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On the Structure of S Space

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§ 1. Introduction.

1. The theory of Fourier transform constitutes an important method in investigating many problems in mathematical analysis and differential equations. In classical theory of the Fourier transform the applicability of this method has been limited by some restrictions on the behaviour at infinity of the functions to be transformed.

L. Schwartz in his book [1], where he introduced the notion of the function space S and treated the Fourier transform as a functional on S, built up the theory of the Fourier transform of the functions which increase not more rapidly than some powers of the independent variables.

Further, Gelfand and Sylov [2] introduced the notion of the subspace $Z=S^0$ of the space S and treated the Fourier transform of function which increases with arbitrary rapidity as a functional on the space Z. (The research of the Fourier transform based on the same idea was made independently by L. Ehrenpreis [3].)

Gelfand and Sylov [2, 4, 5] introduced at the same time the notion of the "*S type*" spaces S_α, S^β, S_α^β, $S_{\alpha,A}$, $S^{\beta,B}$, $S_{\alpha,A}^{\beta,B}$ which are the subspaces of the space S, and investigated the structure of the space S in detail.

To study the structure of the space S as this is important from two points of view.

First we can treat the Fourier or Laplace transform of functions or functionals not only without restriction on their behaviours at infinity but also with less restrictions on other sort of their behaviours.

In the second place we can investigate in detail the natures of the dual spaces of S *type* spaces or their Fourier transforms by using the structure of S spaces.

Thus we can construct the theory of Fourier (§ 5) or Laplace [6, 7] transform in more general form, and apply more extensively the theory of Fourier or Laplace transform to mathematical analysis or differential equations [8].

2. Gelfand and Sylov define those "*S type*" spaces from the standpoints of order of growth of functions and their derivatives. They state also that S_α, S^β, S *can be considered as the limit space of* S_α^β, *i.e.*, $S_\alpha=S_\alpha^\infty$, $S^\beta=S_\infty^\beta$, $S=S_\infty^\infty$, without giving precise definitions of S_α^∞, S_∞^β, and S_∞^∞.

To define S_α^∞, S_β^∞ and S_∞^∞, first we notice the inclusion relations of S_α^β.

$$\text{If } \alpha<\alpha',\ \beta<\beta', \quad \text{then } S_\alpha^\beta \begin{matrix} \subset S_{\alpha'}^\beta \subset S^\beta \subset \\ \subset S_\alpha^{\beta'} \subset S_\alpha \subset \end{matrix} S.$$

Then the natural definitions of the spaces S_α^∞, S_∞^β, S_∞^∞ are as follows:

$$S_\alpha^\infty = \bigcup_\beta S_\alpha^\beta, \quad S_\infty^\beta = \bigcup_\alpha S_\alpha^\beta, \quad S_\infty = \bigcup_\alpha S_\alpha, \quad S^\infty = \bigcup_\beta S^\beta, \quad S_\infty^\infty = \bigcup_{\alpha,\beta} S_\alpha^\beta.$$

If the equalities $S_\alpha=S_\alpha^\infty$, $S^\beta=S_\infty^\beta$, $S=S_\infty^\infty$ were true in this sense of definitions of S_α^∞, S_∞^β and S_∞^∞, then those subspaces S_α^β would give classifications of the space S_α, S^β, or S, and the order of growth* of functions and their derivatives would become characteristic properties of functions to be the elements of the classes in S.

In the present paper we show first (§ 3) that there exist some classes of functions in the space S whose orders of growth* are so rapid that the functions can not be contained in any of those subspaces, which contradicts $S=S_\infty$.

In the second place, we show (§ 4) that there exist functions which can not be bounded by any order of growth* (which contradicts $S_\alpha=S_\alpha^\infty$, $S^\beta=S_\infty^\beta$, $S=S_\infty^\infty$). This fact means that the countable classification of the space S, S_α or S^β by the order of growth of functions and their derivatives is impossible (Theorem 5).

In the third place, using these facts, we show (§ 5) that we can construct some Fourier invariant functional spaces which include the space S' and are larger than S'. The existence of these spaces is contradictory to L. Ehrenpreis' assertion [3] that S' is the furthest Fourier invariant functional space.

§ 2. Notions and properties of S type spaces.

1. Here we cite notions and properties of S type spaces mainly from Gelfand and Sylov's works [2, 4, 5].

We consider the space S of infinitely differentiable functions $\varphi(x)$ which satisfy the condition: $x^k\varphi^{(q)}(x)$ converges to 0 when $|x|\to\infty$ for any non-negative integers q and k.

The neighborhood $V(m, k, \varepsilon)$ of zero in the space S is determined by integers m and k and a positive number ε and consists of all functions of S which satisfy the inequality $(1+x^2)^k|\varphi^{(q)}(x)|\leq\varepsilon$ for all $q\leq m$.

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* Any function $\varphi(x)$ in S satisfies $|x^k\varphi^{(q)}(x)|\leq\bar{m}_{k,q}$ for some sequence of numbers $\{\bar{m}_{k,q}\}$ $(k, q=0, 1, 2, \cdots)$. So the sequence $\{m_{k,q}=\inf \bar{m}_{k,q}\,|\,|x^k\varphi^{(q)}|\leq\bar{m}_{k,q}\}(k, q=0, 1, 2, \cdots)$ corresponds to the function $\varphi(x)$. We call the order of growth of $m_{k,q}$ (as a function of k, q) of φ simply by the order of growth of the function φ and its derivatives.

2. Let $\alpha \geq 0$, and S_α denote the subspace of S whose elements satisfy the inequality $|x^k \varphi^{(q)}(x)| \leq C_q A^\alpha k^{k\alpha}$ for some $A \geq 0$ and for $k, q = 0, 1, 2, \cdots$, where the constant C_q depends on φ and q and is independent of k and α.

Let $S_{\alpha,A}$ denote the subspace of S_α whose elements satisfy the following inequality for any arbitrary positive number δ:

$$|x^k \varphi^{(q)}(x)| \leq C_{q\delta}(A+\delta)^k k^{k\alpha}.$$

According to Gelfand and Sylov [5], this inequality is equivalent to the following inequality: $|\varphi^{(q)}(x)| \leq C'_{q\delta} \exp(-a|x|^{1/\alpha})$ where $a = \alpha/e(A+\delta)^{1/\alpha}$ and $C'_{q\delta} = C_{q\delta} \cdot e^{\alpha e/2}$.

Thus the space S_α, or $S_{\alpha,A}$ consists of functions which are, together with each of their derivatives, of orders of exponential decrease $1/\alpha$ with type a.

The space S_α is the union of the space $S_{\alpha,A}$: $S_\alpha = \bigcup_A S_{\alpha,A}$. For $\alpha = 0$, the space $S_{\alpha,A} = S_{0,A}$ consists of C_0^∞ functions with carriers in the interval $|x| \leq A$.

The topology of the space $S_{\alpha,A}$ is given by the countable system of norms: $||\varphi||_{q\delta} = \sup_{x,k} |x^k \varphi^{(q)}(x)| / (A+\delta)^k k^{k\alpha}$ $(q = 0, 1, 2, \cdots, \ \delta = 1, 1/2, 1/3 \cdots)$. By this topology the space S_{α,A_1} becomes Montel space.

If $A_1 < A_2$, the space S_{α,A_1} is a subspace of the space S_{α,A_2}, and if the sequence $\{\varphi_\nu(x)\}$ converges in the space S_{α,A_1}, then $\{\varphi_\nu(x)\}$ converges also in the space S_{α,A_2}.

We define the convergence in the space S_α as follows: the sequence $\{\varphi_\nu(x)\}$ converges to 0 in S_α if and only if the sequence $\{\varphi_\nu(x)\}$ belongs to a subspace $S_{\alpha,A}$ and $\{\varphi_\nu(x)\}$ converges to 0 in the space $S_{\alpha,A}$.

3. Let $\beta \geq 0$, and S^β denote the subspace of S whose elements satisfy the inequality $|x^k \varphi^{(q)}(x)| \leq C_k B^q q^{\beta q}$ for some $B \geq 0$ and for $k, q = 0, 1, 2, \cdots$, where the constant C_k depends on φ and k and is independent of q and β.

Let $S^{\beta,B}$ denote the subspace of S^β whose elements satisfy the following inequality for any arbitrary positive number ρ:

$$|x^k \varphi^{(q)}(x)| \leq C_{k\rho}(B+\rho)^q q^{q\beta}.$$

The topology of the space $S^{\beta,B}$ is given by the following countable set of norms: $||\varphi||_{k\rho} = \sup_{x,q} |x^k \varphi^{(q)}| / (B+\rho)^q q^{q\beta}$ $(k = 0, 1, 2, \cdots, \ \rho = 1, 1/2 \cdots)$.

By this topology the space $S^{\beta,B}$ becomes complete Montal space. If $B_1 < B_2$, the space S^{β,B_1} is a subspace of the space S^{β,B_2}, and if $\{\varphi_\nu(x)\}$ converges in the space S^{β,B_1}, then it converges in the space S^{β,B_2}.

The space S^β is the union of the space $S^{\beta,B}$: $S^\beta = \bigcup_B S^{\beta,B}$. The topology of the space S^{β} is defined as follows: the sequence $\{\varphi_{\nu}(x)\}$ converges to 0 in S^{β} if and only if all $\varphi_{\nu}(x)$ belong to some subspace $S^{\beta,B}$ and $\{\varphi_{\nu}(x)\}$ converges to 0 in the space $S^{\beta,B}$.

According to Gelfand and Sylov [5], the following structure of S^{β} is shown:

For $0\leq\beta<1$, S^{β} consists of the functions $\varphi(x)$ which satisfy the following conditions:

1) $\varphi(x)$ can be continued analytically for all complex values of the arguments $z=x+iy$.
2) The resulting integral functions $\varphi(z)$ have orders of growth not greater than $1/(1-\beta)$ with type b, i.e., $\varphi(z)$ satisfy the following inequalities: $|x^{k}\varphi(x+iy)|\leq C_{k}' \text{ex} \{b|y|^{1/(1-\beta)}\}$, if $\varphi(x)$ satisfy the inequality $|x^{k}\varphi^{(q)}(x)|\leq C_{k}B^{q}$, where $b=(1-\beta)(Be)^{1/1-\beta}/e$.

For $\beta=1$ it is known also that S^{β} consists of quasianalytic function.

4. Let $\alpha\geq 0$, $\beta\geq 0$ and S_{α} denote the subspace of S whose elements satisfy the inequality $|x^{k}\varphi^{(q)}(x)|\leq CA^{k}B^{q}k^{k\alpha}q^{q\beta}$ for some $A\geq 0$, $B\geq 0$, $C\geq 0$ and for k, $q=0, 1, 2, \cdots$, where the constant C depends on φ.

Let $S_{\alpha,A}^{\beta,B}$ denote the subspace of S_{α}^{β} whose elements satisfy the following inequality for any positive number δ, ρ;

$$|x^{k}\varphi^{(q)}(x)|\leq C_{\delta\rho}(A+\delta)^{k}k^{k\alpha}(B+\rho)^{q}q^{q\beta}.$$

We introduce the system of countable norms in the space $S_{\alpha,A}^{\beta,B}$ as follows: $||\varphi||_{\delta\rho}=\sup_{x,k,q}|x^{k}\varphi^{(q)}(x)|/(A+\delta)^{k}(B+\rho)^{q}k^{k\alpha}q^{q\beta}$ (δ, $\rho=1, 1/2, \cdots$).

By this set of norms $S_{\alpha,A}^{\beta,B}$ becomes complete Montal space [5].

If $A_{1}<A_{2}$, $B_{1}<B_{2}$, then the space $S_{\alpha,A_{1}}^{\beta,B_{1}}$ is a subspace of $S_{\alpha,A_{2}}^{\beta,B_{2}}$, and if the sequence $\{\varphi_{\nu}(x)\}$ converges in the space $S_{\alpha,A_{1}}^{\beta,b_{1}}$, then it converges in the space $S_{\alpha,A_{2}}^{\beta,B_{2}}$.

The space S_{α}^{β} is the union of the space $S_{\alpha,A}^{\beta,B}$, i.e., $S_{\alpha}^{\beta}=\bigcup_{A,B}S_{\alpha,A}^{\beta,B}$. The togology of the space S_{α}^{β} is given as follows: the sequence $\{\varphi_{\nu}(x)\}$ converges to 0 in S_{α}^{β} if and only if all $\varphi_{\nu}(x)$ belong to some subspace $S_{\alpha,A}^{\beta,B}$, and the sequence $\{\varphi_{\nu}(x)\}$ converges to 0 in the space $S_{\alpha,A}^{\beta,B}$.

The space S_{α}^{β} is included in the space S_{α} and S^{β}. If $\varphi(x)\in S_{\alpha}$ satisfies the inequality $|x^{k}\varphi^{(q)}(x)|\leq CA^{k}B^{q}k^{k\alpha}q^{q\beta}$, then $\varphi(x)$ satisfies the inequality $|\varphi^{(q)}(x)|\leq C_{1}B^{q}q^{q\beta}\exp(-a|x|^{1/\alpha})$. Moreover, for $\beta\leq 1$ any function $\varphi(x)\in S_{\alpha}^{\beta}$ can be uniquely continued for all complex values of the arguments $z=x+iy$, and for $\beta<1$, $\varphi(z)$ satisfies the following inequality:

$$|\varphi(x+iy)|\leq C_{2}\exp\{-a|x|^{1/\alpha}+b|y|^{1/(1-\beta)}\}.$$

For $\alpha+\beta<1$, S_{α} consists of only trivial function $\varphi(x)\equiv 0$. S_{0}^{1} and S_{1}^{0} also contain only trivial function. It is shown by Gelfand and Sylov that the following spaces have non trivial functions:

1) $S_{\alpha,A}$, $S^{\beta,B}$ with arbitrary α, β, A, B. (except $S_{0,0}$, $S^{0,0}$);

2) $S_{\alpha,A}^{0,B}$, $S_{0,A}^{\beta,B}$ with any $\alpha>1$, $\beta>1$, A, B, (except $S_{\alpha,A}^{0,0}$, $S_{0,0}^{\beta,B}$);

3) $S_{\alpha,A}^{\beta,B}$ with $\alpha+\beta>1$, A, B;

4) $S_{\alpha,A}^{\beta,B}$ with $\alpha+\beta=1$, $AB\geq\gamma$, where γ is arbitrary positive number.

(The above exceptions in 1) and 2) are not stated in [5]).

According to Gelfand and Sylov [5], the Fourier transform~of these spaces are as follows: $\tilde{S}_\alpha=S^\alpha$, $\tilde{S}^\beta=S_\beta$, $\tilde{S}_\alpha^\beta=S_\beta^\alpha$, and the topology is preserved under these transformations. For example if φ_ν converges in S_α then the Fourier transform $\tilde{\varphi}_\nu$ converges in S^α.

§ 3. Some subspaces which are not included in S_∞, S^∞.

1. As stated in § 2, $\varphi(x)\in S_\alpha$ is a decreasing function of exponential order $1/\alpha$. We shall show here that there exist functions whose orders of decrease are less than any $1/\alpha$, (i.e., 0) and decrease faster than $1/|x|^n$ for any integer n.

These functions belong to the space S, but not to the space S_∞. Thus we shall see $S\neq S_\infty$ and $S\neq S_\infty^\infty$.

Lemma 1. *Let f_0 be the following function:*

$$f_0(x) = \begin{cases} \exp\{-(\log|x|)^\gamma\} & \textit{for} \quad |x|>e\,, \\ 1/e & \textit{for} \quad |x|\leq e\,, \end{cases}$$

where γ is an arbitrary real number which is larger than 1.

Then the function f_0 is continuous in $(-\infty, \infty)$ and has following properties:

(1) *f_0 is a rapidly decreasing function, i.e., for any positive integer n,* $\lim_{x\to\pm\infty}|x^n f_0(x)|=0$

(2) *The order of decreasing, as an exponential type function, is* 0, *i.e.,*

$$\lim_{x\to\pm\infty}\log|\log f_0(x)|/\log|x|=0\,.$$

Proof. For $|x|>e$, we see that $\log|\log f_0(x)|=\gamma\log\log|x|$.

So, for any positive integer n, there exists x_0 such that, if $|x|>|x_0|$, then

$$\log(-\log f_0(x))>\log(n+1)+\log\log|x|=\log\{(n+1)\log|x|\}\,.$$

Hence $-\log f_0(x)>(n+1)\log|x|$, so $|f_0(x)|<|x|^{-(n+1)}$ for $|x|>|x_0|$.

We also see that

$$\lim_{|x|\to\infty}\frac{\log|\log f_0(x)|}{\log|x|}=\lim_{x\to\pm\infty}\frac{\gamma\log\log|x|}{\log|x|}=0\,.$$

We regularize this function f_0, using the following function $\rho_\varepsilon(x)$. Let $\rho_\varepsilon(x)$ be the infinitely differentiable function such that

$$\rho_\varepsilon(x) = \begin{cases} 0 & \text{for } |x| \geq \varepsilon, \\ \dfrac{k}{\varepsilon} \exp\left(\dfrac{-\varepsilon^2}{\varepsilon^2 - x^2}\right) & \text{for } |x| < \varepsilon, \end{cases}$$

where the constant k satisfies $k \int_{-1}^{+1} \exp\left(\frac{-1}{1-x^2}\right) dx = 1$.

Then we can see the following

Lemma 2. *The infinitely differentiable function $f_0(x)*\rho_\varepsilon(x)$ has property* (1) *in Lemma* 1, *together with each of its derivatives. $f(x)*\rho_\varepsilon(x)$ has also property* (2).

Proof. $|(f_0*\rho_\varepsilon)^{(q)}(x)| = |\int_{-\infty}^{+\infty} f_0(x-\xi)\rho_\varepsilon^{(q)}(\xi)\,d\xi|$

$$\leq \int_{-\varepsilon}^{+\varepsilon} |f_0(x-\xi)\rho_\varepsilon^{(q)}(\xi)|\,d\xi \leq \underset{\xi\in[-\varepsilon,+\varepsilon]}{\operatorname{Max}} |f_0(x-\xi)| \cdot \int_{-\varepsilon}^{+\varepsilon} |\rho_\varepsilon^{(q)}(\xi)|\,d\xi$$

$$= \begin{cases} a_q f_0(x-\varepsilon) & \text{for } x > \varepsilon \\ a_q f_0(x+\varepsilon) & \text{for } x < -\varepsilon, \end{cases} \quad \text{where } a_q = \int_{-\varepsilon}^{+\varepsilon} |\rho_\varepsilon^{(q)}(\xi)|\,d\xi.$$

So the function $(f_0*\rho_\varepsilon)^{(q)}(x)$ has property (1) for any non negative integer q.

On the other hand, since $\rho_\varepsilon(\xi) \geq 0$,

$$f_0*\rho_\varepsilon(x) = \int_{-\varepsilon}^{+\varepsilon} f_0(x-\xi)\rho_\varepsilon(\xi)\,d\xi \geq \underset{\xi\in[-\varepsilon,+\varepsilon]}{\operatorname{Min}} f_0(x-\xi) \int_{-\varepsilon}^{+\varepsilon} \rho(_\varepsilon\xi)\,d\xi$$

$$= \begin{cases} f_0(x+\varepsilon) & \text{for } x > 0. \\ f_0(x-\varepsilon) & \text{for } x < 0. \end{cases}$$

So the behaviour of the function $f_0*\rho_\varepsilon$ for $x \to \pm\infty$ is the same as that of the function f_0, and $f_0*\rho_\varepsilon$ has property (2).

Theorem 1. *There exists a function in the space S, which does not belong to the space S_α for any $\alpha \geq 0$.*

Proof. $f_0*\rho_\varepsilon \in S$, since $(f_0*\rho_\varepsilon)^{(q)}(x)$ satisfies the condition (1). On the other hand, as stated in § 2, an C^∞ function φ belongs to the space S_α if and only if φ satisfies the following inequality: $|\varphi^{(q)}(x)| < C_q \exp\{-a|x|^{1/\alpha}\}$. The function $f_0*\rho_\varepsilon$ can not satisfy this condition for $q=0$, since the condition (2) holds for $f_0*\rho_\varepsilon$.

Corollary. $S \supsetneqq S_\infty$, $S \supsetneqq S^\infty$, $S \supsetneqq S_\infty^\infty$.

Proof. The first relation is a direct consequence of Theorem 1. The second relation is obtained from the first relation by Fourier transforms. The third relation is a direct consequence of $S \supset S_{\infty} \supset S_{\infty}^{\infty}$.

2. Let $S_{\log,\gamma}$ denote the subspace of S which consists of all functions φ of S that satisfy the inequality $|\varphi^{(q)}(x)| \leq C_q \exp\{-a \log^{\gamma}|x|\}$ for some positive constant C_q, a. Then $S_{\log,\gamma}$ is a new class which is not included in any S_{α}. The function $f_0 * \rho_{\varepsilon}$ belongs to this space $S_{\log,\gamma}$.

We can introduce also another types of subspaces, replacing the order $A^k k^{k\alpha}$ or $B^q q^{q\beta}$ by some increasing functions of k or q.

For example, let $S_{\bar{\gamma}}$ denote the subspace of S which consists of all functions φ of S that satisfy the inequality $|x^k \varphi^{(q)}(x)| \leq C_q A^k k^{k^{\gamma}}$.

Then for $\gamma > 1$, $S_{\bar{\gamma}}$ is also a new class which is not included in any S_{α}.

There are some mutual relations between these two sorts of subspaces. For example the space $S_{\log,2}$ is included in the space $S_{\bar{2}}$, since the following evaluation holds.

We put $|x|/|A| = \xi$ and $k^{k^2}/\xi^k = f(k)$. Then $(\log f)' = 2k \log k + k - \log \xi$, and $(\log f)'' = 2 \log k + 3 > 0$ for $k \geq 1$. So $\operatorname*{Min}_{k \leq 1, k:\text{real}} f(k) = f(k_0)$ exists and k_0 satisfies $2k_0 \log k_0 + k_0 - \log \xi = 0$ for any fixed ξ. Hence ξ is a monotone increasing function of k_0 and grows to infinity if k_0 grows to infinity. On the other hand, we have

$$\begin{aligned} \rho &\equiv \log(C_q e^{-a \log^2|x|}) = \log C_q - a \log^2|x| \\ &= \log C_q - a\ (\log \xi + \log|A|)^2 \\ &= \log C_q - a\ (2k_0 \log k_0 + k_0 + \log|A|)^2 . \end{aligned}$$

Since $a > 0$, there exists k_0' such that, for any $k_0 > k_0'$, holds $\rho < -k_0^2 \log k_0 - k_0^2$. For k_0 which satisfies $1 \leq k_0 \leq k_0'$, we can find sufficiently large number C_q' such that the following inequality holds:

$$\rho < \log C_q' - k_0^2 \log k_0 - k_0^2 = \log\{C_q' f(k_0)\} .$$

So we see that there exist positive numbers C_q'' and C_q''' such that

$$\begin{aligned} &C_q \exp\{-a \log^2|x|\} < C_q'' \operatorname*{Min}_{k \geq 1, k:\text{real}} f(k) \\ &\leq C_q'' \operatorname*{Min}_{k \geq 1, k:\text{integer}} f(k) \leq C_q''' \operatorname*{Min}_{k \geq 0, k:\text{integer}} \frac{A^k k^{k^2}}{|x|^k} . \end{aligned}$$

This inequality shows $S_{\log,2} \subset S_{\bar{2}}$.

§ 4. Impossibility of countable classification.

1. We show here there exists a function in the space S whose order of growth is larger than any given set of numbers m_{kq} (k, $q=0, 1, 2, \cdots$).

Lemma 3. *For any set of positive numbers* $\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{k-1}, b_k$ *and* σ_k, *there exists a function* $\varphi_k(x) \in S_0$ *such that* (1) *the length of carrier of* φ_k *is less than* σ_k, (2) $|\varphi_k^{(r)}(x)| \leq \varepsilon_r$ *for* $r=0, 1, \cdots, k-1$, *and* (3) $\operatorname*{Max}_x |\varphi_k^{(k)}(x)| \geq b_k$.

Proof. We construct inductively the required functions. For $k=1$ its construction is easy.

Now let $\varepsilon_0, \cdots, \varepsilon_{k-2}, \varepsilon_{k-1}, b_k$ and σ_k be a set of given numbers. Assume by way of induction that there exists a function φ_{k-1} which satisfies $|\varphi_{k-1}^{(r)}| \leq \varepsilon_{r+1}$ ($r=0, 1, \cdots, k-2$), $\operatorname*{Max}_x |\varphi_{k-1}^{(k-1)}(x)| \geq b_k$, and whose carrier is contained in $[0, \sigma_k/2]$.

Let $f_1(x)=\varphi_{k-1}(x)-\varphi_{k-1}(\sigma_k-x)$. Then $f_1(x)$ is of class C_0^∞ and satisfies $\int_0^{\sigma_k} f_1(x)dx=0$.

Let $f_2(x)=\int_0^x f_1(\xi)\,d\xi$. Then $f_2(x)$ is also of class C_0^∞ and $|f_2^{(r)}(x)| = |f_1^{(r-1)}(x)| = |\varphi_{k-1}^{(r-1)}(x)+(-1)^r\varphi_{k-1}^{(r-1)}(\sigma_k-x)|$.

If $|f_2(x)| \leq \varepsilon_0$, we put $\varphi_k(x) \equiv f_2(x)$. Then $\varphi_k(x)$ satisfies all the required condition.

If $\operatorname*{Max}_x |f_2(x)| > \varepsilon_0$, we put $f_3(x)=\varepsilon_0 f_2(x)/\operatorname*{Max}_x |f_2(x)|$, and $\alpha=\varepsilon_0/\operatorname{Max}|f_2(x)|$. Then obviously $|f_3(x)| \leq \varepsilon$, and $\alpha<1$. Now we take number λ that satisfies inequality $\lambda^{k+1} \geq 1/\alpha \geq \lambda^k > 1$, and put $\varphi_k(x) \equiv f_3(\lambda x)$.

Then we can see $\varphi_k^{(r)}(x)=\lambda^r f_3^{(r)}(\lambda x)=\lambda^r\alpha\cdot f_2^{(r)}(\lambda x)$. So,

$$\varphi_k^{(r)}(x) \begin{cases} \leq \varepsilon_0 & \textit{for} \quad r=0\,, \\ \leq \lambda^r\alpha\varepsilon_r \leq \varepsilon_r & \textit{for} \quad r=1, \cdots, k-1\,, \\ \geq \lambda^k \alpha b_k \geq b_k & \textit{for} \quad r=k\,. \end{cases}$$

For the sake of simplicity we have constructed here the required function $\varphi_k(x)$ on $[0, \sigma_k]$. But this process does not depend on the interval and we can construct $\varphi_k(x)$ on any required intervai (of length σ_k).

Theorem 2. *There exists a function* $\varphi(x)$ *such that* $\varphi(x) \in S_0$ *and* $\varphi(x) \overline{\in} S^\infty$. *There exists also a function* $\psi(x)$ *such that* $\psi(x) \in S^0$ *and* $\psi(x) \overline{\in} S_\infty$.

Proof. We use the sequence of intervals $\{I_q\}$;

$$I_q = \{x \mid 3/2^{q+2} \leq x \leq 1/2^q\}\,,$$

and define the sequence of functions $\{\varphi_q(x)\}$ as follows, We take a function $\varphi_0(x)\in C_0^\infty$ with carrier in I_0 which satisfies the inequality $|\varphi_0(x)|\leq 1$. Next taking $\operatorname{Max}|\varphi_0(x)|/2^2$ as ε_0, e as b_1 in Lemma 3, we construct $\varphi_1(x)$ which satisfies the conditions of Lemma 3 and whose carrier is contained in I_1.

Similarly we can define sucessesively $\varphi_q(x)$ $(q=2, 3, 4, \cdots)$, using $\varphi_r(x)$ $(r=0, 1, 2, \cdots, q-1)$ which satisfies following three conditions stated in Lemma 3.

(1) the carrier of $\varphi_r(x)$ is contained in I_r,

(2) we take $\{e^{k^2}\}$ as $\{b_k\}$ in Lemma 3,

(3) we take $\{\varepsilon_{q,r}\}=\operatorname{Max}_x|\varphi_r^{(r)}(x)|\cdot 2^{2(r-q)}$, for $0\leq r<q$.

Now the series $\sum_{q=0}^{\infty}\varphi_q(x)$ converges and defines $\varphi(x)$.

$\varphi(x)$ is obviously infinitely differentiable, in any interval I_r. At the origin $\varphi(x)$ is also infinitely differentiable, since for $x\geq 0$, and $q\geq r$, the graph of $\varphi_q^{(r)}(x)$ is contained between the parabollas $|y|=d_r x^2$, where d_r satisfies $\operatorname{Max}_x|\varphi_r^{(r)}(x)|=d_r(3/2^{r+2})^2$.

Hence $\varphi(x)$ is of class C_0^∞ on the whole straight line. However $\varphi(x)$ can not belong to the space S^∞, since $\operatorname{Max}_x|\varphi^{(q)}(x)|\geq\operatorname{Max}_x|\varphi_q^{(q)}(x)|\geq e^{q^2}$ for any q.

The latter half of theorem is proved if we take the Fourier transform of $\varphi(x)$ as $\psi(x)$.

Theorem 3. *The "S type" space having only upper index cannot coincide with any "S type" space having lower index or both indices except trivial spaces indicated in* § 2, 4.

Especially we have $S_\alpha\neq S_\alpha^\infty$, $S^\beta\neq S_\infty^\beta$, $S^\infty\neq S_\infty^\infty$, $S_\infty\neq S_\infty^\infty$, $S^\infty\neq S_\infty$.

Corollary. $S^\beta\cap(S_\infty^\infty)^C\neq\phi$, $(S^\beta)^C\cap(S_\infty^\infty)\neq\phi$,
$S_\alpha\cap(S_\infty^\infty)^C\neq\phi$, $(S_\alpha)^C\cap(S_\infty^\infty)\neq\phi$.

Proof. By Lemma 3, $\varphi\in S_0\subset S_\alpha\subset S_\infty\subset S$ and $\varphi\overline{\in}S^\infty\supset S^\beta\supset S_\alpha^\beta$, for any $0\leq\beta\leq\infty$, $0\leq\alpha\leq\infty$.

By Fourier transforms we obtain

$\tilde{\varphi}\in S^0\subset S^\beta\subset S^\infty\subset S$ and $\tilde{\varphi}\overline{\in}S_\infty\supset S_\alpha\supset S_\alpha^\beta$ for any $0\leq\beta\leq\infty$, $0\leq\alpha\leq\infty$.

Hence theorem 3 and the corollary hold.

Remark. The corollary to Theorem 1, i.e., $S\neq S^\infty$, $S\neq S_\infty$, $S\neq S_\infty^\infty$ can also be proved similarly by virtue of Lemma 3.

2. Now we turn to the problem of characterization of subspaces by

orders $m_{k,q}$ of growth of functions and their derivatives.

Any $\varphi(x)$ in S satisfies the inequalities $|x^k\varphi^{(q)}(x)|\leq m_{k,q}$ for some sequence of numbers $\{m_{k,q}\}$ $(k, q=0, 1, 2, \cdots)$. The sequences $\{m_{kq}\}$ for the functions in the classes S_α, S^β and S_α^β are of the forms $m_{kq}=C_q(A+\delta)^\alpha k^{k\alpha}$, $m_{kq}=C_k(B+\rho)^q q^{q\beta}$ and $m_{kq}=C(A+\delta)^\alpha(B+\rho)^q k^{k\alpha}q^{q\beta}$ respectively.

We have seen in § 3 that these orders are too low to classify the space S and there exist higher orders of growth, for example $m_{kq}=C_qA^kk^{k^\gamma}$. In the proof of Theorem 2 in this paragraph, we have seen also that there exist functions such that $m_{0q}\geq e^{q^2}$ and $m_{k0}\geq e^{k^2}$. Further, we can see the following

Theorem 4. *There exists a function that has increasing order of growth m_{kq} with arbitrary rapidity of q or k in S, S_α or S^β.*

As a corollary to this theorem we have

Theorem 5. *Elements of S, S_α or S^β cannot be classified into any set of countable classes C_γ $(\gamma=1, 2, \cdots)$ such that $C_\gamma=\{\varphi\,|\,|x^k\varphi^{(q)}(x)|\leq f_\gamma(k, q)\}$ and $S=\bigcup_\gamma C_\gamma$, $S_\alpha=\bigcup_\gamma C_\gamma$ or $S^\beta=\bigcup_\gamma C_\gamma$.*

Proof. In the proof of Theorem 2, we used Lemma 3 selecting $\{e^{k^2}\}$ as the sequence $\{b_k\}$. But, obviously, we can discuss analogously replacing $\{e^{k^2}\}$ by any sequence $\{b_k\}$. Then the constructed function and its Fourier transform satisfy the condition of Theorem 4.

Especially, if we choose $b_q>f_q\ (0, q)$, $(q=0, 1, 2 \cdots)$ then the constructed function φ belongs to S and to S_α but not to $\bigcup_\gamma C_\gamma$. Hence Theorem 5 holds.

§ 5. Some Fourier invariant functional spaces.

1. We shall apply our results to the theory of Fourier transforms of functionals.

According to the theory of distributions of L. Schwartz, the Fourier transform of any tempered distribution (element of the space S') is defined as another tempered distribution.

Now the question arises whether the space S' is the furthest Fourier invariant space or not, i.e., whether there is any space that includes the space S' and is invariant under Fourier transforms or not [3].

H. Yamagata [9], using Theorem 2 of the present paper, showed that the space $D'\cap Z'$ (where Z' means the Fourier transform of the space D') is the Fourier invariant distribution space which includes the space S' and is not equal to S'.

Here, also using our results, we show that there are some larger Fourier invariant functional spaces which include S'. We show also these spaces are dual spaces of some subspaces of S.

Let f_0 be the function given in Lemma 1, and let f_1 be the function $f_1(x)=1/f_0(x)$, i.e.,

$$f_1(x) \equiv \begin{cases} \exp\{(\log|x|)^\gamma\} & \text{for } |x|>e \\ e & \text{for } |x|\le e. \end{cases}$$

Then the function f_1 is continuous in $(-\infty, \infty)$, and has following properties.

(1) f_1 is a rapidly increasing function, i.e., for any positive integer n, $\lim_{x\to\pm\infty} f_1(x)/|x|^n=\infty$.

(2) The order of increase as an exponential type function is 0, i.e.,

$$\lim_{x\to\pm\infty} (\log\log f_1(x))/\log|x| = 0\,.$$

Proof. By similar calculation in the proof Lemma 1, we obtain these results.

Lemma 4. *Any times primitive of $f_1(x)$ satisfies also the property* (1).

Proof. This is trivial from property (1) and $f_1(x)>0$.

Lemma 5. *f_1 does not belong to the space S'.*

Proof. Assume that $f_1\in S'$. Then f_1 is expressed as $f_1=D^q(P(x)f(x))$, where $P(x)$ is a polynoimal and $f(x)$ is a bounded continuous function (L. Schwartz [1]). So $P(x)f(x)=O(x^n)$ for some n.

On the other hand $P(x)f(x)$ is a some times primitive of f_1 and so has the property (1) by Lemma 4. This contradicts the fact that

$P(x)f(x)=O(x^n)$. q.e.d.

Lemma 6. *If the sequence $\varphi_\nu(x)$ is convergent in the space $S_0^\beta(\beta>1)$, then $\varphi_\nu(x)$ is convergent in (D).*

Proof. As stated in § 2, if $\{\varphi_\nu(x)\}$ is convergent in S_0^β, then $\{\varphi_\nu(x)\}\subset S_{0,A}^{\beta,B}$ for suitable $A>0$, $B\ge 0$, and the carriers of φ_ν are contained in the interval $[-A, A]$, and $\{\varphi_\nu(x)\}$ is convergent in $S_{0,A}^{\beta,B}$ where the norm in $S_{0,A}^{\beta,B}$ is given by

$$\|\varphi\|_{\delta\rho} = \sup_{k,q,x}|x^k\varphi^{(q)}(x)|/(A+\delta)^k(B+\rho)^q q^{q\beta}, \quad (\delta, \rho=1, 1/2, \cdots)\,.$$

If $\{\varphi_\nu\}$ converges to 0, then the sequence $\{\varphi_\nu^{(q)}(x)\}$ uniformely (in x) converges to 0, and its carrier is contained in a compact set. Hence,

$\{\varphi_\nu\}$ is convergent in (D).

Lemma 7. *The space D' is contained in the space $(S_0^\beta)'$.*

Proof. This is an immediate result of Lemma 6.

Lemma 8. *f_1 belongs to the space $(S_0^\beta)'$ for any $\beta>1$.*

Proof. Since $f_1(x)$ is a locally summable function,

$$f_1(x)\in D'\subset(S_0^\beta)'.$$

Lemma 9. *f_1 belongs to the space $(S_\alpha^0)'$ for any $\alpha>1$.*

Proof. Let $\varphi(x)$ be a function of S_α^0, then, as stated in § 2, $\varphi(x)$ is continued analytically for all complex values $z=x+iy$, and $\varphi(z)$ satisfies the following inequality: $|\varphi(x+iy)|\leq C\exp\{-a|x|^{1/\alpha}+b|y|\}$.

So, we see that the order of exponential decrease of the function $f_1(x)\varphi(x)$ is $1/\alpha$ by virtue of property (2), and $\langle f_1, \varphi\rangle$ converges.

Let $\{\varphi_\nu(x)\}$ be a convergent sequence in S_α^0, then, as stated in § 2, there exist A and B such that $\{\varphi_\nu\}\subset S_{\alpha,A}^{0,B}$ and $\{\varphi_\nu\}$ converges in the norm in $S_{\alpha,A}^{0,B}$. The element φ_ν of $S_{\alpha,A}^{0,B}$ is an entire function which satisfies the above inequality where a is a fixed number indepent of ν.

So $f_1\varphi_\nu(x)$ are also functions which belong to a fixed exponential type (a) of decreasing order $1/\alpha$.

Since $\inf_k k^{k\alpha}/|\xi|^k\leq C\exp\left(-\frac{\alpha}{e}|\xi|^{1/\alpha}\right)$ for $\alpha>0$ [5, P. 204], it follows that

$$||\varphi||_{\delta\rho}=\sup_{x,k,q}\frac{|x^k\varphi^{(q)}(x)|}{(A+\delta)^k(B+\rho)^q k^{k\alpha}}\geq\sup_{x,q}\frac{\exp[a\{x/(A+\delta)\}^{1/\alpha}]|\varphi^{(q)}(x)|}{C(B+\rho)^q}$$

where $a=\alpha/eA^{1/\alpha}$. Hence it follows that

$$|f_1\varphi_\nu(x)|\leq C'\exp\left\{-a\left|\frac{x}{A+\delta}\right|^{1/\alpha}\right\}\cdot||\varphi_\nu||_{\delta\rho}.$$

The constants A, C', δ, a, α do not depend on ν, and so, if $||\varphi_\nu||\to 0$, then $\langle f_1, \varphi_\nu\rangle\to 0$.

Theorem 6. *For $\beta>1$,*

(1) *$(S_0^\beta)'\cap(S_\beta^0)'$ is a linear functional space which is invariant under Fourier transform.*

(2) *The space $(S_0^\beta)'\cap(S_\beta^0)'$ contains the space S'.*

(3) *$(S_0^\beta)'\cap(S_\beta^0)'\cap(S')^C$ is not empty.*

Proof. (1) Since $\tilde{S}_0^\beta=S_\beta^0$ and the topology is preserved under this transform, we see $\widetilde{(S_0^\beta)'}=(S_\beta^0)'$.

(2) Since $S_0^\beta \subset S$, $S_\beta^0 \subset S$, and both topologies of S_0^β and S_β^0 are finer than that of S, we have this conclusion.

(3) We obtain this from Lemma 5, 8, 9.

Remark. We can also see that $(S_0^\beta)' \cap (S_\beta^0)' \supset D' \cap Z'$, from Lemma 7.

2. We study next some properties of the space $(S_0^\beta)' \cap (S_\beta^0)'$.

We consider the linear sum $(S_0^\beta + S_\beta^0)$ of the spaces S_0^β and S_β^0. Since S_β^0 consists of entire functions and S_0^β consists of C^∞ functions with compact carriers, there is no common element between these two spaces other than the function $\varphi(x) \equiv 0$. So the linear sum $(S_0^\beta + S_\beta^0)$ is the direct sum of two spaces. We define the neighborhood system in this space by the linear sum of the neighborhood of each space.

Now let $(S_0^\beta + S_\beta^0)'$ be the dual space of the space $(S_0^\beta + S_\beta^0)$. Then we can see the following.

Theorem 7. $(S_0^\beta + S_\beta^0)' = (S_0^\beta)' \cap (S_\beta^0)'$.

Proof. The sequence $\{\varphi_\nu\}$ in the space $(S_0^\beta + S_\beta^0)$ converges to φ if and only if $\{\varphi_{\nu_1}\}$, converges to φ_1 in S_0^β and $\{\varphi_{\nu_2}\}$ converges to φ_2 in S_β^0 where $\varphi_\nu = \varphi_{\nu_1} + \varphi_{\nu_2}$ and $\varphi = \varphi_1 + \varphi_2$ are unique expressions and φ_{ν_1}, $\varphi_1 \in S_0^\beta$, φ_{ν_2}, $\varphi_2 \in S_\beta^0$.

So any element T of the space $(S_0^\beta)' \cap (S_\beta^0)'$ is consistently identified with the element $\bar{T}$ of $(S_0^\beta + S_\beta^0)'$ in the following way:

$$< T, \varphi_{\nu_1} >_1 + < T, \varphi_{\nu_2} >_2 = < \bar{T}, \varphi_\nu >.$$

Conversely any element $\bar{T}$ of $(S_0^\beta + S_\beta^0)'$ is identified with an element T of $(S_0^\beta)' \cap (S_\beta^0)'$ in the following way:

$$< T, \varphi_{\nu_1} >_1 = < \bar{T}, \varphi_{\nu_1} > \quad \text{and} \quad < T, \varphi_{\nu_2} >_2 = < \bar{T}, \varphi_{\nu_2} >, \quad \text{q.e.d.}$$

Since the spaces S_0^β and S_β^0 are complete Montel spaces as refered in § 2, the space $(S_0^\beta + S_\beta^0)$ becomes also complete Montel space.

Hence from Theorem 7 we obtain the following

Corollary. *The space $(S_0^\beta)' \cap (S_\beta^0)'$ with bounded convergence topology is a Reflexive montel space.*

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