -\infty < t < \infty.
\]

By the condition (3), \(T_t\) is a one-parameter group:

\[
T_{t + s} = T_t T_s.
\]

In order that \(T_t\) be a measurable flow, it is necessary and sufficient that \(\alpha(t, x)\) satisfies the following conditions, and then \(T_t\) is called a skew product flow with the \(\alpha\)-function \(\alpha(t, x)\):

1. \(\{(x, y) \mid \psi_{\alpha(t, x)} y \in B\} \in \mathcal{M}_X \otimes \mathcal{M}_Y\), for any fixed \(t \in R\), and \(B \in \mathcal{M}_Y\).
2. \(\{(t, x) \mid \psi_{\alpha(t, x)} y \in B\} \in \mathcal{M}_R \otimes \mathcal{M}_X \otimes \mathcal{M}_Y\), for \(B \in \mathcal{M}_Y\).

It is still an open question to the author whether the conditions (4) and (5) are consequences of the conditions (1), (2), and (3) or not. But, for instance, in each of the following cases, it is easily verified that the conditions (4), (5) are satisfied:

1. \(\varphi_t\) is a Borel-measurable flow on \(Y\).
2. \(\mu\{x \mid \alpha(t, x) \in N\} = 0\), for \(N \in \mathcal{B}_R\) such that \(m(N) = 0\), for every fixed \(t \in R\).
(8) $\alpha(t, x)$ takes only countably many values.

Let $\beta(x)$ be a real-valued measurable function on $X$. Then $V(x, y) = (x, \psi_{\beta(x)} y)$ is a measure preserving transformation on $\Omega$, if and only if

$$\{(x, y) \mid \psi_{\beta(x)} y \in B\} \in \mathcal{M}_x \otimes \mathcal{M}_y,$$ for $B \in \mathcal{M}_y$.

The condition (9) is satisfied, for instance, in each of the following cases:

(6) $\psi_t$ is a Borel-measurable flow on $Y$.

(7) $\mu\{x \mid \beta(x) \in N\} = 0$ for $N \in \mathcal{B}$ such that $m(N) = 0$.

(8) $\beta(x)$ takes only countably many values.

If for two $\alpha$-functions $\alpha(t, x)$ and $\gamma(t, x)$ fulfilling (4), (5), there exists a measurable $\beta(x)$ satisfying (9), such that $\alpha(t, x) - \gamma(t, x) = \beta(\varphi, x)$, then between the corresponding skew product flows $T_t$ and $S_t$, there exists the following relation:

$$T_t = V S_t V^{-1}, \text{ where } T_t(x, y) = (\varphi_t x, \psi_{\beta(t, x)} y),$$

$$S_t(x, y) = (\varphi_t x, \psi_{\beta(t, x)} y), V(x, y) = (x, \psi_{\beta(x)} y).$$

In this case, $\alpha(t, x)$ and $\gamma(t, x)$, $T_t$ and $S_t$, are called equivalent to each other. And if $\gamma(t, x) = 0$, that is, if $S_t(x, y) = (\varphi_t x, y)$, $\alpha(t, x)$ ($= \beta(\varphi, x) - \beta(x)$) is called a trivial $\alpha$-function and $T_t$ is called a trivial flow.

**Theorem 2.** Let $X$ be the infinite line, and let $\varphi_t$ be the translation by $t$ on $X$: $\varphi_t x = t + x$. Then for any $\alpha$-function $\alpha(t, x)$, there exists a measurable function $\beta(x)$ on $X$ such that $\alpha(t, x) = \beta(\varphi_t x) - \beta(x)$. [This is the only case in this note in which the total measure of the space is not finite.]

**Proof:** Put $x = 0$ in the condition (3), then we get

$$\alpha(t, s) = \alpha(s + t, 0) - \alpha(s, 0).$$

Considering $s \in \mathcal{B}$ as an element of $X$, we may regard $\beta(s) = \alpha(s, 0)$ as a function defined on $X$. If we replace $s$ in (10) by $x$, we get

$$\alpha(t, x) = \beta(t + x) - \beta(x) = \beta(\varphi_t x) - \beta(x).$$

By applying Fubini's theorem, the condition (2) implies the existence of an $x_0 \in X$, such that $\alpha(t, x_0)$ is a measurable function of $t$. Hence $\beta(t + x_0) - \beta(x_0)$ is a measurable function of $t$. By replacing again the variable $t$ by $x$, we may conclude the measurability of $\beta(x)$.

**Theorem 3.** Let $X$ be the circle of unit length, and let $\varphi_t$ be the
rotation of $X$ by the angle $2\pi t$. Then for any $\alpha$-function $\alpha(t, x)$, there exists a constant $c$ and a measurable function $\beta(x)$ on $X$, such that $\alpha(t, x) - ct = \beta(\phi_t x) - \beta(x)$.

**Proof:** $X$ is $R \pmod{1}$. Hence from the equality (10), $\alpha(s+t, 0) - \alpha(s, 0)$ is a periodic function of $s$ with the period 1;

(12) $\alpha(s+t, 0) - \alpha(s, 0) = \alpha(s+t+1, 0) - \alpha(s+1, 0)$.

(12') $\alpha(s+t+1, 0) - \alpha(s+t, 0) = \alpha(s+1, 0) - \alpha(s, 0)$.

(12') shows that $\alpha(s+1, 0) - \alpha(s, 0)$ is a constant $c$.

Put $\gamma(s) = \alpha(s, 0)$ for $0 \leq s \leq 1$, then for $0 \leq s \leq 1$,

$$\alpha(s+1, 0) = c + \gamma(s), \quad \alpha(s+2, 0) = 2c + \gamma(s), \ldots$$

Therefore we get for $-\infty < s < \infty$,

(13) $\alpha(s, 0) = e [s] + \gamma([s])$,

where $[s]$ is the integral part of $s$, and $\{s\} = s - [s]$.

(13') $\alpha(s, 0) = es + \gamma([s]) - c \{s\}$.

Putting $\beta([s]) = \gamma([s]) - c \{s\}$, from (10) and (13') we get

(14) $\alpha(t, s) = ct + \beta([s + t]) - \beta([s])$.

Considering $s$ as an element of $X$, replace it by $x$, then we get

$$\alpha(t, x) = ct + \beta(\phi_t x) - \beta(x).$$

The same argument as in the proof of Theorem 2 assures the measurability of $\beta(x)$.

A skew product flow $T_t$ of the rotations $\phi_t$ on the circle $X$ and of a Borel-measurable flow $\psi_t$ on a space $Y$ is always equivalent with a direct product flow $S_t$;

$$S_t(x, y) = (\phi_t x, \psi_t y), \text{ where } c \text{ is a constant.}$$

It was shown by S. Kakutani that if $\phi_t$ is an ergodic flow with pure point spectrum on a measure space $X$, the statement of Theorem 3 is valid under the following supplementary conditions (15) and (22):

(15) $\int |\alpha(t, x)|^2 dx < \infty$ for every $-\infty < t < \infty$.

Since $\phi_t$ is an ergodic flow with pure point spectrum, we may expand $\alpha(t, x)$, as a function of $L^2(X)$, by the system of characteristic functions $\theta_p(x)$, $p = 0, 1, 2, \ldots$, where $\theta_p(\phi_t x) = \exp (i \lambda_p t) \theta_p(x)$, $[\lambda_p = 0, \theta_0(x) = 1]$ and

$$\int |\theta_p(x)|^2 dx = 1.$$
(16) \[ \alpha(t, x) = \sum_{p=0}^{\infty} a_p(t) \theta_p(x) \]

where

(17) \[ \sum_p |a_p(t)|^2 < \infty. \]

From (3) and (16), and from the fact that \( \theta_p(q, x) = \exp(i\lambda_p s) \theta_p(x) \),

(18) \[ \sum_p a_p(s) \theta_p(x) + \sum_p a_p(t) \exp(i\lambda_p t) \theta_p(x) = \sum_p a_p(s+t) \theta_p(x). \]

By comparing the coefficients of the functions \( \theta_p(x) \), we get

(19) \[ a_p(s) + \exp(i\lambda_p t) a_p(t) = a_p(s+t). \quad p = 0, 1, 2, \ldots \]

It is easily seen that the measurable solutions of (19) are

(20) \[ a_p(t) = c_p(\exp(i\lambda_p t) - 1), \] where \( c_p \) is a constant, \( p \neq 0 \),

(21) \[ a_0(t) = ct, \] where \( c \) is a constant.

If

(22) \[ \sum_{p=1}^{\infty} |c_p|^2 < \infty \]

holds, then

(23) \[ \beta(x) = \sum_{p=1}^{\infty} c_p \theta_p(x) \]

is a function belonging to \( L^2(X) \).

From (20), (21) and (16), we have

\[ \alpha(t, x) = ct + \beta(q, x) - \beta(x). \]

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