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A MAPPING OF MARTIN BOUNDARY INTO KURAMOCHI BOUNDARY BY MEANS OF POLES

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1. Introduction

Let R be a hyperbolic Riemann surface. Among all kinds of compactifications of R we are especially interested in the Martin's and the Kuramochi's compactifications which are denoted by R^{*M} and R^{*K} respectively. It was shown by Z. Kuramochi [6] that there exists a sequence of points on R converging to a Martin boundary point b and containing two subsequences such that each one tends to a Kuramochi boundary point and their limit points are distinct. We shall state this fact that many Kuramochi boundary points lie upon b. He also showed an example of R in which many boundary points lie upon a Kuramochi boundary point. Thus, we know there exist no quotient relations between R^{*M} and R^{*K}. It would not be too much to say that the difficulties to establish the relation between R^{*M} and R^{*K} arise from this fact. In this paper, we shall study this relationship with the aid of *poles*.

The notion of poles was introduced by M. Brelot [1] and its importance was shown by L. Naïm [8] in her study of axiomatic Dirichlet problem. Let k_b be a minimal positive harmonic function corresponding to a minimal Martin boundary point b. Then, there exists at least one point b' in the Kuramochi boundary for which the reduced function $(k_b)_{\{b'\}}$ is equal to k_b. This point b' is termed the pole of b in the Kuramochi boundary. If the pole of b is b' only, b' is called the *unique pole* of b and is denoted by $\Phi(b)$. In the present paper, the mapping Φ will play a central role.

In §2, the definition and some elementary properties of poles are given. The poles in the Kuramochi boundary are related intimately to Dirichlet problem in R^{*K}, which is stated in Theorem A in §3. From this point of view we may say many Martin boundary points lie upon a pole in the Kuramochi boundary in general; that is, except a set of harmonic measure zero the Kuramochi boundary is covered by the Martin boundary. An HD function u has fine boundary values $\hat{u}$ on the Martin boundary and quasi-continuous extension u^* on the Kuramochi boundary. $\hat{u}$ and u^* are connected with each other by Φ, which is stated in Theorem 4. From this we can also prove Theorem 5, which was partly obtained by J.L. Doob [4]. In §4 the Kuramochi boundary of some subdomain is studied.

By using the result of §4, in §5 we can derive a property of canonical potential which was studied by Y. Kusunoki [7]. In §6 we shall restrict ourselves to the surface on which $MHB(R)=MHD(R)$. In this Riemann surface, roughly speaking, Φ makes correspond the Martin boundary to the Kuramochi boundary one-to-one almost everywhere. In the last section, we consider the boundary property of a sequence of HD functions converging in the Dirichlet norm, which is a consequence of the result of J.L. Doob [4] and ours*.

2. The definition and the elementary properties of poles in the Kuramochi boundary

Let R be a hyperbolic Riemann surface. In the sequel, we shall use the following notations:

$R^{*M}(R^{*K})$: the Martin's (Kuramochi's) compactification of R.
$\Delta^M(\Delta^K)$: the Martin (Kuramochi) boundary of R.
$\omega^M(\omega^K)$: the harmonic measure on $\Delta^M(\Delta^K)$.
Δ_1^M : the set of all minimal boundary points in Δ^M.
$H_f^M(H_{f'}^K)$: the solution of Dirichlet problem corresponding to the boundary function $f(f')$ on $\Delta^M(\Delta^K)$ with respect to $R^{*M}(R^{*K})$.
$\bar{A}(\bar{A}^*)$: the closure of A which is taken in R^{*M} (R^{*K}).

To every $b\in\Delta_1^M$, there corresponds the minimal positive harmonic function k_b. If the reduced function of k_b with respect to a subset A of Δ^K is denoted by $(k_b)_A$, $(k_b)_A$ is equal to k_b or zero and for at least one point $b'\in\Delta^K$, $(k_b)_{\{b'\}}=k_b$ [1]. In this case, the point b' is termed the *pole* of b (of k_b) in Δ^K, and the set of poles of b is denoted by $b^\vee$. As in [5] we see:

Theorem 1. *For $b\in\Delta_1^M$, the following three sets coincide*

(i) *the set of poles of b in Δ^K.*
(ii) $\cap\{\bar{E}^*; E\subset R, R-E$ *is thin at* $b\}$.
(iii) $\cap\{\bar{G}^*; G$ *is an open set in* R *and* $R-G$ *is thin at* $b\}$.

where in the latter two sets closures are taken in R^{*K}.

From the above theorem we see that the set of poles is connected.

We shall denote by $\Delta_1'^M$ the set of all points b, each of which has the *unique* pole in Δ^K. For each $b\in\Delta_1'^M$, the unique pole of b is denoted by $\Phi(b)$. It is known[2]

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* *Added in proof.* Prof. Y. Kusunoki has kindly pointed out that some results, in particular those in §5 and §7, are obtained, by using quite different method, in his article: Y. Kusunoki and S. Mori: *Some remarks on boundary values of harmonic functions with finite Dirichlet integrals*, J. Math. Kyoto Univ. **7** (1968), 315–324.

1) Cf. [1], p. 328, also, [8], p. 256.

2) Cf. [8], th. 33 and cor., p. 260.

(2. 1) $$\omega^M(\Delta^M - \Delta_1'^M) = 0\,,$$

(2. 2) $$\omega^K(\Delta^K - \Phi(\Delta_1'^M)) = 0\,.$$

We have also:

Theorem A. *For every resolutive function f' in Δ^K the function f, defined as $f'\circ\Phi$ on $\Delta_1'^M$ and zero elsewhere, is also resolutive and $H_f^M = H_{f'}^K$.*

This theorem is a consequence of Naïm's theorem ([8], th. 38, p. 267), but in view of the central role of this theorem in our present paper, we shall give here a direct proof.

Proof of Theorem A. Let s be an arbitrary subharmonic function, bounded from above, and $\lim_{a\to b'} s(a) \leq f'(b')$ at each $b' \in \Delta^K$. Since by Theorem 1, for any neighbourhood U' of b' in R^{*K}, $U' \cap R$ is not thin at every point of $\Phi^{-1}(b')$, with the aid of (2.1) we have $s \leq \underline{H}_f^M$ [3]. Hence, $H_{f'}^K \leq \underline{H}_f^M$. Similarly $\bar{H}_f^M \leq H_{f'}^K$, consequently $H_{f'}^K = H_f^M$.

REMARK 1. From Naïm's theorem we can derive moreover $\bar{H}_{f'}^K = \bar{H}_{f'\circ\Phi}^M$ for an arbitrary boundary function f'.

REMARK 2. If $u \in MHB(R)$ [4] and if there exists a set A of $d\omega^M$-harmonic measure zero such that the fine boundary values $\hat{u}$ of u are equal on each $\Phi^{-1}(a') - A$, then the function f' defined as this equal value $\hat{u}(a)$ at each $a' = \Phi(a)$ and zero elsewhere is resolutive and $H_{\hat{u}}^M = H_{f'}^K$.

For an ideal boundary component of Kérékjártó-Stoïlow e, we shall define

(2. 3) $$\Delta_e^M = \bigcap_{n=1}^{\infty} \bar{G}_n\,,$$

(2. 4) $$\Delta_e^K = \bigcap_{n=1}^{\infty} \bar{G}_n^*\,,$$

where $\{G_n\}$ is a determining sequence of e. They are the boundary components.

Theorem 2. *For $b \in \Delta_e^M \cap \Delta_1^M$ the set of poles $b^\vee$ is contained in Δ_e^K.*

Proof. We may take the determining sequence $\{G_n\}$ of e such that

1) $R - G_n$ is a domain,
2) the relative boundary of G_n coincides with the relative boundary of $R - \bar{G}_n$: $\partial G_n = \partial(R - \bar{G}_n)$.

Here, if $\overline{R - G_n} \cap \bar{G}_n \cap \Delta^M \neq \phi$, then $R^{*M} - \partial G_n$ is open and connected. Since the Martin's compactification is of type S ([2], p. 99), we have $R^{*M} - \partial G_n - \Delta^M$

3) Cf. [8], th. 23, p. 245.
4) MHB(R) is the family of all quasi-bounded HP functions; i.e. the smallest monotone family of HP functions containing all bounded harmonic functions.

$=(R-\bar{G}_n)\cup G_n$ is connected, which is a contradiction. Therefore $\overline{R-G_n}\cap \bar{G}_n\cap\Delta^M=\phi$. This means, for every G_n and for every point b of Δ_e^M, there exists a neighbourhood $U(b)$ of b such that $U(b)\cap R\subset G_n$. Accordingly, $R-G_n$ is thin at each point of $\Delta_e^M\cap\Delta_1^M$. Hence, by Theorem 1 $b^\vee\subset\bar{G}_n{}^*$ and finally $b^\vee\subset\Delta_e^K$, which completes the proof.

Corollary. $\Phi(\Delta_1'^M\cap\Delta_e^M)=\Phi(\Delta_1'^M)\cap\Delta_e^K$.

REMARK. The correspondence Φ is not one-to-one in general.

3. Dirichlet problem with respect to the Kuramochi's compactification

In this section we shall consider Dirichlet problem with respect to R^{*K}. It is known that the Kuramochi's compactification is a resolutive compactification ([2], p. 167). In this paper, it is fundamental that the family of all Dirichlet solutions with respect to R^{*K} is identical with $MHD(R)$, i.e. the smallest monotone family of HP functions containing all HD functions. ([2], p. 167). With resolutive function f' on Δ^K, we can associate the harmonic function $H_{f'}^K$. But this correspondence is not one-to-one. To remove this inconvenience, we shall consider two functions which are identical except a set of harmonic measure zero, to be equivalent. Then, there exists a one-to-one correspondence between the Banach space $L^1(\Delta^K;\,d\omega^K)=\left\{f';\int_{\Delta^K}|f|\,d\omega^K<\infty\right\}$ and $MHD(R)$. On account of this convention we may assume $f'\in L^1(\Delta^K,\,d\omega^K)$ vanishes on $\Delta^K-\Phi(\Delta_1'^M)$.

It was shown by Z. Kuramochi [6], that, as the topological space there are no quotient relations between R^{*M} and R^{*K}. But in view of Theorem A, concerning Dirichlet problem, we may consider R^{*K} is covered by R^{*M}. We shall state this in

Theorem 3. *There exists a mapping T from $L^1(\Delta^K,\,d\omega^K)$ into $L^1(\Delta^M,\,d\omega^M)$ such that*

1) *T is linear.*
2) *$f_1'\leq f_2'$ implies $Tf_1'\leq Tf_2'$*[5].
3) $\|f'\|=\|Tf'\|$.

The mapping T is onto if and only if $MHD(R)=MHB(R)$.

Proof. If we associate with $f'\in L^1(\Delta^K,\,d\omega^K)$ the function $\hat{f}$ defined as $f'(\Phi(b))$ on $\Delta_1'^M$ and zero elsewhere, then we have $H_{\hat{f}}^M=H_{f'}^K$. Setting $Tf'=\hat{f}$, we obtain the desired mapping. From

5) This means, of course, the relation "$\leqq$" holds except a set of harmonic measure zero.

$$MHB = \{H^M_f; f \in L^1(\Delta^M, d\omega^M)\},$$
$$MHD = \{H^K_{f'}; f' \in L^1(\Delta^K, d\omega^K)\}$$

we can derive at once the latter half of this theorem.

The image of $L^1(\Delta^K, d\omega^K)$ under the above mapping T is a closed subspace of $L^1(\Delta^M, d\omega^M)$. It seems interesting to the author to what closed subspace of $L^1(\Delta^M, d\omega^M)$ there corresponds the family of resolutive functions on some compactification.

Theorem 4. *For $u \in HD$, the quasi-continuous extension u^* of u on Δ^K belongs to $L^1(\Delta^K, d\omega^K)$ and $H^K_{u^*} = u$.*

Proof. We shall use the same notations as in [2]. In the proof of Constantinescu-Cornea ([2], p. 168) it is assumed that u is bounded, this assumption is not essential however. For some compact exhaustion $\{K_n\}$ of R we have

$$\sum_{n=1}^{\infty} [(u_{n+1} - u_n) \vee (u_n - u_{n+1})] < +\infty, \tag{3.1}$$

and $\{u_n\}$ converges uniformly on every compact set in R, where u_n is the HD-part of Royden's decomposition of a Dirichlet function $u - u^{K_n}$. The functions u^{K_n} are extended continuously on R^{*K}, and if u^{K_n} denotes the extended function again, u^{K_n} is bounded. We have

$$u - u_n = H^K_{u^{K_n}}, \tag{3.2}$$

$$(u_{n+1} - u_n) \vee (u_n - u_{n+1}) = H^K_{|u^{K_{n+1}} - u^{K_n}|}. \tag{3.3}$$

By (3.1) and (3.3) $\sum_{i=1}^{\infty} |u^{K_{i+1}} - u^{K_i}|$ converges $d\omega^K$-a.e.,[6] therefore $\{u^{K_n}\}$ converges $d\omega^K$-a.e..

On the other hand, the quasi-continuous extension u^* of u is defined as

$$u^* = \begin{cases} u^{K_1} + \sum_{i=1}^{\infty} (u^{K_{i+1}} - u^{K_i}) & \text{on } \Delta^K - P \\ 0 & \text{on } P, \end{cases}$$

where P is a set of Kuramochi capacity zero. Since $\omega^K(P) = 0$, we see

$$u^* = u^{K_1} + \sum_{i=1}^{\infty} (u^{K_{i+1}} - u^{K_i}) = u^{K_n} + \sum_{i=n}^{\infty} (u^{K_{i+1}} - u^{K_i}) \qquad d\omega^K\text{-a.e.}$$

and

$$\lim_{n \to \infty} u^{K_n} = u^* \qquad d\omega^K\text{-a.e..}$$

On account of (3.2), for some fixed $a \in R$

6) a.e. means "almost everywhere", that is "except a set of measure zero".

$$u(a)-u_n(a) = H^K_{u^{K_n}}(a) = \int_{\Delta^K} u^{K_n}(b')d\omega_a^K(b') .$$

Since $\lim_{n\to\infty} u_n(a)=0$, we obtain $u(a)=\lim_{n\to\infty}\int_{\Delta^K} u^{K_n} d\omega_a^K$. In view of (3.1), (3.3)

$$\sum_{n=1}^{\infty} |u^{K_n}-u^{K_{n+1}}| \in L^1(\Delta^K, d\omega^K)$$

and from

$$|u^{K_n}| \leq |u^{K_1}| + |u^{K_n}-u^{K_1}| \leq |u^{K_1}| + \sum_{n=1}^{\infty} |u^{K_n}-u^{K_{n+1}}|$$

we have

$$\lim_{n\to\infty}\int_{\Delta^K} u^{K_n} d\omega_a^K = \int_{\Delta^K} \lim_{n\to\infty} u^{K_n} d\omega_a^K$$

i.e. $u^* \in L^1(\Delta^K, d\omega^K)$ and $H^K_{u^*}=u$, q.e.d.

According to Theorem 4 and Theorem A we conclude the following theorem, which was obtained partly by J.L. Doob ([4], th. 6.1).

Theorem 5. *If f is a Dirichlet function, it has a finite fine boundary function $\hat{f}$ $d\omega^M$-almost everywhere on Δ^M. Moreover, if f^* denotes the quasi-continuous extension of f on Δ^K, then $\hat{f}=f^*\circ\Phi$ $d\omega^M$-almost everywhere.*

Proof. Let $f=u+f_0$ be the Royden's decomposition of f, where $u\in HD$ and f_0 is a Dirichlet potential. u has a finite fine boundary function $\hat{f}$ $d\omega^M$-a.e. on Δ^M and $H^M_{\hat{f}}=u$[7]. Since $|f_0|$ is dominated by a potential, f_0 has a fine boundary value zero $d\omega^M$-a.e.[8]. Since a quasi-continuous extension of a Dirichlet potential is zero quasi-everywhere on Δ^K ([2], Satz 17. 10), $f^*=u^*$ $d\omega^K$-a.e. and by Theorem 4, $u=H^K_{f^*}$. Applying Theorem A we have $H^M_{f^*\circ\Phi}=H^M_{\hat{f}}$. This implies the conclusion of our theorem.

4. Kuramochi boundary of subdomains

In this section we shall consider the Kuramochi boundary of a special subdomain, that is, a connected component of the complement of some compact set.

In the sequel, let S be a domain not relatively compact, and let its relative boundary be compact and sufficiently smooth (for instance, it consists of a finite number of piecewise analytic curves).

We shall fix a closed disc K_0 in S and consider the Kuramochi's compactifications of R and S with respect to K_0. Let us denote by $\tilde{g}_a$, $\tilde{g}_a^{(S)}$ the Neumann functions of $R-K_0$ and $S-K_0$ respectively.

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7) Cf. [3], th. 4.1, th. 4.3, pp. 296–7.

8) Cf. [3], th. 4.2, p. 297.

For $a\in S$ we have

$$\tilde{g}_a^{(S)}=\tilde{g}_a-w \tag{4.1}$$

where w is harmonic in S, and except possibly a finite number of points, $\frac{\partial w}{\partial n}=\frac{\partial \tilde{g}_a}{\partial n}$ [9] on ∂S. Let $\{K_n\}$ be a compact exhaustion of R. We obtain

$$\mathrm{D}[w]=\int_{\partial S} w\frac{\partial w}{\partial n}ds\,, \tag{4.2}$$

where $\mathrm{D}[w]$ is the Dirichlet integral of w over $S_0=S-K_0$. In fact, let f be a function in $C^\infty(S)$ such that $f=0$ on $K_n\cap S$ and $f=w$ on $S-K_m$ $(m>n)$. Since for sufficiently large n $w^{K_n\cap S}=w$[10] in each component of S_0-K_n, $\langle dw, df\rangle_{S_0}=0$. And $\langle dw, df\rangle_{S_0}=\langle dw, df\rangle_{S-K_n}=\lim_{m\to\infty}\langle dw, df\rangle_{(K_m-K_n)\cap S}$. On the other hand, by Green's formula we have

$$\langle dw, df\rangle_{(K_m-K_n)\cap S}=\int_{\partial K_m\cap S} w\frac{\partial w}{\partial n}ds\,.$$

Lemma 1. *Let S be a non-compact domain of R whose relative boundary ∂S consists of a finite number of piecewise analytic curves. For $w=\tilde{g}_a-\tilde{g}_a^{(S)}$ and for a closed subset F of S such that $\partial(R-F)$ is compact and $\overline{(S-F)}\cap(\partial S\cup K_0)=\phi$, we have $w^F=w$.*

Proof. If $a\notin\overline{S-F}$, our lemma is derived immediately from the properties of $\tilde{g}_a$ and $\tilde{g}_a^{(S)}$. Assume now $a\in\overline{S-F}$ and set $w_1=w^F$. We can find a closed set F_1 such that $F\subset F_1$ and $w^{F_1}=w$. Since $(w^F)^{F_1}=w^F$ the maximum and the minimum of $w-w_1$ in $S-F_1$ are attained at some points of $\partial(S-F_1)$ respectively. Hence, the harmonic function $w-w_1$ has its maximum and minimum in $S-F$ at some points of $\partial(S-F)$, while $w-w_1=0$ on $\partial(S-F)$, this implies $w\equiv w_1\equiv w^F$, which completes the proof.

Theorem 6. *Let S be a non-compact domain of R whose relative boundary ∂S consists of a finite number of piecewise analytic curves. The Kuramochi's compactification S^{*K} of S is homeomorphic to $\bar{S}^*$, where $\bar{S}^*$ is the closure of S in R^{*K}.*

Proof. Concerning the relative boundary ∂S, the statement of our theorem is evident. Let $\{a_n\}$ be a sequence of points of S converging to a boundary point $b\in\Delta^K$. Since $\{\tilde{g}_{a_n}\}$ converges uniformly to $\tilde{g}_b$ on ∂S, $\{\partial\tilde{g}_{a_n}/\partial n\}$ also converges uniformly to $\partial\tilde{g}_b/\partial n$ on ∂S. If we set $w_n=\tilde{g}_{a_n}-\tilde{g}_{a_n}^{(S)}$, $\{w_n\}$ converges

9) In this and in the following, $\partial/\partial n$ always denotes the outer normal derivative with respect to the related domain.

10) About the definition of $w^{K_n\cap S}$ see [2], p. 155.

uniformly on each compact set in S. To see this, we consider Dirichlet integrals

$$\text{(4. 3)} \qquad D[w_m-w_n] = \int_{\partial S}(w_m-w_n)\frac{\partial(w_m-w_n)}{\partial n}ds = \int_{\partial S}(w_m-w_n)\frac{\partial(\tilde{g}_{a_m}-\tilde{g}_{a_n})}{\partial n}ds\,.$$

Since w_n are uniformly bounded on ∂S, (4. 3) implies $D[w_m-w_n]\to 0$ as $m, n\to\infty$. Combining this with $w_n=0$ on ∂K_0, we see at once $\{w_n\}$ converges to w_0 uniformly on each compact set of S. This means $\{a_n\}$ is at the same time a determining sequence of some boundary point $b'\in S^{*K}-(S\cup\partial S)$.

Conversely, let $\{a_n\}$ be a sequence of points in S converging to $b'\in S^{*K}-(S\cup\partial S)$ in the topology of S^{*K}. Suppose that there exist two subsequences $\{a_{n_\nu'}\}$, $\{a_{n_\nu''}\}$ such that $\{a_{n_\nu'}\}$ converges to $b_1\in\Delta^K$ and $\{a_{n_\nu''}\}$ converges to another point $b_2\in\Delta^K$. As above, $\{w_{n_\nu'}\}$, $\{w_{n_\nu''}\}$ converge to w', w'' respectively. Moreover, we have $\tilde{g}_{b_1}-\tilde{g}_{b_2}\equiv w'-w''$ in S. By the preceding lemma, for sufficiently large compact subset K of R we have $(w'-w'')^{K\cap S}=w'-w''$, whereas $(\tilde{g}_{b_1}-\tilde{g}_{b_2})^{K\cap S}\neq\tilde{g}_{b_1}-\tilde{g}_{b_2}$, which leads us to a contradiction. From this we can easily complete the proof.

The homeomorphic mapping from $\bar{S}^*$ onto S^{*K}, thus established, will be denoted by I.

Corollary. *If P is a subset of $\bar{S}^*-R$ fulfilling $\omega^K(P)=0$, then the harmonic measure of $I(P)$ with respect to S^{*K} is also zero.*

Proof. Let s be a superharmonic function on R, bounded from below and $\varliminf_{a\to b} s(a)\geq\chi_P(b)$ at every $b\in\Delta^K$, where χ_P is the characteristic function of P[11]. It follows immediately $s\geq 0$ and the restriction of s on S belongs to the family defining the upper solution $\bar{H}^{K,S^{*K}}_{\chi_{I(P)}}$ with respect to S^{*K}. Hence, $0\leq\bar{H}^{K,S^{*K}}_{\chi_{I(P)}}\leq\bar{H}^K_{\chi_P}=0$, which completes the proof.

5. Functions of the class HD($R-K$)

In this section, let K be a fixed compact set and let u be in $HD(R-K)$, i.e. u is a harmonic function with finite Dirichlet integral in each component of $R-K$. We shall assume further that $\partial(R-K)$ consists of a finite number of piecewise analytic curves[12]. For definiteness, we shall fix one component S of $R-K$ and consider it a Riemann surface. The Kuramochi boundary of S is divided into two parts. One of them is homeomorphic to a part of $\partial(R-K)$

11) That is, $\chi_P=1$ on P and o elsewhere.

12) It would be easily seen that in the following theorem this restriction is harmless.

and will be denoted by ∂S again. The remaining part of the Kuramochi boundary of S is important for our purpose and is denoted by $\Delta^K_{(S)}$. By Theorem 6, $\Delta^K_{(S)}$ is homeomorphic to $\bar{S}^*-(S\cup\partial S)$.

Lemma 2. *If we denote the quasi-continuous extension of u on $\bar{S}^*$ (with respect to R) by u^* and set $\hat{u}=u^*\circ I^{-1}$ then $H^{S^*}_{\hat{u}}=u$ in S, where $H^{S^*}_{\hat{u}}$ is the Dirichlet solution with respect to S^{*K}.*

Proof. If we remark the following, the proof would be quite same as that of Theorem 4. Let $u-u^{K_n\cap S}=f^{(S)}_n+u^{(S)}_n$[13] be the Royden's decomposition in S, i.e. $f^{(S)}_n$ is a Dirichlet potential in S and $u^{(S)}_n\in HD(S)$. Since $\bar{S}^*-S$ is homeomorphic to $S^{*K}-S$, u^{K_n} can be extended continuously to the Kuramochi boundary of S. Denoting this extended function by $u^{K_n}_{(S)}$, we have also $u-u^{(S)}_n=H^{S^*}_{u^{K_n}_{(S)}}$ and $u=H^{S^*}_{\lim_{n\to\infty} u^{K_n}_{(S)}}$. $u^{K_n}_{(S)}$ converges $d\omega^K_{(S)}$-a.e.[14]. By Theorem 4, $u^*=\lim_{n\to\infty} u^{K_n}\ d\omega^K$-a.e. and by Corollary to Theorem 6, $\hat{u}=u^*\circ I^{-1}=\lim_{n\to\infty} u^{K_n}\circ I^{-1}$ $=\lim_{n\to\infty} u^{K_n}_{(S)}\ d\omega^K_{(S)}$-a.e..

In his paper [7], Y. Kusunoki investigated the properties of canonical potentials: *a canonical potential has a constant quasi-continuous extension quasi-everywhere on each Kuramochi boundary component Δ^K_e.* Concerning this, we get

Theorem 7. *Every canonical potential has a constant fine limit almost everywhere on each Martin boundary component Δ^M_e.*

Proof. A canonical potential u belongs to $HD(R-K)$, where K is compact and may be assumed that $\partial(R-K)$ consists of a finite number of piecewise analytic curves. In each component S of $R-K$, u is a solution of Dirichlet problem $H^{S^*}_{\hat{u}}$ and by the preceding lemma, we can assume u is constant on each boundary component. According to Theorem A, $\hat{u}$ is considered as a Dirichlet solution corresponding to the boundary function φ with respect to the Martin's compactification of S. The Martin boundary of S is divided into two parts, one of which is homeomorphic to ∂S. By the Corollary to Theorem 2 we can assume φ is constant on each boundary component of the remaining part. Hence, u has a constant fine limit almost everywhere on each Martin boundary component not homeomorphic to ∂S. Above reasoning is accomplished related to the Riemann surface S, nevertheless, as is known[15] the same conclusion remains valid for R. Thus, the theorem is proved.

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13) u^{K_n} has the same meaning as in the proof of Theorem 4, that is, $\{K_n\}$ is an exhaustion of R and not of S.

14) $w^{K}{}_{(S)}$ denotes harmonic measure with respect to the Kuramochi's compactification S^{*K}.

15) Cf. [8], th. 15, p. 224.

6. Surfaces on which MHB(R)=MHD(R)

On account of Theorem A, we see that if the mapping Φ, defined in §2, is one-to-one except a set of $d\omega^M$-harmonic measure zero, then $MHB(R)=MHD(R)$. The converse of this is also valid. Before proving this, we shall investigate a special case in which Δ^K contains at least one point with positive harmonic measure.

Theorem 8. (i) *If* $\omega^M(\{b\})>0$ *for* $b\in\Delta_1'^M$, *then* $\omega^K(\{\Phi(b)\})>0$. (ii) *Under the condition* $MHB(R)=MHD(R)$, *if* $\omega^K(\{b'\})>0$ *for* $b'\in\Delta^K$ *then there exists one and only one* $b\in\Delta_1^M$ *such that* $\Phi(b)=b'$ *and* $\omega^M(\{b\})>0$.

Proof. (i) Since $b\in\Phi^{-1}(\Phi(b))$ and $b\in\Delta_1'^M$, we have $\omega^M(\Phi^{-1}(\Phi(b)))>0$. Hence, $H^K_{\chi_{\{\Phi(b)\}}}=H^M_{\chi_{\{\Phi^{-1}(\Phi(b))\}}}>0$, where χ_E denotes the characteristic function of E. This means $\omega^K(\{\Phi(b)\})>0$.

(ii) Suppose that no point of $\Phi^{-1}(b')$ is of harmonic measure positive. Then there exist two measurable subsets E_1, E_2 of $\Phi^{-1}(b)$ such that

$$E_1\cap E_2=\phi\,, \tag{6.1}$$

$$\omega^M(E_i)>0 \qquad (i=1, 2)\,. \tag{6.2}$$

From the assumption $MHB(R)=MHD(R)$ we have $H^M_{\chi_{E_i}}=H^K_{f_i}$ $(i=1, 2)$ for some function f_i. By (6.2), $H^K_{f_i}>0$. Since

$$H^K_{\chi_{\{b'\}}}=H^M_{\chi_{\Phi^{-1}(b')}}\geq H^M_{\chi_{Ei}}=H^K_{f_i}\,,$$

we have $\chi_{\{b'\}}\geq f_i$ $d\omega^K$-a.e.. Hence, $f_i=c_i\chi_{\{b'\}}$, $c_i>0$ $(i=1, 2)$. By (6.1) $H^M_{\chi_{E_1}}\wedge H^M_{\chi_{E_2}}=\int_{\Delta^M}\min(\chi_{E_1}, \chi_{E_2})d\omega^M=0$; this implies $\min(c_1, c_2)=0$, which is a contradiction. Thus, there exists at least one $b\in\Phi^{-1}(b')$ with positive harmonic measure. To prove the uniqueness of such b, suppose there exist two distinct points b_1, b_2 such that $\Phi(b_1)=\Phi(b_2)=b'$, and $\omega^M(\{b_i\})>0$ $(i=1, 2)$. As above, $H^M_{\chi_{\{b_i\}}}=c_iH^K_{\chi_{\{b'\}}}$ for some positive number c_i $(i=1, 2)$, but $0=H^M_{\chi_{\{b_1\}}}\wedge H^M_{\chi_{\{b_2\}}}=\min(c_1, c_2)\,H^K_{\chi_{\{b'\}}}$, which is a contradiction, and the theorem follows.

Since in the statement (i) of Theorem 8 we have not assumed $MHB(R)=MHD(R)$, we have the following corollary which was obtained by Constantinescu-Cornea ([2], Folgesatz, 11.2, p. 123).

Corollary. $U_{HB}\subset U_{HD}$.

Lemma 3. *Suppose* $MHB(R)=MHD(R)$. *For every* $d\omega^M$*-measurable set* A *on* Δ^M *we can find a* $d\omega^K$*-measurable set* A' *such that*

1) $H_A^M = H_{A'}^K$ [16],
2) $A' \subset \Phi(A)$,
3) $\omega^M(A - \Phi^{-1}(A')) = 0$.

Proof. In view of our assumption, we can find $d\omega^K$-measurable set A' fulfilling $H_A^M = H_{A'}^K$. We shall show $\omega^K(A' - \Phi(A)) = 0$. Actually, since $\bar{H}_{A'-\Phi(A)}^K \leq H_{A'}^K = H_A^M$ and since, by the Remark 1 in §2, $H_{A'-\Phi(A)}^K = \bar{H}_{\Phi^{-1}(A'-\Phi(A))}^M \leq H_{\Delta^M - A}^M$ we have

$$0 \leq \bar{H}_{A'-\Phi(A)}^K \leq H_A^M \wedge H_{\Delta^M - A}^M = 0 .$$

This enables us to select A' in the subset of $\Phi(A)$. Next, if there exists $d\omega^M$-measurable set P of positive harmonic measure fulfilling $P \subset A - \Phi^{-1}(A')$, then by what we have proved, we can find a $d\omega^K$-measurable set P', such that $P' \subset \Phi(P)$ and $0 < H_P^M = H_{P'}^K$. From $H_P^M \leq H_A^M = H_{A'}^K$ and $\Phi(P) \cap A' = \phi$, it is derived that $0 < H_P^M \leq H_{A'}^K \wedge H_{P'}^K = 0$. This contradiction tells us $\omega^M(A - \Phi^{-1}(A')) = 0$, and we obtain the assertion of our lemma.

Theorem 9. *In order to be $MHB(R) = MHD(R)$, it is necessary and sufficient that except a set E of $d\omega^M$-harmonic measure zero on Δ^M, Φ defines a one-to-one mapping from $\Delta_1'^M - E$ to its image.*

Proof. Let $\{G_n\}$ be a countable base for Δ^M. For $a \in \Delta^M$ we write $\mathfrak{G}_a = \{G_j;\, a \in G_j\}$. For every G_n the set in Δ^K described in the preceding lemma is denoted by $B(G_n)$. Since $H_{G_n}^M = H_{B(G_n)}^K$, we have $\omega^K(B(G_m) \cap B(G_n)) = 0$ whenever $G_m \cap G_n = \phi$. We set

$$X = \{a \in \Delta_1'^M;\, \Phi(a) \in \bigcap_{G_j \in \mathfrak{G}_a} B(G_j)\}$$

and

$$E' = \cup \{B(G_m) \cap B(G_n)\} ,$$

where the union ranges over all suffices m, n such that $G_m \cap G_n = \phi$. We set also $X' = \Phi(X)$ and $E = \Phi^{-1}(E')$. We shall remark $\omega^M(\Delta^M - X) = \omega^K(\Delta^K - X')$. In fact, if $a \in \Delta_1'^M - X$, then $a' = \Phi(a) \notin \bigcap_{G_j \in \mathfrak{G}_a} B(G_j)$, that is, $a' \notin B(G_{j_0})$ for some $G_{j_0} \in \mathfrak{G}_a$; this means $a \in G_{j_0} - \Phi^{-1}(B(G_{j_0}))$. In other words, $\Delta_1'^M - X \subset \bigcup_{n=1}^{\infty} \{G_n - \Phi^{-1}(B(G_n))\}$ and by the preceding lemma, the latter set is of $d\omega^M$-measure zero. Next, since $\Phi^{-1}(\Delta^K - X') = \Delta_1'^M - \Phi^{-1}(X') \subset \Delta^M - X$, we have $\omega^K(\Delta^K - X') = 0$. If we have $a' \in \Phi(a_1) = \Phi(a_2)$ for distinct points a_1, $a_2 \in X - E$, then $a' \in (\bigcap_{G_j \in \mathfrak{G}_{a_1}} B(G_j)) \cap (\bigcap_{G_j' \in \mathfrak{G}_{a_2}} B(G_j'))$ and $a' \notin E'$. On the other

16) It is denoted by H_A^M the solution $H_{\chi_A}^M$ for brevity, and in the following we shall use this notation throughout.

hand, we find G_k, G_l such that $G_k \in \mathfrak{G}_{a_1}$ and $G_l \in \mathfrak{G}_{a_2}$ and $G_k \cap G_l = \phi$. Then $a' \in B(G_k) \cap B(G_l)$, consequently $a' \in E'$, which is a contradiction. Hence, we see that the mapping Φ gives a one-to-one correspondence from $X-E$ onto $X'-E'$ and that the complements of these sets are of harmonic measure zero respectively. Since, as is stated at the beginning of this section, the converse is trivial we can complete the proof.

We shall use the terminology "canonical domain" as follows: a canonical domain is relatively compact domain whose boundary is composed of a finite number of piecewise analytic curves.

Theorem 10. *In order that the condition $MHB(R)=MHD(R)$ be satisfied, it is necessary and sufficient that for every canonical domain R_1 each component S of $R-R_1$ satisfies the condition $MHB(S)=MHD(S)$.*

Proof. Let S be one of the component of $R-R_1$. The Martin (Kuramochi) boundary of S is decomposed into two parts, one of which is homeomorphic to a simple curve of ∂R_1. The remaining part will be denoted by $\underline{\Delta}^M(\underline{\Delta}^K)$. It is well-known that $\underline{\Delta}^M$ is homeomorphic to a subset of Δ^M and by Theorem 6, $\underline{\Delta}^K$ is also homeomorphic to a subset of Δ^K. If $MHB(R)=MHD(R)$, then in virtue of Theorem 9, we see that except a set of harmonic measure zero, Φ makes correspond Δ^M to Δ^K one-to-one manner. This is also true for between $\underline{\Delta}^M$ and $\underline{\Delta}^K$. Thus, we have $MHB(S)=MHD(S)$. Next, assume that $MHB=MHD$ holds for each component. Let $u \in MHB(R)$. The restriction of u on each component, S say, is a function in $MHB(S)$ and also in $MHD(S)$. This means fine boundary values $\hat{u}$ of u take equal value on $\Phi^{-1}(a')$, from which we conclude $u \in MHD(R)$ and complete the proof.

7. Convergence in the Dirichlet norm

In [4], J.L. Doob proved that for $u \in HD$ if we denote by u' the fine boundary values of u at Δ^M, then we have

$$(7.1) \qquad \int_{\Delta^M} |u|^2 d\omega^M \leq c \cdot \mathrm{D}[u],$$

where c is a constant not depending on u, and $\mathrm{D}[u]$ is the Dirichlet integral of u over R. With regard to this theorem, we can get a relation of boundary functions correspoding to HD functions which converges in the Dirichlet norm.

Theorem 11. *Let u_n, $u \in HD$ $(n=1, 2, \cdots)$ and $\lim_{n\to\infty} \mathrm{D}[u_n-u]=0$. Let $\hat{u}_n$ and $\hat{u}$ be the quasi-continuous extensions of u_n and u respectively. Then, there exists a subsequence $\{\hat{u}_{n_\nu}\}$ of $\{\hat{u}_n\}$ converging to $\hat{u}$ $d\omega^K$-a.e.,* Proof. Without loss of generality, we can assume $u=0$. Let u_n' be the fine boundary values of u_n on Δ^M. We have $u_n'=\hat{u}_n\circ\Phi$ $d\omega^M$-a.e.. Consequently $|u_n'|^2=|\hat{u}_n|^2\circ\Phi$ $d\omega^M$-a.e.. This implies $H^M_{u_n'^2}=H^K_{\hat{u}_n^2}$. By the preliminary remark,

$$\int|\hat{u}_n|^2 d\omega^K=\int|u_n|^2 d\omega^M\leq c\cdot \mathrm{D}[u_n]\,.$$

Hence, there exists a subsequence $\{\hat{u}_{n_\nu}\}$ which converges to zero $d\omega^K$-a.e.. Thus, we obtain the theorem.

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