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<th>A generalization of Magnus’ theorem</th>
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<tr>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 14(2) P.403-P.409</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1977</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/8846">https://doi.org/10.18910/8846</a></td>
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<td>DOI</td>
<td>10.18910/8846</td>
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Let $f(x, y)$ and $g(x, y)$ be polynomials in two variables with integral coefficients. O.H. Keller raised the problem in [1]: If the functional determinant $\partial(f, g)/\partial(x, y)$ is equal to 1, then is it possible to represent $x$ and $y$ as polynomials of $f$ and $g$ with integral coefficients? This problem drew many mathematicians' attention and several attempts have been made by enlarging the coefficient domain to the complex number field $\mathbb{C}$. But no success has been reported yet. On the other hand A. Magnus studied the volume preserving transformation of complex planes and obtained a result which is relevant to Keller's problem ([2]). From his results it is immediately deduced that Keller's problem is answered affirmatively provided one of $f(x, y)$ and $g(x, y)$ has prime degree. For the proof Magnus used recursive formulas. But these formulas are complicated and not easy to handle. In this paper we shall give a simple proof of his theorem based on the notion of quasi-homogeneity for generalized polynomials. Moreover we shall go one step further than he did. Our results ensure that Keller's problem is valid provided one of $f(x, y)$ and $g(x, y)$ has degree 4 or larger degree is of the form $2p$ with an odd prime $p$.

Since a complete solution of Keller's problem is not found yet our paper will be of some interest and worth-while publication.

1. Quasi-homogeneous generalized polynomials

Let $x$ and $y$ be two indeterminates. We shall set $A = \sum_{i,j} C x^i y^j$ where $C$ is the complex number field and $\mathbb{Z}$ is the ring of rational integers. $A$ is a graded ring and the polynomial ring $C[x, y]$ is a graded subring. Hereafter we shall call an element $f(x, y)$ of $A$ a generalized polynomial or simply a $g$-polynomial. We shall denote by $S(f)$ the set of lattice points $(i, j)$ in the real two space $\mathbb{R}^2$ such that the monomial $x^i y^j$ appears in $f(x, y)$ with a non-zero coefficient. $S(f)$ will be called the support of $f(x, y)$. A $g$-polynomial $f(x, y)$ is called a homogeneous $g$-polynomial or a $g$-form if $S(f)$ lies in the straight line of the form $X+Y=m$ where $m \in \mathbb{Z}$ and is called the degree of the $g$-form $f(x, y)$.

*) Supported by Takeda Science Foundation.
We shall use the symbol $S[f]$ to denote the set of monomials $x^i y^j$ such that the lattice point $(i,j)$ is in $S(f)$.

**Proposition 1.** Let $f(x, y)$ and $g(x, y)$ be non-constant $g$-forms of degrees $m$ and $n$ respectively such that the functional determinant $\frac{\partial(f, g)}{\partial(x, y)}$ is equal to zero. We shall define an integer $d$ by the rule: (a) $d$ is equal to the GCD of $|m|$ and $|n|$ if one of $m$ and $n$ is positive, (b) $d$ is equal to the negative of GCD ($|m|$, $|n|$) if both of $m$ and $n$ are negative. We shall set $m/d=m'$ and $n/d=n'$. Then we have the following:

(i) If one of $m$ and $n$ is zero, so is the other and $f(x, y)$ and $g(x, y)$ are $g$-polynomials in one variable $(y/x)$.

(ii) If $mn<0$, then both of $f(x, y)$ and $g(x, y)$ are monomials and there exist a monomial $h(x, y)$ of degree $d$ such that $f=c_i h^{m'}$ and $g=c_j h^{n'}$ where $c_i (i=1, 2)$ are constants.

(iii) If $mn>0$, there exists a $g$-form $h(x, y)$ of degree $d$ such that $f=c_i h^{m'}$ and $g=c_j h^{n'}$.

**Proof.** Assume first $m=0$ and $n \neq 0$. It follows from $\frac{\partial(f, g)}{\partial(x, y)}=0$ that we have $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0$. This is against the assumption. Since a $g$-form of degree zero is necessarily of the form $\sum a_i (y/x)^i$ we get the assertion (i). To prove (ii) we assume $m>0$ and $n<0$ and let $f_1=f^{-n}$ and $g_1=g^n$. Then $\frac{\partial(f_1, g_1)}{\partial(x_1, y_1)}=0$. Since the degrees of $f_1$ and $g_1$ differ only in sign we see immediately that $f_1 \frac{\partial f_1}{\partial x} + g_1 \frac{\partial f_1}{\partial y}=0$, or equivalently, $\frac{\partial(f_1, g_1)}{\partial x}=0$. Similarly we have $\frac{\partial(f_1, g_1)}{\partial y}=0$. Hence $f_1, g_1$ must be a constant. But such a case can occur only when $f_1$, hence $f$, is a monomial because $g_1$ is a $g$-polynomial. The rest follows easily from this. The proof of (iii) will be carried out by a similar device and the detailed proof will be omitted.

**Definition.** A $g$-polynomial $f(x, y)$ is called a quasi homogeneous $g$-polynomial (or simply a quasi $g$-form) if the support $S(f)$ of $f(x, y)$ is contained in the straight line. When the equation of that straight line has the form $Y+\alpha X=\lambda$. We shall say that the quasi $g$-form $f(x, y)$ is $(\alpha)$-homogeneous of degree $\lambda$.

It should be noticed that if $\alpha$ is an irrational number, monomials only can be $(\alpha)$-homogeneous $g$-forms.

**Proposition 2.** Let $f(x, y)$ and $g(x, y)$ be $(\alpha)$-homogeneous $g$-forms of positive degrees $\lambda$ and $\mu$ respectively such that $\frac{\partial(f, g)}{\partial(x, y)}=0$. Assume that $\alpha$ is a rational number $q/p$ with coprime integers $p (>0)$ and $q$. Let $d=\text{GCD}(p \lambda, p \mu)$. Then there exists an $(\alpha)$-homogeneous $g$-form $h(x, y)$ of degree $d/p$ such that $f=c_i h^{m'}$ and $g=c_j h^{n'}$ where $m'=p \lambda/d$, $n'=p \mu/d$ and $c_i (i=1, 2)$ are constants.
Proof. Let \( u, v \) be new indeterminates and let \( x = u^p \) and \( y = v^q \). Then \( F(u, v) = f(u^p, v^q) \) and \( G(u, v) = g(u^p, v^q) \) are \( g \)-forms of degrees \( p\lambda \) and \( p\mu \) respectively. The rest follows easily from Proposition 1.

Let \( \gamma \) be an arbitrary real number. Then we can define a grading on \( A \) in the following way. Let \( \lambda \) be a real number and let \( A_\lambda \) be the vector space over \( \mathbb{C} \) generated by the set of \( \gamma \)-monomials \( x^j y^i \) such that \( j + \gamma i = \lambda \). Then we have \( A = \bigoplus_\lambda A_\lambda \) where the sum is extended over all real numbers contained in the additive subgroup of \( \mathbb{R} \) generated by 1 and \( \gamma \). In case \( \gamma = 1 \) we have the standard grading and its degree function is the ordinary function. The term "homogeneous" is reserved for this standard grading.

**Proposition 3.** Let \( f(x, y) \) and \( g(x, y) \) be \( g \)-polynomials in \( x \) and \( y \) such that \( \partial (f, g) / \partial (x, y) \in \mathbb{C} \). Let \( \alpha \) be any real number and let \( f = \bigoplus f_\lambda \) and \( g = \bigoplus g_\mu \) be the direct sum decomposition by the \( (\alpha) \)-grading. Then we have

\[
\sum_{\lambda + \mu = s} \partial (f_\lambda, g_\mu) \partial (x, y) = 0.
\]

The proof is immediate and will be omitted.

2. Magnus' Theorem

For future reference we shall give Magnus' Theorem in a slightly different formulation from Magnus' original one.

**Theorem 1.** Let \( f(x, y) \) and \( g(x, y) \) be polynomials in two variables \( x \) and \( y \) with complex coefficients and let \( m \) and \( n \) be the degrees of \( f(x, y) \) and \( g(x, y) \) respectively. Assume that the functional determinant \( \partial (f, g) / \partial (x, y) \) is a nonzero constant. If \( \min (m, n) > 1 \), then we have \( \text{GCD}(m, n) > 1 \).

Proof. Assume that \( \text{GCD}(m, n) = 1 \). Let \( f_\alpha \) and \( g_\alpha \) be the degree forms of \( f \) and \( g \) respectively. From proposition 1, there is a linear form, say \( h \), such that \( f_\alpha = \varepsilon h^m \) and \( g_\alpha = \varepsilon h^n \). Without loss of generality we can assume that \( h = x \) and \( \varepsilon = 1 \). We shall pick up a point \( P = (p_1, p_2) \) in \( S(f) \) in the following way. Let \( L \) be the line defined by the equation \( X = m \) and let \( L \) rotate around the point \( M = (m, 0) \) counterclockwise until \( L \) meets a point in \( S(f) \) other than \( M \). Let \( l \) be the line thus obtained. The point in \( S(f) \cap l \) with the smallest \( X \)-coordinate is the desired point \( P \). Pick up a point \( Q = (q_1, q_2) \) in \( S(g) \) in a similar way.

Now assume we have either \( (m > 1) p_2 > 0 \) or \( (n > 1) q_2 > 0 \). Then we easily verify that one of the following situation takes place.

(1) The lines \( MP \) and \( NQ \) are not parallel where \( N = (n, 0) \).

(2) The three points \( P, Q \) and the origin are not collinear.

If the case (a) occurs let
be the equations of the lines \( MP \) and \( NQ \) respectively. Then we have \( a \neq b \).

If \( a > b \) let \( \gamma \) be a real number such that \( a > \gamma > b \). If we choose \( \gamma \) near enough to \( a \), then \( x^{p_1} y^{p_2} \) will have the highest \((\gamma)\)-degree in \( S[f] \) and \( x^\alpha \) will have the highest \((\gamma)\)-degree in \( S[g] \). Hence by Proposition 3, \( \partial(x^{p_1} y^{p_2}, x^\alpha)/\partial(x, y) = np_2 x^{p_1+p_2-1} y^{p_2-1} = 0 \). But this is impossible. Similarly we have a contradiction if \( a < b \).

Now assume the lines \( MP \) and \( NQ \) are parallel, i.e., \( a = b \) then we have the case (2), i.e., \( p_2 q_1 + q_1 p_2 \). Let \( \gamma = a - \varepsilon \) with \( \varepsilon < 0 \). If we choose \( \varepsilon \) small enough, then \( x^{p_1} y^{p_2} \) will have the highest \((\gamma)\)-degree in \( S[f] \) and \( x^{p_1} y^{p_2} \) will have the highest \((\gamma)\)-degree in \( S[g] \). But this contradicts Proposition 3 because we have \( q_1 p_2 \neq q_2 p_1 \).

Thus we have seen that \( p_2 = q_2 = 0 \), i.e., \( f(x, y) \) and \( g(x, y) \) are polynomials in \( x \) alone. But this is impossible because \( \partial(f, g)/\partial(x, y) \) is a non-zero constant, and the proof of Theorem 1 is complete.

For the sake of reference we shall call the method adopted in this proof “the method of rotation of lines around the points \( M \) and \( N \)”.

3. A generalization of Magnus' Theorem

**Theorem 2.** Under the same notations and assumptions as Theorem 1, we have the following: If \( \operatorname{Min}(m, n) > 2 \), then we have \( \operatorname{GCD}(m, n) > 2 \).

Proof. Assume that \( \operatorname{Min}(m, n) > 2 \) and \( \operatorname{GCD}(m, n) = 2 \) and we shall draw a contradiction. Let \( f_m \) and \( g_n \) be degree forms of \( f \) and \( g \) respectively. From Proposition 2 it follows that there exists a quadratic form \( h(x, y) \) such that \( f_m = ah^m \) and \( g_n = bh^n \), where \( m = 2m' \) and \( n = 2n' \). There are two possibilities.

(I) \( h \) is a product of two independent linear forms. In this case we can assume without loss of generalities that \( f_m = (xy)^{m'} \) and \( g_n = (xy)^{n'} \). Apply the method of rotation of lines around the points \( M_1 = (m', n') \) and \( N_1 = (n', m') \). Then we can easily see that any point \((i, j)\) in \( S(f) \) satisfies the condition \( j \leq m' \), and any point \((s, t)\) in \( S(g) \) satisfies the condition \( t \leq n' \).

Now consider the \((0)\)-grading in \( A \). The degree forms of \( f \) and \( g \) are respectively of the forms

\[
\begin{align*}
  f_m^{(0)} &= y^{m'}(a_0 + a_1 x + \ldots + a_{m'-1} x^{m'-1} + x^m) \\
  g_n^{(0)} &= y^{n'}(b_0 + b_1 x + \ldots + b_{n'-1} x^{n'-1} + x^n)
\end{align*}
\]

From Propositions 2 and 3 there is a linear form \( c + x \) such that

\[
\begin{align*}
  f_m^{(0)} &= y^{m'}(c + x)^{m'} \quad \text{and} \quad g_n^{(0)} = y^{n'}(c + x)^{n'}
\end{align*}
\]

If we set \( x_i = c + x \) and consider \( f \) and \( g \) as polynomials in new variables \( x_i \) and...
Then the support \( S_i(f) \) have no point \((i, j)\) with \(j \geq m'\) except the point \((m', m')\). Similarly \( S_i(g) \) have no point \((s, t)\) with \(t \geq n'\). Apply again the method of rotation of lines around the points \(M_i\) and \(N_i\). Then we can see finally that no point \((i, j)\) with \(i < j\) is in \(S(f)\) and no point \((s, t)\) with \(s < t\) is in \(S(g)\). This means that \(f(x, y)\) and \(g(x, y)\) lack the terms \(y^s(s > 1)\). This is impossible because of the assumption \(\partial(f, g)/\partial(x, y)\) is an element of \(C^*\).

\(\text{(II)}\) \(h\) is a power of a linear form: In this case we can assume as before that the degree forms are of the forms \(f_n = x^n\) and \(g_n = x^n\) respectively. Then we can see, following the method of rotations of lines around the point \(M = (m, 0)\) and \(N(n, 0)\), that \(S(f)\) is contained in the region defined by the inequality \(Y + \frac{1}{2}X \leq \frac{m}{2}\) and \((g)\) is in the region \(Y + \frac{1}{2}X \leq \frac{n}{2}\). Consider \((1/2)\)-grading and apply Propositions 2 and 3. Then we see that degree forms of \(f\) and \(g\) by this grading are

\[(ay + x^2)^n'\] and \[(ay - x^2)^n'\]

respectively. If \(a = 0\) we can proceed further and we see that no point \((i, j)\) with \(j > 0\) is in \(S(f)\) and no point \((s, t)\) with \(t > 0\) is in \(S(g)\). This is a contradiction. Hence we must have \(a \neq 0\). Then apply de Jonquière transformation

\[Y_1 = ay + x^2, \quad x_1 = x.\]

Since we have

\[f(x, y) = (ay + x^2)^n' + \sum_{j + i \leq n'} a_{ij} x^i y^j\]

and

\[g(x, y) = (ay + x^2)^n' + \sum_{j + i \leq n'} b_{ij} x^i y^j\]

We easily see that

\[f_i(x_1, y_1) = y_1^{n'} + \sum_{j + i \leq n'} a_{ij} x_1^i y_1^j\]

and

\[g_i(x_1, y_1) = y_1^{n'} + \sum_{j + i \leq n'} b_{ij} x_1^i y_1^j\]

where

\[f_i(x_1, y_1) = f(x_1, a^{-1}(y_1 - x_1^2))\] and \(g_i(x_1, y_1) = g_i(x, a^{-1}(y_1 - x_1^2))\). By the method of (clockwise) rotation of lines around the points \((0, m')\) and \((0, n')\) applied to the pair of polynomials \(f_i(x_1, y_1)\) and \(g_i(x_1, y_1)\), we see that
4. Application to Keller's problem

**Theorem 3.** Let \( f(x, y) \) and \( g(x, y) \) be polynomials of degrees \( m \) and \( n \) respectively with complex coefficients and assume that the functional determinant \( \frac{\partial(f, g)}{\partial(x, y)} \) is a non-zero constant. Then we have \( C[x, y] = C[f, y), g(x, y)] \) in the following three cases:

1. \( m \) or \( n \) is a prime number;
2. \( m \) or \( n \) is 4;
3. \( m = 2p^t \) where \( p \) is an odd prime.

**Proof.** In any case it follows from Theorems 1 and 2 that smaller degree, say \( n \), divides larger degree \( m \). Then from Proposition 2 and 3 the degree forms \( f_m \) and \( g_n \) are related like this, \( f_m = \varepsilon g_n^{m/n} \). Then

\[
f_i = f - (\varepsilon^{m/n}g)^{m/n}
\]

has lower degree than \( f \) and \( \partial(f, g)/\partial(x, y) = \partial(f, g)/\partial(x, y) \) is a non-zero constant. Thus we can use induction on the sum \( m+n \) of degrees to get the conclusion. q.e.d.

Keller's Original problem is also settled in these three cases cited in Theorem 3 because of the following

**Proposition 4.** Let \( f(x, y) \) and \( g(x, y) \) be the polynomials in \( x \) and \( y \) with integer coefficients such that the functional determinant is equal to 1 and \( C[f, g] = C[x, y] \). Then we have necessarily \( Z[f, g] = Z[x, y] \).

**Proof.** It suffices to prove that \( x \) and \( y \) are in \( Z[f, g] \). By assumption we have

\[
x = \sum c_{ij} f^i g^j, \quad c_{ij} \in \mathbb{C}.
\]

If we set

\[
f(x, y) = f_{10}x + f_{01}y + \cdots
\]

\[
g(x, y) = g_{10}x + g_{01}y + \cdots
\]

then the assumption implies that \( f_{10}g_{01} - f_{01}g_{10} = 1 \). Apply the unimodular transformation of variables
Then \( \mathbb{Z}[x, y] = \mathbb{Z}[x', y'] \) and \( f = x' + (\text{higher degree terms}) \) and \( g = y' + (\text{higher degree terms}) \). Hence to prove the assertion we can assume without loss of generalities that linear parts of \( f \) and \( g \) are \( x \) and \( y \) respectively. We shall define a linear order in the set \((i, j)\) of lattice points in \( \mathbb{R}^2 \) by the way: \((i, j) > (i', j')\) if (i) \( i + j > i' + j' \) or (ii) \( i + j = i' + j' \) and \( i > i' \). We shall show that every \( c_{ij} \) is in \( \mathbb{Z} \) by induction on the linear order just defined. Assume every \( c_{i', j'} \) with \((i', j') < (i, j)\) is integer. Then the coefficients of the polynomial

\[
    c_{ij} f^i g^j + c_{i+1, j-1} f^{i+1} g^{j-1} + \cdots + c_{0i+j+1} g^{i+j+1} + \cdots
\]

are integers. In this polynomial \( x^i y^j \) appears once with the coefficient \( c_{ij} \). Hence \( c_{ij} \) must be an integer. Similarly \( y \) is in \( \mathbb{Z}[f, g] \) and the assertion is proved completely.

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References

