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VERTICAL 3-MANIFOLDS IN SIMPLIFIED GENUS-2 TRISECTIONS OF 4-MANIFOLDS

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Abstract

We classify the 3-manifolds obtained as the preimages of arcs on the plane for simplified genus-2 trisection maps, which we call vertical 3-manifolds. Vertical 3-manifolds of $(2, 1)$- and $(2, 2)$-trisection maps can be easily classified. A vertical 3-manifold of a $(2, 0)$-trisection map is a connected sum of finite copies of 6-tuple of vertical 3-manifolds over specific 6 arcs and $S^1 \times S^2$. Consequently, we show that each of the 6-tuples determines the source 4-manifold uniquely up to orientation reversing diffeomorphisms except for the trivial case. We also show that, in contrast to the fact that summands of vertical 3-manifolds of simplified $(2, 0)$-trisection maps are lens spaces, there exist infinitely many simplified $(2, 0)$-4-section maps that admit hyperbolic vertical 3-manifolds.

1. Introduction

A trisection is a decomposition of a closed orientable smooth 4-manifold into three 4-dimensional handlebodies introduced by Gay and Kirby [5]. They proved that any closed orientable smooth 4-manifold has a trisection. While the trisection has a strong meaning as a 4-dimensional analog of the Heegaard splitting of 3-manifolds, it is also deeply related to the study of stable maps as homotopy-deformations of stable maps are used in their proof. The singular value set of a stable map of a trisection is the union of immersed circles with cusps as in Fig.1, where the singular value set in the white boxes consists of immersed curves with only normal double points and without cusps and radial tangencies. This stable map is called a trisection map. The fiber Σ_g over the center point p in the figure is a closed orientable surface of genus g, and this has the highest genus among all regular fibers of the trisection map. Set k to be the number of arcs in Fig.1 connecting to neighboring white boxes without cusps. In this case, the trisection is called a (g, k)-trisection. The vanishing cycles of indefinite folds of the trisection map can be represented by simple closed curves on the center fiber Σ_g. The vanishing cycles of a trisection determines the source 4-manifold up to diffeomorphism. The surface Σ_g with these simple closed curves is called a trisection diagram. To find the usage of trisections, there are several studies of constructing trisections of given 4-manifolds [10, 14, 12]. It is also used for studies of surfaces embedded in 4-manifolds [15, 11]. Classifications of 4-manifolds admitting trisection maps for $g = 1$ is easy, and that for $g = 2$ had been done by Meier and Zupan in [16]. The classification for $g \geq 3$ is still difficult.

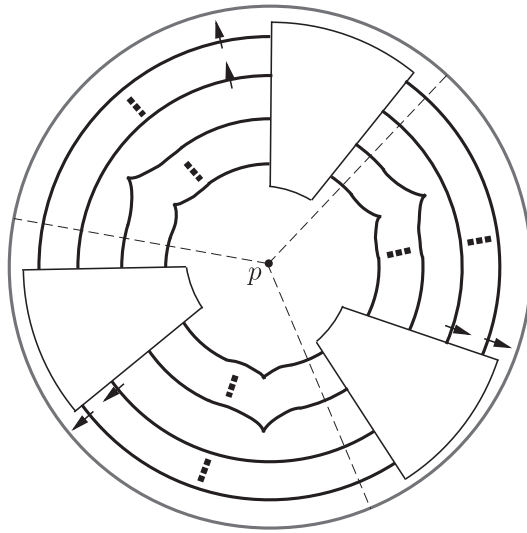On the other hand, it is a long-standing problem in the study of topology of mappings to

Fig. 1. A singular value set of a trisection map.

understand the source manifold of a smooth map from the information of its singularities. In 2009, Baykur proved that any closed 4-manifold admits a smooth map to the 2-sphere with only Lefschetz singularities and a circle of indefinite folds [1]. Recently, Baykur and Saeki proved that any closed 4-manifold admits a stable map to the plane whose singular value set is as in Fig.1 with the singular value set in the white boxes consisting of embedded curves without cusps and radial tangencies. This gives a special case of a trisection, called a simplified (g, k)-trisection. They proved that any closed 4-manifold admitting a $(2, k)$-trisection has a simplified $(2, k)$-trisection. Later, Hayano gave an alternative short proof of this result by observing monodromy diffeomorphisms generated by the Dehn twists along the vanishing cycles [6].

The aim of this paper is to understand relation between trisection maps and the source 4-manifolds from the information of 3-manifolds obtained as the preimages of arcs on $\mathbf{R}^2$. In the study of topology of mappings, Kobayashi studied a stable map whose singular value set consists of two concentric circles, the outer one is the image of definite folds and the inner one is the image of indefinite folds and cusps, where cusps are outward [9]. In his further study [8], he constructed an infinite number of stable maps on each of $S^2 \times S^2$ and $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$ that have the same singular value set but that are not right-left equivalent, using what he called four cusped fans. This theorem is proved by observing the 3-manifolds obtained as the preimage of an arc in the target space.

In this paper, we classify the 3-manifolds obtained as the preimages of arcs on $\mathbf{R}^2$ for a simplified genus-2 trisection map, called vertical 3-manifolds, and study if they determine the source 4-manifold. It is easy to verify that a vertical 3-manifold of a simplified $(2, k)$-trisection for $k = 1, 2$ is a connected sum of finite copies of $S^1 \times S^2$ and it is known that the corresponding 4-manifold is $S^1 \times S^3 \# \mathbf{CP}^2$ or $S^1 \times S^3 \# \overline{\mathbf{CP}^2}$ if $k = 1$ and $\#^2 S^1 \times S^3$ if $k = 2$ [2, 16], see Remark 11. Therefore we focus on the classification of the vertical 3-manifolds of simplified $(2, 0)$-trisection maps.

The classification is the following:

Theorem 1. *Vertical 3-manifolds of simplified $(2, 0)$-trisection maps are*

$$\#^{\ell+\epsilon_1, \ell} L(k^2, k-1) \#^{m, m+\epsilon_2} L(k \pm 1, \pm 1) \#^n S^1 \times S^2 \quad (k \in \mathbf{Z})$$

$$\#^{\ell+\epsilon_1, \ell} L(9, 2) \#^{m, m+\epsilon_2} L(k, 1) \#^n S^1 \times S^2 \quad (k \in \{2, 5\})$$

$$\#^{\ell+\epsilon_1, \ell} L(4, 1) \#^{m, m+\epsilon_2} L(k, 1) \#^n S^1 \times S^2 \quad (k \in \{3, 5\}).$$

Here, $\ell, m, n \in \mathbf{Z}_{\geq 0}$, $\epsilon_i \in \{0, 1\}$ and $\#^{\ell, \ell'} M$ means the connected sum of ℓ copies of an oriented 3-manifold M and ℓ' copies of the mirror image of M. Conversely, any 3-manifold above can be obtained as a vertical 3-manifold of a simplified $(2, 0)$-trisection map.

To prove the theorem, we classify the 6-tuples of vertical 3-manifolds over the 6 arcs in Fig.2. See the beginning of Section 3 for the precise definition of the 6-tuples of vertical 3-manifolds. The vertical 3-manifolds in Theorem 1 are obtained as their connected sums. We use Hayano's argument used in [6] and study the configurations of vanishing cycles of simplified $(2, 0)$-trisection maps.
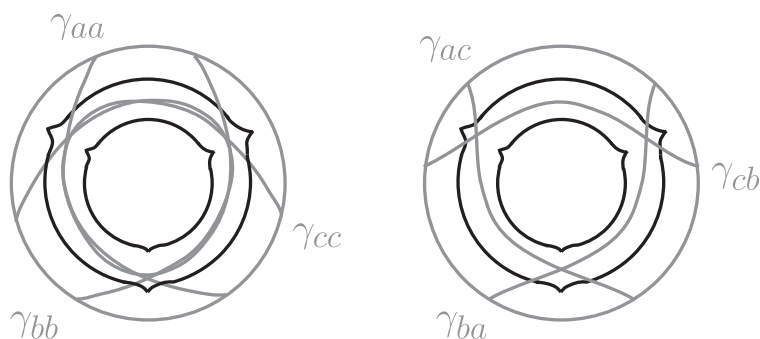

Fig.2. The 6-tuple of arcs.

Using the information obtained in the proof of Theorem 1, we determine the source 4-manifolds from the configurations of vanishing cycles. In consequence, we have the following corollary.

Corollary 2. *The 6-tuple determines the 4-manifold unless it is* $\begin{pmatrix} S^1 \times S^2 & S^3 & S^3 \\ S^3 & L(2, 1) & S^3 \end{pmatrix}$ *or* $\begin{pmatrix} S^1 \times S^2 & S^1 \times S^2 & S^1 \times S^2 \\ S^3 & S^3 & S^3 \end{pmatrix}$. *In the former exceptional case, the 4-manifold is uniquely determined up to orientation reversing diffeomorphisms.*

The detailed correspondence between 6-tuples and 4-manifolds can be found in Theorem 12.

From Theorem 1, we can see that any summand of a vertical 3-manifold of a simplified $(2, 0)$-trisection is a lens space. On the other hand, if we consider 4-manifolds with four sections, named 4-sections (cf. [7]), we can obtain infinitely many different hyperbolic 3-manifolds as vertical 3-manifolds.

Theorem 3. *Suppose $X = \#^2 S^2 \times S^2$ or $\#^2 \mathbf{CP}^2 \#^2 \overline{\mathbf{CP}^2}$. Then there exists a family of simplified $(2, 0)$-4-section maps such that the vertical 3-manifolds over the arc ω in Fig.3 are hyperbolic and mutually non-diffeomorphic.*
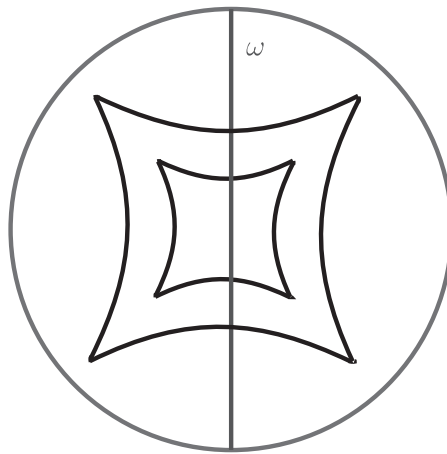
Fig. 3. The singular value set of a simplified $(2, 0)$-4-section map.

To prove the theorem, we use a handle decomposition of the 4-manifold induced by the 4-section map. The infinite sequence of hyperbolic 3-manifolds is given as a sequence of surgered manifolds along hyperbolic 2-bridge links.

The paper is organized as follows: In Section 2, we give the definition of simplified (g, k)-trisection maps and introduce some properties between mapping class groups and vanishing cycles of simplified $(2, 0)$-trisection maps used by Hayano in [6]. In Section 3, we introduce the 6-tuples of vertical 3-manifolds and give their classification. Theorem 1 is proved in Section 4. In Section 5, we determine the 4-manifold for each 6-tuple. Corollary 2 is obtained as a consequence of this result. In Section 6, we give the definition of 4-section maps and prove Theorem 3.

2. Preliminaries

Let X be a closed orientable smooth 4-manifold and $f : X \to \mathbf{R}^2$ be a stable map. Singularities of f are classified into four types: definite folds, indefinite folds, definite cusps and indefinite cusps. The image of indefinite folds is an immersed curve on $\mathbf{R}^2$. Let x be an indefinite fold and choose a short arc on $\mathbf{R}^2$ that intersects the image of indefinite folds transversely at $f(x)$. The fiber changes along this short arc as shown on the left in Fig. 4. The short arc from the point p to the intersection point is called a reference path and the simple closed curve c shrinking to the point x a vanishing cycle of f at x associated with the reference path. The image of definite folds is also an immersed curve on $\mathbf{R}^2$. The fiber changes along a transverse short arc as shown on the middle in Fig. 4. A cusp appears at the endpoints of folds. The image of the singular set near the cusp is a cusped curve as shown on the right in the figure. In the case of trisection maps, the folds adjacent to a cusp are always indefinite. Choose a point p near the image of the cusp and draw a transverse short arc, that is a reference path, from p to each of the two arcs as in the figure. The vanishing

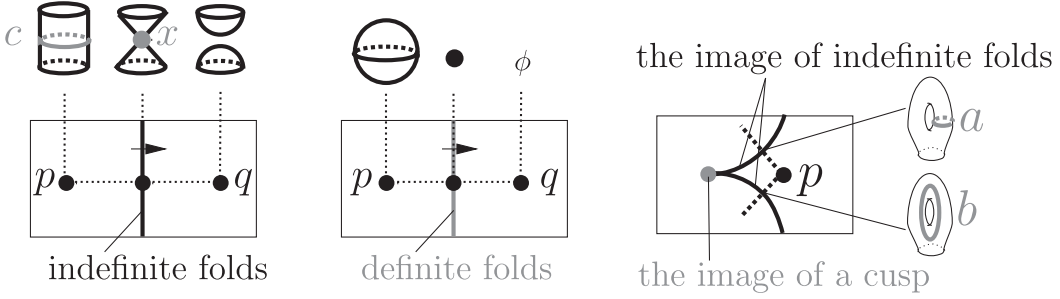cycles a and b for these reference paths intersect transversely at one point on the fiber over p.

Fig. 4. Deformation of fibers near singularities.

DEFINITION 4. A stable map $f : X \to \mathbf{R}^2$ is called a simplified (g, k)-trisection map if the following conditions hold:

- The singular value set of definite folds is a circle, bounding a disk D.
- The closure of the singular value set of indefinite folds consists of g concentric circles on D each of whose inner $g - k$ circles has three outward cusps.
- The preimage of the point at the center is a closed orientable surface of genus g.
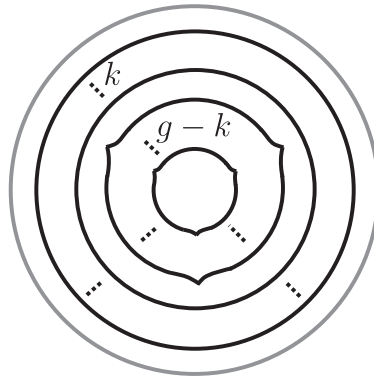
See Fig. 5.

Fig. 5. A simplified (g, k)-trisection map.

In this paper, we mainly study simplified $(2, 0)$-trisection maps. Let e_a, e_b, e_c be the edges of the outer, cusped circle as shown on the left in Fig. 6. The dotted curves in the figure are reference paths. Let p_1 be the branch point of the reference paths shown in the figure and let Σ_1 denote the fiber over p_1. Let γ_μ be a counter-clockwisely oriented circle on $\mathbf{R}^2$ lying between two cusped circles and passing through p_1, see the right figure in Fig. 6. We denote the monodromy diffeomorphism from Σ_1 to itself along γ_μ by μ. By [6, Lemma 3.6], the monodromy μ is divided into the following three cases:

(1) $\mu = \mathrm{id}_{\Sigma_1}$
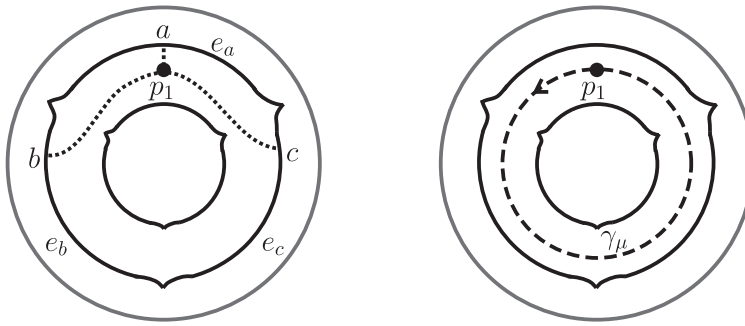
Fig.6. Reference paths and the monodromy μ.

(2) $\mu = t_d^{\pm 1}$
(3) $\mu = t_d^{\pm 4}$,

where d is a simple closed curve on Σ_1 and t_d is the right-handed Dehn twist along d.

   If μ is not the identity map, then we can divide the discussion into the following two cases depending on the configuration of vanishing cycles:

   (A) d is not parallel to any of a, b and c. In this case, it is known in [6] that the map can be deformed so that the closure of the image of indefinite folds consists of a simple closed curve with four cusps as shown on the left in Fig.7.

   (B) d is parallel to one of a, b and c. In this case, it is also known in [6] that the map can be deformed so that the closure of the image of indefinite folds consists of two circles with 3 cusps as shown on the right in Fig.7.
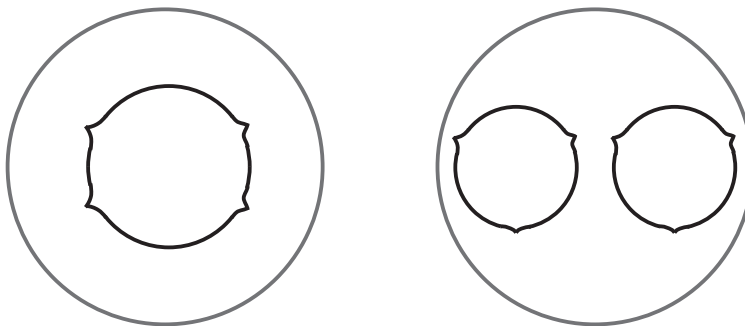

Fig.7. Deformation of simplified trisection maps.

Lemma 5 (See the proof of [6, Theorem 3.9]). *Suppose that μ is not the identity map. Then the following hold:*

   • *If d is in* Case (A), *then one of a, b, c intersects d once transversely.*
   • *If $\mu = t_d^{\pm 4}$, then d is in* Case (B).

3. Classification of vertical 6-tuples

We first introduce the notion of a 6-tuple of vertical 3-manifolds. Let f be a simplified $(2, 0)$-trisection map from an oriented 4-manifold X to $\mathbf{R}^2$. Let γ_{aa} be a properly embedded arc on $f(X) \simeq D^2$ that intersects the image of indefinite folds only at two points on the edge e_a and separates $e_b \cup e_c$ and the inner cusped circle, where e_a, e_b and e_c are the arcs in Fig.6. The arcs γ_{bb}, γ_{cc} are defined similarly. See the left figure in Fig.2. Let γ_{ba} be a properly embedded arc on $f(X) \simeq D^2$ that intersects the image of indefinite folds twice, at a point on e_a and a point on e_b, and separates e_c and the inner cusped circle. The arcs γ_{cb}, γ_{ac} are defined similarly. See the right figure in Fig.2. We set counter-clockwise orientations to these arcs. Set $V_{ij} = f^{-1}(\gamma_{ij})$ for $(i, j) \in \{(a, a), (b, b), (c, c), (b, a), (c, b), (a, c)\}$ and set the orientations of V_{ij} so that it coincides with the product of the orientations of the fiber and the arc γ_{ij}. The orientation reversing diffeomorphism of D^2 sends the 6-tuple $\begin{pmatrix} V_{aa} & V_{bb} & V_{cc} \\ V_{ba} & V_{cb} & V_{ac} \end{pmatrix}$ to $\begin{pmatrix} \bar{V}_{aa} & \bar{V}_{cc} & \bar{V}_{bb} \\ \bar{V}_{ac} & \bar{V}_{cb} & \bar{V}_{ba} \end{pmatrix}$, where $\bar{V}_{ij}$ is the mirror image of V_{ij}. This operation corresponds to the exchange of the labels b and c. We call it a reflection. Note that the reflection reverses the orientation of X and those of the vertical 3-manifolds in the 6-tuple and, since we reverse the orientations of arcs so that they become counter-clockwise after the reflection, it does not change the orientations of the fibers. The rotation of D^2 gives a cyclic permutation of the union of the arcs γ_{ij} of order 3, which maps (a, b, c) to (b, c, a). In this section, we give a classification of the 6-tuple up to reflection and cyclic permutation.

If the 6-tuple is $\begin{pmatrix} S^1 \times S^2 & S^1 \times S^2 & S^1 \times S^2 \\ S^3 & S^3 & S^3 \end{pmatrix}$ then we call it the trivial 6-tuple.

In Section 3.1, we study the case where μ is not the identity. The case where μ is the identity is an obvious case, which will be explained in Section 3.2.

3.1. Case: $\mu \neq \mathrm{id}_{\Sigma_1}$. In this subsection, we assume that μ is not the identity. In Case (A), one of a, b and c intersects d once transversely by Lemma 5.

Theorem 6. *Suppose f is in* Case (A) *and the labels a, b and c are chosen so that a and d intersect once transversely. Then the 6-tuple* $\begin{pmatrix} V_{aa} & V_{bb} & V_{cc} \\ V_{ba} & V_{cb} & V_{ac} \end{pmatrix}$ *is one of the followings up to reflection and cyclic permutation:*

$$\begin{pmatrix} S^3 & S^3 & L((q-1)^2, \epsilon q) \\ S^1 \times S^2 & L(q-2, \epsilon) & L(q, -\epsilon) \end{pmatrix}, \begin{pmatrix} S^3 & L(9, 2\epsilon) & L(4, \epsilon) \\ L(2, 1) & L(5, \epsilon) & S^3 \end{pmatrix},$$

where $q \neq 1$ and $\epsilon \in \{-1, 1\}$.

In Case (B), the curve d is parallel to one of the vanishing cycles a, b and c.

Theorem 7. *Suppose f is in* Case (B) *and the labels a, b and c are chosen so that a is parallel to d. Then the 6-tuple* $\begin{pmatrix} V_{aa} & V_{bb} & V_{cc} \\ V_{ba} & V_{cb} & V_{ac} \end{pmatrix}$ *is one of the followings up to reflection and cyclic permutation:*

$$\begin{pmatrix} S^1 \times S^2 & S^3 & S^3 \\ S^3 & L(1 + \epsilon, 1) & S^3 \end{pmatrix}, \begin{pmatrix} S^1 \times S^2 & L(4, 1) & L(4, 1) \\ S^3 & L(4 + \epsilon, 1) & S^3 \end{pmatrix},$$

where $\epsilon \in \{-1, 1\}$.

Before proving these theorems, we introduce a lemma about Dehn twists. Let T^2 be a torus and $\gamma_1 \cdot \gamma_2$ denote the algebraic intersection number of simple closed curves γ_1 and γ_2 on an oriented surface.

Lemma 8. *Let $\gamma \subset T^2$ be an essential simple closed curve, representing an element $\begin{pmatrix} p \\ q \end{pmatrix}$ in $H_1(T^2; \mathbf{Z})$. Then, the representation matrix of the right-handed Dehn twist t_γ along γ is $\begin{pmatrix} 1 - pq & p^2 \\ -q^2 & 1 + pq \end{pmatrix}$.*

Proof. The map t_γ sends $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} \right) \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 - pq \\ -q^2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} \right) \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p^2 \\ 1 + pq \end{pmatrix}$.   □

In the following proofs, the notation of double sign correspondence is used. The notation $M(T, \alpha, \beta)$ means the closed 3-manifold obtained from a thickening $T \times [0, 1]$ of a torus T by attaching 2-handles along essential simple closed curves α on $T \times \{0\}$ and β on $T \times \{1\}$ and filling the boundary 3-balls.

Proof of Theorem 6.   Draw the dotted arcs on $f(X)$ as in Fig.8. The fiber over p_1 is a torus, denoted by Σ_1, and if we move from p_1 to a point on the upper edge along the dotted curve, then a simple closed curve a_2 on Σ_1 shrinks to a point. The fiber Σ_1 over p_1 is a reference fiber, the dotted curve is a reference path and the simple closed curve a_2 is a vanishing cycle along this reference path. Let b_2, c_2, a'_2, b'_2 and c'_2 be vanishing cycles on Σ_1 along the corresponding reference paths shown in Fig.8. In Case (A), $\mu = t_d^{\pm 1}$ by Lemma 5. Since a_2 and b_2 vanish at the same cusp, choosing the orientations of a_2 and b_2 suitably, we may identify $H_1(\Sigma_1; \mathbf{Z})$ with $\mathbf{Z}^2$ so that a_2 and b_2 represent the elements

$$[a_2] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [b_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since b_2 and c'_2 vanish at the same cusp, choosing the orientation of c'_2 suitably, we may set $[c'_2] = \begin{pmatrix} -1 \\ q \end{pmatrix}$ with $q \in \mathbf{Z}$. Since d intersects a_2 once transversely, choosing the orientation of d suitably, we may set $[d] = \begin{pmatrix} r \\ 1 \end{pmatrix}$ with $r \in \mathbf{Z}$. Then, by Lemma 8, we have $[a'_2] = [t_d^{\mp 1}(a_2)] = \begin{pmatrix} p \\ \pm 1 \end{pmatrix}$, where $p = 1 \pm r$. Since a'_2 and c'_2 vanish at the same cusp, we may set $a'_2 \cdot c'_2 = \epsilon_1 \in \{-1, 1\}$. Then we have $pq = \epsilon_1 \mp 1$ as

$$\epsilon_1 = a'_2 \cdot c'_2 = \det \begin{pmatrix} p & -1 \\ \pm 1 & q \end{pmatrix} = pq \pm 1.$$

First, we study the case where either V_{ba} or V_{ac} is $S^1 \times S^2$. By applying a reflection, which means by exchanging the labels b and c, if necessary, we assume V_{ba} is $S^1 \times S^2$. Since $V_{ba} = M(\Sigma_1, b_2, a'_2)$, a'_2 and b_2 are parallel, meaning $p = 0$. Hence $\epsilon_1 = a'_2 \cdot c'_2 = \pm 1$ and $r = \mp 1$.
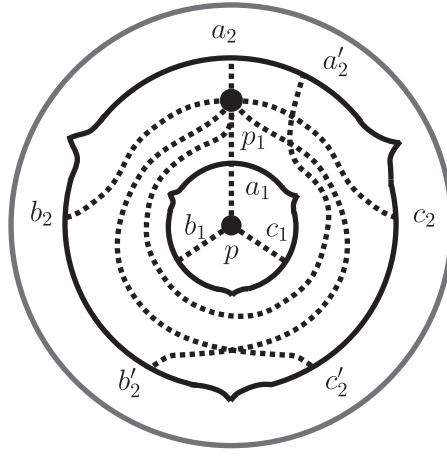
Fig. 8

Suppose $\mu = t_d$. Then $r = -1$. By Lemma 8, we have

$$[b_2'] = [t_d(b_2)] = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$[c_2] = [t_d(c_2')] = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ q \end{pmatrix} = \begin{pmatrix} q-2 \\ 1 \end{pmatrix}.$$

Since it is in Case (A) and $d = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, we have $q \neq 1$. We obtain $V_{bb} = S^3$, $V_{ba} = S^1 \times S^2$, $V_{cb} = L(q-2, 1)$, $V_{ac} = L(q, -1)$. Remark that the orientation of the arc γ_{ij} is counter-clockwise and the orientation of V_{ij} is given so that it coincides with the product orientation of the fiber and γ_{ij}. For $V_{cc} = M(\Sigma, c_2, c_2')$, setting $T = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}$ we have $T[c_2] = \begin{pmatrix} q-2 \\ (q-1)^2 \end{pmatrix}$ and $T[c_2'] = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Hence $V_{cc} = \overline{L((q-1)^2, q-2)} = L((q-1)^2, q)$. Obviously, V_{aa} is S^3. Thus we have the first 6-tuple with $\epsilon = 1$ in the assertion.

Suppose $\mu = t_d^{-1}$. Then $r = 1$. By Lemma 8, we have

$$[b_2'] = [t_d^{-1}(b_2)] = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

$$[c_2] = [t_d^{-1}(c_2')] = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ q \end{pmatrix} = \begin{pmatrix} -2-q \\ -1 \end{pmatrix}.$$

Since it is in Case (A) and $d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we have $q \neq -1$. We obtain $V_{bb} = S^3$, $V_{ba} = S^1 \times S^2$, $V_{cb} = L(q+2, 1)$ and $V_{ac} = L(q, -1)$. For $V_{cc} = M(\Sigma, c_2, c_2')$, setting $T = \begin{pmatrix} 0 & -1 \\ 1 & -2-q \end{pmatrix}$ we have $T[c_2] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T[c_2'] = \begin{pmatrix} -q \\ -(q+1)^2 \end{pmatrix}$. Hence $V_{cc} = L((q+1)^2, q)$. By replacing q into $-q$, we have the 6-tuple

$$\begin{pmatrix} V_{aa} & V_{bb} & V_{cc} \\ V_{ba} & V_{cb} & V_{ac} \end{pmatrix} = \begin{pmatrix} S^3 & S^3 & L((q-1)^2, -q) \\ S^1 \times S^2 & L(q-2, -1) & L(q, 1) \end{pmatrix}$$

with $q \neq 1$. This is the first 6-tuple with $\epsilon = -1$ in the assertion.

Next, we consider the case where both of V_{ba} and V_{ac} are not $S^1 \times S^2$. Since $M(\Sigma_1, b_2, a_2')$ and $M(\Sigma_1, a_2, c_2')$ are not $S^1 \times S^2$, $pq \neq 0$. Hence $pq = \epsilon_1 \mp 1$ is either -2 or 2. In particular, $p \in \{-2, -1, 1, 2\}$.

If $p = 1$, then $r = 0$ as $p = 1 \pm r$. We have $q = \mp 2$ since $q = \epsilon \mp 1$ and $q \neq 0$, and $[d] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Since b_2 is parallel to d, it is not in Case (A).

If $p = -1$, we have $r = \mp 2$. Since $q = \pm 1 - \epsilon_1$ and $q \neq 0$, we have $q = \pm 2$. Since $[d] = \begin{pmatrix} \mp 2 \\ 1 \end{pmatrix}$, the representation matrix of $\mu = t_d^{\pm 1}$ is $\begin{pmatrix} 3 & \pm 4 \\ \mp 1 & -1 \end{pmatrix}$ by Lemma 8. Hence we have

$$[b_2'] = [\mu(b_2)] = \begin{pmatrix} 3 & \pm 4 \\ \mp 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \pm 4 \\ -1 \end{pmatrix},$$

$$[c_2] = [\mu(c_2')] = \begin{pmatrix} 3 & \pm 4 \\ \mp 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ \pm 2 \end{pmatrix} = \begin{pmatrix} 5 \\ \mp 1 \end{pmatrix}.$$

Setting $T = \begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 5 \end{pmatrix}$, we have $T[c_2] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T[c_2'] = \begin{pmatrix} -2 \\ \pm 9 \end{pmatrix}$. Thus we obtain $V_{aa} = S^3$, $V_{bb} = L(\pm 4, -1)$, $V_{cc} = L(\pm 9, -2)$, $V_{ba} = S^3$, $V_{cb} = L(\pm 5, -1)$ and $V_{ac} = L(2, 1)$. This is the reflection of the second 6-tuple in the assertion.

If $p = 2$, then we have $r = \pm 1$. Since $2q = \epsilon_1 \mp 1$ and $q \neq 0$, we have $q = \mp 1$. Since $[d] = \begin{pmatrix} \pm 1 \\ 1 \end{pmatrix}$, the representation matrix of $\mu = t_d^{\pm 1}$ is $\begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 2 \end{pmatrix}$ by Lemma 8. Then we have

$$[b_2'] = [\mu(b_2)] = \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \pm 1 \\ 2 \end{pmatrix},$$

$$[c_2] = [\mu(c_2')] = \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ \mp 1 \end{pmatrix} = \begin{pmatrix} -1 \\ \mp 1 \end{pmatrix}.$$

Since c_2 is parallel to d, it is not in Case (A).

If $p = -2$, then we have $r = \mp 3$. Since $2q = \pm 1 - \epsilon_1$ and $q \neq 0$ we have $q = \pm 1$. Since $[d] = \begin{pmatrix} \mp 3 \\ 1 \end{pmatrix}$, the representation matrix of $\mu = t_d^{\pm 1}$ is $\begin{pmatrix} 4 & \pm 9 \\ \mp 1 & -2 \end{pmatrix}$ by Lemma 8. Then we have

$$[b_2'] = [\mu(b_2)] = \begin{pmatrix} 4 & \pm 9 \\ \mp 1 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \pm 9 \\ -2 \end{pmatrix},$$

$$[c_2] = [\mu(c_2')] = \begin{pmatrix} 4 & \pm 9 \\ \mp 1 & -2 \end{pmatrix}\begin{pmatrix} -1 \\ \pm 1 \end{pmatrix} = \begin{pmatrix} 5 \\ \mp 1 \end{pmatrix}.$$

Thus we obtain $V_{aa} = S^3$, $V_{bb} = L(\pm 9, -2)$, $V_{ba} = L(2, 1)$, and $V_{ac} = S^3$. For V_{cc} and V_{cb}, multiplying $T = \begin{pmatrix} 1 & \pm 4 \\ \pm 1 & 5 \end{pmatrix}$, we have $T[c_2] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T[c_2'] = \begin{pmatrix} 3 \\ \pm 4 \end{pmatrix}$ and $T[b_2] = \begin{pmatrix} \pm 4 \\ 5 \end{pmatrix}$. Hence

$V_{cc} = L(\pm 4, -1)$, $V_{cb} = L(5, \mp 1)$. This is the second 6-tuple in the assertion. This completes the proof. $\square$

Proof of Theorem 7. Let $a_2, b_2, c_2, a_2', b_2'$ and c_2' be the vanishing cycles on Σ_1 and identify $H_1(\Sigma_1; \mathbf{Z})$ with $\mathbf{Z}^2$ so that a_2 and b_2 represent the elements

$$[a_2] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [b_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

as in the proof of Theorem 6. By the assumption, we may set $[d] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence $[a_2'] = [t_d^{\mp 1}(a_2)] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Since b_2 and c_2' vanish at the same cusp and a_2' and c_2' vanish at the same cusp, we may set $[c_2'] = \begin{pmatrix} -1 \\ \epsilon_2 \end{pmatrix}$ with $\epsilon_2 \in \{-1, 1\}$.

Suppose $\mu = t_d^{\pm 1}$. Then we have

$$[b_2'] = [\mu(b_2)] = [t_d^{\pm 1}(b_2)] = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \pm 1 \\ 1 \end{pmatrix},$$

$$[c_2] = [\mu(c_2')] = [t_d^{\pm 1}(c_2')] = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ \epsilon_2 \end{pmatrix} = \begin{pmatrix} -1 \pm \epsilon_2 \\ \epsilon_2 \end{pmatrix}.$$

Thus we obtain $V_{aa} = S^1 \times S^2$, $V_{bb} = S^3$, $V_{cc} = S^3$, $V_{ba} = S^3$, $V_{cb} = L(-1 \pm \epsilon_2, \epsilon_2)$ and $V_{ac} = S^3$. If $\pm \epsilon_2 = 1$ then it is the first 6-tuple in the assertion with $\epsilon = -1$ and if $\pm \epsilon_2 = -1$ then it is that with $\epsilon = -1$.

If $\mu = t_d^{\pm 4}$, then we have

$$[b_2'] = [\mu(b_2)] = [t_d^{\pm 4}(b_2)] = \begin{pmatrix} 1 & \pm 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \pm 4 \\ 1 \end{pmatrix},$$

$$[c_2] = [\mu(c_2')] = [t_d^{\pm 4}(c_2')] = \begin{pmatrix} 1 & \pm 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ \epsilon_2 \end{pmatrix} = \begin{pmatrix} -1 \pm 4\epsilon_2 \\ \epsilon_2 \end{pmatrix}.$$

So we have $V_{aa} = S^1 \times S^2$, $V_{bb} = L(\pm 4, 1)$, $V_{ba} = S^3$, $V_{ac} = S^3$. For V_{cc} and V_{cb}, multiplying $T = \begin{pmatrix} -1 & \pm 4 \\ -\epsilon_2 & -1 \pm 4\epsilon_2 \end{pmatrix}$, we have $T[c_2] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T[c_2'] = \begin{pmatrix} 1 \pm 4\epsilon_2 \\ \pm 4 \end{pmatrix}$ and $T[b_2] = \begin{pmatrix} \pm 4 \\ -1 \pm 4\epsilon_2 \end{pmatrix}$. Hence $V_{cc} = L(\pm 4, 1)$ and $V_{cb} = L(-1 \pm 4\epsilon_2, \pm 4) = L(\pm 4 - \epsilon_2, 1)$. Thus, we obtain

$$\begin{pmatrix} V_{aa} & V_{bb} & V_{cc} \\ V_{ba} & V_{cb} & V_{ac} \end{pmatrix} = \begin{pmatrix} S^1 \times S^2 & L(4, 1) & L(4, 1) \\ S^3 & L(4 \mp \epsilon_2, 1) & S^3 \end{pmatrix}$$

after applying the reflection if necessary. This completes the proof. $\square$

3.2. Case: μ is trivial.

Lemma 9. *If μ is the identity map, then the 6-tuple of vertical manifolds is*

$$\begin{pmatrix} V_{aa} & V_{bb} & V_{cc} \\ V_{ba} & V_{cb} & V_{ac} \end{pmatrix} = \begin{pmatrix} S^1 \times S^2 & S^1 \times S^2 & S^1 \times S^2 \\ S^3 & S^3 & S^3 \end{pmatrix}$$

Proof. Since μ is the identity, the vertical manifold over the arc ω_1 shown on the left in Fig.9 is homeomorphic to the one over the arc ω_1' shown on the right. Therefore, V_{aa} is homeomorphic to $S^1 \times S^2$. The other vertical 3-manifolds can be determined by the same manner.   □
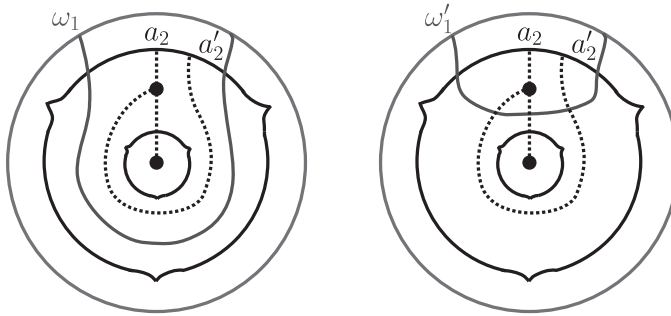

Fig.9. Case: μ is the identity.

4. Proof of Theorem 1

Lemma 10. *Let $f : X \to \mathbf{R}^2$ be a simplified $(2,0)$-trisection map and $\omega \subset f(X)$ be a proper, simple, generic arc. Then, $f^{-1}(\omega)$ is either S^3 or a connected sum of finite copies of $S^1 \times S^2$, V_{ii} and V_{jk} and their mirror images, where (j,k) is one of $(b,a),(c,b)$ and (a,c) and either $i = j$ or $i = k$.*

Proof. We first introduce two local moves shown in Fig.10. The move on the left is named a *cusp move* and the one on the right is a *bigon move*. The vertical manifold $f^{-1}(\gamma_1)$ is homeomorphic to $f^{-1}(\gamma_1')$, and the vertical manifold $f^{-1}(\gamma_2)$ is a connected sum of $f^{-1}(\gamma_2')$ and $S^1 \times S^2$.

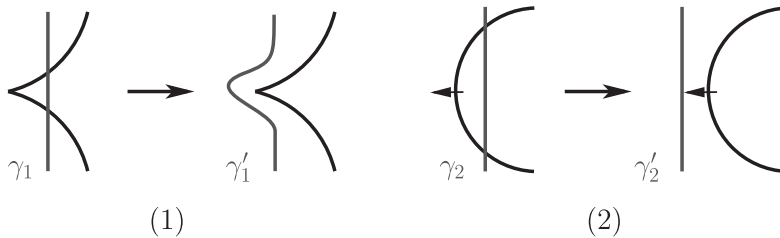Fig.10. (1) a cusp move, (2) a bigon move.

The closure of the image of the indefinite folds consists of two concentric circles with cusps. Let C_1 be the inner circle and C_2 be the outer one. If ω passes near a cusp of C_1 as shown in Fig.10 (1), then we apply a cusp move to ω. If ω and C_1 bound a disc as shown in Fig.10 (2), then we apply a bigon move to ω. Applying these operations inductively, we may assume that ω does not intersect C_1.

Next we focus on the subarcs on ω whose endpoints are on C_2 and which lie on the

annulus between C_2 and the image of definite folds. For an outermost arc of these arcs we apply the operation shown in Fig.11. Suppose that the arc ω is changed into the union of the arcs ω' and ω'' by this operation. Since the preimage of the intersection of an arc and the image of the definite folds is a set of points, the inverse of the above operation induces the following surgery for the 3-manifolds $f^{-1}(\omega')$ and $f^{-1}(\omega'')$; remove a 3-ball from each of $f^{-1}(\omega')$ and $f^{-1}(\omega'')$ and glue the two 2-spheres appearing on the boundaries, which produces a connected sum of $f^{-1}(\omega')$ and $f^{-1}(\omega'')$. We apply this operation inductively so that there are no arcs whose endpoints are on C_2 and which lie on the annulus between C_2 and the image of definite folds.
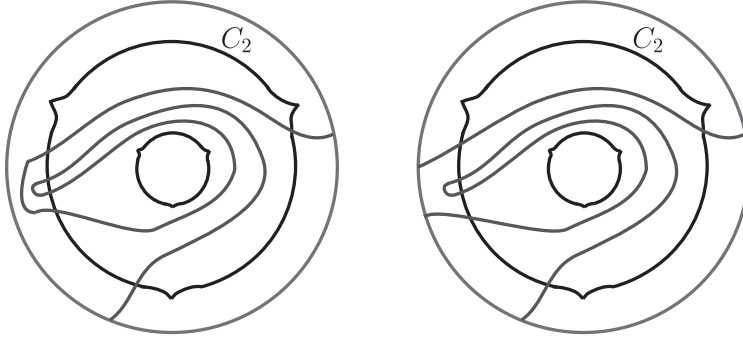

Fig.11. Cutting the arc.

Checking the positions of the arcs $\gamma_{aa}, \gamma_{bb}, \gamma_{cc}, \gamma_{ba}, \gamma_{cb}$ and γ_{ac}, we see that possible combinations are

$$(\gamma_{aa}, \gamma_{ba}), (\gamma_{aa}, \gamma_{ac}), (\gamma_{bb}, \gamma_{ba}), (\gamma_{bb}, \gamma_{cb}), (\gamma_{cc}, \gamma_{cb}), (\gamma_{cc}, \gamma_{ac}),$$

cf. Fig.12 below. Applying cusp and bigon moves to the arcs decomposed by these operations inductively, we see that the preimages of the resulting arcs are finite copies of $S^1 \times S^2$, V_{ii} and V_{jk} and their mirror images. Thus we have the assertion.   □

Proof of Theorem 1. Consider the first 6-tuple

$$\begin{pmatrix} V_{aa} & V_{bb} & V_{cc} \\ V_{ba} & V_{cb} & V_{ac} \end{pmatrix} = \begin{pmatrix} S^3 & S^3 & L((q-1)^2, \epsilon q) \\ S^1 \times S^2 & L(q-2, \epsilon) & L(q, -\epsilon) \end{pmatrix}$$

in Theorem 6. By Lemma 10, the vertical manifolds in this case are connected sums of finite copies of $S^1 \times S^2$, V_{ii} and V_{jk} and their mirror images, where (j, k) is one of $(b, a), (c, b)$ and (a, c) and either $i = j$ or $i = k$. The vertical manifolds corresponding to $(\gamma_{aa}, \gamma_{ba})$ are connected sums of finite copies of $S^1 \times S^2$. The vertical manifolds corresponding to $(\gamma_{aa}, \gamma_{ac})$ are $\#^{\ell+\epsilon_1, \ell} L(q, -\epsilon)$ up to summands $S^1 \times S^2$ and orientation reversal. We may determine the vertical manifolds corresponding to $(\gamma_{bb}, \gamma_{ba})$ and $(\gamma_{bb}, \gamma_{cb})$ by the same manner.

A vertical manifold corresponding to $(\gamma_{cc}, \gamma_{cb})$ is the preimage of the arc in Fig.12 with closing the endpoints of the arcs suitably, where r and s are the numbers of the arcs. Checking orientations of the arcs carefully, we conclude that it is $\#^{\ell+\epsilon_1, \ell} L((q-1)^2, \epsilon q) \#^{m, m+\epsilon_2} L(q-2, \epsilon)$ up to summands $S^1 \times S^2$ and orientation reversal, where $2\ell + \epsilon_1 = s$ and $2m + \epsilon_2 = r$.

For the vertical manifold corresponding to $(\gamma_{cc}, \gamma_{ac})$ we may conclude that it is $\#^{\ell+\epsilon_1, \ell} L((q-1)^2, \epsilon q) \#^{m, m+\epsilon_2} L(q, -\epsilon)$ up to summands $S^1 \times S^2$ and orientation reversal.
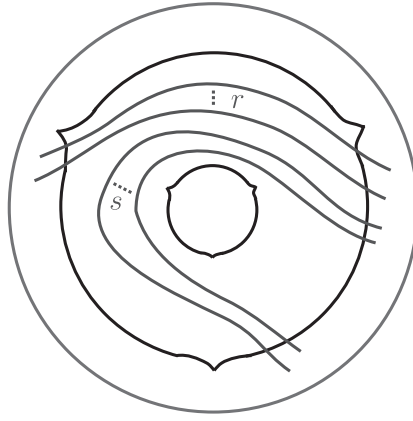
Fig. 12. The arcs corresponding to γ_{cc} and γ_{cb}.

We may determine the vertical 3-manifolds for the other 6-tuples in Theorem 6 and also Theorem 7 by the same manner.

Since a simplified $(2, 0)$-trisection map exists for each of 6-tuples in Theorems 6 and 7, we can conclude that any 3-manifold in the assertion can be obtained as a vertical 3-manifold of a simplified $(2, 0)$-trisection map. This completes the proof. □

Remark 11. A vertical 3-manifold of a simplified $(2, k)$-trisection for $k = 1, 2$ is $\#^n S^1 \times S^2$ since any arc on D^2 can be moved by cusp moves and bigon moves so that it does not intersect the image of indefinite fold and cusp singularities. Note that n is the number of bigon moves.

5. Vertical 3-manifolds determine 4-manifolds

In this section, we determine the 4-manifold for each 6-tuple of vertical 3-manifolds by describing its Kirby diagram. Such observation can be found in [4] in more general setting, where relations between simple wrinkled fibrations and surface diagrams are studied.

Theorem 12. *Let $f : X \to \mathbf{R}^2$ be a simplified $(2, 0)$-trisection map. The 4-manifold X is determined as follows:*

(1) *The 4-manifold X with the 6-tuple $\begin{pmatrix} S^3 & S^3 & L((q-1)^2, \epsilon q) \\ S^1 \times S^2 & L(q-2, \epsilon) & L(q, -\epsilon) \end{pmatrix}$ is $S^2 \times S^2$ if q is even and $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$ if q is odd and $q \neq 1$.*

(2) *The 4-manifold X with $\begin{pmatrix} S^3 & L(9, 2\epsilon) & L(4, \epsilon) \\ L(2, 1) & L(5, \epsilon) & S^3 \end{pmatrix}$ is $\mathbf{CP}^2 \# \mathbf{CP}^2$ if $\epsilon = -1$ and $\overline{\mathbf{CP}^2} \# \overline{\mathbf{CP}^2}$ if $\epsilon = 1$.*

(3) *The 4-manifold X with $\begin{pmatrix} S^1 \times S^2 & L(4, 1) & L(4, 1) \\ S^3 & L(4 + \epsilon, 1) & S^3 \end{pmatrix}$ is $\mathbf{CP}^2 \# \mathbf{CP}^2$ if $\epsilon = 1$ and $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$ if $\epsilon = -1$.*

(4) *The 4-manifold X with $\begin{pmatrix} S^1 \times S^2 & S^3 & S^3 \\ S^3 & S^1 \times S^2 & S^3 \end{pmatrix}$ is $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$.*

(5) *The 4-manifold X with* $\begin{pmatrix} S^1 \times S^2 & S^3 & S^3 \\ S^3 & L(2,1) & S^3 \end{pmatrix}$ *is* $\mathbf{CP}^2 \# \mathbf{CP}^2$ *or* $\overline{\mathbf{CP}^2} \# \overline{\mathbf{CP}^2}$.

REMARK 13. The reflection changes the 6-tuple $\begin{pmatrix} V_{aa} & V_{bb} & V_{cc} \\ V_{ba} & V_{cb} & V_{ac} \end{pmatrix}$ to $\begin{pmatrix} \bar{V}_{aa} & \bar{V}_{cc} & \bar{V}_{bb} \\ \bar{V}_{ac} & \bar{V}_{cb} & \bar{V}_{ba} \end{pmatrix}$. In Case (1) in the above theorem, the position of $L((q-1)^2, \epsilon q)$ in the 6-tuple changes by the reflection. To apply the theorem in this case, we need to apply the reflection if necessary so that $L((q-1)^2, \epsilon q)$ sets in the top-right entry of the 6-tuple, though it is not important since the orientation reversals of $S^2 \times S^2$ and $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$ are again $S^2 \times S^2$ and $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$, respectively. In Case (2), $L(9, 2\epsilon)$ should be set in the top-middle entry of the 6-tuple by applying the reflection if necessary. In Case (3), $L(4,1)$ in the 6-tuple is changed to $L(4,3)$ by the reflection. Therefore, we need to apply the reflection if necessary so that the lens spaces in the top-middle and top-right entries are $L(4,1)$.

Proof. The 6-tuples in Cases (1) and (2) appear only in Case (A). From the proof of Theorem 6, we know

$$[a_2] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [b_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [c_2'] = \begin{pmatrix} -1 \\ q \end{pmatrix}, \quad [a_2'] = \begin{pmatrix} p \\ \pm 1 \end{pmatrix},$$

where $p, q \in \mathbf{Z}$.

In Case (1), the 6-tuple appears when either V_{ba} or V_{ac} is $S^1 \times S^2$. Assuming $V_{ba} = S^1 \times S^2$, we have $p = 0$ and $[a_2'] = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}$. An unwrinkle is a homotopy of a smooth map which replaces an innermost curve with outward three cusps in the singular value set into the image of a Lefschetz singularity, and a sink is a homotopy of a smooth map which replaces the image of a Lefschetz singularity and a part of the image of indefinite folds into a cusp, see [13]. We apply an unwrinkle to the inner, cusped circle on the image of f, move the image of the Lefschetz singularity obtained by the unwrinkle to the indefinite fold between a_2 and a_2' and apply a sink as shown in Fig.13. Describe the simple closed curves a_2, b_2, c_2', a_2' on the standard torus embedded in S^3 and assign over/under information to their intersections in the order of the monodromy starting from b_2, that is b_2, c_2', a_2', a_2. See Fig.14 for the case $q > 0$. The framing coefficients of these simple closed curves are induced by the surface framings. This is a Kirby diagram of the 4-manifold X. Canceling the two dotted circles and a_2 and b_2, we have a Hopf link with framing coefficients ϵq and 0, where ϵ is the sign in Theorem 6. Remark that ϵ is needed since we replaced q by $-q$ in the proof of Theorem 6 with $\epsilon = -1$. Hence X is $S^2 \times S^2$ if q is even and X is $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$ if q is odd. In Case (2), the 6-tuple appears when both of V_{ba} and V_{ac} are not $S^1 \times S^2$. Hence it is either (i) $p = -1$ and $[c_2'] = \begin{pmatrix} -1 \\ \pm 2 \end{pmatrix}$ or (ii) $p = -2$ and $[c_2'] = \begin{pmatrix} -1 \\ \pm 1 \end{pmatrix}$. We apply an unwrinkle and a sink as in the previous case. In Case (2)-(i), $a_2' = \begin{pmatrix} -1 \\ \pm 1 \end{pmatrix}$ and the Kirby diagram of $\overline{X}$ is as shown on the left in Fig.15 if $\mu = t_d$ and on the right if $\mu = t_d^{-1}$. Here the 4-manifold is the mirror image $\overline{X}$ of X, not X itself, since we took the reflection at the end of the proof of Theorem 6 with $p = -1$. Canceling the two dotted circles and a_2 and b_2, we have a Hopf link with framing coefficients ± 1 and ± 2. Therefore, $\overline{X}$ is $\mathbf{CP}^2 \# \mathbf{CP}^2$ if $\mu = t_d$ and it is $\overline{\mathbf{CP}^2} \# \overline{\mathbf{CP}^2}$ if $\mu = t_d^{-1}$.
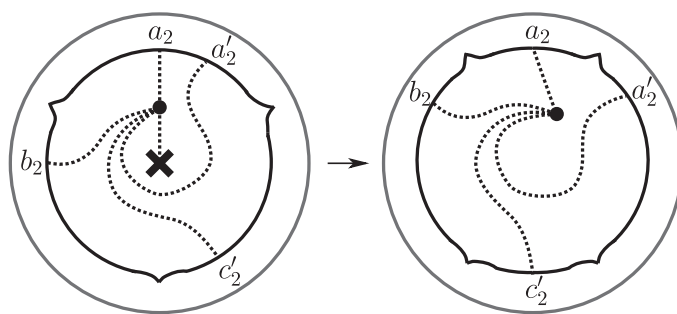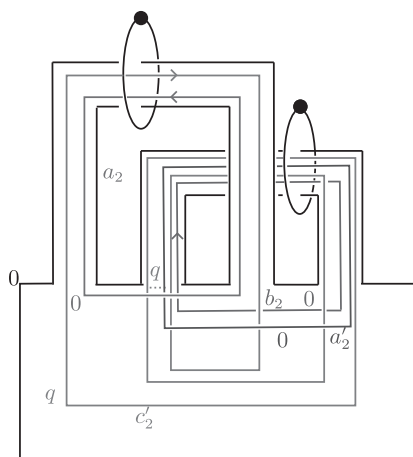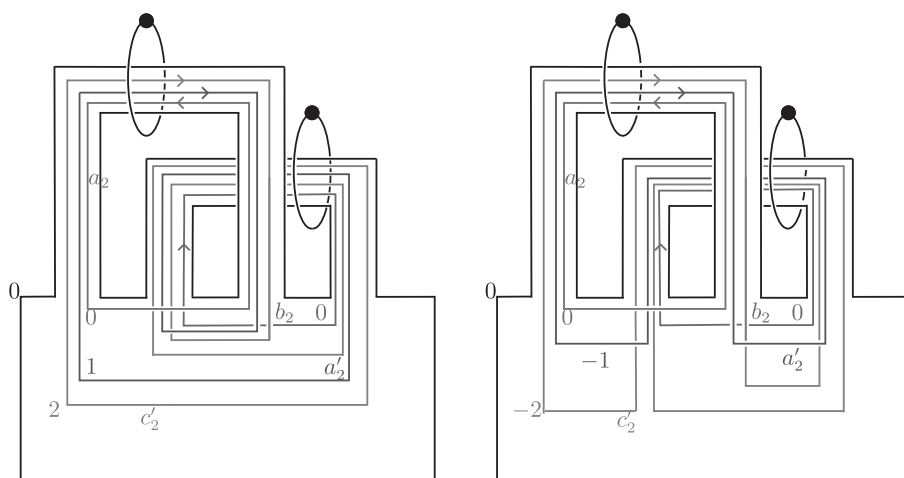
Fig. 13. A homotopy deformation by a sink.

Fig. 14. The Kirby diagram in Case (1) with $q > 0$.

From the proof of Theorem 6, we see that $\mu = t_d$ if $\epsilon = 1$ and $\mu = t_d^{-1}$ if $\epsilon = -1$. This coincides with the assertion in Case (2).

Fig. 15. The Kirby diagrams of $\overline{X}$ in Case (2)-(i).

In Case (2)-(ii), $a_2' = \begin{pmatrix} -2 \\ \pm 1 \end{pmatrix}$ and the Kirby diagram of X is as shown on the left in Fig.16 if $\mu = t_d$ and that on the right if $\mu = t_d^{-1}$. Applying the same argument as in Case (2)-(i) we have the same conclusion.
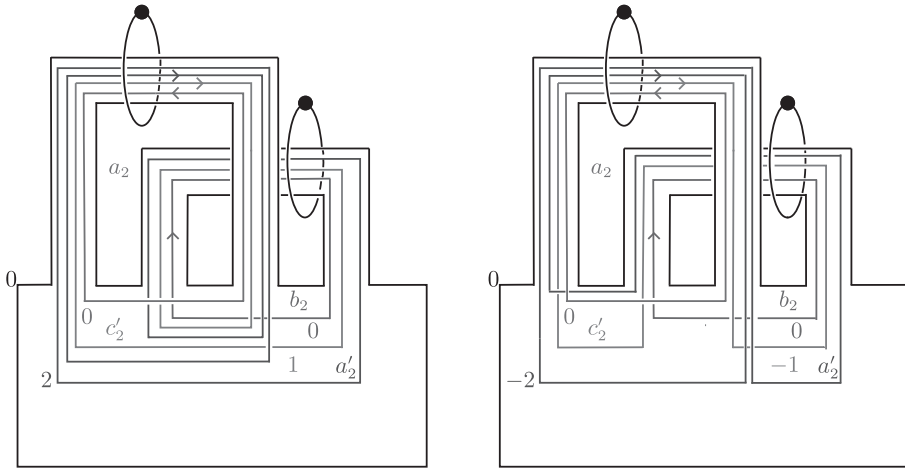

Fig. 16. The Kirby diagrams of X in Case (2)-(ii).

The 6-tuple in Case (3) only appears in Case (B) with $\mu = t_d^{\pm 4}$. We have

$$[a_2] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [b_2] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [c_2'] = \begin{pmatrix} -1 \\ \epsilon_2 \end{pmatrix},$$

where $\epsilon_2 \in \{-1, 1\}$, and $\epsilon_2 = \mp\epsilon$ since $L(4 \mp \epsilon_2, 1) = L(4 + \epsilon, 1)$.

Since it is in Case (B), the map can be deformed so that the closure of the image of indefinite folds consists of two circles with 3 cusps as shown on the right in Fig.7. One of the 3-cusped circles corresponds to the vanishing cycles a_2, b_2, c_2' and it corresponds to a summand $\mathbf{CP}^2$ if $\epsilon_2 = -1$ and $\overline{\mathbf{CP}^2}$ if $\epsilon_2 = 1$. The other 3-cusped circle corresponds to the inner 3-cusped circle before the deformation and it corresponds to a summand $\mathbf{CP}^2$ if $\mu = t_d^4$ since the configuration of the three vanishing cycles of the inner 3-cusped circle is that of $\mathbf{CP}^2$ and to a summand $\overline{\mathbf{CP}^2}$ if $\mu = t_d^{-4}$ since the configuration is that of $\overline{\mathbf{CP}^2}$ as shown in [6, Figure 6]. Suppose that $\mu = t_d^4$. If $\epsilon_2 = -1$, then $\epsilon = 1$ and $X = \mathbf{CP}^2 \# \mathbf{CP}^2$. If $\epsilon_2 = 1$, then $\epsilon = -1$ and $X = \mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$. Suppose that $\mu = t_d^{-4}$. Since $V_{bb} = L(\pm 4, 1)$ in the proof of Theorem 7, we need to apply the reflection so that the entry of V_{bb} becomes $L(4, 1)$. Therefore, if $\epsilon_2 = -1$ then $\epsilon = -1$ and X is the mirror image of $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$, and if $\epsilon_2 = 1$ then $\epsilon = 1$ and X is the mirror image of $\overline{\mathbf{CP}^2} \# \mathbf{CP}^2$. These coincide with the assertion in Case (3).

In Case (4), the 6-tuple appears in Case (B) with $\mu = t_d^{\pm 1}$ and $\pm\epsilon_2 = 1$. Hence X is $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$ by the same observation as in Case (3).

In Case (5), the 6-tuple appears in Case (B) with $\mu = t_d^{\pm 1}$ and $\pm\epsilon_2 = -1$. Hence X is $\mathbf{CP}^2 \# \mathbf{CP}^2$ if $\epsilon_2 = -1$, where $\mu = t_d$, and $\overline{\mathbf{CP}^2} \# \overline{\mathbf{CP}^2}$ if $\epsilon_2 = 1$, where $\mu = t_d^{-1}$. This completes the proof. $\qquad\square$

Corollary 2 stated in the introduction follows from Theorem 12 immediately.

6. A construction of vertical hyperbolic 3-manifolds

In this section, we give simplified $(2,0)$-4-section maps that have hyperbolic vertical 3-manifolds.

DEFINITION 14. A stable map $f : X \to \mathbf{R}^2$ is called a simplified (g,k)-4-section map if the following conditions hold:

- The singular value set of definite folds is a circle, bounding a disk D.
- The closure of the singular value set of indefinite folds consists of g concentric circles on D each of whose inner $g - k$ circles has 4 outward cusps.
- The preimage of the point at the center is a closed orientable surface of genus g.

See Fig.3.

Theorem 15. *Suppose* $X = \#^2 S^2 \times S^2$ *or* $\#^2 \mathbf{CP}^2 \# ^2 \overline{\mathbf{CP}^2}$. *Then there exists an infinite sequence of simplified $(2,0)$-4-section maps* $\{f_i : X \to \mathbf{R}^2\}_{i \in \mathbf{N}}$ *such that their vertical 3-manifolds over the arc ω in* Fig.3 *are hyperbolic and mutually non-diffeomorphic.*

Proof. Let Σ_2 be a closed oriented surface of genus 2 and describe simple closed curves $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$ on Σ_2 as in Fig.17, where the union of simple closed curves γ_1, γ_2 constitutes a 2-bridge link with slope p/q for q being even and Σ_2 is assumed to be embedded in S^3 in the standard position. The curve γ_1 winds l times around the left handle of Σ_2 and the curve γ_2 winds r times around the right one.
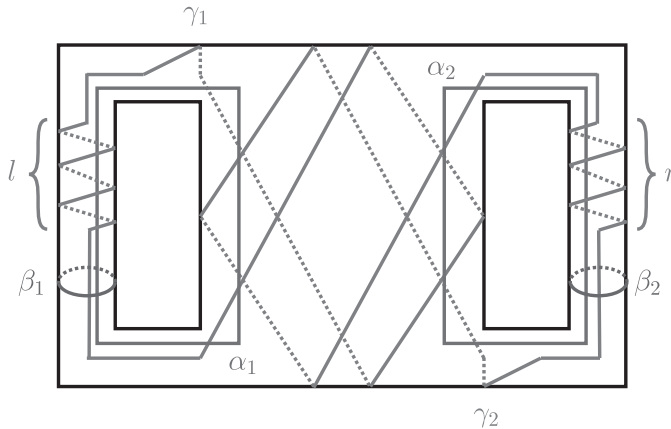

Fig.17. The genus-2 surface Σ_2 embedded in S^3. This is the case $p/q = 3/2$ and $l = r = 3$.

We will make a smooth map whose singular value set is the left half of the image of a simplified $(2,0)$-4-section map as in Fig.18. Identify the fiber over p with Σ_2 and attach 2-handles along $\alpha_1, \beta_1, \gamma_1$ on Σ_2, so that we extend the stable map over a small neighborhood of p to a neighborhood of the union of the reference paths for $\alpha_1, \beta_1, \gamma_1$ from p. Since α_1 and β_1 on Σ_2 intersect at one point transversely we may extend this map beyond the cusp. Since β_1 and γ_1 on Σ_2 intersect at one point transversely we can also extend the map beyond the other cusp. We attach 2-handles along α_2, γ_2, and β_2. By Lemma [6, Lemma 3.2], α_2' and α_2 are isotopic and hence α_2' intersects β_2 once transversely. By the same reason, γ_2'

intersects β_2 once transversely. Hence we may extend the map beyond the two cusps on the outer, cusped half circle. We further extend the map along the vertical dotted arc from p to the top endpoint of the image of definite folds, which corresponds to a 3-handle attaching. Then, attaching a 4-handle, we get a smooth map from a 4-manifold W with boundary whose vanishing cycles are as shown in Fig.17 and whose singular value set is as in Fig.18.
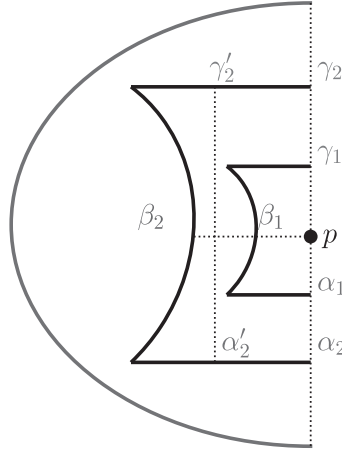

Fig. 18. A 4-manifold W with boundary and a stable map $f : W \to \mathbf{R}^2$.

The 4-manifold W has a handle decomposition into one 0-handle h^0 corresponding to the bottom half of the half disk in Fig.18, two 2-handles $h^2 \cup h^2$ attached along the vanishing cycles γ_1, γ_2 and one 3-handle h^3 and one 4-handle h^4, where the pair (h^3, h^4) is a cancelling pair. Therefore, the Kirby diagram of W is given by the 2-bridge link $\gamma_1 \cup \gamma_2$ with surface framing induced by Σ_2 in the standard position in S^3.

By gluing this smooth map and its mirror we get a closed 4-manifold $W \cup \overline{W}$, which is the double DW of W, with a stable map from DW to $\mathbf{R}^2$ whose singular value set is as shown in Fig.3. The Kirby diagram of DW is obtained from the Kirby diagram of W, which is the 2-bridge link $\gamma_1 \cup \gamma_2$ with surface framing, by adding a meridional curve for each of the link components γ_1 and γ_2 with framing 0. By Kirby calculus, it becomes a disjoint union of two Hopf links one of whose link component has framing 0. Since we can change even/odd of the framing of the other link component by changing r and l, we obtain the 4-manifolds in the assertion.

The 3-manifold on the boundary of W, which is the preimage of ω in the assertion, is obtained from S^3 by applying Dehn surgeries along γ_1 and γ_2. The 2-bridge link $\gamma_1 \cup \gamma_2$ is hyperbolic unless it is a torus link by [17]. Now we assume $|l|$ and $|r|$ are sufficiently large so that the absolute values of the surgery coefficients become sufficiently large. For such l and r the surgered 3-manifolds are hyperbolic by [18]. Moreover, for sufficiently large (l, r) and (l', r), the corresponding hyperbolic 3-manifolds are not mutually diffeomorphic by Thurston's hyperbolic Dehn surgery theorem. This completes the proof. □

REMARK 16. In the above proof, we can show the existence of mutually non-diffeomorphic vertical 3-manifolds by checking the torsions of their homology groups as follows. Let (μ_1, λ_1) and (μ_2, λ_2) be meridian-longitude pairs of the link components γ_1 and γ_2, respec-

tively. Then the first integral homology of the complement of the link $\gamma_1 \cup \gamma_2$ is isomorphic to $\mathbf{Z}\langle\mu_1\rangle \oplus \mathbf{Z}\langle\mu_2\rangle$. For each $k = 1, 2$, the image of the meridian of the solid torus by the Dehn filling for γ_k is given by $r_k\mu_k + \lambda_k$ for $r_k \in \mathbf{Z}$. Set n to be the linking number of γ_1 and γ_2. Then $\lambda_1 = n\mu_2$ and $\lambda_2 = n\mu_1$. Hence the first integral homology of the surgered manifold $f^{-1}(\omega)$ can be written as

$$H_1(f^{-1}(\omega); \mathbf{Z}) = \langle\mu_1, \mu_2 \mid r_1\mu_1 + n\mu_2, \ r_2\mu_2 + n\mu_1\rangle.$$

Choosing an infinite sequence of (r_1, r_2) suitably, we can show the existence of mutually non-diffeomorphic vertical 3-manifolds.

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