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Osaka University
ON IMBEDDING 3-MANIFOLDS INTO 4-MANIFOLDS

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Introduction

We discuss an imbedding problem of a closed, connected, oriented 3-manifold into a given compact connected 4-manifold, which arises from certain signature invariants of 3-manifold associated with its cyclic coverings. Our main result is the following:

Theorem. For any compact, connected (orientable or non-orientable) 4-manifold $W$ (with or without boundary), there exist infinitely many closed, connected, orientable 3-manifolds $M$ which cannot be imbedded in $W$.

For a closed orientable 4-manifold $W$, this is a direct consequence of [8, Theorem 3.2] and, for an orientable 4-manifold $W$ with boundary, we can prove it by using the doubling technique for $W$. Thus the main concern in this paper is for a non-orientable 4-manifold $W$.

The proof of Theorem is given in §3. In §2, a classification of the types of imbeddings of $M$ into a closed 4-manifold $W$ is given. Section 1 is devoted to the calculation of the signatures of the finite cyclic covers of a homology handle $M$. We can express these signatures in terms of the local signatures of $M$ under a certain condition on the Alexander polynomial of $M$, where the Alexander polynomial of a homology handle is defined in the same way as in the case of knots (cf. [3, Definition 1.3]). Let $\sigma_a(M)$ be the local signature of $M$ at $a \in [-1,1]$, which is an analogue of the Milnor signature of a knot (cf. [9]). Let $\sigma^{(n)}(M)$ be the signature of $n$-fold cyclic cover of $M$ (whose definition is given in Section 1 where $\sigma^{(n)}(M)$ is denoted by $\sigma^{(n)}(M_{[a]})$). Then the following will be shown.

Proposition 1.3. If the Alexander polynomial of $M$ has no 2n-th root of unity, then

$$\sigma^{(n)}(M) = \sum_{j=0}^{n-1} (-1)^j \sum_{s_{j+1} < \ldots < s_j} \sigma_s(M),$$
where \(a_j = \cos(j\pi/n), j = 0, 1, \ldots, n\).

This result reveals a connection between the signatures of finite cyclic covers of a homology handle and the local signatures of its infinite cyclic cover. When \(n=2\) the assumption of the above proposition is always satisfied. So we have the following formula, which will be used in §3 to prove Theorem for a non-orientable 4-manifold \(W\).

**Corollary 1.4.** \(\sigma^{(\ell)}(M) = \sum_{-1 < a < 1} \text{sign}(a) \sigma_a(M)\).

Throughout this paper, all manifolds and all maps between manifolds will be assumed to be smooth.

I would like to thank my advisor Professor Akio Kawauchi for suggesting the problem to me and for his advice and encouragement.

1. **Signatures of Finite Cyclic Covers of a Homology Handle**

In this section, we consider the signature of the \(n\)-fold cyclic cover of a homology handle.

Throughout this paper, we use Kawauchi’s notations for signatures and local signatures of a 3-manifold; for a closed oriented 3-manifold \(M\) equipped with an element \(\gamma \in H^1(M; \mathbb{Z})\), \(\sigma(M)\) denotes the signature of \((M, \gamma)\) and \(\sigma^2(M)\), \(a \in [-1, 1]\), denotes the local signature of \((M, \gamma)\) at \(a\). For the definitions of these invariants, see [6] and also [4], [5], [7]. (Local signatures were first considered in [9, Section 5] for the exterior of a knot in \(S^3\).) In this section, \(\mathbb{Z}\langle t\rangle\) (resp. \(\mathbb{R}\langle t\rangle\)) denotes the group ring over the infinite cyclic group \(\langle t\rangle\) generated by \(t\) with coefficient ring the ring \(\mathbb{Z}\) of integers (resp. the field \(\mathbb{R}\) of real numbers).

Now let \(M\) be an oriented homology handle, that is, a compact oriented 3-manifold having the homology isomorphic to that of \(S^2 \times S^1\) (cf. [3]), and \(\gamma\) be a fixed generator of \(H^1(M; \mathbb{Z}) = [M, S^1]\). Using the transversality of a map \(M \to S^1\) representing \(\gamma\), we can find a closed, connected, oriented surface \(V\) in \(M\) representing the Poincaré dual of \(\gamma\). \(V\) is called a leaf of \(\gamma\) (cf. [6]).

We choose an orientation of \(M \times [-1, 1]\) so that \(M \times 1\) with the induced orientation is identified with \(M\). Let \(N(V)\) be a bicollar neighborhood of \(V\) in \(M\). Let \(W = M \times [-1, 1] - \text{int}(N(V) \times [-1/2, 1/2])\) (cf. [7]). There is a natural diffeomorphism \(N(V) \times [-1/2, 1/2] \cong V \times D^2\). Let \(V\) be a handlebody such that \(\partial V\) is diffeomorphic to \(V\). By identifying \(\partial(V \times S^1)\) with \(V \times S^1 = \partial N(V) \times [-1/2, 1/2] \subset W\), we get a compact 4-manifold \(\bar{W} = W \cup V \times S^1\) with boundary diffeomorphic to \(M \cup -M\). By the Pontrjagin/Thom construction, we have an element \(\bar{\eta} \in H^4(\bar{W}; \mathbb{Z})\) such that \(\bar{\eta}|_{M \times 1} = \gamma, \bar{\eta}|_{M \times (-1)} = 0\) and \(\bar{\eta}|_{V \times S^1}\) is represented by the natural projection \(V \times S^1 \to S^1\). Taking a compact, oriented 4-manifold \(W_0\) bounded by \(M\), we can cap the component
$M \times (-1)$ of $\partial \tilde{W}_c$ and finally get a 4-manifold $W = \tilde{W}_c \cup W_0$ with boundary $M$.

Define an element $\gamma \in H^2(W; \mathbb{Z})$ by $\gamma|\tilde{W}_c = \gamma_c$ and $\gamma|W_0 = 0$. Note that $\partial(W, \gamma) = (M, \gamma)$ and $\gamma$ has a leaf $U_\gamma = (V \times [1/2, 1]) \cup (V \times x_0)$, where $x_0 \in S^1$ is the point such that $\partial(V \times x_0) \equiv V \times (1/2) \subset \partial W_c$.

For each positive integer $n$, let $p_n: M_{(n)} \to M$ (resp. $P_n: W_{(n)} \to W$) be the $n$-fold cyclic covering of $M$ (resp. $W$) associated with the mod $n$ reduction $\gamma(n)$ (resp. $\gamma(n)$) of $\gamma$ (resp. $\gamma$). If $f_\gamma: M \to S^1$ (resp. $f_\gamma: W \to S^1$) is a map representing $\gamma$ (resp. $\gamma$), then the covering $p_n: M_{(n)} \to M$ (resp. $P_n: W_{(n)} \to W$) is defined to be the fibered product of $f_\gamma$ (resp. $f_\gamma$) with the natural $n$-fold covering $g_n: S^1 \to S^1, z \mapsto z^n$, where $z \in S^1$ is considered as a complex number with unit norm.

The lift $f_\gamma^{(n)}: M_{(n)} \to S^1$ (resp. $f_\gamma^{(n)}: W_{(n)} \to S^1$) of $f_\gamma$ (resp. $f_\gamma$) by $g_n$ is determined by $\gamma$ (resp. $\gamma$) up to homotopy. The homotopy class of $f_\gamma^{(n)}$ (resp. $f_\gamma^{(n)}$) is denoted by $\hat{\gamma}^{(n)}(\equiv M_{(n)}, \gamma^{(n)})$ and $\hat{\gamma}^{(n)}$ (resp. $\gamma^{(n)}$) has as its leaf a component of the pre-image of $V$ (resp. $U_\gamma$) by the projection $p_n: M_{(n)} \to M$ (resp. $P_n: W_{(n)} \to W$).

Since $W_{(2n)}$ is the 2-fold cyclic cover of $W_{(n)}$ associated with the mod 2 reduction of $\gamma^{(n)}$, we have, by [7, Lemma 4.3],

$$\sigma^{(n)}(M_{(n)}) = \text{sign } W_{(2n)} - 2 \text{ sign } W_{(n)}.$$

To calculate $\text{sign } W_{(m)}$, note that $W_{(m)} = W_{(m)}^{(o)} \cup \tilde{V} \times S^1 \cup \bigcup W_0$, where $W_{(m)}^{(o)}$ denotes the $m$-fold cyclic cover of $W_c$ associated with the mod $m$ reduction of $\tilde{V}$. Since $\partial V \times S^1 = 0$, the Novikov additivity implies $\text{sign } W_{(m)} = \text{sign } W_{(m)}^{(o)} + m \text{ sign } W_0$. Therefore

$$\sigma^{(m)}(M_{(n)}) = \text{sign } W_{(2n)} - 2 \text{ sign } W_{(n)}.$$

Thus the calculation is reduced to that of $\text{sign } W_{(m)}^{(o)}$. For the calculation, we use, instead of $W_{(m)}^{(o)}$, the $m$-fold cyclic branched cover $\tilde{W}_{(m)} = W_{(m)}^{(o)} \cup V \times D^2$ of $M \times [-1, 1] = W_c \cup V \times D^2$ branched along $V \times 0$. Note that, by the Novikov additivity and sign $V \times D^2 = 0$, sign $\tilde{W}_{(m)} = \text{sign } W_{(m)}^{(o)}$.

Let $L: H_1(V; \mathbb{R}) \times H_1(V; \mathbb{R}) \to \mathbb{R}$ be the linking form defined by $L(x, y) = \text{Link}_M(c_x, c_y)$ for $x = [c_x], y = [c_y] \in H_1(V; \mathbb{R})$, where $c_x^+$ denotes the translation of the cycle $c_x$ in the positive normal direction and $\text{Link}_M(c_x, c_y)$ is the linking number of $c_x$ with $c_y^+$ (cf. [6, p. 53 and p.77]). A matrix representing $L$ for some basis of $H_1(V; \mathbb{R})$ is called a linking matrix on $H_1(V; \mathbb{R})$. Let $T: \tilde{W}_{(m)} \to \tilde{W}_{(m)}$ be the natural extension of the generator $T: \tilde{W}_{(m)} \to \tilde{W}_{(m)}$ of the group of covering transformations of the covering $P_n: W_{(m)}^{(o)} \to \tilde{W}_{(m)}$ which is specified by $\eta_{m}|W_c$. Let $\text{Int}_{(m)}: H_2(\tilde{W}_{(m)}; \mathbb{R}) \times H_2(\tilde{W}_{(m)}; \mathbb{R}) \to \mathbb{R}$ be the intersection form on $\tilde{W}_{(m)}$. Take a basis $\{e_1, e_2, \cdots, e_r\}$ for $H_1(V; \mathbb{R})$. By a standard argument due to [11] or [2] and used in [7, Lemma 3.3], we have the following.
Lemma 1.1. There exist elements $\bar{e}_1, \ldots, \bar{e}_r, \bar{e}_{r+1}, \ldots, \bar{e}_s$ in $H_2(\tilde{W}_m; R)$ such that $\bar{e}_1, \ldots, \bar{e}_r, T_* \bar{e}_1, \ldots, T_* \bar{e}_r, \ldots, T_{r+1}^{m-2} \bar{e}_1, \ldots, T_{r+1}^{m-2} \bar{e}_r, \bar{e}_{r+1}, \ldots, \bar{e}_s$ form a basis for $H_2(\tilde{W}_m; R)$ and such that, for $i, j \leq r$ and $p, q = 0, 1, \ldots, m-2$,

$$\text{Int}_{\tilde{c} \circ \sigma} (T_* \bar{e}_i, T_* \bar{e}_j) = \begin{cases} 0 & \text{if } |p-q| > 1, \\ -L(e_i, e_j) & \text{if } p = q + 1, \\ L(e_j, e_i) & \text{if } q = p + 1, \\ L(e_i, e_j) + L(e_j, e_i) & \text{if } p = q. \end{cases}$$

and, for $i = 1, 2, \ldots, s$, $j > r$ and $k = 0, 1, \ldots, m-2$, $\text{Int}_{\tilde{c} \circ \sigma} (T_* \bar{e}_i, \bar{e}_j) = 0$.

Let $\mathcal{E}$ be the subspace of $H_2(\tilde{W}_m; R)$ generated by $T_* \bar{e}_i, i = 1, \ldots, r, j = 0, 1, \ldots, m-2$. It is easily seen that the form $(\text{Int}_{\tilde{c} \circ \sigma} | \mathcal{E}, T_* | \mathcal{E})$ is isomorphic to the symmetric $\mathbb{Z}_m$-form of $L$ defined in [11] (although the coefficient in [11] is rational). Recall that the symmetric $\mathbb{Z}_m$-form of $L$ is the pair $(L^{(m)}, \tau_m)$ of symmetric bilinear form $L^{(m)} \colon H_{m-1} \times H_{m-1} \to R$ and isometry $\tau_m \colon H_{m-1} \to H_{m-1}$ of $L^{(m)}$ of order $m$, defined by

$$L^{(m)}(x, y) = \sum_{i=1}^{m-1} L(\pi_i(x), \pi_i(y)) + L(\pi_i(y), \pi_i(x)) - \sum_{i=1}^{m-2} L(\pi_{i+1}(x), \pi_i(y)) + L(\pi_{i+1}(y), \pi_i(x))$$

and

$$\tau_m(x) = \sum_{i=1}^{m-1} t_{i+1} \pi_i(x) - \sum_{i=1}^{m-1} t_i \pi_{m-i}(x)$$

for $x, y \in H_{m-1}$, where $H_{m-1}$ denotes the $(m-1)$th Cartesian product of the real vector space $H = H_1(V; R)$, and $\pi_i \colon H_{m-1} \to H$ and $t_i \colon H \to H_{m-1}(i = 1, 2, \ldots, m-1)$ are the $i$-th coordinate projection and imbedding respectively.

Thus we have proved the following.

Proposition 1.2. $\sigma^{(\infty)}(M_\gamma(N)) = \text{sign} L^{(m)} - 2 \text{sign} L^{(n)}$.

By using Proposition 1.2, we can express $\sigma^{(\infty)}(M_\gamma(N))$ in terms of local signatures $\sigma_\mathcal{M}(M)$ of $(M, \gamma)$.

Proposition 1.3. If the Alexander polynomial $A_\gamma(t) \in \mathbb{Z}[t]$ of the homology handle $(M, \gamma)$ has no $2n$-th root of unity, then

$$\sigma^{(\infty)}(M_\gamma(N)) = \sum_{j=0}^{n-1} (-1)^j \sum_{a_j \cos(j\pi/n), j=0, 1, \ldots, n} \sigma_\mathcal{M}(M),$$

where $a_j = \cos(j\pi/n), j=0, 1, \ldots, n$.

Since $|A_\gamma(1)| = 1$ for any homology handle $(M, \gamma)$ (cf. [3, Theorem 1.4]),
Corollary 1.4. For any homology handle \((M, \gamma)\),

\[ \sigma^{(2)}(M; \omega) = \sum_{-1 < a < 1} \text{sign}(a) \sigma_a^2(M). \]

To prove Proposition 1.3, we need some lemmas. Let \(H_{\mathbb{C}}^{-1} = H_{\mathbb{C}}^{-1} \otimes \mathbb{C}(m \geq 2)\) and \(L_c^{(m)}: H_{\mathbb{C}}^{-1} \times H_{\mathbb{C}}^{-1} \to \mathbb{C}\) be the Hermitian form of \(L^{(m)}\) in the usual sense (cf. [11, 3.6. Note]). The isometry \(\tau_m: H_{\mathbb{C}}^{-1} \to H_{\mathbb{C}}^{-1}\) of \(L_c^{(m)}\) extends to the isometry (also denoted by \(\tau_m\)) \(H_{\mathbb{C}}^{-1} \otimes \mathbb{C} \to H_{\mathbb{C}}^{-1} \otimes \mathbb{C}\) naturally. Let \(E_m(\zeta)\) be the eigenspace of \(H_c^{-1}\) corresponding to the eigenvalue \(\zeta \in \mathbb{C}\) of \(\tau_m: H_{\mathbb{C}}^{-1} \to H_{\mathbb{C}}^{-1}\).

Lemma 1.5. If \(m = pq, p, q > 0, \) and \(\zeta_p\) is a primitive \(p\)-th root of unity, then

\[ \mu: E_p(\zeta_p) \to E_m(\zeta_p), \quad \mu(x) = \frac{1}{\sqrt{q}} \sum_{j=0}^{q-1} \sum_{i=0}^{p-1} \epsilon^{(m)}_{j+p; i} \pi_i^{(p)}(x) \]

is an isometry between \(L_c^{(p)}|_{E_p(\zeta_p)}\) and \(L_c^{(m)}|_{E_m(\zeta_p)}\), where \(\pi_i^{(p)}: H_c^{-1} \to \mathbb{C}\) and \(\epsilon^{(p)}: H_{\mathbb{C}}^{-1} \to \mathbb{C}\) are the \(j\)-th coordinate projection and imbedding respectively on \(H_{\mathbb{C}}^{-1}\).

Proof. First we show that

\[ \overline{\mu}: E_p(\zeta_p) \to H_{\mathbb{C}}^{-1}, \quad \overline{\mu}(x) = \sum_{j=0}^{p-1} \sum_{i=0}^{q-1} \epsilon^{(m)}_{j+p; i} \pi_i^{(p)}(x) \]

is an injection and the image of \(\overline{\mu}\) is \(E_m(\zeta_p)\). In fact, by solving the equation \(\tau_x z = \zeta_p z(k = p, m)\) directly, we can check that

\[ E_p(\zeta_p) = \left\{ (x, \sum_{j=0}^{1} E_j x, \ldots, \sum_{j=0}^{p-2} E_j x) \in H_{\mathbb{C}}^{-1}; x \in \mathbb{C} = H_{\mathbb{C}} \right\} \]

and \(E_m(\zeta_p) = \overline{\mu}(E_p(\zeta_p))\), from which the injectivity of \(\overline{\mu}\) is obvious.

Since spaces \(E_p(\zeta_p)\) and \(E_m(\zeta_p)\) are such ones as described above, we can easily calculate \(L_c^{(m)}(\overline{\mu}(x), \overline{\mu}(y))\) for \(x, y \in E_p(\zeta_p)\) and have

\[ L_c^{(m)}(\overline{\mu}(x), \overline{\mu}(y)) = qL_c^{(p)}(x, y), \]

which means that \(\mu=(1/\sqrt{q}) \cdot \overline{\mu}\) is an isometry between \(L_c^{(p)}|_{E_p(\zeta_p)}\) and \(L_c^{(m)}|_{E_m(\zeta_p)}\). This completes the proof.

For \(\omega \in \mathbb{C}, |\omega| = 1, \omega \neq 1,\) define a Hermitian form \(L_{(\omega)}: (H \otimes \mathbb{C}) \times (H \otimes \mathbb{C}) \to \mathbb{C}\) by

\[ L_{(\omega)}(x \otimes \alpha, y \otimes \beta) = \alpha \beta((1 - \overline{\omega})L(x, y) + (1 - \omega)L(y, x)) \]
for \(x, y \in H\) and \(\alpha, \beta \in \mathbb{C}\). The following lemma is well-known (cf. [11, 4.7]).

**Lemma 1.6.** Let \(p(\geq 2)\) be an integer. If \(\zeta_p\) is a primitive \(p\)-th root of unity, then the form \(L_{(\zeta_p)}\) is isomorphic to the restriction to \(E_p(\zeta_p)\) of the form \(L_{\mathbb{F}_p}\).

Let \(\omega_2 = x + \sqrt{1-x^2} i \in \mathbb{C}\), \(x \in [-1, 1]\). For any real square matrix \(A\), define a \(t\)-Hermitian \(\mathcal{R}(t)\)-matrix

\[
A^-(t) = (2 - (t + t^{-1})) ((1 - t) A + (1 - t^{-1}) A^T).
\]

Kawauchi [6, §5] considered the "local signatures" \(\sigma_A^\pm(A)\), \(a \in [-1, 1]\), of \(A\) which are defined by \(\sigma_A^+(A) = \lim_{x \to +0} \text{sign} A^-(\omega_x) - \lim_{x \to +0} \text{sign} A^-(\omega_x) \) for \(a \in (-1, 1)\) and \(\sigma_A^-(A) = \lim_{x \to -1} \text{sign} A^-(\omega_x), \sigma_A^+(A) = \text{sign} (A + A^T) - \lim_{x \to -1} \text{sign} A^-(\omega_x).

**Lemma 1.7.** For \(\omega_0(\pm 1)\) satisfying \(\text{rank}_C(A - \omega_x A^T) = \text{rank}_R(\omega_0 (A - t A^T))\),

\[
\text{sign} ((1 - \omega_x) A + (1 - \omega_x) A^T) = \sum_{a \leq s \leq 1} \sigma_A^-(A).
\]

Proof. Note that \(A^-(t) = (1 - t^2)(1 - t^{-1}) (A - t^{-1}) A^T\). Let \(x_1 < x_2 < \cdots < x_n\) be the all points in the interval \((a, 1)\) satisfying \(\text{rank}_C(A - \omega_{x_i} A^T) < \text{rank}_R(\omega_0 (A - t^{-1}) A^T)\). By assumption, \(\text{rank}_C(A - \omega_x A^T) = \text{rank}_R(\omega_0 (A - t^{-1}) A^T)\) on \(x \in [a, 1) - \{x_1, x_2, \ldots, x_r\}\). Then by [6, Corollary 5.2],

\[
\text{sign} A^-(\omega_x) = \lim_{x \to i} \text{sign} A^-(\omega_x),
\]
and

\[
\lim_{x \to u} \text{sign} A^-(\omega_x) = \lim_{x \to u} \text{sign} A^-(\omega_x), \quad i = 1, \ldots, r - 1
\]

Thus

\[
\text{sign} ((1 - \omega_x) A + (1 - \omega_x) A^T) = \text{sign} A^-(\omega_x) = \text{sign} A^-(\omega_x) = \sum_{a \leq s \leq 1} \sigma_A^-(A).
\]

This completes the proof.

1.8. Proof of Proposition 1.3. For simplicity, we use the following notations:

\[
\langle k \rangle_m = E_m(e^{2\pi ik/m}), \quad k = 0, 1, \ldots, m - 1,
\]

\[
\sigma \langle k \rangle_m = \text{sign}(L_{(\omega_0)} | \langle k \rangle_m), \quad k = 0, 1, \ldots, m - 1
\]

\[
s_j = \sum_{s_{j+1} < s < s_j} \sigma^\omega(M), \quad j = 0, 1, \ldots, n - 1.
\]

Note that \(\langle 0 \rangle_m = \{0\}\) for all \(m\). We have to show \(\sigma^\omega(M_{(\omega_0)} = \sum_{j=0}^{n-1} (-1)^j s_j\).

First we consider the case when \(n\) is odd. In this case, \(H_{\mathbb{C}^{-1}}\) and \(H_{\mathbb{C}^{-1}}\)}
split into the orthogonal sums

\[ H_{c}^{2n-1} = \left( \frac{1}{k=1}^{n-1} \langle k \rangle_{2n} \perp \langle -k \rangle_{2n} \right) \perp \langle n \rangle_{2n} \]

and

\[ H_{c}^{-1} = \left( \frac{1}{k=1}^{(a-1)/2} \langle k \rangle_{n} \perp \langle -k \rangle_{n} \right) \]

with respect to \( L_{c}^{(2n)} \) and \( L_{c}^{(3)} \) respectively. By Proposition 1.2 and the fact \((\dagger)\) \( \sigma \langle 2k \rangle_{2n} = \sigma \langle k \rangle_{n} = \sigma \langle q \rangle_{p} \), where \( 0 < q < p, (p, q) = 1 \) and \( q/p = k/n \), which is derived from Lemma 1.5, we have

\[ \sigma^{(2n)}(M_{r}(n)) = \text{sign} L_{c}^{(2n)} - 2\text{sign} L_{c}^{(3)} \]

\[ = \sum_{k=1}^{n-1} 2\sigma \langle k \rangle_{2n} + \sigma \langle n \rangle_{2n} - 2 \sum_{k=1}^{(a-1)/2} 2\sigma \langle k \rangle_{n} \]

\[ = 2 \sum_{k=1}^{(a-1)/2} (\sigma \langle 2k - 1 \rangle_{2n} - \sigma \langle 2k \rangle_{2n} + \sigma \langle n \rangle_{2n}). \]

Note that \( \sigma \langle k \rangle_{m} = \sigma \langle -k \rangle_{m} \) by \( (\dagger) \) and Lemma 1.7. If the Alexander polynomial \( A(t) = \det(A - tA^{T}) \in R[t] \) has no 2n-th root of unity, then, by \( (\dagger) \) and Lemmas 1.6 and 1.7, we have

\[ \sigma \langle k \rangle_{2n} = \text{sign} L_{c}^{(2n)}(A(\sigma^{ik/n}A^{T}) + (1 - e^{-\sigma^{ik/n}}A^{T})) \]

\[ = \sum_{j=0}^{k-1} s_{j}, \]

for all \( k=1, 2, \ldots, n \), where \( A \) is a linking matrix on \( H = H_{1}(V; R) \). So we have \( \sigma \langle 2k - 1 \rangle_{2n} - \sigma \langle 2k \rangle_{2n} = -s_{2k-1}, k=1, 2, \ldots, (n-1)/2 \). Furthermore, by [6, Main Theorem], \( \sigma \langle n \rangle_{2n} = \sigma \langle 1 \rangle_{2} = \sigma^{(2)}(M) = \sum_{j=1}^{n-1} s_{j} \). Therefore

\[ \sigma^{(2n)}(M_{r}(n)) = 2 \sum_{k=1}^{(a-1)/2} (-s_{2k-1}) + \sum_{j=0}^{n-1} s_{j} = \sum_{j=0}^{n-1} (-1)^{j} s_{j}. \]

Next we consider the case when \( n \) is even. In this case, \( H_{c}^{2n-1} \) and \( H_{c}^{-1} \) split into the orthogonal sums

\[ H_{c}^{2n-1} = \left( \frac{1}{k=1}^{n-1} \langle k \rangle_{2n} \perp \langle -k \rangle_{2n} \right) \perp \langle n \rangle_{2n} \]

and

\[ H_{c}^{-1} = \left( \frac{1}{k=1}^{(n-2)/2} \langle k \rangle_{n} \perp \langle -k \rangle_{n} \right) \perp \langle n/2 \rangle_{n} \]
respectively. By the same argument as in the odd case, we have

$$\sigma_0^{(2\ell)}(M_{\gamma(u)}) = \text{sign } L_\ell^{(2n)} - 2\text{sign } L_\ell^{(g)}$$

$$= \left(\sum_{k=1}^{n-1} 2\sigma(k)_{2n} + \sigma(n)_{2n}\right) - 2 \left(\sum_{k=1}^{n-2} 2\sigma(k)_{2n} + \sigma(n/2)_{2n}\right)$$

$$= 2 \left(\sum_{k=1}^{n-2} (\sigma(2k-1)_{2n} - \sigma(2k)_{2n} - \sigma(n-1)_{2n} - \sigma(1)_{2n}\right)$$

$$= 2 (- \sum_{k=1}^{n-2} s_{2k-1} + \sum_{j=0}^{n-2} s_j) - \sum_{j=0}^{n-1} s_j$$

$$= 2 \sum_{j=0}^{n-1} (-1)^j s_j$$

This completes the proof.

**Example 1.9.** Let $k$ be a knot in $S^3$ and $M=M(k)$ denote $S^3$ surgered along $k$ with framing zero. Then $M$ is a homology handle. Let $\tilde{M}$ be the infinite cyclic cover of $M$ associated with any generator $\gamma$ of $H^1(M; \mathbb{Z})$. The quadratic form of $\tilde{M}$ on $H^1(\tilde{M}; \mathbb{R})$ (see [4, p. 186] for the definition) in the present case is non-singular (cf. [5, p. 99]).

If $k$ is a trefoil knot, then $H^1(\tilde{M}; \mathbb{R}) \approx \mathbb{R}^2/(t^2 - t + 1)$. Thus $\sigma_1^{(2)}(M) = \pm 2$ and $\sigma_2^a(M) = 0$ for $a = 1/2$ (cf. [9, Assertion 11] or [5, Lemma 1.4]). By Corollary 1.4, we have $\sigma_0^{(2\ell)}(M_{\gamma(2)}) = \sigma_1^{(2)}(M) = \pm 2$. This result can be obtained from a direct calculation of the quadratic form by using a mapping torus structure of $M(k)_{\gamma(2)}$ (cf. [10, p. 333]). Furthermore, if $k$ is the $g$-fold connected sum of trefoil knot, then the quadratic form of $\tilde{M}$ is the orthogonal sum of $g$ copies of the form of trefoil knot. Thus $\sigma_1^{(2)}(M(k)) = \pm 2g$ and $\sigma_2^a(M(k)) = 0$ for $a = 1/2$.

By Corollary 1.4, $\sigma_0^{(2\ell)}(M(k)_{\gamma(2)}) = \sigma_1^{(2)}(M(k)) = \pm 2g$, which of course coincides with the result obtained from the calculation using the mapping torus structure of $M(k)_{\gamma(2)}$.

### 2. Types of Imbeddings

Throughout this section, $M$ is a closed, connected, oriented 3-manifold and $W$ a closed, connected 4-manifold. We consider imbeddings $f: M \to W$.

First note that $f$ has at least two types according to whether $W - fM$ is connected or not. We say that $f$ is of type I (resp. type II) if $W - fM$ is connected (resp. disconnected). We can characterize the type I or II imbedding by examining the homomorphism $f_*: H_3(M; \mathbb{Z}) \to H_3(W; \mathbb{Z})$. If $f_* = 0$ then $f$ is of type I, and if $f_* = 0$ then $f$ is of type II and $W - fM$ has exactly two components. This is stated in [8] in the case when $W$ is orientable, and Kawauchi's
proof is valid for non-orientable 4-manifold $W$. Note that the coefficient of the (co-)homology in [8, p. 171] is $\mathbb{Z}_2$.

For the rest of this section we assume that $W$ is non-orientable, and classify the types of $f: M \to W$ more in detail. Let $p: W \to W$ be the orientation double covering of $W$.

**Type I imbedding.** A type I imbedding $f$ is called two-sided or one-sided according as the normal bundle of $f$ is trivial or not.

If $f$ is of type I and one-sided (called type $I_1$), we have two cases according as $W - fM$ is orientable or not. These two cases may be characterized by the types of the imbedding $\tilde{M} = p^{-1}(fM) \subset W$. That is, $W - fM$ is non-orientable (resp. orientable) if and only if $\tilde{M} \subset W$ is of type I (resp. type II). Thus we say that $f$ is of type $I_1 - 1$ (resp. type $I_1 - 2$) if $W - fM$ is non-orientable (resp. orientable).

If $f$ is of type I and two-sided (called type $I_2$), then $f$ can be lifted to two imbeddings $\tilde{f}: M \to W$, each of which is of type I. [To see that $\tilde{f}$ is of type I, note that there is a loop $\alpha$ in $W$ which intersects $fM$ transversely in a single point. If $\alpha$ preserves orientation, then one of the lifts of $\alpha$ to $W$ intersects $fM$ transversely in a single point. Thus $\tilde{f}_* = 0: H_3(M; \mathbb{Z}_2) \to H_3(W; \mathbb{Z}_2)$, which means $\tilde{f}$ is of type I. If $\alpha$ reverses orientation, then, by using the loop $p^{-1} \alpha$, we can do the same argument as above and have the same conclusion.]

**Type II imbedding.** Assume $f$ is of type II. Let $W_1, W_2$ be the components of $W - fM$. Since $W$ is non-orientable and $M$ is connected, we have the following two cases:

a) both $W_1$ and $W_2$ are non-orientable,

b) one of $W_1$ and $W_2$ is orientable and the other is non-orientable.

The type II imbedding $f$ can be lifted to two imbeddings $\tilde{f}: M \to W$. Take any one of them. Then it is easily seen that a) (resp. b)) is equal to the condition that $\tilde{f}$ is of type I (resp. $\tilde{f}$ is of type II). From this, in case a) (resp. b)) we say that $f$ is of type II - 1 (resp. type II - 2).

### 3. Proof of Theorem

Throughout this section, for a manifold $X$ with boundary, $DX$ denotes the double of $X$. For a closed oriented 3-manifold $M$ equipped with an element $\gamma \in H^1(M; \mathbb{Z})$, we define $\tau_\gamma(M) = \sum_{\alpha \in \langle a, 1 \rangle} \sigma_\gamma(M)$ for all $a \in [-1, 1]$ (cf. [8]). We denote by $\kappa(M)$ the rank of the kernel of the homomorphism $t - 1: H_1(M; \mathbb{Z}) \to H_1(M; \mathbb{Z})$, where $M$ is the infinite cyclic cover of $M$ associated with $\gamma$ and $t: H_1(M; \mathbb{Z}) \to H_1(M; \mathbb{Z})$ is the automorphism induced from the generator specified by $\gamma$ of the group of covering transformations on $M$ (cf. [8]).

For the rest of this section, $M$ denotes a closed, connected, oriented 3-manifold and $W$ denotes a compact, connected 4-manifold. Let $M^0$ denote the once punctured $M$. Recall that an element $\tilde{\gamma} \in H^1(DM^0; \mathbb{Z})$ is called $\mathbb{Z}_2$-asym-
metric if the mod 2 reduction \( \hat{\gamma}(2) \in H^1(DM^0; \mathbb{Z}_2) \) of \( \hat{\gamma} \) satisfies \( \rho_2(\hat{\gamma}(2)) = \hat{\gamma}(2) \) for the standard reflection \( \rho \) of \( DM^0 \) ([8, p. 179]). Theorem 3.1 of [8] can be extended to the case of orientable 4-manifold \( W \) with boundary.

**Lemma 3.1.** Assume that \( W \) is orientable and \( \partial W \neq \emptyset \). If \( M^0 \) is imbedded in \( W \), then \( \beta_1(M; \mathbb{Z}) \leq \beta_2(W; \mathbb{Z}_2) \) or there is a \( \mathbb{Z}_2 \)-asymmetric indivisible element \( \hat{\gamma} \in H^1(DM^0; \mathbb{Z}) \) such that for all \( a \in [-1, 1] \)

\[
|\tau^2_2(DM^0)| - \kappa_2^1(DM^0) \leq 2\beta_2(W; \mathbb{Z}) .
\]

Proof. Applying [8, Theorem 3.1] to the imbedding \( M^0 \subset W \subset DW \), we have the above conclusion. Note that \( \beta_2(DW; \mathbb{Z}) = 2\beta_2(W; \mathbb{Z}) \), \( \beta_2(DW; \mathbb{Z}_2) = 2\beta_2(W; \mathbb{Z}_2) \) and sign \( DW = 0 \).

We then think of non-orientable case.

**Lemma 3.2.** Assume \( W \) is non-orientable and closed. Let \( f: M \to W \) be an imbedding.

1. If \( f \) is of type \( I_2 \) or \( II-1 \), then \( \beta_1(M; \mathbb{Z}) \leq \beta_2(W; \mathbb{Z}_2) \) or there is a \( \mathbb{Z}_2 \)-asymmetric indivisible element \( \hat{\gamma} \in H^1(DM^0; \mathbb{Z}) \) such that for all \( a \in [-1, 1] \)

\[
|\tau^2_2(DM^0)| - \kappa_2^1(DM^0) \leq \beta_2(W; \mathbb{Z}) + \beta_2(W; \mathbb{Z}_2) .
\]

2. If \( f \) is of type \( II-2 \), then \( 2\beta_1(M; \mathbb{Z}) \leq \beta_2(W; \mathbb{Z}) + \beta_2(W; \mathbb{Z}_2) \) or there is an indivisible element \( \hat{\gamma} \in H^1(M; \mathbb{Z}) \) such that for all \( a \in [-1, 1] \)

\[
|\tau^2_2(M)| - \kappa_2^1(M) \leq \beta_2(W; \mathbb{Z}) + \beta_2(W; \mathbb{Z}_2) .
\]

Proof. Let \( \tilde{W} \) be the orientation double cover of \( W \). As seen in section 2, each imbedding of above types has a lift \( \tilde{f}: M \to \tilde{W} \). Applying [8, Theorems 2.1, 3.1] to \( \tilde{f} \) and noting the following lemma and the fact that sign \( \tilde{W} = 0 \) [because \( \tilde{W} \) admits an orientation-reversing involution], we have the result.

**Lemma 3.3.** Let \( X \) be a compact manifold and \( \tilde{X} \) be any double cover of \( X \). Then \( \beta_k(X; \mathbb{Z}) \leq \beta_k(\tilde{X}; \mathbb{Z}) \leq \beta_k(X; \mathbb{Z}) + \beta_k(X; \mathbb{Z}_2) \) and \( \beta_k(\tilde{X}; \mathbb{Z}_2) \leq 2\beta_k(X; \mathbb{Z}_2) \) for all \( k \).

Proof. By the transfer argument, we have \( \beta_k(X; \mathbb{Z}) \leq \beta_k(\tilde{X}; \mathbb{Z}) \). The inequality \( \beta_k(\tilde{X}; \mathbb{Z}) \leq \beta_k(X; \mathbb{Z}) + \beta_k(X; \mathbb{Z}_2) \) is the case \( d=2 \) of [1, Proposition 1.3]. The inequality \( \beta_k(\tilde{X}; \mathbb{Z}_2) \leq 2\beta_k(X; \mathbb{Z}_2) \) is readily obtained from the exact sequence of Smith homology groups used in the proof of [1, Proposition 1.3].

In the case of type \( I_1 \) imbedding, we cannot use [8, Theorem 2.1, 3.1] as in the proof of Lemma 3.2. But for certain \( M \) an estimation like Lemma 3.2 can be obtained by using the consequence of Section 1. For each positive integer \( r \), consider the class \( \mathcal{A}(r) \) of 3-manifolds consisting of the connected
sums of $r$ homology handles:

$$\mathcal{H}(r) = \{ M = \bigoplus_{i=1}^{r} M_i ; M_i is a 3-manifold with H_*(M_i; \mathbb{Z}) \cong H_*(S^2 \times S^1; \mathbb{Z}), \forall i \} .$$

Especially we have a subclass $\mathcal{H}'(r)$ of $\mathcal{H}(r)$ consisting of all $M = \bigoplus_{i=1}^{r} M_i$ such that each $M_i$ is $S^3$ surgered along a knot with framing zero (cf. Example 1.9). Note that, for any $M \in \mathcal{H}(r)$ and any $\hat{\gamma} \in H^1(M; \mathbb{Z})$, $\tau^{\hat{\gamma}}(M) = \sigma^{\hat{\gamma}}(M)$ and $\kappa^{\hat{\gamma}}(M) = 0$ (cf. [3]). For an (oriented) homology handle $M$, we denote by $\sigma(M)$ (resp. $\sigma_s(M)$) the signature $\sigma^{\hat{\gamma}}(M)$ (resp. the local signature $\sigma_s^{\hat{\gamma}}(M)$) associated with any generator $\hat{\gamma}$ of $H^1(M; \mathbb{Z})$. [Note that $\sigma^{\hat{\gamma}}(M) = \sigma_s^{\hat{\gamma}}(M)$ and $\sigma_s^{\hat{\gamma}}(M) = \sigma_s^{\hat{\gamma}}(M)$.]

**Lemma 3.4.** Let $W$ be as in Lemma 3.2. Let $M = \bigoplus_{i=1}^{r} M_i$ be the connected sum of homology handles $M_i$, $i=1, 2, \ldots, r$. If $M$ is type I, imbedded in $W$, then $r \leq \beta_2(W; \mathbb{Z}_2)$ or there are numbers $(1 \leq i, i_2, \ldots, i_p, i_{p+1}, \ldots, i_q \leq r)$ such that

$$\sum_{j=1}^{r} \sum_{-l \leq i \leq l} \varepsilon_i \sign(a) \sigma(M_{ij}) + \sum_{j=p+1}^{q} \varepsilon_i \sigma(M_{ij}) \leq \beta_2(W; \mathbb{Z}) + \beta_2(W; \mathbb{Z}_2),$$

where $\varepsilon_i = 1$, or $-1$, $j=1, 2, \ldots, q$.

Proof. Assume that $M$ is type $I_1$ imbedded in $W$. We think $M$ is a submanifold of $W$. If $p: W \to W$ is the orientation double covering of $W$, then $M^{(2)} = p^{-1} M \subset W$ is a double cover of $M$.

Since the mod 2 reduction $H^1(M; \mathbb{Z}) \cong \bigoplus_{i=1}^{r} \mathbb{Z} \to H^1(M; \mathbb{Z}_2) \cong \bigoplus_{i=1}^{r} \mathbb{Z}_2$ is onto, any double cover of $M$ is associated with the mod 2 reduction $\psi(2) \in H^1(M; \mathbb{Z}_2)$ of some $\psi \in H^1(M; \mathbb{Z})$. For each $i=1, 2, \ldots, r$, the restriction $\psi(2) | M_i$ is the $\delta_i$ multiple of the generator of $H^1(M_i; \mathbb{Z}_2) \cong \mathbb{Z}_2$, where $\delta_i = 0$ or 1. Thus we denote $\psi(2)$ by $\psi[\delta_1, \ldots, \delta_r]$.

We may assume $\psi(2)$ is the double cover corresponding to $\psi[1, \ldots, 1, 0, \ldots, 0]$ by permuting the indices if necessary. Then $M^{(2)}$ is diffeomorphic to

$$\bigoplus_{i=1}^{m} M^{(2)}_i \cong \bigoplus_{i=m+1}^{r} M_i \cong \bigoplus_{i=m+1}^{r} M_i \cong (\# S^2 \times S^1),$$

where $M^{(2)}_i$ denotes the unique (up to equivalence) double cover of $M_i$.

Put $\hat{M} = (\#_{i=1}^{r} M^{(2)}_i) \# (\#_{i=m+1}^{r} M_i)$. Since $\hat{M}$ can be imbedded into $W$. Applying Theorem 3.1 of [8] and using Lemma 3.3, we have $r \leq \beta_2(W; \mathbb{Z}_2)$ or there is a $\mathbb{Z}_2$-asymmetric indivisible element $\hat{\gamma} \in H^1(D\hat{M}; \mathbb{Z})$ such that $|\sigma^{\hat{\gamma}}(D\hat{M})| \leq \beta_2(W; \mathbb{Z}) + \beta_2(W; \mathbb{Z}_2)$. (Note that $\tau^{\hat{\gamma}}(D\hat{M}) = \sigma^{\hat{\gamma}}(D\hat{M})$.)

Since $D\hat{M} = [\#_{i=1}^{r} (M^{(2)}_i \# - M^{(2)}_i)] \# [\#_{i=m+1}^{r} (M_i \# - M_i)]$, we have

$$\sigma^{\hat{\gamma}}(D\hat{M}) = \sum_{i=1}^{r} \sigma^{\hat{\gamma}}(M^{(2)}_i \# - M^{(2)}_i) \sigma^{\hat{\gamma}}(M_i \# - M_i),$$
where \( \tilde{\eta}_i \) is the restriction of \( \tilde{\eta} \) to the \( i \)-th summand, \( i = 1, 2, \ldots, r \). Let \( \{ i_j; 1 \leq j \leq p \} \) (resp. \( \{ i_j; p+1 \leq j \leq q \}) \) be the set of all integers \( i \) between 1 and \( m \) (resp. \( m+1 \) and \( r \)) such that the restriction \( \tilde{\eta}_i \) of \( \tilde{\eta} \) is still \( \mathbb{Z}_2 \)-asymmetric. Then by [8, Lemma 1.3] we have

\[
\sigma^i(D[M^0]) = \sum_{j=1}^s \varepsilon_j \sigma^i_j(M^{(2)}_i) + \sum_{j=p+1}^q \varepsilon_j \sigma(M_{ij})
\]

for some \( \varepsilon_j \in \{1, -1\}, j = 1, 2, \ldots, q \), where \( \dot{\gamma}^{(2)}_{ij} \in H^*(M_i^{(2)}; \mathbb{Z}) = \mathbb{Z} \) is the element defined, as in section 1, by a generator \( \dot{\gamma}_{ij} \) of \( H^1(M_i; \mathbb{Z}); j = 1, 2, \ldots, p \). Compare the proof of [8, Theorem 3.2]. Since \( \sigma^{\dot{\gamma}^{(2)}_{ij}}(M^{(2)}_i) = \sum_{-1 \leq a \leq 1} \text{sign}(a) \sigma_a(M_i) \) by Corollary 1.4, this implies the inequality (*). This completes the proof.

We now prove Theorem.

3.5. Proof of Theorem for orientable 4-manifold \( W \). Assume that \( W \) is compact, connected and orientable. If \( W \) is closed, then Theorem is an immediate consequence of [8, Theorem 3.2] showing that, for sufficiently large \( r_0 \) and for all \( r > r_0 \), certain elements of \( \mathcal{M}'(r) \) cannot be imbedded in \( W \). If \( W \) is bounded, then Lemma 3.1 implies Theorem by the same argument as the proof of [8, Theorem 3.2].

3.6. Proof of Theorem for non-orientable 4-manifold \( W \). Assume first that \( W \) is closed, connected and non-orientable. Let \( M[g] = S^3 \) surgered along the \( g \)-fold connected sum of trefoil knot with framing zero. Recall that, for any generator \( \dot{\gamma} \in H^1(M[g]; \mathbb{Z}) = \mathbb{Z}, \quad \sigma^\dot{\gamma}(M[g]) = \sigma^{\dot{\gamma}(2)}(M[g][s]) = \pm 2g \) (cf. Example 1.9).

From now on, assume \( M = \#_{i=1}^s M[g_i] \). We show that if \( M \) is imbedded in \( W \), then one of the following conditions holds:

1. \( r \leq \beta_2(W; \mathbb{Z}). \)
2. For some numbers \( (1 \leq i_1, i_2, \ldots, i_s \leq r) \) and for some choice of \( \varepsilon_j \in \{1, -1\}, j = 1, 2, \ldots, s \), the inequality

\[
|\varepsilon_{i_1} g_{i_1} + \varepsilon_{i_2} g_{i_2} + \cdots + \varepsilon_{i_s} g_{i_s}| \leq \beta_2(W; \mathbb{Z}) + \beta_2(W; \mathbb{Z})
\]

holds.

In fact, if \( M \) is type \( I \) imbedded in \( W \), then, by Lemma 3.4, we obtain the desired result. If \( M \) is type \( I_2 \) or \( II-1 \) imbedded in \( W \), then, by Lemma 3.2-(1), we have the above result. Compare the proof of Lemma 3.4. If \( M \) is type \( II-2 \) imbedded in \( W \), then by Lemma 3.2-(2) we have \( r \leq \beta_2(W; \mathbb{Z}) + \beta_2(W; \mathbb{Z}) \) or the above condition (2) holds. Note that for an indivisible element \( \dot{\gamma} \in H^1(M; \mathbb{Z}) \), if \( M[g_{i_j}], j = 1, 2, \ldots, s, 1 \leq i_1 < i_2 < \cdots < i_s \leq r \) are the all summands of \( M \) such that \( \dot{\gamma}|M[g_{i_j}] \) is an odd multiple of a generator of \( H^1(M[g_{i_j}]; \mathbb{Z}) = \mathbb{Z} \), we have
\[ \tau^\mu(M) = \sigma^\mu(M) = 2(\varepsilon_1 g_{i_1} + \varepsilon_2 g_{i_2} + \cdots + \varepsilon_s g_{i_s}) \]

for some \( \varepsilon \in \{1, -1\} \) (cf. [8, Lemma 1.3]).

Thus, if we take \( r_0 = \beta_2(W; \mathbb{Z}_2) \), then for all \( r > r_0 \) and for \( \{ g_i \}_{i=1}^{r} \) such that

\[ g_i \geq \beta_2(W; \mathbb{Z}_2) \quad \text{and} \quad g_i \geq \beta_2(W; \mathbb{Z}_2) + \sum_{j=1}^{i-1} g_j, \quad i = 2, 3, \ldots, r, \]

\( M = \#^{r-1} M[g_i] \) cannot be imbedded in \( W \). This implies Theorem for closed non-orientable 4-manifold \( W \).

To have Theorem for non-orientable 4-manifold \( W \) with boundary, we have only to use the doubling technique as in the orientable case. The proof of Theorem is completed.

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References
