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Osaka University
CONVERGENCE OF RIEMANNIAN MANIFOLDS
AND ALBANESE TORI

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(Received July 19, 1993)

0. Introduction

The purpose of this note is to prove the convergence of the Albanese tori of compact Riemannian manifolds which collapse to a lower dimensional space while keeping their curvatures and diameters bounded.

Given a compact Riemannian manifold $M$, we denote by $\mathcal{H}^1(M)$ the space of harmonic one-forms on $M$ equipped with an inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle \omega, \eta \rangle = \int_M (\omega, \eta) d\mu_M,$$

where $\mu_M$ stands for the normalized Riemannian measure of $M$ with unit mass, $\mu_M = d\text{vol}_M/\text{Vol}(M)$. Let $H^1(M)_\mathbb{Z}$ be a lattice of $H^1(M)$ which consists of harmonic one-forms of integral periods. Dividing the dual space $H^1(M)^*$ by the dual lattice $H^1(M)^*_\mathbb{Z}$, we obtain a flat torus, called the Albanese torus of $M$,

$$\mathcal{A}(M) = H^1(M)^*/H^1(M)^*_\mathbb{Z}.$$

We may view $\mathcal{A}(M)$ as a map of the set of isometry classes of compact Riemannian manifolds $M$ onto that of flat tori.

Given a positive integer $m$, a nonnegative number $x$ and a positive one $D$, we write $\mathcal{H}(m, x, D)$ for the set of isometry classes of compact Riemannian $m$-manifolds $M$ such that the Ricci curvature of $M$ is bounded from below by $-(m-1)x^2$ and the diameter of $M$ is bounded from above by $D$. Then according to Gromov [7], for a Riemannian manifold $M$ in $\mathcal{H}(m, x, D)$, the dimension of the Albanese torus $\mathcal{A}(M)$, namely, the first Betti number $b_1(M)$, has an upper bound depending only on the dimension $m$ of $M$ and $xD$. Using this, we shall show the following

Theorem 1. Given $m \in \mathbb{Z}^+$, $x \geq 0$ and $D > 0$, there is a constant $C$ depending only on $m$ and $xD$ such that
\[ \text{diam}(\mathcal{A}(M)) \leq C \text{diam}(M) \]
for any $M$ in $\mathcal{J}(m, x, D)$.

We would like to ask if the convergence of compact Riemannian manifolds in $\mathcal{J}(m, x, D)$ with respect to the spectral distance would imply the convergence of their Albanese tori. Here we recall the definition of a spectral distance between two compact Riemannian manifolds which was introduced in [9]. Given two compact Riemannian manifolds $M$ and $N$, a mapping $f : M \to N$ is called an $\varepsilon$-spectral approximation if
\[ e^{-\frac{1}{(1+t^2)}|p_M(t, x, y) - p_N(t, f(x), f(y))|} < \varepsilon \]
for all $t > 0$, and for all points $x, y$ of $M$, where $p_M(t, x, y)$ and $p_N(t, u, v)$ denote respectively the heat kernel of $M$ in $L^2(M, \mu_M)$ and that of $N$ in $L^2(N, \mu_N)$. The spectral distance, $SD(M, N)$, between $M$ and $N$ is by definition the lower bound of the positive numbers $\varepsilon$ such that there exist $\varepsilon$-spectral approximations $f : M \to N$ and $h : N \to M$. According to [9], we know that (i) the metric space $(\mathcal{J}(m, x, D), SD)$ is precompact; (ii) the eigenvalues and eigenfunctions of compact Riemannian manifolds in $\mathcal{J}(m, x, D)$ are continuous with respect to the spectral distance in a certain sense; (iii) the topology on $\mathcal{J}(m, x, D)$ induced by the spectral distance is finer than that of measured Hausdorff convergence introduced by Fukaya [3] and hence that of the Gromov-Hausdorff distance. Moreover if we denote by $\mathcal{K}(m, x, D)$ the set of isometry classes of compact Riemannian $m$-manifolds such that the sectional curvatures are bounded by $x^2$ in the absolute values and the diameters are not greater than $D$, then the topologies of the spectral distance and the measured Hausdorff convergence coincide on this set $\mathcal{K}(m, x, D)$. See [3], [8] and [9] for details. We note also that the spectral distance and the Gromov-Hausdorff distance induce the same topology on the set of flat tori.

The following theorem shows that the above question is affirmative if we restrict ourselves to the class $\mathcal{K}(m, x, D)$ for given numbers $m, x,$ and $D$.

Theorem 2. Let $(M_i)$ be a sequence in $\mathcal{K}(m, x, D)$ which converges with respect to the spectral distance. Then the Albanese torus $\mathcal{A}(M_i)$ converges to a flat torus $\mathcal{A}_\infty$ of dimension $n$ with $0 \leq n \leq \lim \inf_{i \to \infty} \partial(M_i)$.

Here in our convention, $\mathcal{A}_\infty$ stands for a point when $n = 0$. We remark also that under the assumption of this theorem, we are able to show the convergence of the Albanese maps in a certain sense. See Section 3 for details.

The proofs of Theorems 1 and 2 are respectively given in Sections 2 and 3. For the latter, we shall basically make use of some results in [8]. In this sense, the
present paper is a continuation of [8].

1. Albanese Tori

In this section, we shall consider a compact Riemannian manifold \( M \) endowed with a certain measure \( \mu \) and define the Albanese torus and the Albanese map of such a pair \((M, \mu)\) (cf. [10], [11]).

Let \( M = (M, g_M) \) be a compact Riemannian manifold of dimension \( m \) and \( \mu \) a measure on \( M \) with smooth density function \( \chi > 0 \). A one-form \( \omega \) on \( M \) is said to be \( \mu\)-harmonic if \( \omega \) is closed and co-closed with respect to \( \mu \), namely, \( d\omega = 0 \) and \( \delta_\mu \omega = -\text{trace}(\nabla \omega) - \omega(\nabla \log \chi) = 0 \).

In other words, \( \mu\)-harmonic one-form \( \omega \) can be expressed locally as the differential of an \( L_\mu \)-harmonic function \( f \), \( \omega = df \). Here a smooth function \( f \) defined on an open set in \( M \) is called an \( L_\mu \)-harmonic function if

\[ L_\mu f = -\frac{1}{\chi} \text{div}(\chi \nabla f) = \Delta f + \nabla \log \chi \cdot f = 0. \]

We denote by \( H^1(M, \mu) \) the space of \( \mu\)-harmonic one-forms on \( M \) and by \( H^1(M, \mu)_Z \) the lattice of \( H^1(M, \mu) \) which consists of one-forms with integral periods. The vector space \( H^1(M, \mu) \) is endowed with an \( L^2 \) inner product \( \langle , \rangle \) defined by

\[ \langle \omega, \eta \rangle_\mu = \int_M (\omega, \eta) d\mu. \]

The norm of \( \omega \in H^1(M, \mu) \) is denoted by \( \| \omega \|_\mu \).

Similarly, given a compact Riemannian manifold \( N \), we say a smooth mapping \( \phi \) of \( M \) into \( N \) is \( \mu\)-harmonic if it satisfies

\[ \tau(\phi) + d\phi(\nabla \log \chi) = 0, \]

where \( \tau(\phi) \) stands for the tension field of the mapping \( \phi \). A \( \mu \)-harmonic mapping \( \phi \) is a stationary point of the energy functional

\[ E_\mu(\phi) = \int_M e(\phi) d\mu. \]

When \( N \) is a circle of length 1, \( N = \mathbb{R}/\mathbb{Z} \), we write \( \mathcal{H}(M, \mu; \mathbb{R}/\mathbb{Z}) \) for the set of \( \mu \)-harmonic mappings of \( M \) into \( \mathbb{R}/\mathbb{Z} \), which forms an additive group in an obvious manner. Since the derivative \( d\phi \) of a smooth mapping \( \phi \) of \( M \) into \( \mathbb{R}/\mathbb{Z} \) may be considered as an integral one-form on \( L \), we have a natural surjective homomorphism \( d \) of \( \mathcal{H}(M, \mu; \mathbb{R}/\mathbb{Z}) \) onto \( H^1(M, \mu)_Z \) whose kernel is the set of constant mappings \( \theta \in \mathbb{R}/\mathbb{Z} \). We note that the homomorphism \( d \) preserves the norms in the sense that

\[ E_\mu(\phi) = \langle d\phi, d\phi \rangle_\mu \]
for $\phi \in \mathcal{A}(M, \mu ; R/\mathbb{Z})$.

The Albanese torus $\mathcal{A}(M, \mu)$ of a pair $(M, \mu)$ is by definition a flat torus derived from dividing the dual space $H^1(M, \mu)^*$ by the dual lattice $H^1(M, \mu)^*$.

$$\mathcal{A}(M, \mu) = H^1(M, \mu)^*/H^1(M, \mu)^*.$$  

Let $\tilde{M}$ be the universal covering of $M$ and $\pi$ the projection of $\tilde{M}$ onto $M$. If we fix a point $\tilde{p}$ of $\tilde{M}$ and take a point $\tilde{p}$ of $\tilde{M}$ with $\pi(\tilde{p}) = p$, then we have a map $\tilde{J}_{M,\mu}$ of $\tilde{M}$ into the dual space $H^1(M, \mu)^*$ defined by

$$\tilde{J}_{M,\mu}(\tilde{\omega}) = \int_{\tilde{p}}^{p} \pi^* \omega.$$  

This map induces a $\mu$-harmonic map $J_{M,\mu}$ of $M$ into $\mathcal{A}(M, \mu)$ (with $J_{M,\mu}(p) = 0$). We call $J_{M,\mu}$ the Albanese map of a pair $(M, \mu)$.

Let $\Omega = \{\omega_1, \ldots, \omega_r\}$ be a basis of $H^1(M, \mu)$ and $\Omega^* = \{\omega_1^*, \ldots, \omega_r^*\}$ the dual basis. Then a diffeomorphism $T_{\alpha}$ of $\mathcal{A}(M, \mu)$ onto $R^r/\mathbb{Z}^r$ is derived from a linear isomorphism of $H^1(M, \mu)^*$ onto $R^r$:

$$\theta_1 \omega_1^* + \cdots + \theta_r \omega_r^* \mapsto (\theta_1, \ldots, \theta_r).$$  

If we set a metric $g_\alpha$ on $R^r/\mathbb{Z}^r$ by

$$g_\alpha = \sum_{\alpha=1, \ldots, r} \langle \omega_\alpha, \omega_\beta^* \rangle \theta_\alpha \theta_\beta,$$

then $T_{\alpha}$ induces an isometry between $\mathcal{A}(M, \mu)$ and $(R^r/\mathbb{Z}^r, g_\alpha)$. Moreover if we take a $\mu$-harmonic map $\psi_\alpha : M \rightarrow R/\mathbb{Z}$ in such a way that

$$\psi_\alpha(p) = 0, \quad \omega_\alpha = d\psi_\alpha,$$

we see that

$$T_{\alpha} J_{M,\mu}(x) = (\psi_1(x), \ldots, \psi_r(x))$$

for $x \in M$. Here we remark that

$$E_\mu(J_{M,\mu}) = b_1(M).$$

Given a pair $(M, \mu)$, we define a symmetric tensor $Ric_{M,\mu}$ by

$$Ric_{M,\mu} = Ric - \frac{1}{\chi} \nabla^2 \chi (= Ric - d \log \chi \otimes d \log \chi - \nabla^2 \log \chi),$$

where $Ric$ (resp., $\chi$) stands for the Ricci tensor of $M$ (resp., the density function of $\mu$, $\mu = \chi d\text{vol}_M$). Given $m$, $x$ and $D$ as before, we denote by $\mathcal{D}_x^m(m, x, D)$ the set of equivalence classes of pairs $(M, \mu)$ such that $\dim M = m$, the diameter $\text{diam}(M)$ of $M \leq D$, $\mu$ has unit mass, and $\text{Ric}_{M,\mu} \geq -(m-1)x^2 g_m$. Here we say two pairs $(M, \mu)$ and $(N, \nu)$ are equivalent when there is an isometry $f : M \rightarrow N$ which preserves the measures, $f_* \mu = \nu$. We remark that the spectral distance $SD$ can be defined on the set of equivalence classes of pairs $(M, \mu)$. See [9] for some properties.
of the metric space \( (\mathcal{M}_m^*(m, \lambda, D), SD) \) as mentioned in Introduction.

In what follows, when \( \mu \) is the canonical Riemannian measure \( \mu_M \) with unit mass, \( \mu_M = d\text{vol}/\text{Vol}(M) \), we omit to indicate the dependency of the measure \( \mu_M \) in some of the above notations (for example, \( \mathcal{A}(M) \) stands for \( \mathcal{A}(M, \mu_M) \)).

2. Proof of Theorem 1

We recall first that there is a positive constant \( C' \) depending only on \( m \) and \( xD \) such that

\[
e(\phi) \leq C'E_{\mu_M}(\phi)
\]

for all \( \phi \in \mathcal{H}(M; R/Z) \), because the energy density \( e(\phi) \) satisfies

\[
\Delta e(\phi) \geq -2(m-1)x^2 e(\phi)
\]

(cf. e.g., [8-a, §4]). In addition, we note that if \( \phi \) is not constant, the energy density \( e(\phi) \) must be greater than or equal to \( 1/4 \text{diam}(M)^2 \) somewhere on \( M \) (otherwise, the distance between \( \phi(p) \) and \( \phi(q) \) in \( R/Z \) for any pair of points \( p, q \) of \( M \) would be less than \( 1/2 \), and hence the harmonic map \( \phi \) should be constant). Therefore we have

\[
E_{\mu_M}(\psi) \geq \frac{1}{4C'\text{diam}(M)^2}
\]

for all nonconstant \( \phi \in \mathcal{H}(M; R/Z) \); in other words,

\[
\langle \omega, \omega \rangle_{\mu_M} \geq \frac{1}{4C'\text{diam}(M)^2}
\]

for all nonzero \( \omega \in H^1(M)_Z \). This implies that the first eigenvalue \( \lambda_1(\mathcal{A}(M)) \) is bounded from below by \( \pi^2/C'\text{diam}(M)^2 \). On the other hand, we know that

\[
\lambda_1(\mathcal{A}(M)) \leq \frac{C''}{\text{diam}(\mathcal{A}(M))^2}
\]

for some constant \( C'' \) depending only on the dimension of \( \mathcal{A}(M) \), and hence on \( m \) and \( xD \), since

\[
b_1(M) \leq \frac{v_m(x^2, 5\text{diam}(M))}{v_m(x^2, \text{diam}(M))},
\]

where \( v_m(x^2, r) \) stands for the volume of a metric ball in the simply connected space form of dimension \( m \) with constant curvature \( x^2 \) (cf. [7]). Thus the assertion of Theorem 1 is clear. This completes the proof of Theorem 1.

Let \( (M, \mu) \) be a pair in \( \mathcal{M}_m^*(m, \lambda, D) \). Then it is not hard to see that the above submean value inequality (2.1) holds for any \( \mu \)-harmonic map \( \phi \in \mathcal{H}(M, \mu; R/Z) \) (cf. e.g., [8-a, §4]). Moreover it follows from the same reason as in deriving (2.2)
that
\[ b_i(M) \leq \frac{v_{m+i}(x^2, 5\text{diam}(M))}{v_{m+i}(x^2, \text{diam}(M))}, \]
because we have a Bishop-Gromov type inequality for the pair \((M, \mu)\)(cf. [9, §2]). Thus Theorem 1 holds for \((M, \mu)\). Namely we have

\textbf{Theorem 1'.} Given \(m, x\) and \(D\) as before, there is a constant \(C'\) depending only on \(m\) and \(xD\) such that
\[ \text{diam}(\mathcal{A}(M, \mu)) \leq C' \text{diam}(M). \]
for all \((M, \mu) \in \mathcal{A}_m^*(m, x, D)\).

\textbf{Remarks.} (1) When \(x=0\) in Theorem 1, the classical Bochner theorem says that any harmonic one-form is parallel, so that \(JM\) is a Riemannian submersion with totally geodesic fibers and in particular the diameter of \(\mathcal{A}(M)\) is less than or equal to the diameter of \(M\) \((C=1\) in Theorem 1\)). This is also true for Theorem 1' (cf. [12]). (2) A slightly different proof of Theorem 1 is presented in [6].

3. Proof of Theorem 2

The proof of the theorem is divided into 4 steps and the same notations as in the preceding sections will be used.

\textbf{Step 1.} We shall start with recalling the notions of convergence of Gromov-Hausdorff distance and measured Hausdorff topology introduced by Gromov [7] and Fukaya [3] respectively. Given a sequence of compact Riemannian manifolds, \([M_i]\), we say that \(M_i\) converges to a compact metric space \(X\) with respect to the Gromov-Hausdorff distance, if there are a sequence of positive numbers \(\{\epsilon_i\}\) with \(\lim_{i \to \infty} \epsilon_i = 0\) and mappings \(f_i : M_i \to X\) such that the \(\epsilon\)-neighborhood of \(f_i(M_i)\) covers \(X\) and \(|d_{M_i}(x, y) - d_x(f_i(x), f_i(y))| < \epsilon\) for all \(x, y\) of \(M_i\). Moreover we say that \(M_i = (M_i, \mu_{M_i})\) converges to a pair \((X, \mu)\) of \(X\) and a Borel measure \(\mu\) on \(X\) with respect to the measured Hausdorff topology, if \(f_i\) are Borel measurable and the push-forward \(f_i_*\mu_{M_i}\) of the normalized Riemannian measure \(\mu_{M_i}\) via \(f_i\) converges to \(\mu\) in the weak* topology.

Let \(M_i\) be a sequence in \(\mathcal{K}(m, x, D)\) which converges to a compact metric space \(M_m\) with respect to the Gromov-Hausdorff distance. Then there is a smooth manifold \(F_m\) with metric of class \(C^{1, a}\) (for any \(a \in (0, 1)\)), on which the orthogonal group \(O(m)\) acts by isometries in such a way that the quotient space \(F_m/O(m)\) is isometric to \(M_m\). In fact, \(F_m\) is a limit of the set of the frame bundles \(FM_i\) of \(M_i\) equipped with a canonical metric so that the action of \(O(m)\) is isometric, the projection of \(FM_i\) onto \(M_i\) is a Riemannian submersion with totally geodesic fibers, and the sectional curvature of \(FM_i\) remains bounded uniformly in \(i\). When \(M_m\) is smooth, there is a fibration \(\Phi_i : M_i \to M_m\) (for large \(i\)) and a sequence of
positive numbers \(\{\varepsilon_i\}\) with \(\lim_{i \to \infty} \varepsilon_i = 0\) satisfying

1. for all \(z \in M_\infty\), the diameter of \(\Phi_i^{-1}(z) \leq \varepsilon_i\);
2. \(\Phi_i\) is an \(\varepsilon_i\)-almost Riemannian submersion, that is, for all \(z \in M_\infty\), \(x \in \Phi_i^{-1}(z)\) and \(X \in T_x M_i\) normal to the fiber \(\Phi_i^{-1}(z)\),
   \[(1 - \varepsilon_i)|d\Phi_i(X)| \leq |X| \leq (1 + \varepsilon_i)|d\Phi_i(X)|;\]
3. the second fundamental form of the submersion \(\Phi_i\) is bounded uniformly in \(i\).

For these assertions, see \([2], [4]\) and \([5]\).

Now as in Theorem 2, we suppose that \(M_i\) converges to \((M_\infty, \mu_\infty)\) with respect to the spectral distance and hence the measured Hausdorff topology. Then we may assume that the push-forward \(\Phi_i^* \mu_{M_i}\) of the canonical Riemannian measure \(\mu_{M_i}\) of \(M_i\) converges to \(\mu_\infty\) in the weak* topology. In case \(M_\infty\) is smooth, the density function \(\chi_\infty\) of \(\mu_\infty\) is a positive function of class \(C^1\). Moreover we may assume that the above submersion \(\Phi_i\) has the following property: for all smooth function \(h\) on \(M_\infty\),

\[\Delta_{M_i} \Phi_i^*(h) - \Phi_i^*(L_{\mu_\infty} h) \leq \varepsilon_i \Phi_i^*(|Ddh| + |dh|).\]

See \([8-a]\) for this and further properties of \(\Phi_i\).

In the following Steps 2 and 3, we consider the case that the limit metric space \(M_\infty\) is a smooth manifold, and assume that the metric of \(M_\infty\) and the density \(\chi_\infty\) are smooth, to avoid some technical argument of approximation. Moreover \(\{\varepsilon_i\}\) stands for a sequence of positive constants which tends to zero as \(i\) goes to infinity.

**Step 2.** Given a \(\mu_\infty\)-harmonic one-form \(\omega \in H^1(M_\infty, \mu_\infty)\), the pull-back \(\Phi_i^* \omega\) can be uniquely expressed as

\[\Phi_i^* \omega = \Gamma_i(\omega) + d\Lambda_i(\omega)\]

according to the orthogonal decomposition of \(d\)-closed one-forms \(Z(M_i)\) of \(M_i\),

\[Z^1(M_i) = H^1(M_i, \mu_{M_i}) \oplus dC^\infty(M_i).\]

Here the function \(\Lambda_i(\omega)\) is chosen in such a way that

\[\int_{M_i} \Lambda_i(\omega) d\mu_i = 0.\]

Now we claim that

\[\Phi_i^* \omega - \Gamma_i(\omega)\big|_{C^0(M_i)} = |d\Lambda_i(\omega)|_{C^0(M_i)} \leq \varepsilon_i\]

for any \(\omega \in H^1(M_\infty, \mu_\infty)\) with unit norm, \(\|\omega\|_{\mu_\infty} = 1\). Indeed, we fix a sufficiently small \(\alpha\) and consider the metric ball \(B_\alpha(p, 3\alpha)\) of \(M_\infty\) around a point \(p\) of radius \(3\alpha\). Let \(f\) be an \(L_\infty\)-harmonic function on \(B_\alpha(p, 3\alpha)\) such that \(\omega = df\) and

\[\int_{B_\alpha(p, 3\alpha)} f d\mu_\infty = 0.\]
Then applying the Poincaré inequality, we have first
\[ \int_{B_{\omega}(p,2a)} |f|^2 d\mu_\omega \leq C_1 \int_{B_{\omega}(p,3a)} |f|^2 d\mu_\omega \leq C_1 \]
for some constant $C_1$. Since $f$ is $L_\infty$-harmonic, it follows from the standard elliptic regularity estimates that
\[ |f|_{C^{2,\alpha}(B_{\omega}(p,2a))} \leq C_2 \]
for some constant $C_2$, where $\alpha \in (0, 1)$. Hence in view of the property (4) of $\Phi_i$, we see that
\[ |\Delta_{M_i}\Phi_i^* f| \leq \varepsilon_i \]
on $\Phi_i^{-1}(B_\omega(p, a))$. This shows that
\[ |\Delta_{M_i} \Lambda_i(\omega)| \leq \varepsilon_i, \]
since $d\Phi_i^* f = \Gamma_i(\omega) + d\Lambda_i(\omega)$ and $\Gamma_i(\omega)$ is harmonic. Finally it follows from the regularity estimates again that
\[ |\Lambda_i(\omega)|_{C^{1,\alpha}(M_i)} \leq \varepsilon_i, \]
where $p \in (1, \infty)$ (cf. [8-a, Lemma 1.3]), and hence
\[ |\Lambda_i(\omega)|_{C^{1,\alpha}(M_i)} \leq \varepsilon_i. \]
This proves (3.1).

Now this estimate (3.1) together with the property (3) of $\Phi_i$ implies that
\[ (1 - \varepsilon_i)\|\omega\|_{\mu_\omega} \leq \|\Gamma_i(\omega)\|_{\mu_i} \leq (1 + \varepsilon_i)\|\omega\|_{\mu_\omega} \]
For all $\omega \in H^1(M_{\omega}, \mu_{\omega})$. In particular, $\Gamma_i$ is injective (for large $i$). We observe further that $\Gamma_i$ maps the lattice $H^1(M_{\omega}, \mu_{\omega})_\mathbb{Z}$ into the lattice $H^1(M_i)_\mathbb{Z}$,
\[ \Gamma_i(H^1(M_{\omega}, \mu_{\omega})_\mathbb{Z}) \subset H^1(M_i)_\mathbb{Z}. \]

**Step 3.** Given any number $K$, Theorem 4.3 in [8-a] says that for large $i$, a harmonic one-form $\xi$ on $M_i$ with integral periods must belong to the image $\Gamma_i(H^1(M_{\omega}, \mu_{\omega})_\mathbb{Z})$, whenever the $L^2$ norm $\|\xi\|_{\mu_i}$ is less than $K$. In other words, there is a positive constant $K_i$ with $\lim_{i \to \infty} K_i = \infty$ such that
\[ \|\xi\|_{\mu_i} \geq K_i \]
for any $\xi \in H^1(M_i)_\mathbb{Z} \setminus \Gamma_i(H^1(M_{\omega}, \mu_{\omega})_\mathbb{Z})$ (if it exists).

Let us now take a basis $\Omega = \{\omega_1, \ldots, \omega_r\}$ of $H^1(M_{\omega}, \mu_{\omega})_\mathbb{Z}(r = b_1(M_{\omega}))$ in such a way that an element $\omega$ of $H^1(M_{\omega}, \mu_{\omega})_\mathbb{Z}$ is a linear combination of $\omega_1, \ldots, \omega_{s-1}$ whenever $\|\omega\|_{\mu_\omega}$ is less than $\|\omega_s\|_{\mu_\omega}$ (cf. [1, Chap. VIII]). Then we choose a basis $\Omega_i = \{\omega_{i,1}, \ldots, \omega_{i,r_i}\}$ of $H^1(M_i)_\mathbb{Z}(r_i = b_1(M_i))$ in such a way that
\[ \omega_{i,s} = \Gamma_i(\omega_s) \quad (s = 1, \ldots, r) \]

and any element \( \omega \) is linearly dependent of \( \omega_{i,1}, \ldots, \omega_{i,s-1} \) whenever \( \|\omega\|_{\nu} \) is less than \( \|\omega_{i,s}\|_{\nu} \) for \( s > r \). We note that

\[ \|\omega_{i,s}\| \geq K_i \]

for \( s > r \) (if \( r_i > r \)).

Let \( \Omega_i^* = \{\omega_i^{*s}\} \quad (s = 1, \ldots, r_i) \) be the dual basis of \( \Omega_i \) and \( \Gamma_i^* : H^1(M_i)^* \rightarrow H^1(M_m, \mu_\omega)^* \) the dual mapping of \( \Gamma_i \). Then \( \Gamma_i^* \) is surjective and its kernel is spanned by \( \omega_{i,s}(s = r + 1, \ldots, r_i) \). Hence \( \Gamma_i^* \) induces a surjective homomorphism, denoted by \( [\Gamma_i^*] \), from the Albanese torus \( \mathcal{A}(M_i) \) of \( M_i \) onto \( \mathcal{A}(M_m, \mu_\omega) \). Then in view of (3.2) and (3.3), \( \mathcal{A}(M_i) \) converges via \( [\Gamma_i^*] \) to \( \mathcal{A}(M_m, \mu_\omega) \) with respect to the Gromov-Hausdorff distance. We observe that \( [\Gamma_i^*] \) is affine, namely the second fundamental form vanishes identically. Moreover if we take a point \( p_m \) of \( M_m \) and choose \( p_i \) as a fixed point of \( M_i \) in such a way that \( \Phi_i(p_i) = p_m \), then the mappings \( J_{M_m,\mu_\omega} \) and \( [\Gamma_i^*] \circ J_M \) are close for large \( i \) in the sense that

\[ \max_{x \in M_i} \text{dis}(J_{M_m,\mu_\omega}(x), [\Gamma_i^*] \circ J_M(x)) \leq \varepsilon_i. \]

To be precise, let \( T_{D_1} : \mathcal{A}(M_i) \rightarrow (R^r/Z^r, g_{D_1}) \) and \( T_D : \mathcal{A}(M_m, \mu_\omega) \rightarrow (R^r/Z^r, g_D) \) respectively be isometries described in Section 1, and let \( \pi_i : R^r/Z^r \rightarrow R^r/Z^r \) be a canonical projection such that \( \pi_i(\theta_i, \ldots, \theta_r) = (\theta_i, \ldots, \theta_r) \). Then \( T_{D_1} \circ J_M \) and \( T_D \circ J_{M_m,\mu_\omega} \) respectively can be expressed as

\[ T_{D_1} \circ J_M = (\psi_{i,1}, \ldots, \psi_{i,r_i}) \]

and

\[ T_D \circ J_{M_m,\mu_\omega} = (\psi_i, \ldots, \psi_r), \]

where \( \psi_{i,s} \) is the harmonic mapping of \( M_i \) to \( R/Z \) corresponding to \( \omega_{i,s} \) and also \( \psi_s \) is the \( \mu_\omega \)-harmonic mapping of \( M_m \) to \( R/Z \) corresponding to \( \omega_s \). We note that for each \( s, 1 \leq s \leq r, \psi_{i,s} \) is homotopic to \( \psi_i \circ \Phi_i \) and further that

\[ |\psi_{i,s} - \psi_s \circ \Phi_i|_{C^{2,a}(M_i)} \leq \varepsilon_i \]

(cf. [8-a, §4]). Thus the mappings \( J_{M_m,\mu_\omega} \circ \Phi_i \) and \( [\Gamma_i^*] \circ J_M \) (for large \( i \)) are close with respect to the \( C^{2,a} \) topology.

Step 4. It remains to prove Theorem 2 in case \( M_m \) is not smooth. In this case, we consider the frame bundle \( FM_i \) of each \( M_i \) equipped with a canonical metric \( \bar{g}_i \) in such a way that the sectional curvature and the diameter are bounded uniformly in \( i \). We denote by \( \rho_i \) the canonical projection of \( FM_i \) onto \( M_i \). Observe that the pull-back \( \rho_i^* \omega \) of a harmonic one-form \( \omega \) on \( M_i \) is harmonic on \( FM_i \) and further that this correspondence preserves the inner products,

\[ \langle \rho_i^* \omega, \rho_i^* \omega' \rangle_{\mu_i} = \langle \omega, \omega' \rangle_{\mu_i}, \]
For this reason, the space of harmonic one-forms $H^1(M_i)$ on $M_i$ endowed with the $L^2$ inner product can be considered as a subspace of $H^1(FM_i)$. In the same way, we identify the lattice $H^1(M_i)_z$ and the group $\mathcal{H}(M_i; R/Z)$ respectively with a sublattice of $H^1(FM_i)_z$ and a subgroup of $\mathcal{H}(FM_i; R/Z)$. Under this identification, an element $\tilde{\psi}$ of $\mathcal{H}(FM_i; R/Z)$ belongs to the subgroup $\mathcal{H}(M_i; R/Z)$ if and only if $\tilde{\psi}$ is $O(m)$-invariant.

In what follows, we suppose that this sequence $\{FM_i\}$ converges in the topology of measured Hausdorff convergence. Let $\tilde{M}_\infty$ and $\tilde{\mu}_\infty$ be the limit space and measure respectively. Then according to Fukaya [4, 5], $\tilde{M}_\infty$ is a smooth manifold with Riemannian metric $\tilde{g}_\infty$ of class $C^{1,\alpha}$, on which the orthogonal group $O(m)$ acts as isometries in such a way that the quotient space $\tilde{M}_\infty/O(m)$ is isometric to $M_\infty$. Moreover there are $O(m)$-equivariant almost Riemannian submersions $\tilde{\Phi}_i: FM_i \to \tilde{M}_\infty$ such that $\tilde{\Phi}_i: FM_i \to \tilde{M}_\infty$ converges to $\tilde{\mu}_\infty$ in the weak* topology, where $\tilde{\mu}_\infty$ stands as before for the normalized Riemannian measure of $FM_i$. We note that the limit measure $\mu_\infty$ on $M_\infty$ coincides with the push-forward $\rho_{\tilde{\mu}_\infty}^* \tilde{\mu}_\infty$ of $\tilde{\mu}_\infty$ via the projection $\rho_{\infty}: \tilde{M}_\infty \to M_\infty$ and the density $\tilde{\kappa}_\infty$ of $\tilde{\mu}_\infty$ with respect to the Riemannian measure of $\tilde{g}_\infty$ is $O(m)$-invariant.

Now perturbing the submersion $\tilde{\Phi}_i$ in the $C^{1,\alpha}$ topology, we can obtain an almost Riemannian submersion $\tilde{\Phi}_i$ of $FM_i$ onto $\tilde{M}_\infty$, to which we can apply the same arguments as in the preceding steps. To be precise, we write first $\mathcal{H}(M_\infty, \mu_\infty; R/Z)$ for the subgroup of $\mathcal{H}(\tilde{M}_\infty, \tilde{\mu}_\infty; R/Z)$ consisting of $O(m)$-invariant $\tilde{\mu}_\infty$-harmonic maps $\tilde{\psi}: \tilde{M}_\infty \to R/Z$. We note that $\mathcal{H}(M_\infty, \mu_\infty; R/Z)$ is determined by the pair $(M_\infty, \mu_\infty)$ itself (cf. [8-a, §4]). Then we denote by $H^1(M_\infty, \mu_\infty)_z$ and $H^1(M_\infty, \mu_\infty)$ respectively the sublattice of $H^1(\tilde{M}_\infty, \tilde{\mu}_\infty)_z$ corresponding to $\mathcal{H}(M_\infty, \mu_\infty; R/Z)$, $H^1(\tilde{M}_\infty, \tilde{\mu}_\infty)_z = d\mathcal{H}(M_\infty, \mu_\infty; R/Z)$, and the vector space spanned by $H^1(\tilde{M}_\infty, \tilde{\mu}_\infty)_z$.

Then we obtain an $O(m)$-invariant $\tilde{\mu}_\infty$-harmonic map $\tilde{J}_{FM_i}: \tilde{M}_\infty \to \mathcal{A}(M_\infty, \mu_\infty)$, from which a Lipschitz map $J_{FM_i}: M_\infty \to \mathcal{A}(M_\infty, \mu_\infty)$ is derived. This map $J_{FM_i}$ is $\mu_\infty$-harmonic on the set of regular points of $M_\infty$. Moreover as we have seen in Steps 2 and 3, $\tilde{\Phi}_i$ (for large $i$) induces a surjective homomorphism $[\tilde{T}_i^*]: \mathcal{A}(FM_i) \to \mathcal{A}(M_\infty, \tilde{\mu}_\infty)$ such that $[\tilde{T}_i^*] \circ J_{FM_i}$ are close in the $C^2,\alpha$ topology. Finally we obtain a surjective homomorphism $[T_i^*]: \mathcal{A}(M_i) \to \mathcal{A}(M_\infty, \mu_\infty)$ from $[\tilde{T}_i^*]$ such that $J_{FM_i} \circ \tilde{\Phi}_i$ and $[T_i^*] \circ J_{FM_i}$ are close in the $C^0$ topology for large $i$, where $\tilde{\Phi}_i: M_i \to M_\infty$ is a Lipschitz map derived from the $O(m)$-equivariant submersion $\tilde{\Phi}_i$. As $i$ goes to infinity, the Albanese torus $\mathcal{A}(M_i)$ converges to the torus $\mathcal{A}(M_\infty, \mu_\infty)$ via the surjective homomorphism $[T_i^*]$. This completes the proof of Theorem 2.

It is possible to apply the same arguments as above to a sequence of certain pairs $(M_i, \mu_i)$ (cf. [8-a, Remark 3.3]). In fact, we can show the following
**Theorem 2.** Let \( \{(M_i, \mu_i)\} \) be a sequence in \( \mathcal{D}^\tau_\mathcal{F}(m, x, D) \) which converges to \( (M_\infty, \mu_\infty) \) with respect to the measured Hausdorff topology. Suppose that the sectional curvature of \( M_i \) is bounded uniformly and also the density function \( \chi_i \) of \( \mu_i \) satisfies

\[ |Dd\chi_i| \leq C \]

for some constant \( C \). Then for large \( i \), there is a surjective homomorphism \( \Theta_i \) of the Albanese torus \( \mathcal{A}(M_i, \mu_i) \) onto a flat torus \( \mathcal{A}(M_\infty, \mu_\infty) \) of dimension \( n \) such that \( 0 \leq n \leq \lim \inf_{i \to \infty} b_i(M_i) \) and \( \mathcal{A}(M_i, \mu_i) \) converges to \( \mathcal{A}(M_\infty, \mu_\infty) \) with respect to the Gromov-Hausdorff distance via \( \Theta_i \).

Moreover there are a Lipschitz map \( \Phi_i \) of \( M_i \) onto \( M_\infty \) through which \( (M_i, \mu_i) \) converges to \( (M_\infty, \mu_\infty) \) and a \( (\mu_\infty\)-harmonic) map \( J_{M_\infty, \mu_\infty} : M_\infty \to \mathcal{A}(M_\infty, \mu_\infty) \) such that \( J_{M_\infty, \mu_\infty} \circ \Theta_i \) and \( \Phi_i \circ J_{M_i, \mu_i} \) are close in the \( C^0 \) topology, namely,

\[
\lim_{i \to \infty} \max_{x \in M_i} \text{dis}(J_{M_\infty, \mu_\infty} \circ \Theta_i(x), \Theta_i \circ J_{M_i, \mu_i}(x)) = 0.
\]

The convergence holds in the \( C^{2,\alpha} \) topology when \( M_\infty \) is a smooth manifold.

As an immediate consequence of Theorem 2', we have the following

**Corollary 3.** Given numbers \( m, x \) and \( D \), and given a flat torus \( T \) of dimension \( n \), there is a constant such that the rank of the Albanese map \( J_M \) of a manifold \( M \) in \( \mathcal{K}(m, x, D) \) is greater than or equal to \( n \), if the Gromov-Hausdorff distance between \( M \) and \( T \) is less than \( r \), and in addition, \( J_M \) is a submersion if \( b_i(M) = n \).

Finally we refer the reader to [12] for some results and problems related to this corollary.

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**References**


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