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ON THE VALUES OF UNIPOTENT CHARACTERS OF FINITE CHEVALLEY GROUPS OF TYPE $E_7$ IN CHARACTERISTIC 2

JONAS HETZ

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Abstract

Let $G$ be a finite group of Lie type. In order to determine the character table of $G$, Lusztig developed the theory of character sheaves. In this framework, one has to find the transformation between two bases for the space of class functions on $G$, one of them being the irreducible characters of $G$, the other one consisting of characteristic functions associated to character sheaves. In principle, this has been achieved by Lusztig and Shoji, but the underlying process involves some scalars which are still unknown in many cases. The problem of specifying these scalars can be reduced to considering cuspidal character sheaves. We will deal with the latter for the specific case where $G = E_7(q)$, and $q$ is a power of the bad prime $p = 2$ for $E_7$.

1. Introduction

Let $G$ be a connected reductive algebraic group over the algebraic closure $k = \overline{F}_p$ of the field with $p$ elements (for a prime $p$). Assume that $G$ is defined over the finite subfield $F_q$ of $k$, where $q$ is a power of $p$, so the $F_q$-rational points on $G$ constitute the corresponding finite group of Lie type $G(q) = G(F_q)$. The theory of character sheaves, due to Lusztig [22]-[26], provides a general procedure by which the problem of determining the character table of $G(q)$ can be tackled. Character sheaves are certain geometric analogues to the irreducible representations of $G(q)$, and they give rise to class functions on $G(q)$, called characteristic functions. Lusztig ([20, 13.7], [26]) conjectured that any such characteristic function coincides up to multiplication by a non-zero scalar with an appropriate almost character of $G(q)$, that is, an explicitly known linear combination of the irreducible characters. Then he provides an algorithm to compute the characteristic functions, at least in principle ([26, §24]). While Lusztig’s conjecture is still open in general (see, e.g., [33]), it has been proven by Shoji ([40], [41]) under the assumption that the centre of $G$ is connected. Even in this case, however, the scalars relating characteristic functions of character sheaves and almost characters need to be specified. Following [27, §3], this task can be reduced to considering cuspidal character sheaves, by means of an induction process for these objects which leads to a parametrisation of the character sheaves on $G$, rather analogous to Harish-Chandra theory for irreducible characters of $G(q)$.

It therefore seems reasonable to consider the individual quasisimple groups of Lie type separately. In those cases, [21] combined with [22]-[26], see also [40, §7], [41, §5], provides a classification of cuspidal character sheaves. We will be concerned with unipotent character sheaves on $G$, that is, the subset of the character sheaves on $G$ corresponding to the unipotent...
characters of $G(q)$ due to Deligne-Lusztig \cite{DeligneLusztig}.

For classical groups of split type, Shoji \cite{Shoji1}, \cite{Shoji2} has shown that the characteristic functions of cuspidal unipotent character sheaves coincide with the corresponding almost characters. As far as exceptional groups are concerned, this statement is only partially verified. E.g., for groups of type $F_4$, it has been proven by Marcelo-Shinoda \cite{MarceloShinoda} for those cuspidal character sheaves whose support contains unipotent elements, in all characteristics. (In type $F_4$, any characteristic, every cuspidal character sheaf is unipotent.) The particular case $F_4$, $p = 2$, is in fact complete, as the respective identities for not unipotently supported cuspidal character sheaves have been verified in \cite[§5]{Geck}. Moreover, by a result of Geck \cite[Proposition 3.4]{Geck}, the computation of the scalars relating characteristic functions of unipotent character sheaves and unipotent almost characters can be reduced to the base case $p = q$. This is particularly useful in small characteristics, as one might hope to settle such cases via direct computations, e.g. by applying computer algebra methods. Several concrete examples are considered in \cite{Geck}, such as the above-mentioned $F_4$, $p = 2$, as well as the split adjoint group of type $E_6$ in characteristic 2, where the problem is solved for all cuspidal character sheaves (there are two of them, and both are unipotent). Note that, in either of these cases, the character table of $G(2)$ is known. However, it is also conjectured in \cite[6.6]{Geck} that an analogous statement will hold for the adjoint groups of type $E_6$ in good characteristic, even though the character table of $G(p)$ is not known here. Finally, \cite{Shoji2} deals with $E_6$, $p = 3$ (both split and non-split).

In this paper, we shall investigate the case where $G$ is a simple Chevalley group of type $E_7$ over $k = \mathbb{F}_2$, defined over $\mathbb{F}_q$, $q$ a power of 2. There are two cuspidal character sheaves on $G$, and both of them are unipotent \cite[Proposition 20.3]{Shoji}. After singling out a specific normalisation of the characteristic functions of these character sheaves, we show that they coincide with the corresponding almost characters of $G(q)$. In order to achieve this, we will, as in \cite{Shoji2}, consider the Hecke algebra associated to $G(q)$ and apply a formula in \cite{Lusztig} which gives a relation between the characters of this Hecke algebra and the principal series unipotent characters of $G(q)$. However, we will need to take into account some more theoretical machinery than in \cite{Shoji2}. This identification of the characteristic functions of cuspidal character sheaves with the respective almost characters immediately determines the values at the regular unipotent elements in $G(q)$ of the two cuspidal unipotent characters and of the two principal series unipotent characters corresponding to the two irreducible characters of the Weyl group of degree 512. More information on character values at unipotent elements is provided by Shoji’s and Lusztig’s algorithms \cite[§5]{Shoji1}, \cite[§24]{Lusztig} for the computation of Green functions (as defined in \cite{DeligneLusztig}). While these algorithms in general involve some unknown scalars, they have been determined for our case recently in \cite{Lusztig2}.

**Notation.** Whenever $G$ is a connected reductive algebraic group over $k = \mathbb{F}_p$, we assume to have fixed a Borel subgroup $B \subseteq G$ as well as a maximal torus $T$ of $G$ which is contained in $B$, and we denote by $W := N_G(T)/T$ the Weyl group of $G$ with respect to $T$. Let $U := R_u(B)$ be the unipotent radical of $B$. Then $B$ is the semidirect product of $U$ and $T$ (with $U$ being normal in $B$).

If, in addition, $G$ is defined over a finite subfield $\mathbb{F}_q \subseteq k$ (where $q$ is a power of $p$) with Frobenius map $F : G \to G$, we write $G(q) = G^F = \{ g \in G \mid F(g) = g \}$ for the corresponding finite group of Lie type. In this case, unless otherwise stated, we will also tacitly assume that
both B and T are F-stable. Thus F induces an automorphism of W which we shall denote again by F.

We will be concerned with characters of G^F in characteristic 0. As usual in the ordinary representation theory of finite groups of Lie type, we consider representations and characters over \( \overline{\mathbb{Q}}_\ell \), an algebraic closure of the field of \( \ell \)-adic numbers, for a fixed prime \( \ell \) different from \( p \). Thus given a finite group \( \Gamma \), let \( CF(\Gamma) \) be the set of class functions \( \Gamma \to \mathbb{Q}_\ell \) and denote by the standard scalar product on \( CF(\Gamma) \), where bar denotes a field automorphism of \( \mathbb{Q}_\ell \) which maps roots of unity to their inverses. Let \( \text{Irr}(\Gamma) \subseteq CF(\Gamma) \) be the subset of irreducible characters of \( \Gamma \). They form an orthonormal basis of \( CF(\Gamma) \) with respect to this scalar product. If \( \Gamma = G^F \), we denote by \( \text{Uch}(G^F) \subseteq \text{Irr}(G^F) \) the subset of unipotent characters, that is, those \( \rho \in \text{Irr}(G^F) \) which satisfy \( \langle \rho, R_w \rangle \neq 0 \) for some \( w \in W \). Here, \( R_w \) is the virtual character defined by Deligne and Lusztig in [5].

2. Character sheaves

Let G be a connected reductive algebraic group over \( k = \mathbb{F}_p \). We begin by very briefly introducing some notions of the theory of character sheaves (only those which are relevant for our purposes), following [22]-[26], [28], [31].

2.1. Denote by \( \mathcal{D}G \) the bounded derived category of constructible \( \mathbb{Q}_\ell \)-sheaves on G, and by \( \mathcal{M}G \) the full subcategory of perverse sheaves on G in the sense of [1]. \( \mathcal{M}G \) is an abelian category. The character sheaves on G are defined ([22, 2.10]) as certain simple objects of \( \mathcal{M}G \) which are equivariant for the conjugation action of G on itself. We fix a set of representatives \( \hat{G} \) for the isomorphism classes of character sheaves on G.

An important subset of the character sheaves on G are the so-called unipotent character sheaves, defined as follows. For \( w \in W \), let \( K_w := K_{w_0}^L \in \mathcal{D}G \) be as defined in [22, 2.4] with respect to the constant local system \( \mathcal{L}_0 = \mathbb{Q}_\ell \) on T. An element of \( \hat{G} \) is called a unipotent character sheaf if it is a constituent of a perverse cohomology sheaf \( p^iH^i(K_w) \) for some \( i \in \mathbb{Z} \), \( w \in W \). We denote by

\[
\hat{G}^{\text{un}} := \{ A \in \hat{G} \mid A \text{ unipotent} \} \subseteq \hat{G}
\]

the unipotent character sheaves in \( \hat{G} \). We also set (see [24, 14.10])

\[
R_\phi^{L_0} := \frac{1}{|W|} \sum_{w \in W} \phi(w) \sum_{i \in \mathbb{Z}} (-1)^{i+\dim G} p^iH^i(K_w) \quad \text{for } \phi \in \text{Irr}(W),
\]

an element of the subgroup of the Grothendieck group of \( \mathcal{M}G \) spanned by the character sheaves. Denote by \( ( \cdot , \cdot ) \) the symmetric \( \mathbb{Q}_\ell \)-bilinear pairing on this subgroup such that for any two character sheaves \( A, A' \) on G we have

\[
(A : A') = \begin{cases} 
1 & \text{if } A \cong A', \\
0 & \text{if } A \not\cong A'.
\end{cases}
\]

Finally, there is the notion of cuspidal character sheaves on a connected reductive group,
see [22, 3.10]. By [22, Proposition 3.12], any cuspidal character sheaf on \( G \) has the form
\[ \text{IC}(\Sigma, \mathcal{E})[\dim \Sigma], \]
where “IC” stands for the intersection cohomology complex due to Deligne-Goresky-MacPherson ([15], [1]), \( \Sigma \) is the inverse image of a conjugacy class under the natural map \( G \to G/\mathbb{Z}(G)^0 \), and \( \mathcal{E} \) is a \( G \)-equivariant irreducible \( \overline{\mathcal{O}}_\ell \)-local system on \( \Sigma \), such that \((\Sigma, \mathcal{E})\) is a cuspidal pair for \( G \) in the sense of [21]. We set
\[ \hat{\mathcal{G}}^o := \{ A \in \hat{G} \mid A \text{ a cuspidal character sheaf} \} \quad \text{and} \quad \hat{\mathcal{G}}^{o,\text{un}} := \hat{\mathcal{G}}^\text{un} \cap \hat{\mathcal{G}}^o. \]
This gives rise to an inductive description of (unipotent) character sheaves, as follows. Let \( L \subseteq G \) be a Levi complement of some parabolic subgroup \( P \subseteq G \). To each complex \( K \in \mathcal{H}_L \) which is equivariant for the conjugation action of \( L \) on itself is associated an induced complex \( \text{ind}^G_{L \subseteq P}(K) \in \mathcal{D}_G \), see [22, 4.1]. If \( A_0 \in \hat{L}^o \), the complex \( \text{ind}^G_{L \subseteq P}(A_0) \) is a semisimple perverse sheaf on \( G \), all its simple direct summands are character sheaves on \( G \), and any character sheaf on \( G \) is a simple direct summand of \( \text{ind}^G_{L \subseteq P}(A_0) \) for some \( L \subseteq P \) as above, and some \( A_0 \in \hat{L}^o \). (In fact, for the latter statement it is enough to consider standard Levi subgroups \( L \) of standard parabolic subgroups \( P \). We will use this in 3.5 below.) Moreover, \( A_0 \) is unipotent if and only if some (or equivalently, every) simple summand of \( \text{ind}^G_{L \subseteq P}(A_0) \) is unipotent ([22, 4.4, 4.8]). In this way, the character sheaves on \( G \) correspond to irreducible modules for the endomorphism algebras (in \( \mathcal{H}_G \)) of various induced complexes as above, using the respective Hom functor (see [23, §10]).

**Proposition 2.2** ([23, §10], [32, 3.8], [40, 5.9]). Let \( L \subseteq G \) be a Levi complement of the parabolic subgroup \( P \subseteq G \). Assume that \( \hat{L}^{o,\text{un}} \) is non-empty and fix some \( A_0 \in \hat{L}^{o,\text{un}} \). Then the endomorphism algebra \( \text{End}_{\mathcal{H}_G}(\text{ind}^G_{L \subseteq P}(A_0)) \) is isomorphic to the group algebra \( \overline{\mathcal{O}}_\ell[W_G(L)] \), where \( W_G(L) := N_G(L)/L \) is the relative Weyl group of \( L \) in \( G \). Hence, we obtain a bijection
\[ \text{Irr}(W_G(L)) \xrightarrow{1-1} \{ A \in \hat{G} \mid A \text{ is a simple direct summand of } \text{ind}^G_{L \subseteq P}(A_0) \}. \]

2.3. From now until the end of this section, we assume that the connected reductive group \( G \) is defined over \( \mathbb{F}_q \) (where \( q \) is a power of \( p \)), with corresponding Frobenius map \( F: G \to G \). For simplicity, we shall also assume that \( F \) induces the identity on \( W \).

Given \( A \in \mathcal{D}_G \), let \( F^*A \in \mathcal{D}_G \) be the inverse image of \( A \) under the Frobenius map \( F \). Suppose that \( F^*A \) is isomorphic to \( A \) and choose an isomorphism \( \varphi: F^*A \to A \). Then \( \varphi \) induces linear maps \( \varphi_{i,g}: \mathcal{H}^i(A) \to \mathcal{H}^i(A) \) for \( i \in \mathbb{Z} \) and \( g \in G^F \), where \( \mathcal{H}^i(A) \) is the stalk at \( g \) of the \( i \)th cohomology sheaf of \( A \), a \( \overline{\mathcal{O}}_\ell \)-vector space of finite dimension. The characteristic function \( \chi_{A,\varphi}: G^F \to \overline{\mathcal{O}}_\ell \) associated with \( A \) (and \( \varphi \)) is defined by (see [23, 8.4])
\[ \chi_{A,\varphi}(g) := \sum_{i \in \mathbb{Z}} (-1)^i \text{Trace}(\varphi_{i,g}, \mathcal{H}^i(A)) \quad \text{for } g \in G^F. \]
This is well-defined since only finitely many of the \( \mathcal{H}^i(A) \) \((i \in \mathbb{Z})\) are non-zero. In the case where \( A \) is a \( G \)-equivariant perverse sheaf, we have \( \chi_{A,\varphi} \in \text{CF}(G^F) \). If, in addition, \( A \) is a simple object of \( \mathcal{H}_G \), then \( \varphi \) (and, hence, \( \chi_{A,\varphi} \)) is unique up to a non-zero scalar. Denote by
\[ \hat{G}^F := \{ A \in \hat{G} \mid F^*A \cong A \} \subseteq \hat{G}. \]
the $F$-stable character sheaves in $\hat{G}$. The following is one of the main results of [26], which we can now state without any restriction on the characteristic in view of [31].

**Theorem 2.4** (Lusztig [26, §25], [31, 3.10]). In the setting of 2.3, there exist isomorphisms $\varphi_A : F^*A \sim A$ (for $A \in \hat{G}^F$) which satisfy the following conditions:

(a) The values of the characteristic functions $\chi_{A,\varphi_A}$ are cyclotomic integers;

(b) $\{\chi_{A,\varphi_A} \mid A \in \hat{G}^F\}$ is an orthonormal basis of $\left(\text{CF}(G^F), \langle \cdot, \cdot \rangle_{G^F}\right)$.

The required properties for the isomorphisms $\varphi_A : F^*A \sim A$ (formulated in [26, 25.1], [24, 13.8]) determine $\varphi_A$ up to multiplication by a root of unity. Sometimes, for $A \in \hat{G}^F$, it will be convenient to just write $\chi_A$ for a characteristic function associated to the character sheaf $A$, without referring to a specific isomorphism $\varphi_A : F^*A \sim A$. Whenever we do this, we tacitly assume that $\chi_A = \chi_{A,\varphi_A}$ for any chosen isomorphism $\varphi_A : F^*A \sim A$ which satisfies these properties.

3. Parametrisation of unipotent characters and unipotent character sheaves

$G$ is a connected reductive algebraic group over $k = \overline{\mathbb{F}}_p$, defined over $\mathbb{F}_q$ with Frobenius map $F : G \to G$. In this section, we assume in addition that the centre of $G$ is connected and that the map induced by $F$ on $W$ is the identity. This assumption ensures that all unipotent character sheaves on $G$ are $F$-stable. In 3.5 below, we will further restrict to the case where $G$ is the simple adjoint group of type $E_7$. There are two natural ways to give “parallel” parametrisations of unipotent characters of $G^F$ and of $(F$-stable) unipotent character sheaves on $G$. We will see however that they are compatible in the case of $E_7$.

3.1. According to [20, Main Theorem 4.23], $\text{Uch}(G^F)$ can be classified in terms of the following data, which only depend on $W$, and not on $p$ or $q$. First, Lusztig describes a partition of the irreducible characters of $W$ into families. To each such family $\mathcal{F} \subseteq \text{Irr}(W)$ is associated a certain finite group $\mathcal{G} = \mathcal{G}_{\mathcal{F}}$ and a set $\mathcal{M}(\mathcal{G})$ consisting of all pairs $(g, \sigma)$, $g \in \mathcal{G}$, $\sigma \in \text{Irr}(C_{\mathcal{G}}(g))$, modulo the relation $(g, \sigma) \sim (hgh^{-1}, h\sigma)$ for $h \in \mathcal{G}$, where $h\sigma$ is the irreducible character of $C_{\mathcal{G}}(hgh^{-1}) = hC_{\mathcal{G}}(g)h^{-1}$ given by composing $\sigma$ with conjugation by $h^{-1}$. $\mathcal{M}(\mathcal{G})$ is equipped with a pairing $\{\cdot, \cdot\} : \mathcal{M}(\mathcal{G}) \times \mathcal{M}(\mathcal{G}) \to \mathbb{Q}_\ell$, defined by

$$\{(g, \sigma), (h, \tau)\} := \frac{1}{|C_{\mathcal{G}}(g)||C_{\mathcal{G}}(h)|} \sum_{x \in \mathcal{G} \atop g\sigma x^{-1}g^{-1} = h\tau x^{-1}h^{-1}} \tau(x^{-1}g^{-1}x)\sigma(xhx^{-1}),$$

for $(g, \sigma), (h, \tau) \in \mathcal{M}(\mathcal{G})$. Given a family $\mathcal{F} \subseteq \text{Irr}(W)$, let $\mathcal{F} \hookrightarrow \mathcal{M}(\mathcal{G}_{\mathcal{F}})$ be the embedding defined in [20, 4.14]. Setting

$$\overline{X}(W) := \bigsqcup_{\mathcal{F} \subseteq \text{Irr}(W) \text{ family}} \mathcal{M}(\mathcal{G}_{\mathcal{F}}),$$

we thus have an embedding

$$\text{Irr}(W) \hookrightarrow \overline{X}(W), \quad \phi \mapsto x_\phi.$$

The pairing $\{\cdot, \cdot\}$ is extended to $\overline{X}(W)$ such that $\{(g, \sigma), (h, \tau)\} := 0$ whenever $(g, \sigma), (h, \tau) \in \overline{X}(W)$ are not in the same $\mathcal{M}(\mathcal{G}_{\mathcal{F}})$. Then $\overline{X}(W)$ parametrises $\text{Uch}(G^F)$ as follows. For
\( \phi \in \text{Irr}(W) \), let
\[
R_\phi := \frac{1}{|W|} \sum_{w \in W} \phi(w) R_w.
\]
Then there is a bijection
\[
(3.3) \quad \text{Uch}(G^F) \sim \tilde{X}(W), \quad \rho \mapsto \tilde{x}_\rho,
\]
such that for any \( \rho \in \text{Uch}(G^F) \) and any \( \phi \in \text{Irr}(W) \), we have
\[
(3.4) \quad \langle \rho, R_\phi \rangle_{G^F} = \Delta(\tilde{x}_\rho)[\tilde{x}_\rho, x_\phi].
\]
Here, \( \Delta(\tilde{x}_\rho) \in \{ \pm 1 \} \) is a sign attached to \( \rho \in \text{Uch}(G^F) \), see [20, 4.21]. The bijection (3.3) is not uniquely determined by this property. In the case where \( G \) is of type \( E_7 \), we will make a definite choice in Remark 3.6 below. Next, for \( x \in \tilde{X}(W) \), a unipotent “almost character” \( R_x \in \text{CF}(G^F) \) is defined by
\[
R_x := \sum_{\rho \in \text{Uch}(G^F)} \langle \tilde{x}_\rho, x \rangle \Delta(\tilde{x}_\rho) \rho.
\]
We then have \( R_\phi = R_{x_\phi} \) for any \( \phi \in \text{Irr}(W) \). Since the “Fourier matrix” \( \Upsilon := ([x, x'])_{x, x' \in \tilde{X}(W)} \) is hermitian and \( \Upsilon^2 \) is the identity matrix [19, §4], we obtain
\[
\langle R_x, R_{x'} \rangle_{G^F} = \begin{cases} 1 & \text{if } x = x', \\ 0 & \text{if } x \neq x' \end{cases} \quad \text{(for } x, x' \in \tilde{X}(W)).
\]
It follows that
\[
(3.5) \quad \rho = \Delta(\tilde{x}_\rho) \sum_{x \in \tilde{X}(W)} \langle \tilde{x}_\rho, x \rangle R_x \quad \text{for } \rho \in \text{Uch}(G^F).
\]
In particular, knowing the values of the unipotent characters of \( G^F \) is equivalent to knowing the values of the \( R_x \) for \( x \in \tilde{X}(W) \).

3.2. On the other hand, by [26, Theorem 23.1] combined with the cleanness results in [31], the set \( \tilde{X}(W) \) in (3.1) also serves as a parameter set for \( \hat{G}^\text{un} \). Namely, there is a bijection
\[
(3.6) \quad \tilde{X}(W) \sim \hat{G}^\text{un}, \quad x \mapsto A_x,
\]
such that for any \( x \in \tilde{X}(W) \) and any \( \phi \in \text{Irr}(W) \), we have
\[
(3.7) \quad (A_x : R^\varphi_0) = \hat{\varepsilon}_A [x, x_\phi].
\]
Here, \( \hat{\varepsilon}_K = (-1)^{\dim G - \dim \text{supp } K} \) for any complex \( K \in \mathcal{D}G \), and \( \text{supp } K \) is the Zariski closure of \( \{ g \in G \mid \mathcal{H}^i_g K \neq 0 \text{ for some } i \in \mathbb{Z} \} \subseteq G \). Once again, this property does not uniquely determine the bijection (3.6). For \( G \) of type \( E_7 \), we will fix a choice in Remark 3.6.

With these notions we can formulate the following theorem of Shoji, which verifies Lusztig’s conjecture under the assumption that \( G \) has connected centre. As mentioned in [11, 2.7], this holds without any conditions on \( p, q \), since the cleanness of cuspidal character sheaves is established in full generality ([31]).
**Theorem 3.3** (Shoji [41, Theorems 3.2, 4.1]). Let $p$ be a prime, $q$ a power of $p$, $G$ a connected reductive group over $\mathbb{F}_p$, defined over $\mathbb{F}_q$ with corresponding Frobenius map $F: G \to G$. Assume that $Z(G)$ is connected and $G/Z(G)$ is simple. Then $\hat{G}^{un} \subseteq \hat{G}^F$ and for any $x \in \overline{X}(W)$, $R_x$ and $\chi_{A_x}$ coincide up to a non-zero scalar.

**Remark 3.4.** In general, any irreducible character of $G^F$ has a corresponding almost character, and the set of all these almost characters is an orthonormal basis for $\text{CF}(G^F)$ ([20, §4]). Shoji formulates and proves Theorem 3.3 for the characteristic function of any $F$-stable character sheaf and the respective almost character. Determining the character table of $G^F$ can thus be reformulated to solving the following two problems (see [29]):

(a) Computing the values of the characteristic functions of $F$-stable character sheaves;

(b) Determining the scalar relating the characteristic function of any $F$-stable character sheaf with the corresponding almost character.

(b) can be reduced to considering cuspidal ($F$-stable) character sheaves, via the induction process for cuspidal character sheaves mentioned in Section 2, see [27, §3]. On the other hand, Lusztig [26, §24] shows that (a) can be reduced to the computation of “generalised Green functions” and he provides an algorithm to compute those functions (refining an earlier algorithm of Shoji in [39]), although it involves some unspecified scalars which are still not known in all cases. We will briefly describe this algorithm in the next section.

3.5. We assume here that $G$ is the simple adjoint group of type $E_7$ over the field $k = \mathbb{F}_p$, defined over $\mathbb{F}_q$, $q$ a power of $p$, with corresponding Frobenius map $F: G \to G$. With respect to this $G$, $T \subseteq B \subseteq G$ and $W$ are as in the introduction. These choices determine the set of roots $R \subseteq X(T) = \text{Hom}(T, k^\times)$, as well as the positive roots $R^+ \subseteq R$ and the simple roots $\Pi := \{\alpha_1, \ldots, \alpha_7\} \subseteq R^+$. We choose the order of $\alpha_1, \ldots, \alpha_7$ in such a way that the Dynkin diagram of $E_7$ is as follows:

$$E_7$$

\[
\begin{array}{ccccccc}
\alpha_1 & - & - & - & - & - & \alpha_7 \\
\alpha_2 & & & & & & \\
\end{array}
\]

$W$ can be viewed as a Coxeter group of type $E_7$ with Coxeter generators $S = \{s_1, \ldots, s_7\}$, where $s_i$ denotes the reflection in the hyperplane orthogonal to $\alpha_i$. We will describe the parametrisations of $\text{Uch}(G^F)$ and $\hat{G}^{un}$ in terms of Harish-Chandra series, following Lusztig [32, §3]. (Note that all the results in loc. cit. hold whenever $G/Z(G)$ is simple or $\{1\}$, and we will implicitly use this as far as subgroups of $G$ corresponding to irreducible subgraphs of the Dynkin diagram of type $E_7$ are concerned.)

To any Coxeter system $(W, S)$ is attached ([32, 3.1]) a certain finite (possibly empty) set $\Xi_W^\circ$. If $W$ is irreducible, $\Xi_W^\circ$ is contained in the set of roots of unity in $\overline{G}^\circ$. Let

$$\Xi_W := \left\{ (J, \epsilon, \zeta) \mid J \subseteq S, \epsilon \in \text{Irr}(W^{S/J}), \zeta \in \Xi_W^\circ \right\},$$

where $W_J := \langle J \rangle \subseteq W$ and $W^{S/J}$ is the subgroup of $W$ generated by the involutions $\sigma_s := w_0^{J \cup \{s\}} w_0^J w_0^{J \cup \{s\}}$ for $s \in S \setminus J$ (for $J' \subseteq S$, we denote by $w_0^{J'}$ the longest element in $W_{J'}$). Then both $(W_J, J)$ and $\left\{ \sigma_s \mid s \in S \setminus J \right\}$ are Coxeter systems, see [17], [18].
(3.8) From now on we shall fix, once and for all, a square root \( \sqrt{p} \) of \( p \) in \( \overline{\mathbb{Q}}_f \).

Furthermore, when \( q = p^f (f \geq 1) \), we write \( \sqrt{q} := \sqrt{p^f} \).

Now consider \( \rho \in \text{Uch}(G^F) \). By [18, 3.25, 3.26], there exist \( J \subseteq S \) and a cuspidal unipotent character \( \rho_0 \in \text{Uch}(L_j^F) \) such that \( \langle \rho, R_{L_j}^G(\rho_0) \rangle_{G^F} \neq 0 \). For \( J \subseteq S \), we denote by \( L_j \) the unique Levi complement of the standard parabolic subgroup \( P_j := BW_jB \subseteq G \) such that \( T \subseteq L_j \), and \( R_{L_j}^G \) is Harish-Chandra induction.) After having chosen \( \sqrt{p} \) in (3.8), there is a natural isomorphism between the endomorphism algebra \( \text{End}_{\mathcal{O}_{G^F}}(R_{L_j}^G(\rho_0)) \) and the group algebra \( \overline{\mathbb{Q}}_l[W^{S/J}] \) (in analogy to 2.2; note that \( W^{S/J} \) is isomorphic to \( W_G(L_j) \)). Hence, among the constituents of \( R_{L_j}^G(\rho_0) \), \( \rho \) is naturally parametrised by some \( \epsilon \in \text{Irr}(W^{S/J}) \) (see also [20, Corollary 8.7]). Moreover, denote by \( \lambda_\rho \) the eigenvalue of Frobenius of \( \rho \) (as defined in [20, \S 11]). Then we have \( \lambda_\rho = \lambda_{\rho_0} \in \Xi_W^* \) and the assignment \( \rho \mapsto (J, \epsilon, \lambda_\rho) \) gives a well-defined bijection

\[
\text{Uch}(G^F) \overset{\sim}{\to} \Xi_W.
\]

In particular, the cuspidal unipotent characters correspond to the elements of \( \Xi_W^* \). On the other hand, let \( A \in \hat{G}^{\text{un}} \). By 2.1, 2.2, there exist \( J \subseteq S \), \( A_0 \in \hat{L}_j^{\epsilon, \text{un}} \), such that \( A \) is a simple summand of \( \text{ind}_{L_j^\epsilon \mathcal{P}_j}(A_0) \), and \( A \) is parametrised by some \( \epsilon \in \text{Irr}(W^{S/J}) \). Furthermore, to \( A \) is associated a root of unity \( \lambda_A \) by means of a certain “twisting operator” defined via Shintani descent, see [40, 1.16, Theorem 3.3]. Another definition of \( \lambda_A \) is given in [32, 3.6]. Then we have \( \lambda_A = \lambda_{A_0} \) (see [40, 3.6]), and this is an element of \( \Xi_W^0 \). The assignment \( (J, \epsilon, \lambda_A) \mapsto A \) gives a well-defined bijection

\[
\Xi_W \overset{\sim}{\to} \hat{G}^{\text{un}}.
\]

In particular, the unipotent cuspidal character sheaves correspond to the elements of \( \Xi_W^0 \).

Remark 3.6. We keep the setting of 3.5. In particular, \( Z(G) \) is connected and \( F \) acts trivially on \( W \). Recall from 3.1, 3.2, that the bijections \( \text{Uch}(G^F) \overset{\sim}{\to} \overline{X}(W) \) and \( \overline{X}(W) \overset{\sim}{\to} \hat{G}^{\text{un}} \) are not uniquely determined by the conditions (3.4), (3.7), respectively. We will now make a definite choice following [20, 11.2], see also [6, \S 6].

Given \( \rho \in \text{Uch}(G^F) \), the family \( \mathcal{F} \subseteq \text{Irr}(W) \) for which \( \mathcal{M}(\mathcal{F}) \) contains \( \overline{\chi}_\rho \) is independent of a chosen bijection \( \text{Uch}(G^F) \overset{\sim}{\to} \overline{X}(W) \) in (3.3) satisfying (3.4). Consider an element \( x = (g, \sigma) \in \mathcal{M}(\mathcal{F}) \subseteq \overline{X}(W) \). If \( \mathcal{F} \) contains the irreducible characters of \( W \) of degree 512, and if \( x \) is not in the image of the embedding \( \mathcal{F} \hookrightarrow \mathcal{M}(\mathcal{F}) \), we set \( \tilde{\lambda}_x := \zeta \sigma(g)/\sigma(1) \), where \( \zeta \) is a primitive fourth root of unity in \( \overline{\mathbb{Q}}_f \). In any other case, let \( \tilde{\lambda}_x := \sigma(g)/\sigma(1) \). Then it turns out that the bijection (3.3) can be chosen in such a way that \( \lambda_\rho = \tilde{\lambda}_\rho \) for any \( \rho \in \text{Uch}(G^F) \), and it is almost uniquely determined by this condition along with (3.4), the only ambiguity arises from the two elements of \( \text{Irr}(W) \) of degree 512. This can be removed as follows. We have a natural embedding \( \text{Irr}(W) \hookrightarrow \Xi_W, \phi \mapsto (\emptyset, \phi, 1) \). Then we require that the embeddings \( \text{Irr}(W) \hookrightarrow \Xi_W \overset{\sim}{\to} \text{Uch}(G^F) \) and \( \text{Irr}(W) \hookrightarrow \overline{X}(W) \overset{\sim}{\to} \text{Uch}(G^F) \) coincide. This does not interfere with our choices above, in view of [20, Proposition 12.6], and it uniquely specifies the bijection \( \text{Uch}(G^F) \overset{\sim}{\to} \overline{X}(W) \).

Now let \( A \in \hat{G}^{\text{un}} \) and let \( (J, \epsilon, \lambda_A) \in \Xi_W, \rho \in \text{Uch}(G^F) \) be the elements associated to \( A \) in 3.5. By [32, 3.10], we have \( \langle \rho, R_\phi \rangle_{G^F} = (A : R_\phi^{L_j}) \) for any \( \phi \in \text{Irr}(W) \). Moreover, \( \Delta(\overline{\chi}_\rho) = \hat{\epsilon}_A \),
so (3.4) implies \((A : R_{\phi}^{Z_{\phi}}) = \hat{\xi}_{A}[\bar{x}_{\phi}, x_{\phi}]\) for any \(\phi \in \text{Irr}(W)\). Hence, the assignment \(\bar{x}_{\phi} \mapsto A\) gives rise to a bijection \(\tilde{X}(W) \rightarrow \tilde{G}^{un}\) satisfying (3.7). Thus we can and will from now on always assume that the bijections \(Uch(G^F) \rightarrow \tilde{X}(W) \rightarrow \tilde{G}^{un}\) in 3.1, 3.2 are chosen in this way. In particular, the diagram

\[
\begin{array}{ccc}
Uch(G^F) & \sim & \tilde{X}(W) \\
\uparrow & & \uparrow \\
\text{Irr}(W) & \sim & \tilde{G}^{un} \\
\downarrow & & \downarrow \\
\mathcal{S}_W & \sim &
\end{array}
\]

is commutative.

4. The generalised Springer correspondence

In this section, \(G\) may be assumed to be an arbitrary connected reductive algebraic group over \(\overline{\mathbb{F}}_p\), as in the introduction. We will focus on those character sheaves whose support contains unipotent elements. In [26, §24], Lusztig provides an algorithm for computing the values of their characteristic functions in principle, using intersection cohomology complexes [21]. Further references for the following are [43, 1.1-1.3], [49, §3, §4].

4.1. Denote by \(\mathcal{N}_G\) the set of all pairs \((\mathcal{O}, \mathcal{E})\) where \(\mathcal{O}\) is a unipotent conjugacy class in \(G\) and \(\mathcal{E}\) is the isomorphism class of an irreducible \(G\)-equivariant \(\overline{\mathbb{Q}}_l\)-local system on \(\mathcal{O}\). For a given \(\mathcal{O}\), these local systems on \(\mathcal{O}\) naturally correspond to irreducible characters of the group \(A(u) := C_G(u)/C_G^e(u)\), where \(u\) is any fixed element of \(\mathcal{O}\). To \((\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G\) is associated \(([21, \text{Theorem 6.5}]) a triple \((L, \mathcal{O}_0, \mathcal{E}_0)\) consisting of a closed subgroup \(L \subseteq G\) which is the Levi complement of some parabolic subgroup of \(G\), a unipotent conjugacy class \(\mathcal{O}_0\) in \(L\) and an \(L\)-equivariant \(\overline{\mathbb{Q}}_l\)-local system \(\mathcal{E}_0\) on \(\mathcal{O}_0\) such that \((Z(L)^c \times \mathcal{O}_0, 1 \otimes \mathcal{E}_0)\) is a cuspidal pair for \(L\) in the sense of [21, 2.4]. Denoting by \(\mathcal{M}_G\) the set of all those triples up to \(G\)-conjugacy, the corresponding map \(\tau : \mathcal{N}_G \rightarrow \mathcal{M}_G\) is surjective, so we have a decomposition

\[
\mathcal{N}_G = \bigsqcup_{(L, \mathcal{O}_0, \mathcal{E}_0) \in \mathcal{M}_G} \tau^{-1}(L, \mathcal{O}_0, \mathcal{E}_0)
\]

of \(\mathcal{N}_G\) into non-empty subsets. The elements in a given \(\tau^{-1}(L, \mathcal{O}_0, \mathcal{E}_0)\) can be distinguished as follows. The triple \((L, \mathcal{O}_0, \mathcal{E}_0)\) gives rise to a semisimple perverse sheaf \(K_{(L, \mathcal{O}_0, \mathcal{E}_0)} \in \mathcal{M}_G\) defined as a certain intersection cohomology complex in [21, §4], see also [23, §8]. Then for any \(\iota = (\mathcal{O}_i, \mathcal{E}_i) \in \tau^{-1}(L, \mathcal{O}_0, \mathcal{E}_0)\) there is (up to isomorphism) exactly one simple direct summand \(A_{\iota} \mid_{G^{un}}\) of \(K_{(L, \mathcal{O}_0, \mathcal{E}_0)}\) which satisfies

\[
A_{\iota} \mid_{G^{un}} \cong IC(\mathcal{O}_i, \mathcal{E}_i)[\dim \mathcal{O}_i + \dim Z(L)^c]
\]

\((G^{un} \subseteq G\) is the unipotent subvariety of \(G\), and each simple direct summand of \(K_{(L, \mathcal{O}_0, \mathcal{E}_0)}\) arises in this way from some \(\iota \in \tau^{-1}(L, \mathcal{O}_0, \mathcal{E}_0)\), see [21, 6.5], [26, 24.1]. On the other hand, the endomorphism algebra \(\mathcal{A} := \text{End}_{\mathcal{M}_G}(K_{(L, \mathcal{O}_0, \mathcal{E}_0)})\) is isomorphic to \(\mathcal{Q}_l[\text{Irr}(W_G(L))]\) ([21, Theorem 9.2]). Thus the isomorphism classes of simple direct summands of \(K_{(L, \mathcal{O}_0, \mathcal{E}_0)}\) are naturally parametrised by \(\text{Irr}(W_G(L))\). Hence, we obtain a bijection
\[ \mathcal{N}_G \cong \bigsqcup_{(L, \mathcal{O}_0, \mathcal{E}_0) \in \mathcal{M}_G} \text{Irr}(W_G(L)), \]

which is called the generalised Springer correspondence. The problem of explicitly determining this correspondence can be reduced to the case where \( G \) is almost simple of simply-connected type, and has been solved by Lusztig and Spaltenstein, except for a few minor cases in type \( E_6 \) with \( p \neq 3 \), and in type \( E_8 \) with \( p = 3 \) ([21], [35], [47], [34]). In particular, the generalised Springer correspondence is completely determined in the case of \( E_7 \), see [47].

4.2. Let us now bring the \( \mathbb{F}_q \)-rational structure on \( G \) into the picture, see [26, §24]. So we assume that \( G \) is defined over \( \mathbb{F}_q \), with Frobenius map \( F : G \to G \). \( F \) acts naturally on both \( \mathcal{N}_G \) and \( \mathcal{M}_G \) via

\[ \mathcal{N}_G \to \mathcal{N}_G, \quad (\mathcal{O}, \mathcal{E}) \mapsto (F^{-1}(\mathcal{O}), F^*\mathcal{E}), \]

and

\[ \mathcal{M}_G \to \mathcal{M}_G, \quad (L, \mathcal{O}_0, \mathcal{E}_0) \mapsto (F^{-1}(L), F^{-1}(\mathcal{O}_0), F^*\mathcal{E}_0), \]

respectively. If \( L \) is \( F \)-stable, it is also clear that \( F \) acts on \( W_G(L) \) and, hence, on \( \text{Irr}(W_G(L)) \), by

\[ \text{Irr}(W_G(L)) \to \text{Irr}(W_G(L)), \quad \epsilon \mapsto \epsilon \circ F. \]

Let \( \mathcal{N}_G^F \subseteq \mathcal{N}_G \), \( \mathcal{M}_G^F \subseteq \mathcal{M}_G \), \( \text{Irr}(W_G(L))^F \subseteq \text{Irr}(W_G(L)) \) be the respective subsets of fixed points under these actions (where, in terms of the local systems, this is only meant up to isomorphism). The generalised Springer correspondence (4.2) is compatible with the above actions, thus it induces a bijection

\[ \mathcal{N}_G^F \cong \bigsqcup_{(L, \mathcal{O}_0, \mathcal{E}_0) \in \mathcal{M}_G^F} \text{Irr}(W_G(L))^F. \]

For any \( \iota = (\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G^F \), the complex \( A_\iota \) is \( F \)-stable. More precisely, let \( \tau(\iota) = (L, \mathcal{O}_0, \mathcal{E}_0) \in \mathcal{M}_G^F \). We may choose an isomorphism \( F^*\mathcal{E}_0 \sim \mathcal{E}_0 \) which induces a map of finite order at the stalk of \( \mathcal{E}_0 \) at any element of \( \mathcal{O}_0^F \). Such a choice induces an isomorphism \( F^*K(L, \mathcal{O}_0, \mathcal{E}_0) \sim K(L, \mathcal{O}_0, \mathcal{E}_0) \), which in turn gives rise to an isomorphism \( \varphi_{A_\iota} : F^*A_\iota \sim A_\iota \), as described in [26, 24.2]. Once \( \varphi_{A_\iota} \) is fixed, it determines an isomorphism \( \psi_{\iota} : \mathcal{E}_\iota \sim \mathcal{E}_\iota \) (via [26, (24.2.2)]), and for any \( g \in \mathcal{O}^F \), the induced map \( \psi_{\iota,g} : \mathcal{E}_\iota \to \mathcal{E}_\iota \) on the stalk of \( \mathcal{E} \) at \( g \) is of finite order. Now define two functions \( Y_\iota, X_\iota : G_{\text{uni}}^F \to \overline{\mathbb{Q}}_q \) by

\[ Y_\iota(g) := \begin{cases} \text{Trace}(\psi_{\iota,g}, \mathcal{E}_\iota) & \text{if } g \in \mathcal{O}^F, \\ 0 & \text{if } g \notin \mathcal{O}^F \end{cases} \]

and

\[ X_\iota(g) := (-1)^{\dim \mathcal{O}^F + \dim Z(L)^e} q^{-d} \chi_{A_\iota \varphi_{A_\iota}}(g) \]

for \( g \in G_{\text{uni}}^F \), where \( d = \frac{1}{2}(\dim \text{supp} A_\iota - \dim \mathcal{O}^F - \dim Z(L)^e) \). Then both \( Y_\iota \) and \( X_\iota \) are invariant under the conjugation action of \( G^F \) on \( G_{\text{uni}}^F \).
Theorem 4.3 (Lusztig [26, §24]). In the setting of 4.2, the following hold.

(a) The functions $Y_\iota$, $\iota \in \mathcal{M}_G^F$, form a basis for the vector space consisting of all functions $G_{\text{uni}}^F \to \overline{Q}_\ell$ which are invariant under $G^F$-conjugacy.

(b) There is a system of equations

$$X_\iota = \sum_{\iota' \in \mathcal{M}_G^F} p_{\iota',\iota} Y_{\iota'} \quad (\iota \in \mathcal{M}_G^F),$$

for some uniquely determined $p_{\iota',\iota} \in \mathbb{Z}$.

Note that the restrictions (23.0.1) on the characteristic of $k$ can be removed since the cleanness of cuspidal character sheaves has been established in complete generality [31].

Remark 4.4. In the setting of Theorem 4.3, if we define a total order $\preceq$ on $\mathcal{M}_G^F$ in such a way that for $\iota = (\mathcal{O}, \mathcal{E})$, $\iota' = (\mathcal{O}', \mathcal{E}') \in \mathcal{M}_G^F$ we have $\iota' \preceq \iota$ whenever $\mathcal{O}' \subseteq \mathcal{O}$ (the latter condition defines a partial order on the set of unipotent classes of $G$), the matrix $(p_{\iota',\iota})_{\iota',\iota \in \mathcal{M}_G^F}$ has upper unitriangular shape, as follows immediately from the definitions. Lusztig [26, 24.4] provides an algorithm for computing this matrix $(p_{\iota',\iota})$, which entirely relies on combinatorial data. This algorithm is implemented in CHEVIE [37] and is accessible via the functions UnipotentClasses and ICCTable. Hence, the computation of the characteristic functions $\chi_{A_\iota,\phi_{\iota_0}} (\iota \in \mathcal{M}_G^F)$ at unipotent elements is reduced to the computation of the functions $Y_\iota$, $\iota \in \mathcal{M}_G^F$. It should be noted, however, that the latter is still a non-trivial task since it is difficult to describe the isomorphisms $\phi_{A_\iota}$ (and thus the $\psi_{\iota}$) explicitly. This has been accomplished for classical groups (in any characteristic) by Shoji, [43], [44], [46]. As far as exceptional groups are concerned, this problem is not yet solved.

Remark 4.5. In the setting of 4.1, let us fix a triple $(L, \mathcal{O}_0, \mathcal{E}_0) \in \mathcal{M}_G$ and consider the complex $K_{(L, \mathcal{O}_0, \mathcal{E}_0)} \in \mathcal{M}_G$. By [21, 4.5], we have a canonical isomorphism

$$K_{(L, \mathcal{O}_0, \mathcal{E}_0)} \cong \text{ind}^G_{L \subseteq P}(\text{IC}(\Sigma_0, \mathcal{E}_0)[\dim \Sigma_0]),$$

where $\Sigma_0 = Z(L)\cdot \mathcal{E}_0$.

Here, $A_0 := \text{IC}(\Sigma_0, \mathcal{E}_0)[\dim \Sigma_0]$ is a cuspidal character sheaf on $L$. In particular, since the definition of $K_{(L, \mathcal{O}_0, \mathcal{E}_0)}$ does not involve the choice of a parabolic subgroup with Levi complement $L$, this shows that $\text{ind}^G_{L \subseteq P}(A_0)$ is independent of $P$, and we can just write $\text{ind}^G_{L}(A_0)$ instead. Next, as the elements of $\mathcal{M}_G$ are taken up to $G$-conjugacy, we may assume that $L$ is a standard Levi subgroup of some standard parabolic subgroup $P$ of $G$ (compare 3.5), that is, there exists some $J \subseteq S$ such that $P = P_J = BWJ$, and $L = L_J$ is the unique Levi complement of $P_J$ containing $T$.

5. The scalars $\xi_x$ for cuspidal $A_x$ in $E_7$, $p = 2$

The notation and assumptions are as in 3.5, and in addition we set $p := 2$. In particular, $G$ has trivial centre and we can apply Theorem 3.3. So there are scalars $\xi_x \in \overline{Q}_\ell$ such that

$$(5.1) \quad R_x = \xi_x A_\iota \quad \text{for } x \in \overline{X}(W).$$

Since $\langle R_x, R_x \rangle = 1 = \langle A_\iota, A_\iota \rangle$ for any $x \in \overline{X}(W)$, $A \in \mathcal{G}^\text{un}$, we know that $\xi_x \overline{\xi}_x = 1$ for any $x \in \overline{X}(W)$. In this section, we determine the scalars $\xi_x$ for the cuspidal (unipotent) character sheaves $A_x$, after having chosen specific isomorphisms $F^*A_x \to A_x$ for the corresponding
Let $a_0$ be the image of $u_0$ in $A(u_0) = C_G(u_0)/C_G^0(u_0)$ under the natural map $C_G(u_0) \to A(u_0)$. Then $A(u_0)$ is a cyclic group of order 4 generated by $a_0$ (see [38, §4] and [7, 14.15, 14.18]), and the automorphism of $A(u_0)$ induced by $F$ is the identity. Hence, the elements of $A(u_0)$ correspond to the $G^F$-conjugacy classes contained in $O^F_{\text{reg}}$ (see, for instance, [8, 4.3.6]). In particular, there are 4 such classes and every irreducible $G$-equivariant $\bar{Q}_l$-local system on $O_{\text{reg}}$ is one-dimensional. For $a \in A(u_0)$, we choose some $x \in G$ such that $x^{-1}F(x) \in C_G(u_0)$ has image $a \in A(u_0)$. Then $u_a := xu_0x^{-1} \in O^F_{\text{reg}}$ (defined only up to $G^F$-conjugacy) represents the $G^F$-conjugacy class corresponding to $a$. If $a$ is the trivial element of $A(u_0)$, we will always assume that the representative for the corresponding $G^F$-conjugacy class is $u_0$. Thus $\{u_{a_0} \mid 0 \leq i \leq 3\}$ is a set of representatives for the $G^F$-conjugacy classes inside $O^F_{\text{reg}}$. Singling out the element $u_0 \in O^F_{\text{reg}}$ as above (or, more precisely, the $G^F$-conjugacy class of $u_0$) gives rise to specific isomorphisms $\varphi_{A_i} : F^* A_i \sim A_i$ with respect to the two cuspidal character sheaves $A_i \in \hat{G}^\text{un} \,(i = 1, 2)$, as follows. Let $\varphi_i$ be the $F$-stable irreducible $G$-equivariant $\bar{Q}_l$-local system on $O_{\text{reg}}$ such that $A_i \cong \text{IC}(\text{G}_\text{uni}, \varphi_i)[\dim O_{\text{reg}}]$. By [26, 24.1], we have

$$H^s(A_i)|_{G_{\text{uni}}} \cong \begin{cases} \varphi_i & \text{if } s = -d, \\ 0 & \text{if } s \neq -d, \end{cases}$$

where $d = \dim O_{\text{reg}}$. Hence, for $u \in O^F_{\text{reg}}$, the stalk $H^{-d}_u(A_i) \cong \varphi_{i,u}$ is one-dimensional. Using [26, 25.1], the $\varphi_{A_i}$ may thus be chosen in such a way that, in particular, for any $u \in O^F_{\text{reg}}$, the induced map $\varphi_{i,u} : O^F_{\text{reg}} \to \varphi_{i,u}$ is given by scalar multiplication with $\sqrt{q}^7$ times a root of unity. We can therefore modify the $\varphi_{A_i}$ by a root of unity, if necessary, such that the induced map on the stalk $\varphi_{i,u_0}$ is just scalar multiplication with $\sqrt{q}^7$, and this uniquely specifies our choices for $\varphi_{A_i} : F^* A_i \sim A_i \,(i = 1, 2)$. Denoting by $\varphi_{\varphi_i} : F^* \varphi_i \sim \varphi_i$ the restriction of $\varphi_{A_i}$, we then simply have

$$\chi_{A_i, \varphi_{A_i}}(u) = \chi_{\varphi_i, \varphi_{\varphi_i}}(u) \quad \text{for } u \in O^F_{\text{reg}}, \ i = 1, 2,$$

The latter function can be computed explicitly, as described in [30, 19.7]. Namely, assigning to a local system $\varphi$ on $O_{\text{reg}}$ its stalk $\varphi_{u_0}$ at $u_0$ gives a bijection between the isomorphism classes of $F$-stable $G$-equivariant irreducible local systems $\varphi$ on $O_{\text{reg}}$ and $F$-invariant simple $\bar{Q}_l[A(u_0)]$-modules. For $i = 1, 2$, let $\sigma_i \in \text{Irr}(A(u_0))^F$ be the character corresponding to the local system $\varphi_i$ on $O_{\text{reg}}$. The $\sigma_i$ are the two faithful irreducible characters of $A(u_0)$. We number the cuspidal character sheaves $A_i$ (and, correspondingly, the $\varphi_i, \sigma_i$) in such a way that $\sigma_1(a_0) = \zeta, \sigma_2(a_0) = -\zeta$, where $\zeta$ is the primitive fourth root of unity in $\bar{Q}_l$ chosen in
Remark 3.6. Using the cleanliness results in [31], \( \chi_{A_i} \), vanishes outside \( \mathcal{O}_{\text{reg}}^F \). Hence, for \( g \in G^F \), we obtain the following table, where \( q^{7/2} = \sqrt{q}^7 \).

<table>
<thead>
<tr>
<th>( g \notin \mathcal{O}_{\text{reg}}^F )</th>
<th>( g = u_0 )</th>
<th>( g = u_{a_0} )</th>
<th>( g = u_{a_0^2} )</th>
<th>( g = u_{a_0^3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_{A_1} )</td>
<td>0</td>
<td>( q^{7/2} )</td>
<td>( q^{7/2} \zeta )</td>
<td>( -q^{7/2} \zeta )</td>
</tr>
<tr>
<td>( \chi_{A_2} )</td>
<td>0</td>
<td>( q^{7/2} )</td>
<td>( -q^{7/2} \zeta )</td>
<td>( -q^{7/2} \zeta )</td>
</tr>
</tbody>
</table>

We can now formulate the result.

**Proposition 5.2.** In the setting of 5.1, let \( x_1, x_2 \) be the elements of \( \overline{X}(W) \) such that \( A_i = A_{x_i}, i = 1, 2 \). \( \langle x_1, x_2 \rangle \) is given by the parametrisation (3.6), which is uniquely determined by the requirements in Remark 3.6. Then we have

\[
R_{x_i} = \chi_{A_i} \text{ for } i = 1, 2.
\]

In other words, with the choices for \( \varphi_{A_i} : F^*A_i \rightarrow A_i \) made in 5.1 \((i = 1, 2)\), the scalars \( \xi_{x_1}, \xi_{x_2} \) in (5.1) are both 1.

The proof will be given in 5.4–5.9 below. We start with the following simple observation.

**Lemma 5.3.** Consider the element \( u_0 = u_{1}(1) \cdot u_{2}(1) \cdots u_{7}(1) \in \mathcal{O}_{\text{reg}}^F \) defined in (5.2). For any permutation \( \pi \) of \( \{1, 2, \ldots, 7\} \), there is an element \( u \in U^F \) such that

\[
u \cdot u_0 \cdot u^{-1} = u_{\pi(1)}(1) \cdot u_{\pi(2)}(1) \cdots u_{\pi(7)}(1).
\]

In particular, \( u_0 \) is conjugate to \( u_0^{-1} \) in \( U^F \subseteq G^F \).

Proof. First consider the Weyl group \( W \) of \( G \), viewed as the irreducible finite Coxeter group of type \( E_7 \) with simple reflections \( S = \{s_1, s_2, \ldots, s_7\} \) (see 3.5). It is well-known that any two Coxeter elements of \( (W, S) \) are conjugate in \( W \), that is, for any permutation \( \pi \) of \( \{1, 2, \ldots, 7\} \) there exists some \( w \in W \) such that

\[
w \cdot (s_1 \cdot s_2 \cdots s_7) \cdot w^{-1} = s_{\pi(1)} \cdot s_{\pi(2)} \cdots s_{\pi(7)}.
\]

More precisely, [3, §1] provides an algorithm to compute such an element \( w \) which is only based on the facts that an element \( s_i \) \((1 \leq i \leq 7)\) in the first (respectively, last) position of a given Coxeter word can be shifted to the last (respectively, first) position by means of conjugation with \( s_i \), and that \( s_i, s_j \) \((1 \leq i, j \leq 7, i \neq j)\) commute if and only if they are not linked in the Coxeter graph of \( (W, S) \). But the analogous statements hold for the \( u_i(1), 1 \leq i \leq 7 \), so we can just mimic the proof of [3, Lemma 1.4] to obtain an element \( u \in U^F \) (a product of certain \( u_i(1), 1 \leq i \leq 7 \); note that \( u_i(1) = u_i(-1) \) since \( h \) has characteristic 2) such that

\[
u \cdot u_0 \cdot u^{-1} = u_{\pi(1)}(1) \cdot u_{\pi(2)}(1) \cdots u_{\pi(7)}(1).
\]

In particular, since \( u_0^{-1} = u_7(1) \cdot u_6(1) \cdots u_1(1), u_0 \) is conjugate to \( u_0^{-1} \) in \( U^F \subseteq G^F \). \( \square \)

5.4. Let us denote by \( \rho_x \in \text{Uch}(G^F) \) the unipotent character corresponding to \( x \in \overline{X}(W) \) in (3.3). The discussion in Remark 3.6 together with [40, 3.8] (or [32, 3.6]) shows that \( \rho_{x_1} = E_7[\zeta], \rho_{x_2} = E_7[-\zeta] \) are the two cuspidal unipotent characters in Carter’s table [2, pp. 482-483]. We have
\[ R_{x_1} = \sum_{x \in \overline{\mathcal{X}(W)}} \{x, x_1\} \Delta(x) \rho_x = -\frac{1}{2} \rho_{x_1} + \frac{1}{2} \rho_{x_2} + \sum_{x \in \overline{\mathcal{X}(W) \backslash \{x_1, x_2\}}} \{x, x_1\} \rho_x, \]

\[ R_{x_2} = \sum_{x \in \overline{\mathcal{X}(W)}} \{x, x_2\} \Delta(x) \rho_x = -\frac{1}{2} \rho_{x_2} + \frac{1}{2} \rho_{x_1} + \sum_{x \in \overline{\mathcal{X}(W) \backslash \{x_1, x_2\}}} \{x, x_2\} \rho_x. \]

(Note that \( \Delta(x_1) = \Delta(x_2) = -1 \), see [20, 4.14].) Now consider the character \( \overline{\rho}_{x_1} \), defined by

\[ \overline{\rho}_{x_1}(g) := \rho_{x_1}(g) \quad \text{for } g \in G^F. \]

\( \overline{\rho}_{x_1} \) is again a cuspidal unipotent (irreducible) character of \( G^F \). Since the character field of the two cuspidal unipotent characters of groups of type \( E_7 \) contains non-real elements (see [9, 5.4, Table 1]), we conclude \( \overline{\rho}_{x_1} = \rho_{x_2} \). In view of Lemma 5.3 and since \( \{x, x_1\} = \{x, x_2\} \) for any \( x \in \overline{\mathcal{X}(W) \backslash \{x_1, x_2\}} \), we obtain

\[ \xi_{x_1} q^{7/2} = \xi_{x_1} \chi_1(u_0) = R_{x_1}(u_0) = R_{x_2}(u_0) = \xi_{x_2} \chi_2(u_0) = \xi_{x_2} q^{7/2}, \]

which also equals \( \overline{R}_{x_2}(u_0) \) (using once again Lemma 5.3). We deduce \( \xi_{x_1} = \xi_{x_2} = \overline{\xi}_{x_2} \) and thus \( \xi_{x_1} = \xi_{x_2} \in \{\pm 1\} \), since \( \xi_{x_1} \xi_{x_2} = 1 \).

5.5. In order to determine the sign \( \xi := \xi_{x_1} = \xi_{x_2} \in \{\pm 1\} \), we want to apply a formula in [10, 3.6(b)], as in [16]. However, we will need more theoretical ingredients than in [16], most notably the generalised Springer correspondence (see Section 4). We consider the Hecke algebra of the group \( G^F \) with its BN-pair \( (B^F, N_G(T)^F) \), that is, the endomorphism algebra

\[ \mathcal{H} := \text{End}_{\mathbb{Q}_\ell}(\mathbb{Q}_\ell[G^F/B^F])^{\text{opp}} \]

(“opp” stands for the opposite algebra). \( \mathcal{H} \) has a \( \mathbb{Q}_\ell \)-basis \( \{T_w \mid w \in W\} \), where

\[ T_w : \mathbb{Q}_\ell[G^F/B^F] \rightarrow \mathbb{Q}_\ell[G^F/B^F], \quad xB^F \mapsto \sum_{yB^F \in G^F/B^F} yB^F, \]

for \( w \in W \). Recall from 3.5 that \( S = \{s_1, \ldots, s_7\} \) is a set of Coxeter generators for \( W \). Denote by \( \ell : W \rightarrow \mathbb{Z}_{\geq 0} \) the length function of \( W \) with respect to \( S \). Then the multiplication in \( \mathcal{H} \) is determined by the following equations.

\[ T_s \cdot T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) = \ell(w) + 1 \\ QT_{sw} + (q-1)T_w & \text{if } \ell(sw) = \ell(w) - 1 \end{cases} \quad \text{for } s \in S, \ w \in W. \]

The irreducible characters of \( W \) naturally parametrise the isomorphism classes of irreducible modules of \( \mathcal{H} \), see [4, §11D]. Given \( \phi \in \text{Irr}(W) \), let \( V_\phi \) be the corresponding module of \( \mathcal{H} \), and let \( \rho_\phi \in \text{Uch}(G^F) \) be the image of \( \phi \) under the map \( \text{Irr}(W) \hookrightarrow \Xi_W \twoheadrightarrow \text{Uch}(G^F) \), see Remark 3.6. By [10, 3.6] and [14, §8.4], we have

\[ \sum_{\phi \in \text{Irr}(W)} \rho_\phi(g) \text{Tr}(T_w, V_\phi) = \frac{|O_g \cap B^F wB^F| \cdot |C_{G^F}(g)|}{|B^F|} \]

for any \( g \in G^F \) and \( w \in W \), where \( w \in N_G(T)^F \) is a representative of \( w \), \( O_g \subseteq G^F \) denotes
the $G^F$-conjugacy class of $g$ and $\text{Tr}(T_w, V_\phi)$ is the trace of the linear map on $V_\phi$ defined by $T_w$. The character table of $H$ is contained in CHEVIE [13], so the numbers $\text{Tr}(T_w, V_\phi)$ are known. We want to apply (5.3) with $g = u_0$ to get hold of the sign $\xi$. In order to achieve this, we thus need some information about the values $\rho_\phi(u_0)$ for $\phi \in \text{Irr}(W)$. By (3.5) and (5.1), we have

$$\rho_\phi = \Delta(x_\phi) \sum_{x \in \overline{X}(W)} \{x_\phi, x\} R_x = \sum_{x \in \overline{X}(W)} \{x_\phi, x\} \xi_\phi \chi_{Ax}.$$

**Lemma 5.6.** Let $x \in \overline{X}(W)$ be such that, via the generalised Springer correspondence (4.3), $A_x$ arises from a pair $t = (\mathcal{O}_t, \mathcal{O}) \in \mathcal{N}_G^F$ where $\mathcal{O} \neq \mathcal{O}_{\text{reg}}$. Then

$$\chi_{Ax}(u_0) = R_x(u_0) = 0.$$

Proof. By (4.1), the support of $A_x|_{\text{cusp}}$ is $\overline{\mathcal{O}}$ which is a union of conjugacy classes of $G$ and therefore does not contain any regular unipotent elements in case $\mathcal{O} \neq \mathcal{O}_{\text{reg}}$, as $\mathcal{O}_{\text{reg}}$ is the unique class of $G$ whose closure is $G_{\text{uni}}$. So we have $\chi_{Ax}(u_0) = 0$ and then also $R_x(u_0) = 0$ in view of Theorem 3.3. □

5.7. Let us now describe the images of the pairs $t = (\mathcal{O}_{\text{reg}}, \mathcal{O}) \in \mathcal{N}_G^F$ under the map (4.3), following Spaltenstein [47, p. 331]. Since $F$ acts trivially on $A(u_0) \cong \mathbb{Z}/4\mathbb{Z}$, there are (up to isomorphism) four $F$-stable irreducible $G$-equivariant local systems on the regular unipotent class $\mathcal{O}_{\text{reg}}$. We will write $\mathcal{O}_i^i (1 \leq i \leq 4)$ for the local system corresponding to the irreducible character of $A(u_0)$ which takes the value $\xi^i$ at $u_0 = u_0 C G(u_0) \in A(u_0)$, where $\xi$ is the primitive fourth root of unity in $\mathbb{Q}_{\ell}$ specified in 5.1, 3.6. First consider $t = (\mathcal{O}_{\text{reg}}, \mathcal{O}^i) \in \mathcal{N}_G^F$. The corresponding element of $\mathcal{M}_G^F$ is $(T, \{1, \overline{\mathcal{O}}_i\})$, and among the simple summands of $K_{(T,\{1,\overline{\mathcal{O}}_i\})}$, the character sheaf $A_x \cong A_t(\mathcal{O}^i, 1)$ is parametrised by the trivial character $1$ of $W = W_G(T)$, whence $R_x = R_1 W = 1_{G^F}$. Next, the pairs $(\mathcal{O}_{\text{reg}}, \mathcal{O}^i \vee \mathcal{O})$ correspond to $(G, \mathcal{O}_{\text{reg}}, \pm \xi)$, so they give rise to the two cuspidal (unipotent) character sheaves $A_t(\mathcal{O}_{\text{reg}}, \pm \xi)$. With the arrangements in 5.1, 5.2, we have $A_{x_1} \cong A_t(\mathcal{O}^i, \mathcal{O})$ and $A_{x_2} \cong A_t(\mathcal{O}^i, -\mathcal{O})$. Finally, to the pair $(\mathcal{O}_{\text{reg}}, \mathcal{O}^{-1}) \in \mathcal{N}_G^F$ is associated the triple $(L, \mathcal{O}, 0, \mathcal{O}) \in \mathcal{M}_G^F$ where $L \subseteq G$ is the Levi complement of some parabolic subgroup of $G$ such that $L/\mathbb{Z}(L)^0$ is simple of type $D_4$, and $(\mathcal{O}, 0, \mathcal{O})$ is the unique cuspidal pair for $L$, see [21, 15.2]. Among the simple summands of $K_{(L,\mathcal{O},\mathcal{O}^0)}$, the character sheaf $A_t(\mathcal{O}_{\text{reg}}, \mathcal{O}^{-1})$ is described by the trivial character of $W_G(L) \cong W(B_3)$. Let $x_0 \in \overline{X}(W)$ be such that $A_{x_0} \cong A_t(\mathcal{O}_{\text{reg}}, \mathcal{O}^{-1})$, and let $\mathcal{F} \subseteq \text{Irr}(W)$ be the family for which $x_0 \in \mathcal{M}(\mathcal{G}, \mathcal{F})$. Then, for $\phi \in \mathcal{F}$, we have

$$\rho_\phi(u_0) = \sum_{x \in \mathcal{M}(\mathcal{G}, \mathcal{F})} \{x_\phi, x\} R_x(u_0) = \{x_\phi, x_0\} R_{x_0}(u_0).$$

Using the notation of [2, p. 482], we obtain

$$\rho_{\phi_{50,3}}(u_0) = -\rho_{\phi_{50,1}}(u_0) = -\rho_{\phi_{21,6}}(u_0) = \frac{1}{2} R_{x_0}(u_0).$$

Now, considering the two characters $\phi_{512,11}, \phi_{512,12} \in \text{Irr}(W)$ of degree 512, we see that

$$\rho_{\phi_{512,11}}(u_0) = [x_{\phi_{512,11}}, x_1] \xi x_{Ax_1}(u_0) + [x_{\phi_{512,11}}, x_2] \xi x_{Ax_2}(u_0) = \xi q^{7/2},$$

$$\rho_{\phi_{512,12}}(u_0) = [x_{\phi_{512,12}}, x_1] \xi x_{Ax_1}(u_0) + [x_{\phi_{512,12}}, x_2] \xi x_{Ax_2}(u_0) = -\xi q^{7/2}.$$
For any \( \phi \in \text{Irr}(W) \) which is not one of \( 1_W = \phi_{1,0}, \phi_{56,3}, \phi_{35,4}, \phi_{21,6}, \phi_{512,11}, \phi_{512,12} \), we have \( \rho_\phi(u_0) = 0 \). We evaluate the left side of (5.3) with \( g = u_0 \) and

\[
w = w_c := s_1 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \in W,
\]
a Coxeter element of \( W \). (In fact any other Coxeter element of \( W \) would lead to the same result, see \([14, 8.2.6]\).) We obtain

\[
q^7 + \frac{1}{2} R_{x_0}(u_0) \left( \text{Tr}(T_{w_c}, V_{\phi_{56,3}}) - \text{Tr}(T_{w_c}, V_{\phi_{35,4}}) - \text{Tr}(T_{w_c}, V_{\phi_{21,6}}) \right) + 2\xi q^7.
\]

Here, \( \text{Tr}(T_{w_c}, V_{\phi_{56,3}}) - \text{Tr}(T_{w_c}, V_{\phi_{35,4}}) - \text{Tr}(T_{w_c}, V_{\phi_{21,6}}) = 2q^5 \), so we have

\[
(5.5) \sum_{\phi \in \text{Irr}(W)} \rho_\phi(u_0) \text{Tr}(T_{w_c}, V_\phi) = q^7 (1 + 2\xi) + q^5 R_{x_0}(u_0).
\]

**Lemma 5.8.** As in 5.7 (and with the notations there), let \( x_0 \in \overline{X}(W) \) be such that \( A_{x_0} \cong A_{(\mathcal{O}_{\text{reg}}, \mathcal{E}^{-1})} \). Then there is a sign \( \delta \in \{\pm 1\} \) such that

\[
R_{x_0}(u_0) = \delta q^2.
\]

Proof. We keep the notation of 5.7 and set \( \iota := (\mathcal{O}_{\text{reg}}, \mathcal{E}^{-1}) \in \mathcal{M}_G^F \). The character sheaf \( A_{x_0} \cong A_t \in \mathcal{G}^\text{uni} \) is an irreducible direct summand of \( K := K_{(L, \mathcal{E}_0, \mathcal{E}_0^\circ)} \), where \( L \subseteq G \) is the Levi complement of some parabolic subgroup of \( G \) such that \( L/Z(L)^\circ \) is simple of type \( D_4 \), and \( (\mathcal{O}_0, \mathcal{E}_0^\circ) \) is the unique cuspidal pair for \( L \). Among the simple summands of \( K, A_t \) is uniquely characterised by the property

\[
A_t|_{G_{\text{uni}}} \equiv \text{IC}(G_{\text{uni}}, \mathcal{E}^{-1})[\dim \mathcal{O}_{\text{reg}} + \dim Z(L)^\circ],
\]

see (4.1). Now consider the functions \( X_t, Y_t \) defined in 4.2. Using Theorem 4.3 and Remark 4.4, we see that

\[
\chi_{A_t, \varphi_{A_t}}(u_0) = -X_t(u_0) = -Y_t(u_0) = -\text{Trace}(\psi_t, u_0, \mathcal{E}_0^\circ)^{-1}.
\]

In order to meet the requirements of Theorem 2.4 (see \([26, 25.1]\)), we need to modify the isomorphism \( \varphi_{A_t} : F^* A_t \sim A_t \), as follows. Let \( \Sigma_0 := Z(L)^\circ \mathcal{O}_0 \). By \([25, 19.3]\), the image of \( \Sigma_0 \) (or \( \mathcal{O}_0 \)) under the canonical map \( L \rightarrow L/Z(L)^\circ \) is the regular unipotent class of \( L/Z(L)^\circ \). As in \([11, 4.4]\), we may single out a specific element \( g_1 \in (\Sigma_0/Z(L)^\circ)^F \). Following \([27, 3.2]\), the choice of \( g_1 \) then gives rise to an isomorphism \( \varphi_0 : F^* \mathcal{O}_0 \sim \mathcal{O}_0 \) by requiring that it induces on the stalk of \( \mathcal{E}_0^\circ \) at \( g_1 \) (or, more precisely, at the preimage of \( g_1 \) under \( L \rightarrow L/Z(L)^\circ \)) the map given by scalar multiplication with \( q^{(\dim (L/Z(L)^\circ)) - \dim \mathcal{O}_0^\circ/2} = q^2 \).

Once \( \varphi_0 \) is fixed, it naturally induces an isomorphism \( F^* K \sim K \), which in turn determines an isomorphism \( \varphi_{A_{x_0}} : F^* A_{x_0} \sim A_{x_0} \), see again \([27, 3.2]\). Then \( \varphi_{A_{x_0}} \) satisfies the requirements formulated in \([26, 25.1]\), so it is a valid choice for Theorem 2.4. In particular, we have

\[
\langle \chi_{A_{x_0}, \varphi_{A_{x_0}}}, X A_{x_0}, \varphi_{A_{x_0}} \rangle = 1.
\]

On the other hand, recall from 4.2 that the definition of \( \varphi_{A_t} : F^* A_t \sim A_t \) is based on the choice of an isomorphism \( F^* \mathcal{E}_0 \sim \mathcal{E}_0 \) which induces a map of finite order on the stalk of \( \mathcal{E}_0^\circ \) at any element of \( \mathcal{O}_0^\circ \). Modifying this isomorphism by multiplication with a root of unity, if necessary, we may thus achieve that \( \varphi_{A_{x_0}} = q^2 \cdot \varphi_{A_t} \). Now, by Theorem 3.3, we have

\[
R_{x_0} = \xi_{x_0} X A_{x_0}, \varphi_{A_{x_0}}
\]
for some \( \xi_{x_0} \in \overline{Q}_\ell \) such that \( \xi_{x_0} \overline{\xi}_{x_0} = 1 \). So we obtain
\[
R_{\xi_{x_0}}(u_0) = \xi_{x_0} \chi_{A_{x_0}, \varphi_{A_{x_0}}}(u_0) = \xi_{x_0} q^2 \cdot \chi_{A_{x_0}, \varphi_{A_{x_0}}}(u_0) = -\xi_{x_0} q^2 \cdot \text{Trace}(\psi_{t, u_0}, \epsilon_{u_0}^{-1}) = \delta q^2,
\]
where \( \delta := -\xi_{x_0} \text{Trace}(\psi_{t, u_0}, \epsilon_{u_0}^{-1}) \in \overline{Q}_\ell \). Since the local system \( \epsilon_{u_0}^{-1} \) is one-dimensional and \( \psi_{t, u_0} : \epsilon_{u_0}^{-1} \rightarrow \epsilon_{u_0}^{-1} \) has finite order, the number \( \text{Trace}(\psi_{t, u_0}, \epsilon_{u_0}^{-1}) \) is a root of unity in \( \overline{Q}_\ell \), so \( \delta \) satisfies \( \delta \bar{\delta} = 1 \), that is, \( \bar{\delta} = \delta^{-1} \). Using Lemma 5.3, we conclude
\[
\delta q^2 = R_{\xi_{x_0}}(u_0) = R_{\xi_{x_0}}(u_0^{-1}) = \bar{\delta} q^2,
\]
so we must have \( \delta = \bar{\delta} = \delta^{-1} \in \{ \pm 1 \} \), as desired. \( \square \)

5.9. In view of Lemma 5.8, (5.3) and (5.5), we have
\[
\frac{|O_{u_0} \cap B^F \hat{w}_c B^F|}{|B^F|} = q^7 (1 + 2\xi + \delta),
\]
where \( \xi \in \{ \pm 1 \} \) is as in 5.5, \( \delta \in \{ \pm 1 \} \) is as in Lemma 5.8, and \( \hat{w}_c \in N_{G}(T)^F \) is a representative of \( w_c \in W \). We claim that \( O_{u_0} \cap B^F \hat{w}_c B^F \neq \emptyset \). Indeed, with the notation of 3.5, consider the longest element \( w_0 \) of \( W \) with respect to the usual length function on \( W \) determined by the simple roots \( \Pi = \{ \alpha_1, \ldots, \alpha_7 \} \subseteq R^+ \). \( W \) is a Coxeter group with generators \( \{ s_1, \ldots, s_7 \} \), and there is a natural action of \( \hat{W} \) on the roots \( R \). Using some standard properties of finite Coxeter groups (see, e.g., [48, Appendix]), the element \( w_0 \) is then characterised by the property \( w_0(\Pi) = -R^+ \), so we have \( -w_0(\Pi) = \Pi \). More precisely, \( -w_0 \) defines a graph automorphism of the Dynkin diagram of \( E_7 \), but the only such automorphism is the identity, so \( -w_0(\alpha_i) = \alpha_i \) for \( 1 \leq i \leq 7 \). Let us choose a representative \( \hat{w}_0 \in N_{G}(T)^F \) of \( w_0 \in W \). For \( 1 \leq i \leq 7 \), we have \( \hat{w}_0 U_{c_i} \hat{w}_0^{-1} = U_{-c_i} \), so \( \hat{w}_0 u_i(1) \hat{w}_0^{-1} \in U_{-c_i} \). Now
\[
U_{-c_i} \subseteq L_{\{ s_i \}} \subseteq P_{\{ s_i \}} = B \cup B_{s_i} B
\]
(see, e.g., [2, 2.6.2]). Since \( U_{-c_i} \cap B = \{1\} \), we must have \( \hat{w}_0 u_i(1) \hat{w}_0^{-1} \in B_{s_i} B \). As \( w_c = s_1 \cdots s_7 \) is a reduced expression for \( w_c \), we know that
\[
B_{s_1} B \cdot B_{s_2} B \cdots B_{s_7} B = B w_c B,
\]
and so
\[
\hat{w}_0 \cdot u_0 \cdot \hat{w}_0^{-1} \in O_{u_0} \cap B w_c B = O_{u_0} \cap B^F \hat{w}_c B^F.
\]
Hence, the left side of (5.6) is strictly positive. Thus (5.6) implies that \( \xi \neq -1 \), so we must have \( \xi = +1 \). This proves Proposition 5.2.

Remark 5.10. As described in 5.1, the isomorphisms \( \varphi_A : F^* A_i \sim A_i \) \( (i = 1, 2) \) are determined by choosing one of the four \( G^F \)-conjugacy classes inside \( G_{\text{reg}}^F \). The proof of Proposition 5.2 relies on the following two properties of the \( G^F \)-conjugacy class \( O_{u_0} \) of \( u_0 \):

(1) Any element of \( O_{u_0} \) is \( G^F \)-conjugate to its inverse, that is, \( O_{u_0} = O_{u_0}^{-1} \).
(2) We have \( O_{u_0} \cap B^F \hat{w}_c B^F \neq \emptyset \).

Once \( \xi \) is specified, a similar computation as in 5.9 shows that \( O_{\hat{w}_0^{-1}} \cap B^F \hat{w}_c B^F = \emptyset \). Hence,
among the four $G^F$-classes inside $O^F_{\text{reg}}$, the class $O_{u_0}$ is uniquely determined by (1) and (2) above.

5.11. We conclude by giving the values of the (unipotent) characters in the family

$$\mathcal{F}_0 = \{\rho_{\phi_{512,11}}, \rho_{\phi_{512,12}}, E_7[\zeta], E_7[-\zeta]\} \subseteq \overline{X}(W)$$

at the regular unipotent elements. Using (3.5), 5.6 and 5.2, we have for $u \in O_{\text{reg}}^F$:

$$\rho_{\phi_{512,11}}(u) = -\rho_{\phi_{512,12}}(u) = (\chi_{A_1, \varphi_{A_1}}(u) + \chi_{A_2, \varphi_{A_2}}(u))/2;$$

$$E_7[\zeta](u) = -E_7[-\zeta](u) = -(\chi_{A_1, \varphi_{A_1}}(u) - \chi_{A_2, \varphi_{A_2}}(u))/2.$$

The values of the $\chi_{A_i, \varphi_{A_i}}$ are given at the end of 5.1. We thus get the following table, where $u \sim u_a$ means that $u$ lies in the $G^F$-conjugacy class corresponding to $a \in A(u_0)$ as described in 5.1.

<table>
<thead>
<tr>
<th>$u \sim u_0$</th>
<th>$u \sim u_{a_0}$</th>
<th>$u \sim u_{a_0^2}$</th>
<th>$u \sim u_{a_0^3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{\phi_{512,11}}(u)$</td>
<td>$q^{1/2}$</td>
<td>0</td>
<td>$-q^{1/2}$</td>
</tr>
<tr>
<td>$\rho_{\phi_{512,12}}(u)$</td>
<td>$-q^{7/2}$</td>
<td>0</td>
<td>$q^{7/2}$</td>
</tr>
<tr>
<td>$E_7<a href="u">\zeta</a>$</td>
<td>0</td>
<td>$-q^{1/2} \zeta$</td>
<td>0</td>
</tr>
<tr>
<td>$E_7<a href="u">-\zeta</a>$</td>
<td>0</td>
<td>$q^{7/2} \zeta$</td>
<td>0</td>
</tr>
</tbody>
</table>

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Note added in proof. In the meantime, the remaining open problems for type $F_4$ and $E_6$, as well as $E_7$ in characteristic $p > 2$, have been solved in a preprint by Geck, see arXiv:2105.00722

References


IDSR - Lehrstuhl für Algebra
Universität Stuttgart
Pfaffenwaldring 57, D–70569 Stuttgart
Germany
e-mail: jonas.hetz@mathematik.uni-stuttgart.de