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DEFORMATIONS OF REDUCIBLE SL($n, \mathbb{C}$) REPRESENTATIONS OF FIBERED 3-MANIFOLD GROUPS

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Abstract

Let M_ϕ be a surface bundle over a circle with monodromy $\phi : S \to S$. We study deformations of certain reducible representations of $\pi_1(M_\phi)$ into $\mathrm{SL}(n, \mathbb{C})$, obtained by composing a reducible representation into $\mathrm{SL}(2, \mathbb{C})$ with the irreducible representation $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{SL}(n, \mathbb{C})$. In particular, we show that under certain conditions on the eigenvalues of ϕ^*, the reducible representation is contained in a $(n+1+k)(n-1)$ dimensional component of the representation variety, where k is the number of components of ∂M_ϕ. This result applies to mapping tori of pseudo-Anosov maps with orientable invariant foliations whenever 1 is not an eigenvalue of the induced map on homology, where the reducible representation is also a limit of irreducible representations.

1. Introduction

Suppose that $S = S_{g,p}$ is a surface of genus g with $p \geq 1$ punctures, where $2g + p > 2$. Then S admits a hyperbolic structure. If $\phi : S \to S$ is a homeomorphism, we can form the mapping torus $M_\phi = S \times [0, 1]/(x, 1) \sim (\phi(x), 0)$. Whenever λ^2 is an eigenvalue of $\phi^* : H^1(S) \to H^1(S)$ with eigenvector $(a_1, \ldots, a_{2g+p-1})^T$ with respect to a generating set $\{[\gamma_1], \ldots, [\gamma_{2g+p-1}]\}$ of $H^1(S)$, we obtain a reducible representation $\rho_\lambda : \pi_1(M_\phi) \to \mathrm{SL}(2, \mathbb{C})$ by defining,

$$\rho_\lambda(\gamma_i) = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix},$$

$$\rho_\lambda(\tau) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

where τ is the generator of the fundamental group of the S^1 base of the fiber bundle $S \to M_\phi \to S^1$. (Recall that a representation $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ is *reducible* if the image $\rho(G)$ preserves a proper subspace of $\mathbb{C}^n$, and otherwise is called *irreducible*.)

When M_ϕ is the complement of a knot K in S^3, this observation was originally made by Burde [1] and de Rham [3]. Furthermore, the Alexander polynomial is the characteristic polynomial of ϕ^*, so the condition on λ is equivalent to the condition that λ^2 is a root of the Alexander polynomial $\Delta_K(t)$. It was shown in [6] that the non-abelian, metabelian, reducible representation ρ_λ is the limit of irreducible representations if λ^2 is a simple root of $\Delta_K(t)$. Heusener and Medjerab [5] have also shown using an inductive argument that the conclusion still holds in $\mathrm{SL}(n, \mathbb{C})$, $n \geq 3$, if ρ_λ is composed with the irreducible representation

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$r_n : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(n,\mathbb{C})$. These results apply even if the knot complement is not fibered, as long as λ^2 is a simple root of $\Delta_K(t)$.

In this paper, we show that reducible $\mathrm{SL}(n,\mathbb{C})$ representations of fibered 3-manifolds groups obtained as the composition $\rho_{\lambda,n} = r_n \circ \rho_\lambda$ can be deformed to irreducible representations using a more direct calculation of the deformation space using coordinates for $\mathfrak{sl}(n,\mathbb{C})$. If the punctures form a single orbit under ϕ and the mapping torus M_ϕ is the complement of a fibered knot, then the results of [6] and [5] apply. The main result in Theorem 1.1 also covers the cases where M_ϕ is the complement of a fibered link L with $k \geq 2$ components $L_1, \ldots, L_k$, or a k-cusped fibered manifold which is not a link complement. In the statement of Theorem 1.1, $\bar{\phi}$ is the homeomorphism on $\bar{S} = S_{g,0}$ obtained from ϕ by filling in the p punctures of $S_{g,p}$. This defines a homeomorphism $\bar{\phi} : \bar{S} \to \bar{S}$.

Theorem 1.1. *Suppose that λ^2 is a simple eigenvalue of ϕ^*. If $|\lambda| \neq 1$, $\bar{\phi}^* : H^1(\bar{S}) \to H^1(\bar{S})$ does not have 1 as an eigenvalue, and if for each $2 \leq j \leq n-1$, we have that λ^{2j} is not an eigenvalue of ϕ^*, then $\rho_{\lambda,n}$ is a smooth point of the representation variety $R(\pi_1(M_\phi), \mathrm{SL}(n,\mathbb{C}))$, contained in a unique component of dimension $(n+1+k)(n-1)$.*

Note that for a knot complement, the Alexander polynomial satisfies $\Delta_K(1) = \pm 1$. Hence for a knot complement, the condition that $\bar{\phi}^* : H^1(\bar{S}) \to H^1(\bar{S})$ does not have 1 as an eigenvalue (in the fibered case) or the corresponding condition that 1 is not a root of $\Delta_K(t)$ (in the non-fibered case) is automatically satisfied. For a generic mapping torus, a fixed point of $\bar{\phi}^*$ implies that the closed manifold obtained as the mapping torus of $\bar{\phi}$ has second Betti number at least 2, in which case the manifold fibers over a circle in infinitely many ways [17]. Heuristically, this leads to more infinitesimal deformations. When the local dimension of infinitesimal dimensions is higher than half the dimension of $H^1(\partial M_\phi)$, the standard techniques using Poincaré duality to show smoothness of the space of representations cannot be used. Whether the reducible representation can be obtained as a limit of irreducible representations in this case is unknown.

When ϕ is a pseudo-Anosov element of the mapping class group, λ^2 is the dilatation factor of ϕ, and the p punctures are exactly the singular points of the invariant foliations of ϕ, $\rho_\lambda = \rho_{\lambda,2}$ is shown to have deformations to irreducible representations under some additional conditions on the eigenvalues of $\bar{\phi}^*$, the map on the closed surface S_g, in [8]. We show that under the same hypotheses, the same holds for $\rho_{\lambda,n}$ when $n > 2$.

Theorem 1.2. *Suppose that λ^2 is the dilatation of a pseudo-Anosov map ϕ such that the stable and unstable foliations are orientable, and the singular points coincide with the punctures of S. Suppose also that 1 is not an eigenvalue of $\bar{\phi}^*$. Then $\rho_{\lambda,n}$ is a limit of irreducible $\mathrm{SL}(n,\mathbb{C})$ representations and is a smooth point of $R(\pi_1(M_\phi), \mathrm{SL}(n,\mathbb{C}))$, contained in a unique component of dimension $(n+1+k)(n-1)$.*

In Section 2, we give the basic definitions and background about representations of $\mathrm{SL}(2,\mathbb{C})$ into $\mathrm{SL}(n,\mathbb{C})$. Section 3 discusses the general theory of deformations, and Section 4 contains the main results, including relevant cohomological calculations.

2. Representations into $\mathrm{SL}(n,\mathbb{C})$

For notational convenience, we denote $\mathrm{SL}(n) = \mathrm{SL}(n,\mathbb{C})$, $\mathfrak{sl}(n) = \mathfrak{sl}(n,\mathbb{C})$, $\mathrm{GL}(n) = \mathrm{GL}(n,\mathbb{C})$, and $\Gamma_\phi = \pi_1(M_\phi)$. Note that we have the following identities in $\mathrm{SL}(2)$:

$$\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & a+b\\ 0 & 1\end{pmatrix},$$
(2.1)
$$\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}^{-1} = \begin{pmatrix}1 & \lambda^2 a\\ 0 & 1\end{pmatrix}.$$

Thus, if λ^2 is an eigenvalue of $\phi^* : H^1(S) \to H^1(S)$, $\{[\gamma_1], \dots, [\gamma_{2g+p-1}]\}$ generate $H^1(S)$, and $(a_1, \dots, a_{2g+p-1})^T$ is an eigenvector for λ^2, we can define

$$\rho_\lambda(\gamma_i) = \begin{pmatrix}1 & a_i\\ 0 & 1\end{pmatrix},$$
$$\rho_\lambda(\tau) = \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}.$$

Since $\pi_1(\Gamma_\phi)$ is a semi-direct product of the free group $\pi_1(S) = \langle\gamma_1, \dots, \gamma_{2g+p-1}\rangle$ with $\pi_1(S^1) = \langle\tau\rangle$ satisfying the relations $\tau\gamma_i\tau^{-1} = \phi(\gamma_i)$ and ϕ^* maps the vector $(a_1, \dots, a_{2g+p-1})^T$ to $\lambda^2(a_1, \dots, a_{2g+p-1})^T$, the identities (2.1) imply that this defines a representation $\rho_\lambda : \Gamma_\phi \to$ SL(2).

We now describe representations of SL(2) into SL(n), which we will compose with ρ_γ to obtain representations $\Gamma_\phi \to$ SL(n). A more general version of the discussion in this section can be found in [5, Section 4].

Let $R = \mathbb{C}[X, Y]$ be the polynomial algebra on two variables. We have an action of SL(2) on R by,

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \cdot X = dX - bY,$$
$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \cdot Y = -cX + aY,$$

for $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in$ SL(2). Let $R_{n-1} \subset R$ denote the n-dimensional subspace of homogenous polynomials of degree $n-1$, generated by $X^{\ell-1}Y^{n-\ell}, 1 \le \ell \le n$. The action of SL(2) leaves R_{n-1} invariant, turning R_{n-1} into a SL(2) module, and we obtain a representation r_n : SL(2) $\to$ GL(R_{n-1}). We can identify R_{n-1} with $\mathbb{C}^n$ by identifying the basis elements $\{X^{\ell-1}Y^{n-\ell}\}$ with the standard basis elements $\{e_\ell\}$ of $\mathbb{C}^n$. The induced isomorphism turns r_n into a representation SL(2) $\to$ GL(n) $\cong$ GL(R_{n-1}), which we will also call r_n. The representation r_n is *rational*, that is the coefficients of the matrix coordinates of $r_n\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ are polynomials in a, b, c, d.

We have the following two well-known results about r_n.

Lemma 2.1 ([16, Lemma 3.1.3 (ii)])**.** *The representation r_n is irreducible.*

Lemma 2.2 ([16, Lemma 3.2.1])**.** *Any irreducible rational representation of* SL(2, $\mathbb{C}$) *is conjugate to some r_n.*

It is easy to check that r_n maps the unipotent matrices $\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ c & 1\end{smallmatrix}\right)$ to unipotent elements of SL(R_{n-1}), and the diagonal element $\left(\begin{smallmatrix}a & 0\\ 0 & a^{-1}\end{smallmatrix}\right)$ is mapped to the diagonal element diag$(a^{n-1}, a^{n-3}, \dots, a^{-n+1})$. Since these elements generate SL(2), the image of r_n lies in

$\mathrm{SL}(R_{n-1}) \cong \mathrm{SL}(n)$.

We now define $\rho_{\lambda,n} = r_n \circ \rho_\lambda$. As we will only be considering the case when λ^2 is a simple eigenvalue of ϕ^* and the above lemmas imply the uniqueness of r_n, this gives a well-defined and unique (up to conjugation) representation $\rho_{\lambda,n} : \Gamma_\phi \to \mathrm{SL}(n)$.

By composing $\rho_{\lambda,n}$ with the adjoint representation, we also obtain an action of Γ_ϕ on $\mathfrak{sl}(n)$, turning it into a Γ_ϕ module. The following decomposition is a consequence of the Clebsch–Gordan formula (see, for example, [11, Lemma 1.4]).

Lemma 2.3. *With the Γ_ϕ module structure, $\mathfrak{sl}(n) \cong \oplus_{j=1}^{n-1} R_{2j}$.*

This decomposition will be used to calculate the infinitesimal deformations of $\rho_{\lambda,n}$.

3. Infinitesimal deformations

In this section, let M be a 3-manifold with finitely many torus boundary components $\partial M = \sqcup_{i=1}^k T_i$ and $\Gamma = \pi_1(M)$. For each boundary torus T_i, the inclusion map $\iota : T_i \to M$ induces a map from $\pi_1(T_i)$ to a conjugacy class of subgroups isomorphic to $\pi_1(T_i) \cong \mathbb{Z}\times\mathbb{Z}$ in $\pi_1(M)$. To each boundary component T_i, we associate $\pi_1(T_i)$ with a representative subgroup Δ_i in Γ. Let $R(\Gamma, \mathrm{SL}(n)) = \mathrm{Hom}(\Gamma, \mathrm{SL}(n))$ be the variety of representations of Γ into $\mathrm{SL}(n)$ and $X(\Gamma, \mathrm{SL}(n)) = R(\Gamma, \mathrm{SL}(n)) /\!\!/ \mathrm{SL}(n)$ be the $\mathrm{SL}(n)$ character variety, where the quotient is the GIT quotient as $\mathrm{SL}(n)$ acts by conjugation.

Suppose $\rho : \Gamma \to \mathrm{SL}(n)$ is a representation. The group of twisted cocycles $Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$ is defined as the set of maps $z : \Gamma \to \mathfrak{sl}(n)$ that satisfy the twisted cocycle condition

$$z(ab) = z(a) + \mathrm{Ad}_{\rho(a)} z(b), \tag{3.1}$$

which can be interpreted as the derivative of the homomorphism condition for a smooth family of representation ρ_t at ρ. The derivative of the triviality condition that ρ_t is a smooth family of representations obtained by conjugating ρ gives the coboundary condition,

$$z(\gamma) = u - \mathrm{Ad}_{\rho(\gamma)} u, \tag{3.2}$$

and $B^1(\Gamma; \mathfrak{sl}(n)_\rho)$ is defined as the set of coboundaries, or the cocycles satisfying Equation (3.2). The quotient is defined to be

$$H^1(\Gamma; \mathfrak{sl}(n)_\rho) = Z^1(\Gamma; \mathfrak{sl}(n)_\rho)/B^1(\Gamma; \mathfrak{sl}(n)_\rho).$$

Weil [18, 9] has noted that $Z^1(\Gamma; \mathfrak{sl}(n)_\rho)$ contains the tangent space to $R(\Gamma, \mathrm{SL}(n))$ at ρ as a subspace. The following tools can be used to determine if the representation variety is smooth at ρ so that we can study the space of cocycles to determine the first order behavior of deformations of a representation ρ. In the following proposition, $C^1(\Gamma; \mathfrak{sl}(n)_\rho)$ denotes the set of cochains $\{c : \Gamma \to \mathfrak{sl}(n)\}$.

Proposition 3.1 ([5, Lemma 3.2], [6, Proposition 3.1])**.** *Let $\rho \in R(\Gamma, \mathrm{SL}(n))$, $u_i \in C^1(\Gamma; \mathfrak{sl}(n)_\rho)$, $1 \le i \le j$ be given, and $\mathbb{C}[[t]]$ denote the set of formal power series in t with coefficients in $\mathbb{C}$. If*

$$\rho^j(\gamma) = \exp\left(\sum_{i=1}^{j} t^i u_i(\gamma)\right)\rho(\gamma)$$

is a homomorphism into $\mathrm{SL}(n, \mathbb{C}[[t]])$ modulo t^{j+1}, then there exists an obstruction class

$\zeta_{j+1}^{(u_1,\ldots,u_k)} \in H^2(\Gamma; \mathfrak{sl}(n)_\rho)$ *such that:*

(1) *There is a cochain* $u_{j+1} : \Gamma \to \mathfrak{sl}(n)$ *such that*

$$\rho^{j+1}(\gamma) = \exp\left(\sum_{i=1}^{j+1} t^i u_i(\gamma)\right)\rho(\gamma)$$

is a homomorphism modulo t^{j+2} *if and only if* $\zeta_{j+1} = 0$.

(2) *The obstruction* ζ_{j+1} *is natural, i.e. if* f *is a homomorphism then* $f^*\rho^j := \rho^j \circ f$ *is also a homomorphism modulo* t^{j+1} *and* $f^*(\zeta_{j+1}^{(u_1,\ldots,u_j)}) = \zeta_{j+1}^{(f^*u_1,\ldots,f^*u_j)}$.

We will apply the previous proposition to the restriction map ι^* on cohomology, which is induced by the inclusion map $\iota : \partial M \to M$. As ∂M consists of a disjoint union of tori, we will need to understand $H^1(\Delta_i; \mathfrak{sl}(n)_{r_n\circ\rho})$. Recall that a *hyperbolic* element of SL(2) is an element that acts on $\mathbb{H}^3$ with no fixed points in $\mathbb{H}^3$ and two fixed points on $\partial\mathbb{H}^3$. Such elements are characterized by being conjugate in SL(2) to a diagonal matrix with distinct eigenvalues that are not on the unit circle.

Lemma 3.2. *Suppose* $\rho : \mathbb{Z}\times\mathbb{Z} \to \mathrm{SL}(2)$ *contains a hyperbolic element in its image. Then* $\dim H^1(\mathbb{Z}\times\mathbb{Z}; \mathfrak{sl}(n)_{r_n\circ\rho}) = 2(n-1)$.

Proof. Suppose $\gamma \in \mathbb{Z}\times\mathbb{Z}$ such that $\rho(\gamma)$ is a hyperbolic element in SL(2). Then, up to conjugation,

$$\rho(\gamma) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

for some $|a| > 1$. The image of such an element under the irreducible representation $r_n : \mathrm{SL}(2) \to \mathrm{SL}(n)$ is conjugate to a diagonal matrix with n distinct eigenvalues. Hence any nearby representation $\rho' : \mathbb{Z}\times\mathbb{Z} \to \mathrm{SL}(n)$ is conjugate to a diagonal matrix with distinct entries. In other words, up to coboundary, we can assume that any class $[z] \in H^1(\mathbb{Z}\times\mathbb{Z}; \mathfrak{sl}(n)_{r_n\circ\rho})$ has the form of a diagonal matrix $z(\gamma) = \mathrm{diag}(y_1, y_2, \ldots, y_n)$ where $\mathrm{tr}\, z(\gamma) = 0$. Since for any other $\gamma' \in \mathbb{Z}\times\mathbb{Z}$, we have that γ' commutes with γ, $z(\gamma')$ must also be diagonal, so the dimension of $H^1(\mathbb{Z}\times\mathbb{Z}; \mathfrak{sl}(n)_{r_n\circ\rho})$ is $2(n-1)$. □

Lemma 3.3. *Let* $\rho : \pi_1(M) \to \mathrm{SL}(2)$ *be a non-abelian representation such that* $\rho(\Delta_i)$ *contains a hyperbolic element for each subgroup* Δ_i *of* $\pi_1(M)$ *associated to a boundary component* T_i *of* ∂M. *If* $\dim H^1(\Gamma; \mathfrak{sl}(2)_{r_n\circ\rho}) = k(n-1)$ *where* k *is the number of components of* ∂M, *then* $\iota^* : H^2(M; \mathfrak{sl}(n)_{r_n\circ\rho}) \to H^2(\partial M; \mathfrak{sl}(n)_{r_n\circ\rho})$ *is injective.*

Proof. We have the cohomology exact sequence for the pair $(M, \partial M)$,

$$\begin{array}{ccccc} & H^1(M,\partial M) & \longrightarrow & H^1(M) & \xrightarrow{\alpha} & H^1(\partial M) \\ \xrightarrow{\beta} & H^2(M,\partial M) & \longrightarrow & H^2(M) & \xrightarrow{\iota^*} & H^2(\partial M) \\ \longrightarrow & H^3(M,\partial M) & \longrightarrow & \cdots & & \end{array}$$

where all cohomology groups are taken to be with the twisted coefficients $\mathfrak{sl}(n)_{r_n\circ\rho}$. A standard Poincaré duality argument [6, 7, 13] implies that α has half-dimensional image in $H^1(\partial M)$. By Lemma 3.2,

$$\dim H^1(\Delta_i) = 2(n-1),$$

as long as $\rho(\Delta_i)$ contains a hyperbolic element for each i. We can identify $H^1(\partial M) \cong \oplus_{i=1}^k H^1(\Delta_i)$, which has dimension $2k(n-1)$. Since $H^1(M) \cong H^1(\Gamma)$ has dimension $k(n-1)$, then α is injective. Since β is dual to α under Poincaré duality, then β is surjective. This implies that ι^* is injective. □

We now utilize the previous facts to determine sufficient conditions for deforming representations.

Proposition 3.4. *Let $\rho : \Gamma \to \mathrm{SL}(2)$ be a non-abelian representation such that $\rho(\Delta_i)$ contains a hyperbolic element for each subgroup Δ_i. If $H^1(\Gamma; \mathfrak{sl}(2)_{r_n\circ\rho}) = k(n-1)$ where k is the number of components of ∂M, then $r_n \circ \rho$ is a smooth point of the representation variety $R(\Gamma, \mathrm{SL}(n))$, and it is contained in a unique component of dimension $(n+1+k)(n-1) - \dim H^0(\Gamma; \mathfrak{sl}(n)_{r_n\circ\rho})$.*

Proof. We begin by showing that every cocycle in $Z^1(\Gamma; \mathfrak{sl}(n)_{r_n\circ\rho})$ is integrable.

Suppose we have $u_1, \ldots, u_j : \Gamma \to \mathfrak{sl}(n)$ such that

$$\rho_n^j(\gamma) = \exp\left(\sum_{i=1}^{j} t^i u_i(\gamma)\right)\rho(\gamma)$$

is a homomorphism modulo t^{j+1}. By Lemma 3.2 and [14], the restriction of ρ_n to Δ_i is a smooth point of the representation variety $R(\Delta_i, \mathrm{SL}(n))$. Hence $\rho_n^j|_{\pi_1(T_i)}$ extends to a formal deformation of order $j+1$ by the formal implicit function theorem (see [6], Lemma 3.7). This implies that the restriction of $\zeta_{j+1}^{(u_1,\ldots,u_j)}$ to each component $H^2(T_i) < H^2(\partial M)$ vanishes.

As $H^2(\partial M) = \oplus_{i=1}^k H^2(T_i)$, hence, $\iota^* \zeta_{j+1}^{(u_1,\ldots,u_j)} = \zeta_{j+1}^{(\iota^* u_1,\ldots,\iota^* u_j)} = 0$. The injectivity of ι^* follows from Lemma 3.3 and implies that $\zeta_{j+1}^{(u_1,\ldots,u_j)} = 0$. Hence, the homomorphism can be extended to a deformation $(r_n \circ \rho)^{j+1}$ of order $j+1$, and inductively to a formal deformation $(r_n \circ \rho)^\infty$.

Applying [6, Proposition 3.6] to the formal deformation $(r_n \circ \rho)^\infty$ results in a convergent deformation. Hence, $r_n \circ \rho$ is a smooth point of the representation variety.

As in [5], we note that the exactness of

$$1 \to H^0(\Gamma; \mathfrak{sl}(n)_{r_n\circ\rho}) \to \mathfrak{sl}(n)_{r_n\circ\rho} \to B^1(\Gamma; \mathfrak{sl}(n)_{r_n\circ\rho})$$

implies that

$$\dim B^1(\Gamma; \mathfrak{sl}(n)_{r_n\circ\rho}) = n^2 - 1 - \dim H^0(\Gamma; \mathfrak{sl}(n)_{r_n\circ\rho}).$$

Thus, we conclude that the local dimension of $R(\Gamma, \mathrm{SL}(n))$ is

$$\dim Z^1(\Gamma; \mathfrak{sl}(n)_{r_n\circ\rho}) = (n+1+k)(n-1) - \dim H^0(\Gamma; \mathfrak{sl}(n)_{r_n\circ\rho}).$$

That it is in a unique component follows from [6, Lemma 2.6]. □

4. Deforming $\rho_{\lambda,n}$

We will now show that $\rho_{\lambda,n}$ satisfies the conditions in Proposition 3.4 so that $\rho_{\lambda,n}$ can be deformed within a neighborhood of representations. This will entail a computation of the dimension of the cohomology group $H^1(\Gamma_\phi; \mathfrak{sl}(n)_{\rho_{\lambda,n}})$. By the decomposition in Lemma 2.3, the cohomology group $H^1(\Gamma_\phi; \mathfrak{sl}(n)_{\rho_{\lambda,n}})$ is a direct sum,

$$H^1(\Gamma_\phi; \mathfrak{sl}(n)_{\rho_{\lambda,n}}) \cong \bigoplus_{j=1}^{n-1} H^1(\Gamma_\phi; R_{2j}),$$

so it suffices to compute the dimensions of $H^1(\Gamma_\phi; R_{2j})$, for $1 \le j \le n-1$.

To simplify the computations which follow, we give a presentation of Γ_ϕ with an additional generator γ_{2g+p}. We will choose $\gamma_1, \dots, \gamma_{2g}$ to be standard generators of the fundamental group for the closed surface S_g, and $\gamma_{2g+1}, \dots, \gamma_{2g+p}$ to be curves around the p punctures of S. Then $\pi_1(\Gamma_\phi)$ has a presentation of the form:

$$\left\langle \gamma_1, \dots, \gamma_{2g+p}, \tau \;\middle|\; \tau\gamma_i\tau^{-1} = \phi(\gamma_i), \prod_{i=1}^{g} [\gamma_{2i-1}, \gamma_{2i}] = \prod_{s=1}^{p} \gamma_{2g+s} \right\rangle.$$

With these generators for $\pi_1(S)$, $\phi^* : H^1(S) \to H^1(S)$ can be written as a block matrix,

$$\begin{pmatrix} [\bar{\phi}^*] & [*] \\ 0 & [P] \end{pmatrix},$$

where $\bar{\phi}^* : H^1(\bar{S}) \to H^1(\bar{S})$ is the induced map on the first cohomology of the closed surface $\bar{S}$ obtained by filling in the p punctures of S, and $P = (p_{ij})$ is a permutation matrix denoting the permutation of the punctures of S under ϕ. In particular, $p_{jk_j} = 1$ if and only if $\tau\delta_j\tau^{-1}$ is conjugate to δ_{k_j}, and $p_{jk_j} = 0$ otherwise. The matrix $\bar{\phi}^*$ is a symplectic matrix preserving the intersection form ω on $\bar{S}$. The eigenvalues of P are roots of unity, with 1 occurring as an eigenvalue once for each cycle in the permutation.

We now compute the cohomological dimension of $H^1(\Gamma_\phi; R_{2j})$. The argument uses similar ideas to [8, Theorem 4.1] using the generators $X^{\ell-1}Y^{2j-\ell}$, $\ell = 0, \dots, 2j$, of R_{2j} and is equivalent up to a coordinate change when $j = 1$.

Proposition 4.1. *Let $\phi : S \to S$ be a homeomorphism, with λ^2 a simple eigenvalue of ϕ^*. Suppose also that $|\lambda| \ne 1$, $\bar{\phi}^* : H^1(\bar{S}) \to H^1(\bar{S})$ does not have 1 as an eigenvalue, and for each $2 \le j \le n-1$, we have that λ^{2j} is not an eigenvalue of ϕ^*. Then for each j, $1 \le j \le n-1$, $\dim H^1(\Gamma_\phi; R_{2j}) = k$ where k is the number of components of ∂M_ϕ.*

Proof. Let $z \in Z^1(\Gamma_\phi, R_{2j})$. Then z is determined by its values on $\gamma_1, \dots, \gamma_{2g+p}$, and τ, subject to the cocycle condition (3.1) imposed by the relations in Γ_ϕ. These can be computed via the Fox calculus [9, Chapter 3]. Differentiating the relations

$$\tau\gamma_i\tau^{-1} = \phi(\gamma_i),$$

yields

$$\frac{\partial[\phi(\gamma_i)\tau\gamma_i^{-1}\tau^{-1}]}{\partial\gamma_i} = \frac{\partial\phi(\gamma_i)}{\partial\gamma_i} - \phi(\gamma_i)\tau\gamma_i^{-1} = \frac{\partial\phi(\gamma_i)}{\partial\gamma_i} - \tau, \tag{4.1}$$

$$\frac{\partial[\phi(\gamma_i)\tau\gamma_i^{-1}\tau^{-1}]}{\partial\gamma_h} = \frac{\partial\phi(\gamma_i)}{\partial\gamma_h}, \quad i \neq h,$$
$$\frac{\partial[\phi(\gamma_i)\tau\gamma_i^{-1}\tau^{-1}]}{\partial\tau} = \phi(\gamma_i) - \phi(\gamma_i)\tau\gamma_i^{-1}\tau^{-1} = \phi(\gamma_i) - 1.$$

A cocycle z then must satisfy the set of equations for $1 \leq i \leq 2g + p$ of the form

$$\sum_{h=1}^{2g+p} \frac{\partial[\phi(\gamma_i)\tau\gamma_i^{-1}\tau^{-1}]}{\partial\gamma_h} \cdot z(\gamma_h) + \frac{\partial[\phi(\gamma_i)\tau\gamma_i^{-1}\tau^{-1}]}{\partial\tau} \cdot z(\tau) = 0. \tag{4.2}$$

With respect to the basis X^0Y^{2j}, X^1Y^{2j-1}, $\dots$, $X^{2j}Y^0$ for R_{2j}, the values $z(\gamma_i)$ can be expressed in coordinates $(x_{i,\ell})$, where $x_{i,\ell}$ is the coefficient of $X^\ell Y^{2j-\ell}$ for $z(\gamma_i)$. We similarly express $z(\tau)$ in the coordinates $x_{0,\ell}$, $0 \leq \ell \leq 2j$ with $x_{0,\ell}$ being the $X^\ell Y^{2j-\ell}$ coefficient of $z(\tau)$. Direct calculation shows that

$$\begin{aligned} \rho(\gamma_i) \cdot X^\ell Y^{2j-\ell} &= (X - a_iY)^\ell Y^{2j-\ell} = \sum_{m=0}^{\ell} (-a_i)^m \binom{\ell}{m} X^{\ell-m} Y^{2j-\ell+m}, \\ \rho(\tau) \cdot X^\ell Y^{2j-\ell} &= (\lambda^{-1}X)^\ell (\lambda Y)^{2j-\ell} = \lambda^{2j-2\ell} X^\ell Y^{2j-\ell}. \end{aligned} \tag{4.3}$$

The set of coboundaries can be computed from Equation (3.2) as the set of cocycles z' satisfying,

$$z'(\gamma_i) = \sum_{\ell=0}^{2j} b_\ell X^\ell Y^{2j-\ell} - b_\ell(X - a_iY)^\ell Y^{2j-\ell} = \sum_{\ell=0}^{2j}\sum_{m=1}^{\ell} -b_\ell(-a_i)^m \binom{\ell}{m} X^{\ell-m} Y^{2j-\ell+m},$$
$$z'(\tau) = \sum_{\ell=0}^{2j} (b_\ell - \lambda^{2j-2\ell} b_\ell) X^\ell Y^{2j-\ell},$$

where $b_0, \dots, b_{2j} \in \mathbb{C}$ parametrize the set $B^1(\Gamma_\phi, R_{2j})$ of coboundaries. In particular, adding the appropriate coboundary z' to z, we can assume $x_{0,\ell} = 0$ for $\ell \neq j$, so that $z(\tau)$ has the form

$$z(\tau) = x_{0,j} X^j Y^j.$$

Then z is determined by a vector

$$\vec{v} = (x_{1,0}, \dots, x_{2g+p,0}, \dots, x_{0,j}, x_{1,j}, \dots, x_{2g+p,j}, \dots, x_{1,2j}, \dots, x_{2g+p,2j})^T$$

in the kernel of a block matrix $A = (A_{\alpha,\beta})$ where the entries in the i-th row of $A_{\alpha,\beta}$ are the coefficients of the terms $x_{*,\beta} X^\alpha Y^{2j-\alpha}$ in Equation (4.2). Since the image under ρ of any word w in $\{\gamma_i, \gamma_i^{-1}\}_{i=1}^{2g+p}$ has the form

$$\rho(w) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

for some $a \in \mathbb{C}$, then the previous calculations in Equations (4.3) imply that $A_{\alpha,\beta} = \mathbf{0}$ for $\beta < \alpha$. Moreover, when $\alpha \neq j$, $A_{\alpha,\alpha}$ is a square matrix, and we note that the coefficient of $X^\alpha Y^{2j-\alpha}$ in $\rho(\gamma_i) \cdot X^\alpha Y^{2j-\alpha}$ is 1, so that in Equation (4.2), the coefficient of $x_{h,\alpha}$ in the $X^\alpha Y^{2j-\alpha}$ term is the signed number of times that γ_h appears in the word $\phi(\gamma_i)$. In addition, Equation (4.2) will contain a single $-\tau \cdot z(\gamma_i)$ term, so that $A_{\alpha,\alpha} = \phi^* - \lambda^{2j-2\alpha} I$ when $\alpha \neq j$. We also see that

$$A_{j,j} = \left(\begin{matrix}0\\ \vdots\\ 0\end{matrix} \;\left[\phi^* - I\right]\right).$$

Since $z(\tau) = x_{0,j}X^jY^j$, direct calculation also shows that for some matrix K,

$$A_{j-1,j} = \left(\begin{array}{c|c} -j\lambda^2 a_1 & \\ \vdots & K \\ -j\lambda^2 a_{2g+p} & \end{array}\right).$$

As λ^2 is a simple eigenvalue, $\bar{\phi}^*$ is symplectic, and the eigenvalues of P are roots of unity, $\phi^* - \lambda^2 I$ and $\phi^* - \lambda^{-2}I$ have 1 dimensional kernel. Furthermore, since 1 is not an eigenvalue of $\bar{\phi}^*$, $\phi^* - I$ has kernel whose dimension is equal to the number of disjoint cycles of the permutation of the punctures. This is equal to the number of components of ∂M_ϕ. In addition, since $\lambda^{2j-2\alpha}$ is not an eigenvalue of ϕ^* for $\alpha \neq j-1, j, 1$, the kernel of $A_{\alpha,\alpha}$ is trivial in these cases. Hence, the kernel of A has dimension at most $2 + k + 1$, where

$$k = \#\text{ of components of } \Sigma = \#\text{ of components of } \partial M_\phi.$$

The additional dimension comes from the possible contribution to the kernel from the first column of $A_{(j-1),j}$. Consider the submatrix

$$U = \left(\begin{array}{c|c} A_{j-1,j-1} & A_{j-1,j} \\ \hline \mathbf{0} & A_{j,j} \end{array}\right)$$

$$= \left(\begin{array}{c|c|c} & -j\lambda^2 a_1 & \\ \phi^* - \lambda^2 I & \vdots & K \\ & -j\lambda^2 a_{2g+n} & \\ \hline & 0 & \\ \mathbf{0} & \vdots & \phi^* - I \\ & 0 & \end{array}\right).$$

If $\mathrm{null}(A) > 2 + k$, then we must have that $\mathrm{null}(U) > k + 1$.

Since λ^2 is a simple eigenvalue of ϕ^* and $(a_1, \dots, a_{2g+p})^T$ is an eigenvector of the λ^2 eigenspace, $(a_1, \dots, a_{2g+p})^T$ is not in the image of $\phi^* - \lambda^2 I$. Hence, for any $x = (x_{1,j}, \dots, x_{2g+p,j})^T$ in the kernel of $\phi^* - I$, there is a unique $x_{0,j}$ such that $Kx - x_{0,j}(a_1, \dots, a_{2g+p})^T$ is in the image of $\phi^* - \lambda^2 I$. Therefore, $\mathrm{null}(U) = k + 1$.

Hence $\mathrm{null}(A) = 2 + k$. However, the solution arising from the kernel of $\phi^* - \lambda^2 I$ is the eigenvector

$$(0, \dots, 0, x_{1,j}, \dots, x_{2g+p,j}, 0, \dots, 0)^T = (0, \dots, 0, a_1, \dots, a_{2g+p}, 0, \dots, 0)^T$$

which is a coboundary. So we have that $\dim H^1(\Gamma_\phi; R_{2j}) \leq k+1$. Finally, there is one further redundancy since

$$\prod_{i=1}^{g}[\gamma_{2i-1}, \gamma_{2i}] = \prod_{s=1}^{p}\gamma_{2g+s}.$$

From the $\phi^* - I$ in $A_{j,j}$, we can see that $x_{j,2g+1}, \dots, x_{j,2g+p}$ can be freely chosen as long as $x_{j,2g+s} = x_{j,2g+t}$ whenever γ_{2g+s} and γ_{2g+t} are in the same cycle of P. Since $|\lambda| \neq 1$, for any eigenvector of ϕ^*, $a_{2g+1} = \cdots = a_{2g+p} = 0$, so the $X^j Y^j$ coefficient of $z(\Pi_{s=1}^n \gamma_{2g+s})$ can be chosen to be any quantity

$$x_{j,2g+1} + \cdots + x_{j,2g+p}. \tag{4.4}$$

The relation $\Pi_{i=1}^g [\gamma_{2i}, \gamma_{2i+1}] = \Pi_{s=1}^p \gamma_{2g+s}$ relates the sum in Equation (4.4) to the $X^j Y^j$ coefficient of $\Pi_{i=1}^g [\gamma_{2i}, \gamma_{2i+1}]$, which has no dependence on $x_{j,2g+s}$, for $1 \leq s \leq p$. This imposes a 1-dimensional relation on the space of cocycles, and we conclude that

$$\dim H^1(\Gamma_\phi, R_{2j}) = k. \qquad \square$$

We now prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. By Lemma 2.3, $\mathfrak{sl}(n)$ is the direct sum of R_{2j}, $j = 1, \dots, n-1$. The conditions on the eigenvalues of ϕ^* and Proposition 4.1 imply that for each j, $\dim H^1(\Gamma_\phi; R_{2j}) = k$. Hence $\dim H^1(\Gamma_\phi, \mathfrak{sl}(n)_{\rho_{\lambda,n}}) = k(n-1)$. By Proposition 3.4, this implies smoothness of $R(\Gamma_\phi, \mathrm{SL}(n))$ at $\rho_{\lambda,n}$. Since $\rho_{\lambda,n}$ is non-abelian, it has trivial infinitesimal centralizer, so $H^0(\Gamma_\phi; R_2) = 0$, so that the local dimension is $(n+1+k)(n-1)$. $\square$

We obtain the special case in Theorem 1.2 when λ^2 is the dilatation of a pseudo-Anosov map ϕ.

Proof of Theorem 1.2. When the stable and unstable foliations of ϕ are orientable, it is a well-known fact that the dilatation is a simple eigenvalue and the largest eigenvalue of ϕ^* (see [4], [10], [12]). Hence, ϕ satisfies the conditions of Theorem 1.1.

From [8], we know that there are hyperbolic deformations of $\rho_\lambda = \rho_{\lambda,2}$, which are irreducible representations since they correspond to hyperbolic structures. The composition of these deformations with the irreducible representation r_n then provides nearby deformations of $\rho_{\lambda,n}$ which are also irreducible. $\square$

5. Description of deformations

Recall that the action of Γ_ϕ on $\mathfrak{sl}(n)$ is given by composing $\rho_{\lambda,n}$ with the adjoint representation. That is, for $\gamma \in \Gamma_\phi$ and $c \in \mathfrak{sl}(n)$,

$$\gamma \cdot c = \mathrm{Ad}_{\rho_{\lambda,n}(\gamma)}(c) = \rho_{\lambda,n}(\gamma)\, c\, \rho_{\lambda,n}(\gamma)^{-1}.$$

Let E_j denote the j-th standard basis vector for $\mathbb{C}^n$. Then every element of $\mathfrak{sl}(n)$ is a linear combination of the matrices $E_{j,\ell} = E_j \cdot E_\ell^T$. In order to obtain a useful description of the action of Γ_ϕ on $\mathfrak{sl}(n)$, it suffices to compute the action of γ on $E_{j,\ell}$ for a set of generators of Γ_ϕ. By direct calculation,

(5.1)
$$\gamma_i \cdot E_{j,\ell} = r_n\left(\begin{bmatrix}1 & a_i\\ 0 & 1\end{bmatrix}\right)E_j \cdot E_\ell^T r_n\left(\begin{bmatrix}1 & a_i\\ 0 & 1\end{bmatrix}\right)^{-1} = \begin{pmatrix}(-a_i)^{j-1}\binom{j-1}{\ell-1}\\ (-a_i)^{j-2}\binom{j-1}{\ell-2}\\ \vdots\\ (-a_i)^0\binom{j-1}{0}\\ 0\\ \vdots\\ 0\end{pmatrix}\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ (a_i)^1\binom{\ell}{1}\\ (a_i)^2\binom{\ell+1}{2}\\ \vdots\\ (a_i)^{n-\ell}\binom{n-1}{n-\ell}\end{pmatrix}^T$$

$$\begin{aligned}\tau \cdot E_{j,\ell} &= r_n\left(\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}\right)E_j \cdot E_\ell^T r_n\left(\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}\right)^{-1}\\ &= \lambda^{2(n-j+1)}E_j \cdot E_\ell^T \lambda^{-2(n-\ell+1)} = \lambda^{2(\ell-j)}E_{j,\ell}.\end{aligned}$$

Notably, the actions of Γ_ϕ on the (j,ℓ)-coordinates of $\mathfrak{sl}(n)$ have no contributions to all rows $> j$ and all columns $< \ell$. Applying analogous calculations as in the proof of Proposition 4.1 to the relations in Γ_ϕ, we find that if $z : \Gamma_\phi \to \mathfrak{sl}(n)_{\rho_{\lambda,n}}$ is a cocycle and $z_{j,\ell}(\gamma_i)$ is the (j,ℓ)-coordinate of $z(\gamma_i)$, then the vector

$$\vec{v}_{n,1} = \begin{pmatrix} z_{n,1}(\gamma_1)\\ \vdots \\ z_{n,1}(\gamma_{2g+p})\end{pmatrix} = (z_{n,1}(\gamma_i))$$

is a solution to $(\phi^* - \lambda^{-2(n-1)}I)\vec{v}_{n,1} = \mathbf{0}$. Since $\lambda^{-2(n-1)}$ is not an eigenvalue of ϕ^*, it follows that $\vec{v}_{n,1} = \mathbf{0}$.

Since $\vec{v}_{n,1} = \mathbf{0}$, when the relations in Γ_ϕ applied to z are restricted to the $(n-1,1)$-coordinate and the $(n,2)$-coordinate, we obtain that $\vec{v}_{n-1,1} = (z_{n-1,1}(\gamma_i))$ and $\vec{v}_{n,2} = (z_{n,2}(\gamma_i))$ are solutions to $(\phi^* - \lambda^{-2(n-2)}I)\vec{v} = \mathbf{0}$. A straightforward induction combined with Equations (5.1) then shows that $z_{j,\ell}(\gamma_i) = 0$ for all $j > \ell + 1$ while $\vec{v}_{j,\ell} = (z_{j,\ell}(\gamma_i))$ is a λ^{-2}-eigenvector of ϕ^* when $j = \ell + 1$, i.e. the subdiagonal entries of $z(\gamma_i) \in \mathfrak{sl}(n)$ are coordinates from eigenvectors of ϕ^*, and all other entries below the diagonal are 0. This provides $n-1$ generators of cocycles. The others come from the 1-eigenspaces of ϕ^* when applying the cocycle conditions to the diagonal entries of $z(\gamma_i)$.

We have that $\mathfrak{sl}(n)$ can be associated with the tangent space to SL(n) at the identity, and multiplying $z(\gamma_i)$ by $\rho_{\lambda,n}(\gamma_i)$ gives the derivative at $\rho_{\lambda,n}(\gamma_i)$. The previous calculations then imply that if $\rho_t : \Gamma_\phi \to \mathrm{SL}(n)$ is a path of representations such that $\rho_0 = \rho_{\lambda,n}$, then the subdiagonal entries of $\rho_t'(\gamma_i)$ at $t = 0$ are equal to the subdiagonal entries of $z(\gamma_i)$. Hence, for each j,ℓ, there exists at least one i for which $z_{j,\ell}(\gamma_i) \neq 0$.

Note that in the case that λ^2 is the dilatation of a pseudo-Anosov map ϕ as in Theorem 1.2, the subdiagonal entries of the irreducible representations obtained by deforming ρ_λ in SL(2) and composing with r_n to obtain a deformation of $\rho_{\lambda,n}$ necessarily satisfy certain relations. In particular, the first derivatives of the subdiagonal entries would have to be fixed multiples of entries of the λ^{-2}-eigenvector determined by the irreducible representation r_n. As described above, the deformations in SL(n) allow the derivatives to be freely chosen multiples of the $n-1$ generators, so there are deformations which are not from deformations of ρ_λ that are composed with r_n. Since the set of irreducible representations is an open subset

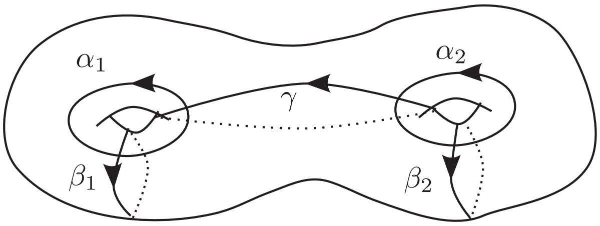

Fig. 1. The curves $\alpha_1, \alpha_2, \beta_1, \beta_2$ which form the basis for $H_1(S)$, and γ.

of the space of $R(\pi_1(M_\phi), \mathrm{SL}(n))$ (see, for example, [2, Lemma 1.4.2], [15, Proposition 27]), this also implies there are nearby irreducible representations which are not from composing deformations of ρ_λ with r_n.

6. Example

The genus 2 example $\phi : S_{2,2} \to S_{2,2}$ from [8], obtained from taking the left Dehn twists $T_{\beta_1}, T_{\beta_2}, T_\gamma$, followed by the right Dehn twists $T_{\alpha_1}^{-1}, T_{\alpha_2}^{-1}$, satisfies the hypotheses of Theorem 1.2. Each component of $S_2 \setminus \{\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma\}$ contains one of the two punctures. The map on cohomology $\bar{\phi}^*$ has two simple eigenvalues $\lambda_1^2 = \frac{5+\sqrt{21}}{2}$ and $\lambda_2^2 = \frac{3+\sqrt{5}}{2}$, along with their reciprocals λ_1^{-2} and λ_2^{-2}. The reducible representations $\rho_{\lambda_i,n}$ are smooth points of $R(\Gamma_\phi, \mathrm{SL}(n))$, each on a component of dimension $(n+3)(n-1)$. There is a two-dimensional family of irreducible representations in $X(\Gamma_\phi, \mathrm{SL}(n))$, which is the image of a two-dimensional family of irreducible representations in $X(\Gamma_\phi, \mathrm{SL}(2))$ under r_n, limiting to $\rho_{\lambda_1,n}$.

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