| | |
|---|---|
| Title | Convergence of Riemannian manifolds and Albanese tori |
| Author(s) | Kasue, Atsushi |
| Citation | Osaka Journal of Mathematics. 1995, 32(3), p. 677-688 |
| Version Type | VoR |
| URL | https://doi.org/10.18910/8847 |
| rights | |
| Note | |

Kasue, A
Osaka J. Math.
**32** (1995), 677-688

# CONVERGENCE OF RIEMANNIAN MANIFOLDS AND ALBANESE TORI

ATSUSHI KASUE*

(Received July 19, 1993))

## 0. Introduction

The purpose of this note is to prove the convergence of the Albanese tori of compact Riemannian manifolds which collapse to a lower dimensional space while keeping their curvatures and diameters bounded.

Given a compact Riemannian manifold $M$, we denote by $H^1(M)$ the space of harmonic one-forms on $M$ equipped with an inner product $\langle\ ,\ \rangle$ defined by

$$\langle\omega,\ \eta\rangle=\int_M(\omega,\ \eta)d\mu_M,$$

where $\mu_M$ stands for the normalized Riemannian measure of $M$ with unit mass, $\mu_M=d\mathrm{vol}_M/\mathrm{Vol}(M)$. Let $H^1(M)_Z$ be a lattice of $H^1(M)$ which consists of harmonic one-forms of integral periods. Dividing the dual space $H^1(M)^*$ by the dual lattice $H^1(M)^*_Z$, we obtain a flat torus, called the Albanese torus of $M$,

$$\mathscr{A}(M)=H^1(M)^*/H^1(M)^*_Z.$$

We may view $\mathscr{A}(M)$ as a map of the set of isometry classes of compact Riemannian manifolds $M$ onto that of flat tori.

Given a positive integer $m$, a nonnegative number $\varkappa$ and a positive one $D$, we write $\mathscr{S}(m,\varkappa,D)$ for the set of isometry classes of compact Riemmanian $m$-manifolds $M$ such that the Ricci curvature of $M$ is bounded from below by $-(m-1)\varkappa^2$ and the diameter of $M$ is bounded from above by $D$. Then according to Gromov [7], for a Riemannian manifold $M$ in $\mathscr{S}(m,\ \varkappa,\ D)$, the dimension of the Albanese torus $\mathscr{A}(M)$, namely, the first Betti number $b_1(M)$, has an upper bound depending only on the dimension $m$ of $M$ and $\varkappa D$. Using this, we shall show the following

*partly supported by Grant-in-Aid for Scientific Research, The Ministry of Education, Science and Culture, Japan.

**Theorem 1.** *Given $m \in Z^+$, $\varkappa \geq 0$ and $D > 0$, there is a constant $C$ depending only on $m$ and $\varkappa D$ such that*

$$\operatorname{diam}(\mathcal{A}(M)) \leq C \operatorname{diam}(M)$$

*for any $M$ in $\mathcal{S}(m, \varkappa, D)$.*

We would like to ask if the convergence of compact Riemannian manifolds in $\mathcal{S}(m, \varkappa, D)$ with respect to the spectral distance would imply the convergence of their Albanese tori. Here we recall the definition of a *spectral distance* between two compact Riemannian manifolds which was introduced in [9]. Given two compact Riemannian manifolds $M$ and $N$, a mapping $f: M \to N$ is called an *$\varepsilon$-spectral approximation* if

$$e^{-(1/t+t)}|p_M(t, x, y) - p_N(t, f(x), f(y))| < \varepsilon$$

for all $t > 0$, and for all points $x$, $y$ of $M$, where $p_M(t, x, y)$ and $p_N(t, u, v)$ denote respectively the heat kernel of $M$ in $L^2(M, \mu_M)$ and that of $N$ in $L^2(N, \mu_N)$. The spectral distance, $SD(M, N)$, between $M$ and $N$ is by definition the lower bound of the positive numbers $\varepsilon$ such that there exist $\varepsilon$-spectral approximations $f: M \to N$ and $h: N \to M$. According to [9], we know that (i) the metric space ($\mathcal{S}(m, \varkappa, D)$, $SD$) is precompact; (ii) the eigenvalues and eigenfunctions of compact Riemannian manifolds in $\mathcal{S}(m, \varkappa, D)$ are continuous with respect to the spectral distance in a certain sense; (iii) the topology on $\mathcal{S}(m, \varkappa, D)$ induced by the spectral distance is finer than that of measured Hausdorff convergence introduced by Fukaya [3] and hence that of the Gromov-Hausdorff distance. Moreover if we denote by $\mathcal{K}(m, \varkappa, D)$ the set of isometry classes of compact Riemmanian $m$-manifolds such that the sectional curvatures are bounded by $\varkappa^2$ in the absolute values and the diameters are not greater than $D$, then the topologies of the spectral distance and the measured Hausdorff convergence coincide on this set $\mathcal{K}(m, \varkappa, D)$. See [3], [8] and [9] for details. We note also that the spectral distance and the Gromov-Hausdorff distance induce the same topology on the set of flat tori.

The following theorem shows that the above question is affirmative if we restrict ourselves to the class $\mathcal{K}(m, \varkappa, D)$ for given numbers $m$, $\varkappa$, and $D$.

**Theorem 2.** *Let $\{M_i\}$ be a sequence in $\mathcal{K}(m, \varkappa, D)$ which converges with respect to the spectral distance. Then the Albanese torus $\mathcal{A}(M_i)$ converges to a flat torus $\mathcal{A}_\infty$ of dimension $n$ with $0 \leq n \leq \liminf_{i \to \infty} b_1(M_i)$.*

Here in our convention, $\mathcal{A}_\infty$ stands for a point when $n=0$. We remark also that under the assumption of this theorem, we are able to show the convergence of the Albaness maps in a certain sense. See Section 3 for details.

The proofs of Theorems 1 and 2 are respectively given in Sections 2 and 3. For the latter, we shall basically make use of some results in [8]. In this sense, the present paper is a continuation of [8].

## 1. Albanese Tori

In this section, we shall consider a compact Riemannian manifold $M$ endowed with a certain measure $\mu$ and define the Albanese torus and the Albanese map of such a pair $(M, \mu)$ (cf. [10], [11]).

Let $M=(M, g_M)$ be a compact Riemannian manifold of dimension $m$ and $\mu$ a measure on $M$ with smooth density function $\chi>0$. A one-form $\omega$ on $M$ is said to be *$\mu$-harmonic* if $\omega$ is closed and co-closed with respect to $\mu$, namely, $d\omega=0$ and

$$\delta_\mu \omega := -\operatorname{trace}(\nabla \omega) - \omega(\nabla \log \chi) = 0.$$

In other words, $\mu$-harmonic one-form $\omega$ can be expressed locally as the differential of an $L_\mu$-harmonic function $f$, $\omega=df$. Here a smooth function $f$ defined on an open set in $M$ is called an $L_\mu$-harmonic function if

$$L_\mu f := \frac{1}{\chi}\operatorname{div}(\chi \nabla f) = \Delta f + \nabla \log \chi \cdot f = 0.$$

We denote by $H^1(M, \mu)$ the space of $\mu$-harmonic one-forms on $M$ and by $H^1(M, \mu)_Z$ the lattice of $H^1(M, \mu)$ which consists of one-forms with integral periods. The vector space $H^1(M, \mu)$ is endowed with an $L^2$ inner product $\langle\ ,\ \rangle$ defined by

$$\langle \omega, \eta \rangle_\mu = \int_M (\omega, \eta)\, d\mu.$$

The norm of $\omega \in H^1(M, \mu)$ is denoted by $\|\omega\|_\mu$.

Similarly, given a compact Riemannian manifold $N$, we say a smooth mapping $\psi$ of $M$ into $N$ is *$\mu$-harmonic* if it satisfies

$$\tau(\psi) + d\psi(\nabla \log \chi) = 0,$$

where $\tau(\psi)$ stands for the tension field of the mapping $\psi$. A $\mu$-harmonic mapping $\psi$ is a stationary point of the energy functional

$$E_\mu(\psi) = \int_M e(\psi)\, d\mu.$$

When $N$ is a circle of length 1, $N=R/Z$, we write $\mathscr{H}(M, \mu; R/Z)$ for the set of $\mu$-harmonic mappings of $M$ into $R/Z$, which forms an additive group in an obvious manner. Since the derivative $d\phi$ of a smooth mapping $\phi$ of $M$ into $R/Z$ may be considered as an integral one-form on $L$, we have a natural surjective homomorphism $d$ of $\mathscr{H}(M, \mu; R/Z)$ onto $H^1(M, \mu)_Z$ whose kernel is the set of constant mappings $\theta \in R/Z$. We note that the homomorphism $d$ preserves the norms in the sense that

$$E_\mu(\psi) = \langle d\psi, d\psi \rangle_\mu$$

for $\psi \in \mathcal{H}(M, \mu; R/Z)$.

The Albanese torus $\mathcal{A}(M, \mu)$ of a pair $(M, \mu)$ is by definition a flat torus derived from dividing the dual space $H^1(M, \mu)^*$ by the dual lattice $H^1(M, \mu)^*_Z$,

$$\mathcal{A}(M, \mu) = H^1(M, \mu)^* / H^1(M, \mu)^*_Z.$$

Let $\tilde{M}$ be the universal covering of $M$ and $\pi$ the projection of $\tilde{M}$ onto $M$. If we fix a point $p$ of $M$ and take a point $\tilde{p}$ of $\tilde{M}$ with $\pi(\tilde{p}) = p$, then we have a map $\tilde{J}_{M,\mu}$ of $\tilde{M}$ into the dual space $H^1(M, \mu)^*$ defined by

$$\tilde{J}_{M,\mu}(\tilde{x})(\omega) = \int_{\tilde{p}}^{\tilde{x}} \pi^* \omega.$$

This map induces a $\mu$-harmonic map $J_{M,\mu}$ of $M$ into $\mathcal{A}(M, \mu)$ (with $J_{M,\mu}(p) = 0$). We call $J_{M,\mu}$ the Albanese map of a pair $(M, \mu)$.

Let $\Omega = \{\omega_1, \ldots, \omega_r\}$ be a basis of $H^1(M, \mu)_Z$ and $\Omega^* = \{\omega_1^*, \ldots, \omega_r^*\}$ the dual basis. Then a diffeomorphism $T_\Omega$ of $\mathcal{A}(M, \mu)$ onto $R^r/Z^r$ is derived from a linear isomorphism of $H^1(M, \mu)^*$ onto $R^r$:

$$\theta_1 \omega_1^* + \cdots + \theta_r \omega_r^* \to (\theta_1, \ldots, \theta_r).$$

If we set a metric $g_\Omega$ on $R^r/Z^r$ by

$$g_\Omega = \sum_{\alpha,\beta=1,\cdots,r} \langle \omega_\alpha^*, \omega_\beta^* \rangle_\mu d\theta_\alpha d\theta_\beta,$$

then $T_\Omega$ induces an isometry between $\mathcal{A}(M, \mu)$ and $(R^r/Z^r, g_\Omega)$. Moreover if we take a $\mu$-harmonic map $\psi_\alpha : M \to R/Z$ in such a way that

$$\psi_\alpha(p) = 0, \quad \omega_\alpha = d\psi_\alpha,$$

we see that

$$T_\Omega \circ J_{M,\mu}(x) = (\psi_1(x), \ldots, \psi_r(x))$$

for $x \in M$. Here we remark that

$$E_\mu(J_{M,\mu}) = b_1(M).$$

Given a pair $(M, \mu)$, we define a symmetric tensor $Ric_{M,\mu}$ by

$$Ric_{M,\mu} = Ric_M - \frac{1}{\chi} \nabla^2 \chi (= Ric_M - d \log \chi \otimes d \log \chi - \nabla^2 \log \chi),$$

where $Ric_M$ (resp., $\chi$) stands for the Ricci tensor of $M$ (resp., the density function of $\mu$, $\mu = \chi d\mathrm{vol}_M$). Given $m$, $\kappa$ and $D$ as before, we denote by $\mathcal{S}_w^*(m, \kappa, D)$ the set of equivalence classes of pairs $(M, \mu)$ such that $\dim M = m$, the diameter $\mathrm{diam}(M)$ of $M \leq D$, $\mu$ has unit mass, and $Ric_{M,\mu} \geq -(m-1)\kappa^2 g_M$. Here we say two pairs $(M, \mu)$ and $(N, \nu)$ are equivalent when there is an isometry $f : M \to N$ which preserves the measures, $f_* \mu = \nu$. We remark that the spectral distance $SD$ can be defined on the set of equivalence classes of pairs $(M, \mu)$. See [9] for some properties of the metric space $(\mathcal{S}_w^*(m, \varkappa, D), SD)$ as mentioned in Introduction.

In what follows, when $\mu$ is the canonical Riemannian measure $\mu_M$ with unit mass, $\mu_M = d\mathrm{vol}/\mathrm{Vol}(M)$, we omit to indicate the dependency of the measure $\mu_M$ in some of the above notations (for example, $\mathcal{A}(M)$ stands for $\mathcal{A}(M, \mu_M)$).

## 2. Proof of Theorem 1

We recall first that there is a positive constant $C'$ depending only on $m$ and $\varkappa D$ such that

$$e(\psi) \leq C' E_{\mu_M}(\psi) \tag{2.1}$$

for all $\psi \in \mathcal{H}(M\,;\ R/Z)$, because the energy density $e(\psi)$ satisfies

$$\Delta e(\psi) \geq -2(m-1)\varkappa^2 e(\psi)$$

(cf. e.g., [8-a, §4]). In addition, we note that if $\psi$ is not constant, the energy density $e(\psi)$ must be greater than or equal to $1/4\ \mathrm{diam}(M)^2$ somewhere on $M$ (otherwise, the distance between $\psi(p)$ and $\psi(q)$ in $R/Z$ for any pair of points $p$, $q$ of $M$ would be less than $1/2$, and hence the harmonic map $\psi$ should be constant). Therefore we have

$$E_{\mu_M}(\psi) \geq \frac{1}{4C'\mathrm{diam}(M)^2}$$

for all nonconstant $\psi \in \mathcal{H}(M\,;\ R/Z)$; in other words,

$$\langle \omega, \omega \rangle_{\mu_M} \geq \frac{1}{4C'\mathrm{diam}(M)^2}$$

for all nonzero $\omega \in H^1(M)_Z$. This implies that the first eigenvalue $\lambda_1(\mathcal{A}(M))$ is bounded from below by $\pi^2/C'\mathrm{diam}(M)^2$. On the other hand, we know that

$$\lambda_1(\mathcal{A}(M)) \leq \frac{C''}{\mathrm{diam}(\mathcal{A}(M))^2}$$

for some constant $C''$ depending only on the dimension of $\mathcal{A}(M)$, and hence on $m$ and $\varkappa D$, since

$$b_1(M) \leq \frac{v_m(\varkappa^2,\ 5\mathrm{diam}(M))}{v_m(\varkappa^2,\ \mathrm{diam}(M))}, \tag{2.2}$$

where $v_m(\varkappa^2, r)$ stands for the volume of a metric ball in the simply connected space form of dimension $m$ with constant curvature $\varkappa^2$ (cf. [7]). Thus the assertion of Theorem 1 is clear. This completes the proof of Theorem 1.

Let $(M, \mu)$ be a pair in $\mathcal{S}_w^*(m, \varkappa, D)$. Then it is not hard to see that the above submean value inequality (2.1) holds for any $\mu$-harmonic map $\psi \in \mathcal{H}(M, \mu\,;\ R/Z)$ (cf. e.g., [8-a, §4]). Moreover it follows from the same reason as in deriving (2.2) that

$$b_1(M) \leq \frac{v_{m+1}(\varkappa^2, 5\mathrm{diam}(M))}{v_{m+1}(\varkappa^2, \mathrm{diam}(M))},$$

because we have a Bishop-Gromov type inequality for the pair $(M, \mu)$(cf. [9, §2]). Thus Theorem 1 holds for $(M, \mu)$. Namely we have

**Theorem 1′.** *Given $m$, $\varkappa$ and $D$ as before, there is a constant $C'$ depending only on $m$ and $\varkappa D$ such that*

$$\mathrm{diam}(\mathcal{A}(M, \mu)) \leq C'\mathrm{diam}(M).$$

*for all $(M, \mu) \in \mathcal{S}_w^*(m, \varkappa, D)$.*

REMARKS. (1) When $\varkappa=0$ in Theorem 1, the classical Bochner theorem says that any harmonic one-form is parallel, so that $J_M$ is a Riemannian submersion with totally geodesic fibers and in particular the diameter of $\mathcal{A}(M)$ is less than or equal to the diameter of $M$ ($C=1$ in Theorem 1). This is also true for Theorem 1' (cf. [12]). (2) A slightly different proof of Theorem 1 is presented in [6].

### 3. Proof of Theorem 2

The proof of the theorem is divided into 4 steps and the same notations as in the preceding sections will be used.

**Step 1.** We shall start with recalling the notions of convergence of Gromov-Hausdorff distance and measured Hausdorff topology introduced by Gromov [7] and Fukaya [3] respectively. Given a sequence of compact Riemannian manifolds, $\{M_i\}$ , we say that $M_i$ converges to a compact matric space $X$ with respect to the Gromov-Hausdorff distance, if there are a sequence of positive numbers $\{\varepsilon_i\}$ with $\lim_{i\to\infty} \varepsilon_i=0$ and mappings $f_i: M_i \to X$ such that the $\varepsilon$-neighborhood of $f_i(M_i)$ covers $X$ and $|d_{M_i}(x, y) - d_X(f_i(x), f_i(y))| < \varepsilon$ for all $x$, $y$ of $M_i$. Moreover we say that $M_i = (M_i, \mu_{M_i})$ converges to a pair $(X, \mu)$ of $X$ and a Borel measure $\mu$ on $X$ with respect to the measured Hausdorff topology, if $f_i$ are Borel measurable and the push-forward $f_{i*}\mu_{M_i}$ of the normalized Riemannian measure $\mu_{M_i}$ via $f_i$ converges to $\mu$ in the weak* topology.

Let $M_i$ be a sequence in $\mathcal{K}(m, \varkappa, D)$ which converges to a compact metric space $M_\infty$ with respect to the Gromov-Haussdorff distance. Then there is a smooth manifold $F_\infty$ with metric of class $C^{1,\alpha}$ (for any $\alpha \in (0, 1)$), on which the orthogonal group $O(m)$ acts by isometries in such a way that the quotient space $F_\infty/O(m)$ is isometric to $M_\infty$. In fact, $F_\infty$ is a limit of the set of the frame bundles $FM_i$ of $M_i$ equipped with a canonical metric so that the action of $O(m)$ is isometric, the projection of $FM_i$ onto $M_i$ is a Riemannian submersion with totally geodesic fibers, and the sectional curvature of $FM_i$ remains bounded uniformly in $i$. When $M_\infty$ is smooth, there is a fibration $\Phi_i: M_i \to M_\infty$ (for large $i$) and a sequence of positive numbers $\{\varepsilon_i\}$ with $\lim_{i\to\infty}\varepsilon_i=0$ satisfying

(1) for all $z\in M_\infty$, the diameter of $\Phi_i^{-1}(z)\le\varepsilon_i$;

(2) $\Phi_i$ is an $\varepsilon_i$-almost Riemannian submersion, that is, for all $z\in M_\infty$, $x\in\Phi_i^{-1}(z)$ and $X\in T_xM_i$ normal to the fiber $\Phi_i^{-1}(z)$,

$$(1-\varepsilon_i)|d\Phi_i(X)|\le|X|\le(1+\varepsilon_i)|d\phi_i(X)|;$$

(3) the second fundamental form of the submersion $\Phi_i$ is bounded uniformly in $i$.

For these assertions, see [2], [4] and [5].

Now as in Theorem 2, we suppose that $M_i$ converges to $(M_\infty, \mu_\infty)$ with respect to the spectral distance and hence the measured Hausdorff topology. Then we may assume that the push-forward $\Phi_{i*}\mu_{M_i}$ of the canonical Riemannian measure $\mu_{M_i}$ of $M_i$ converges to $\mu_\infty$ in the weak* topology. In case $M_\infty$ is smooth, the density function $\chi_\infty$ of $\mu_\infty$ is a positive function of class $C^{1,\alpha}$. Moreover we may assume that the above submersion $\Phi_i$ has the following property: for all smooth function $h$ on $M_\infty$,

$$|\Delta_{M_i}\Phi_i^*(h)-\Phi_i^*(L_{\mu_\infty}h)|\le\varepsilon_i\Phi_i^*(|Ddh|+|dh|). \tag{4}$$

See [8-a] for this and further properties of $\Phi_i$.

In the following Steps 2 and 3, we consider the case that the limit metric space $M_\infty$ is a smooth manifold, and assume that the metric of $M_\infty$ and the density $\chi_\infty$ are smooth, to avoid some technical argument of approximation. Moreover $\{\varepsilon_i\}$ stands for a sequence of positive constants which tends to zero as $i$ goes to infinity.

**Step 2.** Given a $\mu_\infty$-harmonic one-form $\omega\in H^1(M_\infty, \mu_\infty)$, the pull-back $\Phi_i^*\omega$ can be uniquely expressed as

$$\Phi_i^*\omega=\Gamma_i(\omega)+d\Lambda_i(\omega)$$

according to the orthogonal decomposition of $d$-closed one-forms $Z(M_i)$ of $M_i$,

$$Z^1(M_i)=H^1(M_i, \mu_{M_i})\oplus dC^\infty(M_i).$$

Here the function $\Lambda_i(\omega)$ is chosen in such a way that

$$\int_{M_i}\Lambda_i(\omega)d\mu_i=0.$$

Now we claim that

$$|\Phi_i^*\omega-\Gamma_i(\omega)|_{C^0(M_i)}=|d\Lambda_i(\omega)|_{C^0(M_i)}\le\varepsilon_i \tag{3.1}$$

for any $\omega\in H^1(M_\infty, \mu_\infty)$ with unit norm, $\|\omega\|_{\mu_\infty}=1$. Indeed, we fix a sufficiently small $a$ and consider the metric ball $B_\infty(p, 3a)$ of $M_\infty$ around a point $p$ of radius $3a$. Let $f$ be an $L_\infty$-harmonic function on $B_\infty(p, 3a)$ such that $\omega=df$ and

$$\int_{B_\infty(p,3a)}fd\mu_\infty=0.$$

Then applying the Poincaré inequality, we have first

$$\int_{B_\infty(p,3a)}|f|^2 d\mu_\infty \leq C_1 \int_{B_\infty(p,3a)}|df|^2 d\mu_\infty \leq C_1$$

for some constant $C_1$. Since $f$ is $L_\infty$-harmonic, it follows from the standard elliptic regularity estimates that

$$|f|_{C^{2,\alpha}(B_\infty(p,2a))} \leq C_2$$

for some constant $C_2$, where $\alpha \in (0, 1)$. Hence in view of the property (4) of $\Phi_i$, we see that

$$|\Delta_{M_i}\Phi_i^* f| \leq \varepsilon_i$$

on $\Phi_i^{-1}(B_\infty(p, a))$. This shows that

$$|\Delta_{M_i}\Lambda_i(\omega)| \leq \varepsilon_i,$$

since $d\Phi_i^* f = \Gamma_i(\omega) + d\Lambda_i(\omega)$ and $\Gamma_i(\omega)$ is harmonic. Finally it follows from the regularity estimates again that

$$|\Lambda_i(\omega)|_{W^{2,p}(M_i,\, \mu_{M_i})} \leq \varepsilon_i,$$

where $p \in (1, \infty)$ (cf. [8-a, Lemma 1.3]), and hence

$$|\Lambda_i(\omega)|_{C^{1,\alpha}(M_i)} \leq \varepsilon_i.$$

This proves (3.1).

Now this estimate (3.1) together with the property (3) of $\Phi_i$ implies that

$$(1-\varepsilon_i)\|\omega\|_{\mu_\infty} \leq \|\Gamma_i(\omega)\|_{\mu_i} \leq (1+\varepsilon_i)\|\omega\|_{\mu_\infty} \tag{3.2}$$

For all $\omega \in H^1(M_\infty, \mu_\infty)$. In particular, $\Gamma_i$ is injective (for large $i$). We observe further that $\Gamma_i$ maps the lattice $H^1(M_\infty, \mu_\infty)_Z$ into the lattice $H^1(M_i)_Z$,

$$\Gamma_i(H^1(M_\infty, \mu_\infty)_Z) \subset H^1(M_i)_Z.$$

**Step 3.** Given any number $K$, Theorem 4.3 in [8-a] says that for large $i$ , a harmonic one-form $\xi$ on $M_i$ with integral periods must belong to the image $\Gamma_i(H^1(M_\infty, \mu_\infty)_Z)$, whenever the $L^2$ norm $\|\xi\|_{\mu_i}$ is less than $K$. In other words, there is a positive constant $K_i$ with $\lim_{i\to\infty} K_i = \infty$ such that

$$\|\xi\|_{\mu_i} \geq K_i$$

for any $\xi \in H^1(M_i)_Z \setminus \Gamma_i(H^1(M_\infty, \mu_\infty)_Z)$ (if it exists).

Let us now take a basis $\Omega = \{\omega_1, \ldots, \omega_r\}$ of $H^1(M_\infty, \mu_\infty)_Z (r = b_1(M_\infty))$ in such a way that an element $\omega$ of $H^1(M_\infty, \mu_\infty)_Z$ is a linear combination of $\omega_1, \ldots, \omega_{s-1}$ whenever $\|\omega\|_{\mu_\infty}$ is less than $\|\omega_s\|_{\mu_\infty}$ (cf. [1, Chap. VIII]). Then we choose a basis $\Omega_i = \{\omega_{i,1}, \ldots, \omega_{i,r_i}\}$ of $H^1(M_i)_Z (r_i = b_1(M_i))$ in such a way that

$$\omega_{i,s}=\Gamma_i(\omega_s) \quad (s=1, \dots, r)$$

and any element $\omega$ is linearly dependent of $\omega_{i,1}, \dots, \omega_{i,s-1}$ whenever $\|\omega\|_{\mu_i}$ is less than $\|\omega_{i,s}\|_{\mu_i}$ for $s>r$. We note that

$$\|\omega_{i,s}\|\geq K_i \tag{3.3}$$

for $s>r$ (if $r_i>r$).

Let $\Omega_i^*=\{\omega_{i,s}^*\}$ $(s=1, \dots, r_i)$ be the dual basis of $\Omega_i$ and $\Gamma_i^*$: $H^1(M_i)^*\to H^1(M_\infty, \mu_\infty)^*$ the dual mapping of $\Gamma_i$. Then $\Gamma_i^*$ is surjective and its kernel is spanned by $\omega_{i,s}(s=r+1, \dots, r_i)$. Hence $\Gamma_i^*$ induces a surjective homomorphism, denoted by $[\Gamma_i^*]$, from the Albanese torus $\mathcal{A}(M_i)$ of $M_i$ onto $\mathcal{A}(M_\infty, \mu_\infty)$. Then in view of (3.2) and (3.3), $\mathcal{A}(M_i)$ converges via $[\Gamma_i^*]$ to $\mathcal{A}(M_\infty, \mu_\infty)$ with respect to the Gromov-Hausdorff distance. We observe that $[\Gamma_i^*]$ is affine, namely the second fundamental form vanishes identically. Moreover if we take a point $p_\infty$ of $M_\infty$ and choose $p_i$ as a fixed point of $M_i$ in such a way that $\Phi_i(p_i)=p_\infty$, then the mappings $J_{M_\infty,\mu_\infty}$ and $[\Gamma_i^*]\circ J_{M_i}$ are close for large $i$ in the sense that

$$\max_{x\in M_i} \operatorname{dis}(J_{\mu_\infty}\circ\Phi_i(x), [\Gamma_i^*]\circ J_i(x))\leq\varepsilon_i.$$

To be precise, let $T_{\Omega_i}$: $\mathcal{A}(M_i)\to(R^{r_i}/Z^{r_i}, g_{\Omega_i})$ and $T_\Omega$: $\mathcal{A}(M_\infty, \mu_\infty)\to(R^r/Z^r, g_\Omega)$ respectively be isometries described in Section 1, and let $\pi_i$: $R^{r_i}/Z^{r_i}\to R^r/Z^r$ be a canonical projection such that $\pi_i(\theta_1, \dots, \theta_{r_i})=(\theta_1, \dots, \theta_r)$. Then $T_{\Omega_i}\circ J_{M_i}$ and $T_\Omega\circ J_{M_\infty,\mu_\infty}$ respectively can be expressed as

$$T_{\Omega_i}\circ J_{M_i}=(\psi_{i,1},\dots,\psi_{i,r_i})$$

and

$$T_\Omega\circ J_{M_\infty,\mu_\infty}=(\psi_1, \dots, \psi_r),$$

where $\psi_{i,s}$ is the harmonic mapping of $M_i$ to $R/Z$ corresponding to $\omega_{i,s}$ and also $\psi_s$ is the $\mu_\infty$-harmonic mapping of $M_\infty$ to $R/Z$ corresponding to $\omega_s$. We note that for each $s$, $1\leq s\leq r$, $\psi_{i,s}$ is homotopic to $\psi_i\circ\Phi_i$ and further that

$$|\psi_{i,s}-\psi_s\circ\Phi_i|_{C^{2,\alpha}(M_i)}\leq\varepsilon_i$$

(cf. [8-a, §4]). Thus the mappings $J_{M_\infty,\mu_\infty}\circ\Phi_i$ and $[\Gamma_i^*]\circ J_{M_i}$ (for large $i$) are close with respect to the $C^{2,\alpha}$ topology.

**Step 4.** It remains to prove Theorem 2 in case $M_\infty$ is not smooth. In this case, we consider the frame bundle $FM_i$ of each $M_i$ equipped with a canonical metric $\bar{g}_i$ in such a way that the sectional curvature and the diameter are bounded uniformly in $i$. We denote by $\rho_i$ the canonical projection of $FM_i$ onto $M_i$. Observe that the pull-back $\rho_i^*\omega$ of a harmonic one-form $\omega$ on $M_i$ is harmonic on $FM_i$ and further that this correspondence preserves the inner products,

$$\langle\rho_i^*\omega, \rho_i^*\omega'\rangle_{\bar{\mu}_i}=\langle\omega, \omega'\rangle_{\mu_i}.$$

For this reason, the space of harmonic one-forms $H^1(M_i)$ on $M_i$ endowed with the $L^2$ inner product can be considered as a subspace of $H^1(FM_i)$. In the same way, we identify the lattice $H^1(M_i)_Z$ and the group $\mathcal{H}(M_i; R/Z)$ respectively with a sublattice of $H^1(FM_i)_Z$ and a subgroup of $\mathcal{H}(FM_i; R/Z)$. Under this identification, an element $\bar{\psi}$ of $\mathcal{H}(FM_i; R/Z)$ belongs to the subgroup $\mathcal{H}(M_i; R/Z)$ if and only if $\bar{\psi}$ is $O(m)$-invariant.

In what follows, we suppose that this sequence $\{FM_i\}$ converges in the topology of measured Hausdorff convergence. Let $\bar{M}_\infty$ and $\bar{\mu}_\infty$ be the limit space and measure respectively. Then according to Fukaya [4, 5], $\bar{M}_\infty$ is a smooth manifold with Riemannian metric $\bar{g}_\infty$ of class $C^{1,\alpha}$, on which the orthogonal group $O(m)$ acts as isometries in such a way that the quotient space $\bar{M}_\infty/O(m)$ is isometric to $M_\infty$. Moreover there are $O(m)$-equivariant almost Riemannian submersions $\tilde{\Phi}_i$: $FM_i \to \bar{M}_\infty$ such that $\tilde{\Phi}_{i*}\bar{\mu}_{FM_i}$ converges to $\bar{\mu}_\infty$ in the weak* topology, where $\bar{\mu}_{FM_i}$ stands as before for the normalized Riemannian measure of $FM_i$. We note that the limit measure $\mu_\infty$ on $M_\infty$ coincides with the push-forward $\rho_{\infty*}\bar{\mu}_\infty$ of $\bar{\mu}_\infty$ via the projection $\rho_\infty$: $\bar{M}_\infty \to M_\infty$ and the density $\bar{\chi}_\infty$ of $\bar{\mu}_\infty$ with respect to the Riemannian measure of $\bar{g}_\infty$ is $O(m)$-invariant.

Now perturbing the submersion $\tilde{\Phi}_i$ in the $C^{1,\alpha}$ topology, we can obtain an almost Riemannian submersion $\bar{\Phi}_i$ of $FM_i$ onto $\bar{M}_\infty$, to which we can apply the same arguments as in the preceding steps. To be precise, we write first $\mathcal{H}(M_\infty, \mu_\infty; R/Z)$ for the subgroup of $\mathcal{H}(\bar{M}_\infty, \bar{\mu}_\infty; R/Z)$ consisting of $O(m)$-invariant $\bar{\mu}_\infty$-harmonic maps $\bar{\psi}$ : $\bar{M}_\infty \to R/Z$. We note that $\mathcal{H}(M_\infty, \mu_\infty; R/Z)$ *is determined by the pair* $(M_\infty, \mu_\infty)$ *itself* (cf. [8-a, §4]). Then we denote by $H^1(M_\infty, \mu_\infty)_Z$ and $H^1(M_\infty, \mu_\infty)$ respectively the sublattice of $H^1(\bar{M}_\infty, \bar{\mu}_\infty)_Z$ corresponding to $\mathcal{H}(M_\infty, \mu_\infty; R/Z)$, $H^1(\bar{M}_\infty, \bar{\mu}_\infty)_Z = d\mathcal{H}(M_\infty, \mu_\infty; R/Z)$, and the vector space spanned by $H^1(M_\infty, \mu_\infty)_Z$. Set

$$\mathcal{A}(M_\infty, \mu_\infty) = H^1(M_\infty, \mu_\infty)^* / H^1(M_\infty, \mu_\infty)^*_Z.$$

Then we obtain an $O(m)$-invariant $\bar{\mu}_\infty$-harmonic map $\bar{J}_{M_\infty,\mu_\infty}$: $\bar{M}_\infty \to \mathcal{A}(M_\infty, \mu_\infty)$, from which a Lipschitz map $J_{M_\infty,\mu_\infty}$: $M_\infty \to \mathcal{A}(M_\infty, \mu_\infty)$ is derived. This map $J_{M_\infty,\mu_\infty}$ is $\mu_\infty$-harmonic on the set of regular points of $M_\infty$. Moreover as we have seen in Steps 2 and 3, $\bar{\Phi}_i$ (for large $i$) induces a surjective homomorphism $[\bar{\Gamma}_i^*]$: $\mathcal{A}(FM_i) \to \mathcal{A}(F_\infty, \bar{\mu}_\infty)$ such that $\bar{J}_{M_\infty,\mu_\infty} \circ \bar{\Phi}_i$ and $[\bar{\Gamma}_i^*] \circ J_{FM_i}$ are close in the $C^{2,\alpha}$ topology. Finally we obtain a surjective homomorphism $[\Gamma_i^*]$: $\mathcal{A}(M_i) \to \mathcal{A}(M_\infty, \mu_\infty)$ from $[\bar{\Gamma}_i^*]$ such that $J_{M_\infty,\mu_\infty} \circ \Phi_i$ and $[\Gamma_i^*] \circ J_{M_i}$ are close in the $C^0$ topology for large $i$, where $\Phi_i$: $M_i \to M_\infty$ is a Lipschitz map derived from the $O(m)$-equivariant submersion $\tilde{\Phi}_i$. As $i$ goes to infinity, the Albanese torus $\mathcal{A}(M_i)$ converges to the torus $\mathcal{A}(M_\infty, \mu_\infty)$ via the surjective homomorphism $[\Gamma_i^*]$. This completes the proof of Theorem 2.

It is possible to apply the same arguments as above to a sequence of certain pairs $(M_i, \mu_i)$ (cf. [8-a, Remark 3.3]). In fact, we can show the following

**Theorem 2.** *Let $\{(M_i, \mu_i)\}$ be a sequence in $\mathcal{S}^*_w(m, x, D)$ which converges to $(M_\infty, \mu_\infty)$ with respect to the measured Hausdorff topology. Suppose that the sectional curvature of $M_i$ is bounded uniformly and also the density function $\chi_i$ of $\mu_i$ satisfies*

$$|Dd\chi_i| \leq C$$

*for some constant C. Then for large i, there is a surjective homomorphism $\Theta_i$ of the Albanese torus $\mathcal{A}(M_i, \mu_i)$ onto a flat torus $\mathcal{A}(M_\infty, \mu_\infty)$ of dimension n such that $0 \leq n \leq \liminf_{i\to\infty} b_1(M_i)$ and $\mathcal{A}(M_i, \mu_i)$ converges to $\mathcal{A}(M_\infty, \mu_\infty)$ with respect to the Gromov-Hausdorff distance via $\Theta_i$.*

*Moreover there are a Lipschits map $\Phi_i$ of $M_i$ onto $M_\infty$ through which $(M_i, \mu_i)$ converges to $(M_\infty, \mu_\infty)$ and a ($\mu_\infty$-harmonic) map $J_{M_\infty,\mu_\infty}: M_\infty \to \mathcal{A}(M_\infty, \mu_\infty)$ such that $J_{M_\infty,\mu_\infty} \circ \Phi_i$ and $\Phi_i \circ J_{M_i,\mu_i}$ are close in the $C^0$ topology, namely,*

$$\lim_{i\to\infty} \max_{x \in M_i} \operatorname{dis}(J_{M_\infty,\mu_\infty} \circ \Phi_i(x),\ \Theta_i \circ J_{M_i,\mu_i}(x)) = 0.$$

*The convergence holds in the $C^{2,\alpha}$ topology when $M_\infty$ is a smooth manifold.*

As an immediate consequence of Theorem 2', we have the following

**Corollary 3.** *Given numbers m, x and D, and given a flat torus T of dimension n , there is a constant such that the rank of the Albanese map $J_M$ of a manifold M in $\mathcal{K}(m, x, D)$ is greater than or equal to n, if the Gromov-Hausdorff distance between M and T is less than r, and in addition, $J_M$ is a submersion if $b_1(M) = n$.*

Finally we refer the reader to [12] for some results and problems related to this corollary.

## References


[1] J.W.S. Cassel: An introduction to the geometry of numbers, Springer-Verlag, Berlin-Hidelberg-New York, 1959.

[2] J. Cheeger, K. Fukaya and M. Gromov: *Nilpotent structures and invariant metrics on collapsed manifolds*, J. Amer. Math. Soc. **5** (1992), 327-372.

[3] K. Fukaya: *Collapsing Riemannian manifolds and eigenvalues of the Laplace operator*, Invent. Math. **87** (1987), 517-547.

[4] K. Fukaya: *A boundary of the set of the Riemannian manifolds with bounded curvatures and diameters*, J. Differential Geom. **28** (1988), 1-21.

[5] K. Fukaya: *Collapsing Riemannian manifolds to ones of lower dimension II*, J. Math. Soc. Japan **41** (1989), 333-356.

[6] I. Fukuyama: Master Thesis, Osaka University, 1993.

[7] M. Gromov: Structure mētrique pour les variētés riemanniennes (rēdige par J. Lafontaine and P. Pansu), Cedic/Fernand Nathan, Paris, 1981.

[8] A. Kasue: (a) *Measured Hausdorff convergence of Riemannian manifolds and Laplace operators*, Osaka J. Math. **30** (1993), 613-651; (b)—II, Complex Geometry (ed. by G. Komatsu and Y. Sakane), Marcel Dekker, New York, 1992, 97-111.

[9] A. Kasue and H. Kumura: *Spectral convergence of Riemannian manifolds*, Tôhuku Math. J. **46** (1994), 147-179.

[10] A. Lichnerowicz: *Applications harmoniques et variétés kählériennes*, Symp. Math. **III** Bologna (1970), 341-402.

[11] A. Lichnerowicz: *Variétés kählériennes á première classe de Chern nonnegative et variétés riemanniennes á courbure de Ricci généralisée nonnegative*, J. Differential Geom. **6** (1971), 47-94.

[12] T. Yamaguchi: Manifolds of almost nonnegative Ricci curvature, J. Differential Geom. **28** (1988), 157-167.

Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku, Osaka 558,
Japan