Taniguchi, K.
Osaka J. Math.
21 (1984), 169-224

MULTI-PRODUCTS OF FOURIER INTEGRAL OPERATORS AND THE FUNDAMENTAL SOLUTION FOR A HYPERBOLIC SYSTEM WITH INVOLUTIVE CHARACTERISTICS

Dedicated to the memory of Professor Hitoshi Kumano-go

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(Received July 6, 1982)

Introduction. Let $\mathcal{L}$ be a hyperbolic system with the diagonal principal part

$$(1) \qquad \mathcal{L}=D_t-\begin{bmatrix}\lambda_1(t,X,D_x) & & 0\\ & \ddots & \\ 0 & & \lambda_l(t,X,D_x)\end{bmatrix}+(b_{mk}(t,X,D_x)).$$

In order to consider the propagation of singularities of solutions of an equation $\mathcal{L}U(t)=0$, we frequently employ a method of constructing the fundamental solution $\boldsymbol{E}(t,s)$ and investigating its properties. In Kumano-go-Taniguchi-Tozaki [11] and Kumano-go-Taniguchi [10] the fundamental solution $\boldsymbol{E}(t,s)$ of the hyperbolic system $\mathcal{L}$ has been constructed in the form

$$(2) \qquad \boldsymbol{E}(t,s)=\boldsymbol{I}_\phi(t,s)+\int_s^t \boldsymbol{I}_\phi(t,\theta)\{\boldsymbol{W}_\phi(\theta,s) \\ +\sum_{\nu=2}^\infty\int_s^\theta\int_s^{t_1}\cdots\int_s^{t_{\nu-2}}\boldsymbol{W}_\phi(\theta,t_1)\boldsymbol{W}_\phi(t_1,t_2)\cdots \\ \times\boldsymbol{W}_\phi(t_{\nu-1},s)dt_{\nu-1}\cdots dt_1\}d\theta \qquad (t_0=\theta),$$

where $\boldsymbol{I}_\phi(t,s)$ and $\boldsymbol{W}_\phi(t,s)$ are $l\times l$ matrices of Fourier integral operators $P_\phi(t,s)$ defined by $P_\phi(t,s)u=\int e^{i\phi(t,s;x,\xi)}p(t,s;x,\xi)\hat{u}(\xi)d\xi$. The expression (2) is obtained by constructing, first, an approximate fundamental solution $\boldsymbol{I}_\phi(t,s)$ and next applying the method of the successive approximation. When we want to derive some properties of $\boldsymbol{E}(t,s)$ from (2), it is necessary to estimate the multi-product

$$(3) \qquad \tilde{Q}_{\nu+1}=P_{1,\phi_1}P_{2,\phi_2}\cdots P_{\nu+1,\phi_{\nu+1}}$$

of Fourier integral operators P_{j,ϕ_j}. In the present paper, we will show an estimate of $\tilde{Q}_{\nu+1}$ and apply it to reduce $\boldsymbol{E}(t, s)$ of (2) to a finite sum expression

$$(4) \qquad \boldsymbol{E}(t, s) = \boldsymbol{W}_{\Phi}^{0}(t, s)+\sum_{\nu=2}^{l}\int_s^t\int_s^{t_1}\cdots\int_s^{t_{\nu-2}} \boldsymbol{W}_{\nu,\Phi_\nu}^{0}(t, t_1, \cdots, t_{\nu-1}, s)dt_{\nu-1}\cdots dt_1 \qquad (t_0 = t)$$

when the operator (1) is involutive. The expression (4) gives us information on the propagation of singularities.

Let $S_{\rho,\delta}^{m}$ $(-\infty<m<\infty,\ 0\leqq\delta\leqq\rho\leqq 1,\ \delta<1)$ denote a class of symbols $p(x, \xi)$ of pseudo-differential operators in R^n which is defined in Definition 1.1 of Chap. 2 in [8], and set $S_\rho^m=S_{\rho,1-\rho}^m$ for $1/2\leqq\rho\leqq 1$, $S_{\rho,\delta}^{\infty}=\bigcup_m S_{\rho,\delta}^m$ and $S^{-\infty}=\bigcap_m S_1^m$. The class $S_{\rho,\delta}^m$ is a Fréchet space with semi-norms

$$(5) \qquad |p|_{l_1,l_2}^{(m)}=\max_{|\alpha|\leqq l_1,|\beta|\leqq l_2}\sup_{x,\xi}\{|p_{(\beta)}^{(\alpha)}(x, \xi)|\langle\xi\rangle^{-(m-\rho|\alpha|+\delta|\beta|)}\},$$

where $p_{(\beta)}^{(\alpha)}(x, \xi)=\partial_\xi^\alpha D_x^\beta p(x, \xi)$, $D_x^\beta=(-i)^{|\beta|}\partial_x^\beta$ and $\langle\xi\rangle=\sqrt{1+|\xi|^2}$ (c.f. § 1 of Chap. 2 and (1.13) of Chap. 7 in [8]). Let $\mathcal{P}_\rho(\tau, l)$ $(0\leqq\tau<1,\ 1/2\leqq\rho\leqq 1,\ l=0, 1, 2, \cdots)$ be the class of phase functions $\phi(x, \xi)$ such that $J(x, \xi)\equiv\phi(x, \xi)-x\cdot\xi$ $(x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n)$ satisfy $J_{(\beta)}^{(\alpha)}\in S_\rho^{1-|\alpha|}$ for $|\alpha|+|\beta|\leqq 2$ and

$$(6) \qquad \|J\|_l\equiv\sum_{|\alpha+\beta|\leqq 2+l}\sup_{x,\xi}\{|J_{(\beta)}^{(\alpha)}(x, \xi)|\langle\xi\rangle^{-(1-|\alpha|+(1-\rho)(|\alpha+\beta|-2)_+)}\}\leqq\tau,$$

where $a_+=\max(a, 0)$ for a real a. We set $\mathcal{P}_\rho(\tau)=\mathcal{P}_\rho(\tau, 0)$. For $\phi(x, \xi)\in\mathcal{P}_\rho(\tau)$ and $p(x, \xi)\in S_{\rho,\delta}^m$ we define a Fourier integral operator $P_\phi=p_\phi(X, D_x)$ with phase function $\phi(x, \xi)$ and symbol $\sigma(P_\phi)=p(x, \xi)$ by

$$(7) \qquad P_\phi u=\int e^{i\phi(x,\xi)}p(x, \xi)\hat{u}(\xi)đ\xi \qquad \textit{for} \quad u\in\mathcal{S}.$$

Here, $đ\xi=(2\pi)^{-n}d\xi$, $\mathcal{S}$ is the Schwartz space of rapidly decreasing functions on R^n and $\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx$ is the Fourier transform of $u(x)$. In (7) P_ϕ is a pseudo-differential operator when $\phi(x, \xi)=x\cdot\xi$. In this case we write $P_\phi=p_\phi(X, D_x)$ simply by $P=p(X, D_x)$ and we often say that P is a pseudo-differential operator in $S_{\rho,\delta}^m$. For $p_j(x, \xi)\in S_{\rho,\delta}^{m_j}$ $(j=1, 2, \cdots)$ we say that $\{p_j\}$ is bounded in $\{S_{\rho,\delta}^{m_j}\}$ if the set $\{|p_j|_{l_1,l_2}^{(m_j)}\}$ of semi-norms $|p_j|_{l_1,l_2}^{(m_j)}$ is bounded for any l_1 and l_2.

Concerning the multi-products of Fourier integral operators the following is shown in Kumano-go-Taniguchi [10] for the case $\rho>1/2$[1)].

Let $\phi_j(x, \xi)$ belong to $\mathcal{P}_\rho(\tau_j)$ and let $p_j(x, \xi)$ belong to $S_\rho^{m_j}$ $(j=1, 2, \cdots)$. Suppose that

1) Their proof is also valid for $\rho=1/2$.

(*) $\sum_{j=1}^{\infty}\tau_j\leqq\tau^0$ *for some positive constant* τ^0 *and for* $J_j(x, \xi)\equiv\phi_j(x, \xi)-x\cdot\xi$ *the set* $\{J_{j(\beta)}^{(\alpha)}/\tau_j\}$ *is bounded in* $S_\rho^{1-|\alpha|}$ *when* $|\alpha+\beta|\leqq 2$.

Then, for any ν *the multi-product* $\tilde{Q}_{\nu+1}$ *of* (3) *is a Fourier integral operator* $Q_{\nu+1,\Phi_{\nu+1}}$ *with a phase function* $\Phi_{\nu+1}(x, \xi)$ *in* $\mathcal{P}_\rho(c_0\bar{\tau}_{\nu+1})$ $(\bar{\tau}_{\nu+1}=\tau_1+\tau_2+\cdots+\tau_{\nu+1})$ *for some constant* c_0 *and with a symbol* $q_{\nu+1}(x, \xi)$ *in* $S_\rho^{\bar{m}_{\nu+1}}$ *for* $\bar{m}_{\nu+1}=m_1+m_2+\cdots m_{\nu+1}$. (c.f. *Theorem* 2.3 *of* [10]).

The result we want to show on $\tilde{Q}_{\nu+1}$ of (3) is the following:

Theorem 1. *Suppose that* $\phi_j(x, \xi)$ *belongs to* $\mathcal{P}_\rho(\tau_j, \tilde{l}_0)$, $j=1, 2, \cdots$, *and* (*) *holds, where* $\tilde{l}_0$ *is an integer determined only by* ρ *and* n. *Then, for each bounded set* $\{p_j\}$ *in* $\{S_\rho^{m_j}\}$ *there exists a constant* C_0 *such that the set* $\{C_0^{-\nu}q_{\nu+1}\}$ *is bounded in* $\{S_\rho^{\bar{m}_{\nu+1}}\}$ *if we assume* $\sum_{j=1}^{\infty}|m_j|<\infty$.

Concerning estimates of multi-products (3) Kumano-go–Taniguchi [10] gave only operator norms in Sobolev spaces, but they did not show estimates of symbols. To obtain their estimates they used essentially asymptotic expansions of products of Fourier integral operators, and it seems to us that it is almost impossible to obtain the estimates including the case $\rho=1/2$.

In order to prove Theorem 1 we must employ a method completely different from [10]. First we show the fact that there exist pseudo-differential operators R and R' in S_ρ^0 such that

$$(8)\qquad \begin{cases} I_\phi R I_{\phi^*}=I\,, \\ I_{\phi^*}R' I_\phi=I \end{cases}$$

hold for a phase function $\phi(x, \xi)$ in $\mathcal{P}_\rho(\bar{\tau}, \tilde{l}_0)$ if $\bar{\tau}$ is small enough, where I_ϕ [resp. I_{ϕ^*}] is the Fourier [resp. conjugate Fourier] integral operator with phase function $\phi(x, \xi)$ and symbol 1. Then, the multi-product (3) can be written in the form

$$(9)\qquad \begin{cases} \text{i)}\quad \tilde{Q}_{\nu+1}=P_1'P_2'\cdots P_{\nu+1}'I_{\Phi_{\nu+1}}\,, \\ \text{ii)}\quad \tilde{Q}_{\nu+1}=I_{\Phi_{\nu+1}}P_1''P_2''\cdots P_{\nu+1}'' \end{cases}$$

with pseudo-differential operators P_j' and P_j'' in $S_\rho^{m_j}$ $(j=1, 2, \cdots)$. Thus, the problem to estimate the symbol $q_{\nu+1}(x, \xi)$ of a multi-product $\tilde{Q}_{\nu+1}=Q_{\nu+1,\Phi_{\nu+1}}$ of Fourier integral operators is reduced to the problem of obtaining an estimate of a multi-product of pseudo-differential operators. Therefore, it is the key point in the proof of Theorem 1 to show the existence of pseudo-differential operators R and R' verifying (8). To show their existence, it is necessary to obtain a sharp estimate of symbols of multi-products of pseudo-differential operators. Our theorem concerning multi-products of pseudo-differential operators is the following.

Theorem 2. *Let* $0\leqq\delta\leqq\rho\leqq 1$, $\delta<1$ *and let* $p_j(x, \xi)\in S^{m_j}_{\rho,\delta}$, $j=1, 2, \cdots$. *Consider the multi-product*

$$(10) \qquad Q_{\nu+1}=P_1P_2\cdots P_{\nu+1}$$

of $P_j=p_j(X, D_x)$. *Denote by* $q_{\nu+1}(x, \xi)$ *the symbol of* $Q_{\nu+1}$. *Then, there exists a constant* A *determined only by* δ *and* n *such that* $M\equiv\sum_{j=1}^{\infty}|m_j|<\infty$ *and the boundedness of* $\{p_j\}$ *in* $\{S^{m_j}_{\rho,\delta}\}$ *imply the boundedness of* $\{C_O^{-\nu}q_{\nu+1}\}$ *in* $\{S^{\overline{m}_{\nu+1}}_{\rho,\delta}\}$ $\overline{m}_{\nu+1}=m_1+\cdots+m_{\nu+1}$) *with*

$$(11) \qquad C_O=A\max_j|p_j|^{(m_j)}_{n+1,l_0}$$

for

$$(12) \qquad l_0=[n/(1-\delta)+1]\,.$$

Since the $(\nu+1)$-st power $P^{\nu+1}$ of a pseudo-differential operator P with symbol $p(x, \xi)$ in $S^0_{\rho,\delta}$ satisfies

$$(13) \qquad |\sigma(P^{\nu+1})|^{(0)}_{l_1,l_2}\leqq C_{l_1,l_2}(A|p|^{(0)}_{n+1,l_0})^{\nu}\,,$$

we get immediately

Theorem 3. *Assume that* $p(x, \xi)$ *in* $S^0_{\rho,\delta}$ *satisfies*

$$(14) \qquad |p|^{(0)}_{n+1,l_0}<1/A$$

for the constant A *in Theorem* 2. *Then, the inverse* Q *of the operator* $I-P$ *exists and is a pseudo-differential operator in* $S^0_{\rho,\delta}$ *represented by the Neumann series* $\sum_{\nu=0}^{\infty}P^{\nu}$.

The existence of R and R' in (8) is derived by applying Theorem 3 to $I_{\phi*}I_{\phi}-I$ and $I_{\phi}I_{\phi*}-I$. Concerning the estimate of multi-products of pseudo-differential operators Kumano-go obtained in [6] a semi-norm estimate

$$(15) \qquad |q_{\nu+1}|^{(\overline{m}_{\nu+1})}_{(l_1,l_2)}\leqq C_{l_1,l_2}{}^{\nu}\,.$$

The estimate (15) is effectively used for the construction of the fundamental solution of a parabolic equation (see, for example, § 4 of Chap. 7 in [8]) and also used for the L^2-boundedness of a pseudo-differential operator (see [6]). But the estimate (15) is not sufficient for the proof of the convergence of the Neumann series $\sum_{\nu=0}^{\infty}P^{\nu}$. Hence, we need the estimate (13) sharper than (15). Using this estimate (13) we prove the convergence of the Neumann series.

We like to emphasize that by virtue of Theorem 2 the inverse of a pseudo-differential operator may be obtained only by the symbol calculus when (14) is satisfied. In [1] Beals has proved that the inverse of a pseudo-differential operator is also a pseudo-differential operator, but he showed it by a discussion in Sobolev spaces, not by the symbol calculus. In Appendix of [8] Kumano-go has given another proof of the convergence of Neumann series by using the commutator theory and the symbol calculus.

Now, we return to the problem of the reduction of the fundamental solution $\boldsymbol{E}(t, s)$ of (2) for $\mathcal{L}$ to the expression (4). Let $M^0([0, T]; S^m_\rho((k)))$ [resp. $M([0, T]; S^m_\rho((k)))$] be the set of symbols $p(t, x, \xi)$ such that $p^{(\alpha)}_{(\beta)}(t, x, \xi)$ [resp. $\partial^\gamma_t p^{(\alpha)}_{(\beta)}(t, x, \xi)$ for any γ] are bounded in $S^{m-|\alpha|}_\rho$ for any $t\in[0, T]$ when $|\alpha|+|\beta|\leqq k$; and we also set $M^0([0, T]; S^m_\rho)=M^0([0, T]; S^m_\rho((0)))$ and $M([0, T]; S^m_\rho)=M([0, T]; S^m_\rho((0)))$ (for details, see Definition 2.4 and Definition 3.1). In the present paper, we shall consider a system (1) under the following condition (I) or (II).

(I) The characteristic roots $\lambda_m(t, x, \xi)$ belong to $M^0([0, T]; S^1_\rho((2)))\cap C^1([0, T]\times R^{2n}_{x,\xi})$ and the symbols $b_{mk}(t, x, \xi)$ in (1) belong to $M^0([0, T]; S^0_\rho)$. For any m and k there exists a continuous function $a_{m,k}(t)$ such that the Poisson bracket $\{\tau-\lambda_m, \tau-\lambda_k\}$ of $\tau-\lambda_m$ and $\tau-\lambda_k$ satisfies

$$\{\tau-\lambda_m, \tau-\lambda_k\} = a_{m,k}(t)(\lambda_m-\lambda_k)\,. \tag{16}$$

(II) The characteristic roots $\lambda_m(t, x, \xi)$ belong to $M([0, T]; S^1_\rho((3)))$ and the symbols $b_{mk}(t, x, \xi)$ in (1) belong to $M([0, T]; S^0_\rho)$. For any m and k there exist real symbols $a_{m,k}(t, x, \xi)$ and $a'_{m,k}(t, x, \xi)$ with

$$\begin{cases} a_{m,k}(t, x, \xi)\in M([0, T]; S^0_\rho((1)))\,, \\ a'_{m,k}(t, x, \xi)\in M([0, T]; S^0_\rho) \end{cases} \tag{17}$$

such that

$$\{\tau-\lambda_m, \tau-\lambda_k\} = a_{m,k}(t, x, \xi)(\lambda_m-\lambda_k)+a'_{m,k}(t, x, \xi) \tag{18}$$

holds.

By using Theorem 1 and the commutative law for $\#$-products of phase functions (Theorem 3.9) we obtain

Theorem 4. *Under the condition* (I) *or* (II) *the fundamental solution* $\boldsymbol{E}(t, s)$ *of* (2) *can be reduced to the expression* (4).

The expression (4) of $\boldsymbol{E}(t, s)$ gives us much information on the propagation of singularities of the solution of $\mathcal{L}U(t)=0$. For example, the estimate of singularities obtained in [13] follows immediately from (4) (see Corollary 4.5). Concerning the expression (4) of $\boldsymbol{E}(t, s)$ Ludwig-Granoff [12], Hata [2] and

Nosmas [14] obtained it only for $\rho=1$. They constructed it by a method of solving transport equations. On the other hand, Kumano-go, Taniguchi and Tozaki have proved in [10]–[11] Theorem 4 without solving transport equations under a stronger assumption than (I), that is, $\rho=1$ and $a_{m,k}(t)$ in (16) are identically zero, and Morimoto [13] has also obtained it under the assumption (I) with $\rho=1$ and C^∞-functions $a_{m,k}(t)$ in (16). We note that Ichinose [5] also showed Theorem 4 in the case of $l=2$, $\rho>1/2$ and (I).

Theorem 4 with $\rho<1$ makes us possible to treat hyperbolic equations with characteristic roots which are not necessary C^∞ differentiable. Namely, with the aid of the approximation theory by [9] the hyperbolic equations can be reduced to the hyperbolic systems (1) with symbols in the class $S_\rho^\infty=S_{\rho,1-\rho}^\infty$ (see [5], for details). The less differentiable the characteristic roots are, the smaller ρ ($\geqq 1/2$) we need. For example, we consider a hyperbolic operator L_1 in R_x^2:

$$(19) \qquad L_1 = D_t^2 - a_k(x)(D_{x_1}{}^2 + D_{x_2}{}^2),$$

where $a_k(x)$ ($k\geqq 2$) is a C^∞-function satisfying

$$\begin{cases} a_k(x) = x_1{}^{2k} + x_2{}^{2k} \quad (|x|\leqq 1), \qquad = 2 \quad (|x|\geqq 2), \\ 0<a_0\leqq a_k(x)\leqq 2 \quad (|x|\geqq 1) \qquad \textit{for some } a_0 . \end{cases}$$

The operator L_1 has characteristic roots $\lambda_\pm(x,\xi)=\pm\sqrt{a_k(x)}\,|\xi|$ which are C^{k-1}-class with Lipschitz derivatives of $(k-1)$-st order for $|\xi|\geqq 1$. The operator (19) with $k\geqq 5$ was considered in [9] and (19) with $k=4$ in [5]. Including the cases $k=2$ and 3 we shall show that (19) can be reduced to a system (1) with symbols in S_ρ^∞ for $\rho=1-1/k$ and investigate the propagation of singularities. For the case $k=2$ we need $\rho=1/2$. Other examples which can be reduced to the system with $\rho=1/2$ are

$$(20) \qquad L_2 = D_t^2 - a(x_1)^2(D_{x_1}{}^2 + a(x_1)^2 D_{x_2}{}^2),$$

$$(21) \qquad L_3 = D_t^2 - 2a(x_1)^2 D_{x_1} D_t - a(x_1)^6 D_{x_2}{}^2,$$

where $a(x_1)$ is a C^∞-function in $R_{x_1}^1$ satisfying

$$\begin{cases} a(x_1) = x_1 \quad (|x_1|\leqq 1), \quad =\pm 2 \qquad (x_1 \gtrless 2), \\ 0<a_0\leqq |a(x_1)|\leqq 2 \quad (|x_1|\geqq 1) \qquad \textit{for some } a_0 . \end{cases}$$

The reduction of L_j ($j=1, 2, 3$) to the system (1) and the information on the propagation of singularities are given at the end of Section 4.

The outline of the present paper is the following: In Section 1 we shall study multi-products of pseudo-differential operators. Section 2 is devoted to the proof of Theorem 1. In Section 3 we shall prove the commutative law for $\#$-products of phase functions and in Section 4 we shall construct the fundamental solution of (1) and prove Theorem 4.

The author would like to thank Prof. H. Kumano-go for his kind suggestions and constant help. The author would like to express his gratitude to Prof. M. Ikawa for his helpful advice and refinement of this paper, and also to Prof. K. Shinkai and Prof. N. Ideka for their kind help and encouragements.

1. Multi-products of pseudo-differential operators and Neumann series. Let $(x^0, \tilde{x}^\nu)=(x^0, x^1, \cdots, x^\nu)$ be a $(\nu+1)$-tuple of points $x^0, x^1, \cdots, x^\nu$ in R^n_x and $\tilde{\xi}^{\nu+1}=(\xi^1, \cdots, \xi^{\nu+1})$ be a $(\nu+1)$-tuple of points $\xi^1, \cdots, \xi^{\nu+1}$ in R^n_ξ.

DEFINITION 1.1. Let $0\leqq\delta\leqq\rho\leqq1$, $\delta<1$ and let $\tilde{m}_{\nu+1}=(m_1, \cdots, m_{\nu+1})$ be a real vector. We say that a C^∞-function $p(x^0, \tilde{x}^\nu, \tilde{\xi}^{\nu+1}, x^{\nu+1})=p(x^0, \xi^1, x^1, \xi^2, \cdots, x^\nu, \xi^{\nu+1}, x^{\nu+1})$ in $R^{(2\nu+3)n}$ belongs to a multiple symbol class $S^{\tilde{m}_{\nu+1}}_{\rho,\delta}$ when

$$(1.1)\qquad |\partial^{\alpha^1}_{\xi^1}\cdots\partial^{\alpha^{\nu+1}}_{\xi^{\nu+1}}D^{\beta^0}_{x^0}D^{\beta^1}_{x^1}\cdots D^{\beta^{\nu+1}}_{x^{\nu+1}}p(x^0, \tilde{x}^\nu, \tilde{\xi}^{\nu+1}, x^{\nu+1})|$$
$$\leqq C_{\tilde{\alpha}^{\nu+1},\beta^0,\tilde{\beta}^{\nu+1}}\prod_{j=1}^{\nu+1}\langle\xi^j\rangle^{m_j-\rho|\alpha^j|}\langle\xi^1\rangle^{\delta|\beta^0|}\prod_{j=1}^{\nu}\langle\xi^j;\,\xi^{j+1}\rangle_\delta^{|\beta^j|}\langle\xi^{\nu+1}\rangle^{\delta|\beta^{\nu+1}|}$$

holds for any $(\nu+1)$-tuple $\tilde{\alpha}^{\nu+1}=(\alpha^1, \cdots, \alpha^{\nu+1})$ and $(\nu+2)$-tuple $(\beta^0, \tilde{\beta}^{\nu+1})=(\beta^0, \beta^1, \cdots, \beta^{\nu+1})$ of multi-indices $\alpha^1, \cdots, \alpha^{\nu+1}$ and $\beta^0, \beta^1, \cdots, \beta^{\nu+1}$ of R^n, where $\langle\xi;\,\xi'\rangle_\delta=\langle\xi\rangle^\delta+\langle\xi'\rangle^\delta$. For $p(x^0, \tilde{x}^\nu, \tilde{\xi}^{\nu+1}, x^{\nu+1})\in S^{\tilde{m}_{\nu+1}}_{\rho,\delta}$ we define semi-norms $|p|^{(\tilde{m}_{\nu+1})}_{l_1,l_2}$ by

$$(1.2)\qquad |p|^{(\tilde{m}_{\nu+1})}_{l_1,l_2}=\max\inf\{C_{\tilde{\alpha}^{\nu+1},\beta^0,\tilde{\beta}^{\nu+1}} \text{ of } (1.1)\}\,,$$

where the maximum is taken over all $(\tilde{\alpha}^{\nu+1}, \beta^0, \tilde{\beta}^{\nu+1})$ satisfying $|\alpha^j|\leqq l_1$ $(j=1, \cdots, \nu+1)$ and $|\beta^j|\leqq l_2$ $(j=0, 1, \cdots, \nu+1)$.

REMARK. The multiple symbol class was introduced in Kumano-go [6]. But the semi-norms (1.2) are slightly different from semi-norms (2.4) of [6]. Corresponding the multiple symbol class, the class $S^m_{\rho,\delta}$ in Introduction is often called a single symbol class.

For $p(x^0, \tilde{x}^0, \tilde{\xi}^{\nu+1}, x^{\nu+1})=p(x^0, \xi^1, x^1, \xi^2, \cdots, x^\nu, \xi^{\nu+1}, x^{\nu+1})$ in $S^{\tilde{m}_{\nu+1}}_{\rho,\delta}$ $p(X, D_x, X^1, D_{x^1}, \cdots, X^\nu, D_{x^\nu}, X^{\nu+1})$ denotes a pseudo-differential operator P defined by

$$(1.3)\qquad (Pu)(x^0)=O_s-\iint \exp\{i\sum_{j=1}^{\nu+1}(x^{j-1}-x^j)\cdot\xi^j\}$$
$$\times p(x^0, \xi^1, x^1, \xi^2, \cdots, x^\nu, \xi^{\nu+1}, x^{\nu+1})$$
$$\times u(x^{\nu+1})dx^1\cdots dx^{\nu+1}đ\xi^1\cdots đ\xi^{\nu+1}\qquad \textit{for}\quad u\in\mathcal{S}\,,$$

and $\sigma(P)=p(x^0, \tilde{x}^\nu, \tilde{\xi}^{\nu+1}, x^{\nu+1})$ is called a symbol of P. Here, the right hand side of (1.3) is the oscillatory integral defined in Section 6 of Chap. 1 in [8].

Throughout this paper, we shall often use the result there. Following Kumano-go [8], we write $p(X, D_x, X^1)$, $p(X, D_x, X^1, D_{x^1})$ and $p(X, D_x, X^1, D_{x^1}, X^2)$ by $p(X, D_x, X')$, $p(X, D_x, X', D_{x'})$ and $p(X, D_x, X', D_{x'}, X'')$, respectively. For $p(x^0, \xi^1, x^1, \cdots, \xi^\nu, x^\nu, \xi^{\nu+1}, x^{\nu+1})$ in $S_{\rho,\delta}^{\bar{m}_{\nu+1}}$ we write

$$(1.4)\qquad p_L(x, \xi, x') = O_s-\iint e^{-i\psi} p(x, \xi+\eta^1, x+y^1, \cdots, \xi+\eta^\nu, x+y^\nu, \xi, x') d\tilde{y}^\nu d\tilde{\eta}^\nu,$$

called a simplified symbol of $p(x^0, \xi^1, x^1, \cdots, \xi^\nu, x^\nu, \xi^{\nu+1}, x^{\nu+1})$, where

$$(1.5)\qquad \psi = \sum_{j=1}^{\nu} y^j\cdot(\eta^j-\eta^{j+1}) = \sum_{j=1}^{\nu}(y^j-y^{j-1})\cdot\eta^j \qquad (y^0=\eta^{\nu+1}=0),$$

$\tilde{y}^\nu=(y^1, \cdots, y^\nu)\in R^{n\nu}_{\tilde{y}^\nu}$, $\tilde{\eta}^\nu=(\eta^1, \cdots, \eta^\nu)\in R^{n\nu}_{\tilde{\eta}^\nu}$ and $d\tilde{y}^\nu d\tilde{\eta}^\nu=dy^1\cdots dy^\nu d\eta^1\cdots d\eta^\nu$. It is well-known that $p_L(x, \xi, x')$ belongs to $S_{\rho,\delta}^{\bar{m}_{\nu+1}}$ ($\bar{m}_{\nu+1}=m_1+m_2+\cdots+m_{\nu+1}$) and

$$(1.6)\qquad p(X, D_x, X^1, \cdots, X^\nu, D_{x^\nu}, X^{\nu+1}) = p_L(X, D_x, X') \qquad (c.f.\ \S 2\ of\ Chap.\ 7\ of\ [8]).$$

Now, we begin to *prove Theorem* 2. It is well-known from (2.6) and (2.8) of Chap. 7 in [8] that the symbol $q_{\nu+1}(x, \xi)$ of the multi-product $Q_{\nu+1}=P_1P_2\cdots P_{\nu+1}$ has the form

$$(1.7)\qquad q_{\nu+1}(x, \xi) = O_s-\int e^{-i\psi}\prod_{j=1}^{\nu+1} p_j(x+y^{j-1}, \xi+\eta^j)d\tilde{y}^\nu d\tilde{\eta}^\nu \qquad (y^0=\eta^{\nu+1}=0)$$

with ψ in (1.5). Differentiating $q_{\nu+1}(x, \xi)$ with respect to x and ξ we have

$$(1.8)\qquad q_{\nu+1}{}^{(\alpha)}_{(\beta)}(x, \xi) = \sum\nolimits_{\alpha,\beta,\nu+1}\frac{\alpha!\beta!}{\tilde{\alpha}^{\nu+1}!\tilde{\beta}^{\nu+1}!}q_{\nu+1,((\tilde{\alpha}^{\nu+1},\tilde{\beta}^{\nu+1}))}(x, \xi)$$

for

$$(1.9)\qquad q_{\nu+1,((\tilde{\alpha}^{\nu+1},\tilde{\beta}^{\nu+1}))}(x, \xi) = O_s-\iint e^{-i\psi}\prod_{j=1}^{\nu+1} p_j{}^{(\alpha^j)}_{(\beta^j)}(x+y^{j-1}, \xi+\eta^j)d\tilde{y}^\nu d\tilde{\eta}^\nu.$$

Here, $\tilde{\alpha}^{\nu+1}!=\alpha^1!\cdots\alpha^{\nu+1}!$, $\tilde{\beta}^{\nu+1}!=\beta^1!\cdots\beta^{\nu+1}!$ for $\tilde{\alpha}^{\nu+1}=(\alpha^1, \cdots, \alpha^{\nu+1})$, $\tilde{\beta}^{\nu+1}=(\beta^1, \cdots, \beta^{\nu+1})$ and $\sum_{\alpha,\beta,\nu+1}$ means that the summation is taken over all $(\tilde{\alpha}^{\nu+1}, \tilde{\beta}^{\nu+1})$ satisfying $\alpha^1+\cdots+\alpha^{\nu+1}=\alpha$ and $\beta^1+\cdots+\beta^{\nu+1}=\beta$. Note that (1.9) means

$$(1.10)\qquad Q_{\nu+1,((\tilde{\alpha}^{\nu+1},\tilde{\beta}^{\nu+1}))} = P_1{}^{(\alpha^1)}_{(\beta^1)}P_2{}^{(\alpha^2)}_{(\beta^2)}\cdots P_{\nu+1}{}^{(\alpha^{\nu+1})}_{(\beta^{\nu+1})}.$$

Set

$$(1.11)\qquad \varepsilon_0 = (1-\delta)(l_0-n/(1-\delta)) \quad (>0)$$

for the integer l_o in (12). In this section l_o and ε_o always mean the numbers defined by (12) and (1.11).

Proposition 1.2. *Let $\{m_j\}$ and $\{m'_j\}$ be sequences satisfying*

$$\varepsilon_1 \equiv \sum_{j=1}^{\infty} |m_j| < \varepsilon_o , \tag{1.12}$$

$$M' \equiv \sum_{j=1}^{\infty} |m'_j| < \infty , \tag{1.13}$$

$$N^0 \equiv \textit{the number of } \{j;\ m'_j > 0\} < \infty . \tag{1.14}$$

Let $Q^\circ_{\nu+1} = P^\circ_1 P^\circ_2 \cdots P^\circ_{\nu+1}$ for $P^\circ_j = p^\circ_j(X, D_x)$, $p^\circ_j(x, \xi) \in S^{m_j+m_j}_{\rho,\delta}$. Then, there exist a constant $\tilde{A}_O$ independent of M', N^0 and ν and a constant C depending only on M', N^0, n and δ (but independent of ν) such that the symbol $q^\circ_{\nu+1}(x, \xi)$ of $Q^\circ_{\nu+1}$ satisfies

$$\begin{aligned} &|q^\circ_{\nu+1}(x, \xi)| \\ &\leqq C\tilde{A}_O{}^\nu \max_{\substack{k_j=0,1 \\ k_2+\cdots k_{\nu+1} \leqq N^0+1}} \{ |p^\circ_1|^{(m_1+m'_1)}_{n+1,0} \prod_{j=2}^{\nu+1} |p^\circ_j|^{(m_j+m'_j)}_{n+1,\, l_o+k_j\mu^0} \} \langle\xi\rangle^{\bar{m}_{\nu+1}+\bar{m}'_{\nu+1}} \end{aligned} \tag{1.15}$$

with $\mu^0 = [M'/(1-\delta)]^*$, $\bar{m}_{\nu+1} = m_1 + \cdots + m_{\nu+1}$ and $\bar{m}'_{\nu+1} = m'_1 + \cdots + m'_{\nu+1}$. Here, for a real a we denote by $[a]^*$ the smallest integer not less than a.

Admitting this proposition, we apply it to each multi-product (1.10) by setting $P^\circ_j = P_{j(\beta_j)}^{(\alpha_j)}$. Take an integer N_o satisfying

$$\sum_{j=N_o+1}^{\infty} |m_j| < \varepsilon_o \tag{1.16}$$

and set for fixed $\tilde{\alpha}^{\nu+1} = (\alpha^1, \cdots, \alpha^{\nu+1})$ and $\tilde{\beta}^{\nu+1} = (\beta^1, \cdots, \beta^{\nu+1})$

$$\begin{cases} \mu_j = \begin{cases} 0 & j \leqq N_o , \\ m_j & j > N_o , \end{cases} \\ \mu'_j = \begin{cases} m_j - \rho|\alpha^j| + \delta|\beta^j| & j \leqq N_o , \\ -\rho|\alpha^j| + \delta|\beta^j| & j > N_o . \end{cases} \end{cases} \tag{1.17}$$

Then, $\mu_j + \mu'_j = m_j - \rho|\alpha^j| + \delta|\beta^j|$ and for $p^\circ_j = p_{j(\beta^j)}^{(\alpha^j)}$ the set $\{p^\circ_j\}$ satisfies the assumption of Proposition 1.2 with $\{m_j\}$ and $\{m'_j\}$ replaced by $\{\mu_j\}$ and $\{\mu'_j\}$, respectively. The number N^0 in the proposition does not exceed $N_o + \delta^*|\beta|$ if we set $\delta^* = [\delta]^*$, that is, $\delta^* = 0$ when $\delta = 0$ and $\delta^* = 1$ when $0 < \delta < 1$. Hence, we obtain from (1.15)

$$\begin{aligned} &|q_{\nu+1,((\tilde{\alpha}^{\nu+1},\tilde{\beta}^{\nu+1}))}(x, \xi)| \\ &\leqq C_{\alpha,\beta}\tilde{A}_O{}^\nu \max \{ \prod_{j=1}^{\nu+1} |p_{j(\beta^j)}^{(\alpha^j)}|^{(m_j-\rho|\alpha^j|+\delta|\beta^j|)}_{n+1,\, l_o+k_j l''} ; \end{aligned} \tag{1.18}$$

$$\kappa=(k_1,\cdots,k_{\nu+1})\in K_{\nu+1}(N_o+\delta^*|\beta|+1)\}$$
$$\times\langle\xi\rangle^{\bar m_{\nu+1}-\rho|\alpha|+\delta|\beta|}$$
$$\leqq C_{\alpha,\beta}\tilde A_O{}^{\nu}\max\prod_{j=0}^{\nu+1}|p_j|^{(m_j)}_{n+1+k_j|\alpha|,l_o+k_j(l''+|\beta|)};$$
$$\kappa\in K_{\nu+1}(N_o+|\alpha|+|\beta|+\delta^*|\beta|+1)\}$$
$$\times\langle\xi\rangle^{\bar m_{\nu+1}-\rho|\alpha|+\delta|\beta|}\qquad (l''=[(M+\rho|\alpha|+\delta|\beta|)/(1-\delta)]^*)$$

with a constant $C_{\alpha,\beta}$ depending only on α and β, where $K_{\nu+1}(l)=\{\kappa=(k_1,\cdots,k_{\nu+1});\ k_j=0,1,\sum_{j=1}^{\nu+1}k_j\leqq l\}$. If we use $\sum_{\alpha,\beta,\nu+1}\alpha!\beta!/(\tilde\alpha^{\nu+1}!\tilde\beta^{\nu+1}!)=(\nu+1)^{|\alpha|+|\beta|}$, we obtain from (1.8) and (1.18)

$$(1.19)\qquad |q_{\nu+1}{}^{(\alpha)}_{(\beta)}(x,\xi)|\leqq C_{\alpha,\beta}\tilde A_O{}^{\nu}(\nu+1)^{|\alpha|+|\beta|}$$
$$\times\max\{\prod_{j=1}^{\nu+1}|p_j|^{(m_j)}_{n+1+k_j|\alpha|,l_o+k_j(l''+|\beta|)};$$
$$\kappa\in K_{\nu+1}(N_o+|\alpha|+|\beta|+\delta^*|\beta|+1)\}$$
$$\times\langle\xi\rangle^{\bar m_{\nu+1}-\rho|\alpha|+\delta|\beta|}.$$

For any fixed $\tilde\sigma>1$ we take a constant C_l independent of ν such that for all ν

$$(\nu+1)^l\leqq C_l\tilde\sigma^{\nu}.$$

Combining this with (1.19) we get

$$(1.20)\qquad |q_{\nu+1}|^{(\bar m_{\nu+1})}_{l_1,l_2}$$
$$\leqq C_{l_1,l_2}A^{\nu}\max_{\kappa\in K_{\nu+1}(N_o+\tilde l+1)}\prod_{j=1}^{\nu+1}|p_j|^{(m_j)}_{n+1+k_jl_1,l_o+k_j(l''+l_2)}$$
$$(\tilde l=l_1+l_2+\delta^*l_2\,,\ l''=[(M+\rho l_1+\delta l_2)/(1-\delta)]^*)\,,$$

if we set $A=\tilde A_O\tilde\sigma$ and $C_{l_1,l_2}=\max_{|\alpha|\leqq l_1,|\beta|\leqq l_2}(C_{\alpha,\beta}C_{|\alpha+\beta|})$. Consequently, for the constant C_O in (11) the set $\{C_O{}^{-\nu}q_{\nu+1}\}$ is bounded in $\{S^{\bar m_{\nu+1}}_{\rho,\delta}\}$. This concludes the proof of Theorem 2.

REMARK. In Proposition 1.2 the constant $\tilde A_O$ depends also on ε_1. But, when N_o in (1.16) satisfies $\sum_{j=N_o+1}^{\infty}|m_j|\leqq\varepsilon_o/2$, we can take the constant A in Theorem 2 depending only on n and δ.

As a special case of Theorem 2 we have

Corollary 1.3. *Let P be a pseudo-differential operator with symbol $p(x,\xi)$ in $S^0_{\rho,\delta}$. Then, the symbol $q_{\nu+1}(x,\xi)$ of the $(\nu+1)$-st power $Q_{\nu+1}=P^{\nu+1}$ of P satisfies for any l_1 and l_2*

$$(1.21)\qquad |q_{\nu+1}|^{(0)}_{l_1,l_2} \leqq C_{l_1,l_2} C_0^{\nu} (|p|^{(0)}_{n+1+l_1,\tilde{l}'})^{l_1+l_2+\delta^* l_2+1}$$
$$(\tilde{l}' = l_0 + [(\rho l_1 + l_2)/(1-\delta)]^*,\ \delta^* = [\delta]^*)$$

if $\nu \geqq l_1 + l_2 + \delta^* l_2 + 1$ *holds. The constant* C_0 *is determined by*

$$(1.22)\qquad C_0 = A|p|^{(0)}_{n+1,l_0}$$

for the constant A *in Theorem* 2.

Using this corollary we *prove Theorem* 3. Set

$$q_\nu(x, \xi) = \sigma(P^\nu)\,.$$

Suppose (14). Then, taking account of (1.21) the series $\sum_{\nu=0}^{\infty} q_\nu(x, \xi)$ converges to a symbol $q(x, \xi)$ in $S^0_{\rho,\delta}$ because of $C_0 < 1$. Since

$$(1.23)\qquad \begin{cases} (\sum_{j=1}^{\nu} Q_j)(I-P) = I - Q_{\nu+1} \\ (I-P)(\sum_{j=1}^{\nu} Q_j) = I - Q_{\nu+1} \end{cases}$$

hold, by tending ν in (1.23) to the infinity we see that the pseudo-differential operator $Q = q(X, D_x)$ is the inverse of $I-P$.

We note that the symbol $q(x, \xi)$ has a semi-norm estimate

$$(1.24)\qquad |q|^{(0)}_{l_1,l_2} \leqq C_{l_1,l_2} (\max(|p|^{(0)}_{n+1+l_1,\tilde{l}'}, 1))^{l_1+l_2+\delta^* l_2+1}$$
$$(\delta^* = [\delta]^*,\ \tilde{l}' = l_0 + [(\rho l_1 + l_2)/(1-\delta)]^*)$$

with a constant C_{l_1,l_2} depending only on $A|p|^{(0)}_{n+1,l_0}$, l_1 and l_2. In fact, writing

$$q(x, \xi) = \sum_{\nu=0}^{N} q_\nu(x, \xi) + \sum_{\nu=N+1}^{\infty} q_\nu(x, \xi) \qquad (N = l_1 + l_2 + \delta^* l_2)$$

we obtain (1.24) by applying (1.21) to the second term.

We turn to the proof of Proposition 1.2.

DEFINITION 1.4. For an integer N we say that a symbol $p(x, \xi, x') \in S^m_{\rho,\delta}$ belongs to a class $SX^m_{\rho,\delta\,;\,N}$ when $p(x, \xi, x')$ satisfies

$$(1.25)\qquad |\partial^\alpha_\xi D^\beta_x D^{\beta'}_{x'} p(x, \xi, x')| \leqq C_{\alpha,\beta,\beta'} \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta+\beta'|} (1 + \langle\xi\rangle^\delta |x - x'|)^{-N}\,.$$

For $p(x, \xi, x') \in SX^m_{\rho,\delta\,;\,N}$ we define semi-norms $||p||^{(m)}_{l_1,l_2,l'_2\,;\,N}$ by

$$(1.26)\qquad ||p||^{(m)}_{l_1,l_2,l'_2\,;\,N} = \max \inf \{C_{\alpha,\beta,\beta'} \text{ of } (1.25)\}\,,$$

where the maximum is taken over all (α, β, β') satisfying $|\alpha| \leqq l_1$, $|\beta| \leqq l_2$ and $|\beta'| \leqq l'_2$. Then, $SX^m_{\rho,\delta\,;\,N}$ is a Fréchet space.

Lemma 1.5. *Setting*

$$L_\delta = (1+\langle\xi\rangle^{2\delta}|x-x'|^2)^{-1}(1-i\langle\xi\rangle^{2\delta}(x-x')\cdot\nabla_\xi)$$

we define for an integer N *a mapping* F_N *from a single symbol class* $S^m_{\rho,\delta}$ *to a class* $SX^m_{\rho,\delta;N}$ *by*

(1.27) $\quad F_N(p)(x,\xi,x') = (L_\delta{}^t)^N p(x,\xi) \qquad$ *for* $p(x,\xi)\in S^m_{\rho,\delta}$,

where $\nabla_\xi={}^t(\partial/\partial\xi_1,\cdots,\partial/\partial\xi_n)$ *and* $L_\delta{}^t$ *denotes a transposed operator of* L_δ. *Then, we have*

$$(1.28)\qquad \begin{cases} F_N(p)(x,\xi,x')\in SX^m_{\rho,\delta;N}, \\ \|F_N\|(p)^{(m)}_{l_1,l_2,l_2';N} \leqq C_{l_1,l_2,l_2',N}|p|^{(m)}_{l_1+N,l_2} \end{cases}$$

with a constant $C_{l_1,l_2,l_2',N}$ *independent of m and*

(1.29) $\quad F_N(p)(X,D_x,X') = p(X,D_x)$.

Proof. From (1.27) we get (1.28) easily. So, we have only to prove (1.29). For simplicity we denote $p^{(N)}(x,\xi,x')=F_N(p)(x,\xi,x')$. Set $\tilde{L}_\delta=(1+\langle\xi+\eta\rangle^{2\delta}|y|^2)^{-1}(1+i\langle\xi+\eta\rangle^{2\delta}y\cdot\nabla_\eta)$. Then, we have $p^{(N)}(x,\xi+\eta,x+y)=(\tilde{L}_\delta{}^t)^N p(x,\xi+\eta)$. Hence, using $\tilde{L}_\delta e^{-iy\cdot\eta}=e^{-iy\cdot\eta}$ we obtain

$$\begin{aligned}(p^{(N)})_L(x,\xi) &= O_s\text{-}\iint e^{-iy\cdot\eta}p^{(N)}(x,\xi+\eta,x+y)dy\,d\eta \\ &= O_s\text{-}\iint e^{-iy\cdot\eta}(\tilde{L}_\delta{}^t)^N p(x,\xi+\eta)dy\,d\eta \\ &= O_s\text{-}\iint e^{-iy\cdot\eta}p(x,\xi+\eta)dy\,d\eta \\ &= p(x,\xi).\end{aligned}$$

This proves (1.29). Q.E.D.

Lemma 1.6. *Let* δ *satisfy* $0\leqq\delta<1$. *Then, the following hold*:

i) *Set*

$$\mathcal{J}(\xi,\xi') = (1+\langle\xi'\rangle^{-\delta}|\xi-\xi'|)^{-1/(1-\delta)}(\langle\xi'\rangle^{-\delta}\langle\xi;\xi'\rangle_\delta)^{1/(1-\delta)}.$$

Then, we have for μ *with* $|\mu|\leqq 1$

(1.30) $\quad \mathcal{J}(\xi,\xi')\langle\xi\rangle^\mu \leqq C_1\langle\xi'\rangle^\mu$.

ii) *Set*

$$\mathcal{J}(\xi,\xi') = (1+\langle\xi'\rangle^{-\delta}|\xi-\xi'|)^{-\delta}(\langle\xi'\rangle^{-\delta}\langle\xi;\xi'\rangle_\delta).$$

Then, we have for θ *with* $0\leqq\theta\leqq 1$

(1.31) $\quad \mathcal{J}(\xi,\xi')\leqq C_2(\langle\xi\rangle^\delta\langle\xi'\rangle^{-\delta})^{(1-\delta)\theta}$.

Proof. i) First, we assume $|\xi-\xi'|\leqq\langle\xi'\rangle/2$. Then, we get $(1/2)\langle\xi'\rangle\leqq\langle\xi\rangle\leqq2\langle\xi'\rangle$ and get (1.30) immediately. So, we may assume $|\xi-\xi'|\geqq\langle\xi'\rangle/2$. Then, from $\langle\xi;\xi'\rangle_\delta\leqq5|\xi-\xi'|^\delta$ we get

$$\begin{aligned}\mathcal{J}(\xi,\xi')\langle\xi\rangle^\mu&\leqq C_1'(\langle\xi'\rangle^{-\delta}|\xi-\xi'|)^{-1/(1-\delta)}(\langle\xi'\rangle^{-\delta}|\xi-\xi'|^\delta)^{1/(1-\delta)}\langle\xi\rangle^\mu\\&=C_1'|\xi-\xi'|^{-1}\langle\xi\rangle^\mu\leqq C_1\langle\xi'\rangle^\mu.\end{aligned}$$

Hence, we get (1.30).

ii) By the same way as in i), we get (1.31) when $|\xi-\xi'|\leqq\langle\xi'\rangle/2$. So, we may assume $|\xi-\xi'|\geqq\langle\xi'\rangle/2$. Then, we have

$$\begin{aligned}\mathcal{J}(\xi,\xi')&\leqq C_2(\langle\xi'\rangle^{-\delta}|\xi-\xi'|)^{-\delta}(\langle\xi'\rangle^{-\delta}|\xi-\xi'|^\delta)\\&=C_2\langle\xi'\rangle^{-\delta(1-\delta)}\leqq C_2(\langle\xi\rangle^\delta\langle\xi'\rangle^{-\delta})^{(1-\delta)\theta}.\end{aligned}$$

Hence, we get (1.31). Q.E.D.

The following proposition is the first step to the study of multi-products of pseudo-differential operators.

Proposition 1.7. *Let $\{m_j\}_{j=1}^\infty$ be a sequence of real numbers satisfying* (1.12). *Suppose that a multiple symbol $p_{\nu+1}(x^0,\tilde{x}^\nu,\tilde{\xi}^{\nu+1})$ in $S^{\tilde{m}_{\nu+1}}_{\rho,\delta}$, $\tilde{m}_{\nu+1}=(m_1,\cdots,m_{\nu+1})$, satisfies*

$$\begin{aligned}(1.32)\quad&|D_{x^1}^{\beta^1}\cdots D_{x^\nu}^{\beta^\nu}p_{\nu+1}(x^0,\tilde{x}^\nu,\tilde{\xi}^{\nu+1})|\\&\leqq B\prod_{j=1}^{\nu+1}\langle\xi^j\rangle^{m_j}\prod_{j=1}^{\nu}\langle\xi^j;\xi^{j+1}\rangle_\delta^{|\beta^j|}\prod_{j=1}^{\nu}(1+\langle\xi^j\rangle^\delta|x^{j-1}-x^j|)^{-(n+1)}\\&\qquad\text{for}\quad|\beta^j|\leqq l_o,\ j=1,\cdots,\nu.\end{aligned}$$

Then, for a simplified symbol $(p_{\nu+1})_L(x,\xi)$ of $p_{\nu+1}(x^0,\tilde{x}^\nu,\tilde{\xi}^{\nu+1})$ an estimate

$$(1.33)\quad|(p_{\nu+1})_L(x,\xi)|\leqq A_0{}^\nu B\langle\xi\rangle^{\bar{m}_{\nu+1}}\qquad(\bar{m}_{\nu+1}=m_1+\cdots+m_{\nu+1})$$

holds for a constant A_0 determined only by n, δ and ε_1.

Proof. Integrating the oscillatory integral (1.4) for $p=p_{\nu+1}$ by parts with respect to $\tilde{y}^\nu$ we have

$$\begin{aligned}(1.34)\quad(p_{\nu+1})_L(x,\xi)&=(r_{\nu+1})_L(x,\xi)\\&=O_s-\int e^{-i\psi}r_{\nu+1}(x,\xi+\eta^1,x+y^1,\cdots,\xi+\eta^\nu,x+y^\nu,\xi)d\tilde{y}^\nu d\tilde{\eta}^\nu\end{aligned}$$

for a multiple symbol

$$\begin{aligned}(1.35)\quad r_{\nu+1}(x^0,\tilde{x}^\nu,\tilde{\xi}^{\nu+1})&=\prod_{j=1}^{\nu}(1+\langle\xi^{j+1}\rangle^{-2\delta}|\xi^j-\xi^{j+1}|^2)^{-l_o}\\&\times\prod_{j=1}^{\nu}(1-i\langle\xi^{j+1}\rangle^{-2\delta}(\xi^j-\xi^{j+1})\cdot\nabla_{x^j})^{l_o}p_{\nu+1}(x^0,\tilde{x}^\nu,\tilde{\xi}^{\nu+1}).\end{aligned}$$

Expanding $(1-i\langle\xi^{j+1}\rangle^{-2\delta}(\xi^j-\xi^{j+1})\cdot\nabla_{x^j})^{l_o}$ by the polynomial thoerem and applying (1.32) to the derivatives of $p_{\nu+1}$, we have

$$(1.36)\qquad |r_{\nu+1}(x^0, \tilde{x}^\nu, \xi^{\nu+1})| \leqq BA_1{}^\nu G_\nu(x^0, \tilde{x}^\nu, \xi^{\nu+1})$$

for a constant A_1 depending only on n and δ, where

$$(1.37)\qquad \begin{aligned} &G_\nu(x^0, \tilde{x}^\nu, \tilde{\xi}^{\nu+1}) \\ &\quad = \prod_{j=1}^{\nu} (1+\langle\xi^j\rangle^\delta |x^{j-1}-x^j|)^{-(n+1)} \\ &\quad\quad \times \prod_{j=1}^{\nu} \{(1+\langle\xi^{j+1}\rangle^{-\delta}|\xi^j-\xi^{j+1}|)^{-l_o}(\langle\xi^{j+1}\rangle^{-\delta}\langle\xi^j;\xi^{j+1}\rangle_\delta)^{l_o} \\ &\quad\quad\quad \times\langle\xi^j\rangle^{m_j}\}\langle\xi^{\nu+1}\rangle^{m_{\nu+1}}. \end{aligned}$$

Set

$$(1.38)\qquad \begin{cases} \tilde{\varepsilon}_o = \varepsilon_o - \varepsilon_1 \qquad (<0), \\ l'_o = (n+\tilde{\varepsilon}_o)/(1-\delta) \end{cases}$$

and set

$$(1.39)\qquad \begin{cases} H_\nu(\tilde{\xi}^{\nu+1}) = \{\prod_{j=1}^{\nu} \mathcal{I}(\xi^j, \xi^{j+1}) \prod_{j=1}^{\nu+1} \langle\xi^j\rangle^{m_j/\varepsilon_1}\}^{\varepsilon_1}, \\ \tilde{H}_\nu(\tilde{\xi}^{\nu+1}) = \{\prod_{j=1}^{\nu} \mathcal{J}(\xi^j, \xi^{j+1})\}^{l'_o} \end{cases}$$

with $\mathcal{I}(\xi, \xi')$ and $\mathcal{J}(\xi, \xi')$ in the preceding lemma. Then, since $l_o = l'_o + \varepsilon_1/(1-\delta) = (n+\tilde{\varepsilon}_o) + \delta l'_o + \varepsilon_1/(1-\delta)$ we have

$$(1.40)\qquad \begin{aligned} G_\nu(x^0, \tilde{x}^\nu, \tilde{\xi}^{\nu+1}) &= \prod_{j=1}^{\nu} (1+\langle\xi^j\rangle^\delta |x^{j-1}-x^j|)^{-(n+1)} \\ &\quad\times \prod_{j=1}^{\nu} (1+\langle\xi^{j+1}\rangle^{-\delta}|\xi^j-\xi^{j+1}|)^{-(n+\tilde{\varepsilon}_o)} H_\nu(\tilde{\xi}^{\nu+1})\tilde{H}_\nu(\tilde{\xi}^{\nu+1}). \end{aligned}$$

Using Lemma 1.6–i) repeatedly, we have

$$\begin{aligned} &\mathcal{I}(\xi^1, \xi^2)\langle\xi^1\rangle^{m_1/\varepsilon_1} \leqq C_1\langle\xi^2\rangle^{m_1/\varepsilon_1} && (|m_1/\varepsilon_1| \leqq 1), \\ &\mathcal{I}(\xi^2, \xi^3)\langle\xi^2\rangle^{\bar{m}_2/\varepsilon_1} \leqq C_1\langle\xi^3\rangle^{\bar{m}_2/\varepsilon_1} && (|\bar{m}_2/\varepsilon_1| \leqq 1), \\ &\cdots\cdots\cdots \\ &\mathcal{I}(\xi^\nu, \xi^{\nu+1})\langle\xi^\nu\rangle^{\bar{m}_\nu/\varepsilon_1} \leqq C_1\langle\xi^{\nu+1}\rangle^{\bar{m}_\nu/\varepsilon_1} && (|\bar{m}_\nu/\varepsilon_1| \leqq 1). \end{aligned}$$

Then, we have

$$(1.41)\qquad H_\nu(\tilde{\xi}^{\nu+1}) \leqq C_1{}^{\nu\varepsilon_1}\langle\xi^{\nu+1}\rangle^{\bar{m}_{\nu+1}}.$$

Applying Lemma 1.6–ii) with $\theta = n/(n+\tilde{\varepsilon}_o)$ (<1), we have

$$(1.42)\qquad \begin{aligned} \tilde{H}_\nu(\tilde{\xi}^{\nu+1}) &\leqq C_2{}^{\nu l'_o}\{\prod_{j=1}^{\nu} (\langle\xi^j\rangle^\delta\langle\xi^{j+1}\rangle^{-\delta})^{(1-\delta)\theta}\}^{l'_o} \\ &= C_2{}^{\nu l'_o}\prod_{j=1}^{\nu} (\langle\xi^j\rangle^{n\delta}\langle\xi^{j+1}\rangle^{-n\delta}) \end{aligned}$$

by virtue of (1.38). Consequently, we have

$$(1.43)\qquad G_\nu(x^0, \tilde{x}^\nu, \tilde{\xi}^{\nu+1}) \leqq A_2{}^\nu \prod_{j=1}^{\nu} \{(1+\langle\xi^j\rangle^\delta |x^{j-1}-x^j|)^{-(n+1)}\langle\xi^j\rangle^{n\delta}\} \times \prod_{j=1}^{\nu} \{(1+\langle\xi^{j+1}\rangle^{-\delta}|\xi^j-\xi^{j+1}|)^{-(n+\tilde{\varepsilon}_o)} \times\langle\xi^{j+1}\rangle^{-n\delta}\}\langle\xi^{\nu+1}\rangle^{\bar{m}_{\nu+1}}$$

with a constant A_2 determined only by n, δ and ε_1. Set

$$(1.44)\qquad W_\nu \equiv W_\nu(\xi) = \iint \prod_{j=1}^{\nu} \{(1+\langle\xi+\eta^j\rangle^\delta |y^{j-1}-y^j|)^{-(n+1)}\langle\xi+\eta^j\rangle^{n\delta}\} \times \prod_{j=1}^{\nu} \{(1+\langle\xi+\eta^{j+1}\rangle^{-\delta}|\eta^j-\eta^{j+1}|)^{-(n+\tilde{\varepsilon}_o)} \times\langle\xi+\eta^{j+1}\rangle^{-n\delta}\} d\tilde{y}^\nu d\tilde{\eta}^\nu \qquad (y^0=\eta^{\nu+1}=0).$$

Then, from (1.34), (1.36) and (1.43) we have

$$(1.45)\qquad |(p_{\nu+1})_L(x,\xi)| \leqq B(A_1A_2)^\nu W_\nu \langle\xi\rangle^{\bar{m}_{\nu+1}}.$$

Since W_ν has an estimate

$$W_\nu \leqq A_3'^\nu \int \prod_{j=1}^{\nu} \{(1+\langle\xi+\eta^{j+1}\rangle^{-\delta}|\eta^j-\eta^{j+1}|)^{-(n+\tilde{\varepsilon}_o)}\langle\xi+\eta^{j+1}\rangle^{-n\delta}\} d\tilde{\eta}^\nu \leqq A_3'^\nu A_3''^\nu \equiv A_3{}^\nu$$

with constants A_3', A_3'' and A_3 $(=A_3'A_3'')$ independent of ν, we get (1.33) from (1.45) if we set $A_O = A_1A_2A_3$. Q.E.D.

We fix a C^∞-function $\chi_o(\xi)$ satisfying

$$0\leqq\chi_o\leqq 1, \quad \chi_o(\xi)=1 \ (|\xi|\leqq 1/4), \quad \chi_o(\xi)=0 \ (|\xi|\geqq 1/2).$$

Set for $p_j(x,\xi,x')\in S^{m_j}_{\rho,\delta}$, $j=1, 2$, and $\varepsilon\in(0, 1]$

$$(1.46)\qquad \begin{cases} q_0^\varepsilon(x,\xi,x',\xi',x'') = p_1(x,\xi,x')\chi_0^\varepsilon(\xi,\xi')p_2(x',\xi',x''), \\ q_1^\varepsilon(x,\xi,x',\xi',x'') = p_1(x,\xi,x')\chi_1^\varepsilon(\xi,\xi')p_2(x',\xi',x'') \end{cases}$$

and

$$(1.47)\qquad (q_1^\varepsilon(\mu))(x,\xi,x',\xi',x'') = \{-i|\xi-\xi'|^{-2}(\xi-\xi')\cdot\nabla_{x'}\}^\mu q_1^\varepsilon(x,\xi,x',\xi',x'') \qquad (\mu=0, 1, \cdots),$$

where $\chi_0^\varepsilon(\xi,\xi')=\chi_o((\xi-\xi')/(\varepsilon\langle\xi'\rangle))$ and $\chi_1^\varepsilon(\xi,\xi')=1-\chi_0^\varepsilon(\xi,\xi')$. We define for pseudo-differential operators $P_j=p_j(X, D_x, X')$, $j=1, 2$, the products $P_1\Box_k^\varepsilon P_2$ $(k=0, 1)$ and $P_1\Box_{1,\mu}^\varepsilon P_2$, $(\mu=0, 1, \cdots)$ by

$$(1.48)\qquad \begin{cases} P_1\square_k^\varepsilon P_2 = q_k^\varepsilon(X, D_x, X', D_{x'}, X''), & k=0, 1, \\ P_1\square_{1,\mu}^\varepsilon P_2 = (q_1^\varepsilon(\mu))(X, D_x, X', D_{x'}, X''), & \mu = 0, 1, \cdots. \end{cases}$$

We also denote for $P_j=p_j(X, D_x, X')$, $j=1, 2$, the pseudo-differential operator with symbol $p_1(x, \xi, x')p_2(x, \xi, x')$ by $P_1\odot P_2$. Then, we have obtained the following results in Section 2 of [16].

Proposition 1.8. i) *It holds that*

$$(1.49)\qquad P_1P_2 = P_1\square_0^\varepsilon P_2 + P_1\square_1^\varepsilon P_2,$$

$$(1.50)\qquad P_1\square_{1,\mu}^\varepsilon P_2 = P_1\square_1^\varepsilon P_2 \qquad (\mu = 0, 1, \cdots).$$

ii) *Let $p_j(x, \xi, x')$ belong to $SX^{m_j}_{\rho,\delta;N_j}$ $(j=1, 2)$. Set for real numbers s, s_1 and s_2*

$$\begin{cases} Q_0^\varepsilon = (P_1\odot\Lambda^s)\square_0^\varepsilon(\Lambda^{-s}\odot P_2), \\ Q_1^\varepsilon = (P_1\odot\Lambda^{s_1})\square_1^\varepsilon(\Lambda^{s_2}\odot P_2), \\ Q_1^\varepsilon(\mu) = (P_1\odot\Lambda^{s_1})\square_{1,\mu}^\varepsilon(\Lambda^{s_2}\odot P_2), \end{cases}$$

where $\Lambda^s=\langle D_x\rangle^s$. Then, $q_k^\varepsilon(x, \xi, x', \xi', x'')=\sigma(Q_k^\varepsilon)$, $k=0,1$ and $(q_1^\varepsilon(\mu))(x, \xi, x', \xi', x'')=\sigma(Q_1^\varepsilon(\mu))$ satisfy for any β, β', β'' with $|\beta'|\leqq l$, $|\beta|\leqq l$ and $|\beta''|\leqq l$

$$(1.51)\qquad \begin{aligned} &|D_x^\beta D_{x'}^{\beta'} D_{x''}^{\beta''} q_0^\varepsilon(x, \xi, x', \xi', x'')| \\ &\leqq \{(1+\varepsilon)^{|s|}\|p_1\|^{(m_1)}_{0,l,l;N_1}\|p_2\|^{(m_2)}_{0,l,l;N_2}\} \\ &\quad\times\langle\xi\rangle^{m_1+\delta|\beta|}\langle\xi;\xi'\rangle_\delta^{|\beta'|}\langle\xi'\rangle^{m_2+\delta|\beta''|} \\ &\quad\times(1+\langle\xi\rangle^\delta|x-x'|)^{-N_1}(1+\langle\xi'\rangle^\delta|x'-x''|)^{-N_2}, \end{aligned}$$

$$(1.52)\qquad \begin{aligned} &|D_x^\beta D_{x'}^{\beta'} D_{x''}^{\beta''} q_1^\varepsilon(x, \xi, x', \xi', x'')| \\ &\leqq \{\|p_1\|^{(m_1)}_{0,l,l;N_1}\|p_2\|^{(m_2)}_{0,l,l;N_2}\} \\ &\quad\times\langle\xi\rangle^{m_1+\delta|\beta|+s_{1+}}\langle\xi;\xi'\rangle_\delta^{|\beta'|}\langle\xi'\rangle^{m_2+\delta|\beta''|+s_{2+}} \\ &\quad\times(1+\langle\xi\rangle^\delta|x-x'|)^{-N_1}(1+\langle\xi'\rangle^\delta|x'-x''|)^{-N_2}, \end{aligned}$$

$$(1.53)\qquad \begin{aligned} &|D_x^\beta D_{x'}^{\beta'} D_{x''}^{\beta''} (q_1^\varepsilon(\mu))(x, \xi, x', \xi', x'')| \\ &\leqq \{C_{l,\mu,\varepsilon}\|p_1\|^{(m_1)}_{0,l,l+\mu;N_1}\|p_2\|^{(m_2)}_{0,l+\mu,l;N_2}\} \\ &\quad\times\langle\xi\rangle^{m_1+\delta|\beta|}\langle\xi;\xi'\rangle_\delta^{|\beta'|}\langle\xi'\rangle^{m_2+\delta|\beta''|} \\ &\quad\times(1+\langle\xi\rangle^\delta|x-x'|)^{-N_1}(1+\langle\xi'\rangle^\delta|x'-x''|)^{-N_2} \end{aligned}$$

when $\mu\geqq(s_{1+}+s_{2+})/(1-\delta)$.

REMARK. A product $P_1\odot P_2$ is denoted as $P_1\otimes P_2$ in [16] and a slightly different estimates are derived there, but (1.51)–(1.53) follow by their proof.

Now, we are prepared to *prove Proposition* 1.2. We devide the proof into four steps.

I) Let F_{n+1} be a mapping from $S^m_{\rho,\delta}$ to $SX^m_{\rho,\delta;n+1}$ defined in Lemma 1.5 with $N=n+1$. Denote for simplicity

$$(1.54)\qquad p'_j(x,\xi,x')=F_{n+1}(p^\circ_j)(x,\xi,x')\qquad \textit{for}\quad j\leqq\nu\,.$$

From Lemma 1.5 we note that $p'_j(x,\xi,x')$ belongs to $SX^{m_j+m'_j}_{\rho,\delta;n+1}$ and satisfies

$$(1.55)\qquad \|p'_j\|^{(m_j+m'_j)}_{0,l_2,l'_2;n+1}\leqq C_{l_2,l'_2}|p^\circ_j|^{(m_j+m'_j)}_{n+1,l_2}$$

for a constant C_{l_2,l'_2} independent of j. From (1.54) and (1.29) we can write

$$(1.56)\qquad Q^\circ_{\nu+1}=P'_1P'_2\cdots P'_\nu P^\circ_{\nu+1}\,.$$

Set

$$K_\nu=\{\kappa=(k_1,k_2,\cdots,k_\nu);\,k_j=0,1\}\,.$$

Then, from (1.49) we have

$$(1.57)\qquad Q^\circ_{\nu+1}=\sum_{\kappa\in K_\nu}Q_{\nu+1,(\kappa)}\Lambda^{\bar{m}'_{\nu+1}}$$

for

$$(1.58)\qquad Q_{\nu+1,(\kappa)}=P'_1\square^{\varepsilon}_{k_1}P'_2\square^{\varepsilon}_{k_2}\cdots\square^{\varepsilon}_{k_{\nu-1}}P'_\nu\square^{\varepsilon}_{k_\nu}P^\circ_{\nu+1}\Lambda^{-\bar{m}'_{\nu+1}}\,.$$

II) Set $\kappa^0\equiv\kappa^0_\nu=(0,0,\cdots,0)\in K_\nu$ and consider $Q_{\nu+1,(\kappa^0)}$ in (1.58). We set

$$(1.59)\qquad \begin{cases} p''_j(x,\xi,x')=p'_j(x,\xi,x')\langle\xi\rangle^{-m'_j}\ (\in SX^{m_j}_{\rho,\delta;n+1}) & (1\leqq j\leqq\nu)\,,\\ p''_{\nu+1}(x,\xi)=p^\circ_{\nu+1}(x,\xi)\langle\xi\rangle^{-m'_{\nu+1}}\ (\in S^{m_{\nu+1}}_{\rho,\delta})\,. \end{cases}$$

Then, we can write $Q_{\nu+1,(\kappa^0)}$ in the form

$$Q_{\nu+1,(\kappa^0)}=(P''_1\odot\Lambda^{m'_1})\square^{\varepsilon}_0(\Lambda^{-m'_1}\odot P''_2\odot\Lambda^{\bar{m}'_2})\square^{\varepsilon}_0\cdots \cdot\square^{\varepsilon}_0(\Lambda^{-\bar{m}'_{\nu-1}}\odot P''_\nu\odot\Lambda^{\bar{m}'_\nu})\square^{\varepsilon}_0(\Lambda^{-\bar{m}'_\nu}\odot P''_{\nu+1})\,.$$

Let $q_{\nu+1,(\kappa^0)}(x^0,\tilde{x}^\nu,\tilde{\xi}^{\nu+1})$ be the multiple symbol corresponding to $Q_{\nu+1,(\kappa^0)}$ and set

$$(1.60)\qquad B_{\nu+1,(\kappa^0)}=(1+\varepsilon)^{M'\nu}\|p''_1\|^{(m_1)}_{0,0,l_0;n+1}\prod_{j=2}^{\nu}\|p''_j\|^{(m_j)}_{0,l_0,l_0;n+1}|p''_{\nu+1}|^{(m_{\nu+1})}_{0,l_0}\,.$$

Then, applying (1.51) we obtain for $|\beta^j|\leqq l_0$ $(j=1,\cdots,\nu)$,

$$(1.61)\qquad |D^{\beta^1}_{x^1}\cdots D^{\beta^\nu}_{x^\nu}q_{\nu+1,(\kappa^0)}(x^0,\tilde{x}^\nu,\tilde{\xi}^{\nu+1})| \leqq B_{\nu+1,(\kappa^0)}\prod_{j=1}^{\nu+1}\langle\xi^j\rangle^{m_j}\prod_{j=1}^{\nu}\langle\xi^j;\xi^{j+1}\rangle_\delta^{|\beta^j|}\prod_{j=1}^{\nu}(1+\langle\xi^j\rangle^\delta|x^{j-1}-x^j|)^{-(n+1)}\,,$$

that is, $q_{\nu+1,(\kappa^0)}(x^0, \tilde{x}_\nu, \xi^{\nu+1})$ satisfies (1.32) with $B=B_{\nu+1,(\kappa^0)}$. Hence, we can apply Proposition 1.7 to obtain

$$(1.62) \qquad |(q_{\nu+1,(\kappa^0)})_L(x, \xi)| \leqq A_0{}^\nu B_{\nu+1,(\kappa^0)}\langle\xi\rangle^{\overline{m}_{\nu+1}}.$$

III) Next, we consider

$$(1.63) \qquad Q_{\nu+1,(\kappa)} = P'_1\square^{\varepsilon}_{k_1}P'_2\square^{\varepsilon}_{k_2}\cdots\square^{\varepsilon}_{k_{\nu-1}}P'_\nu\square^{\varepsilon}_{k_\nu}P^{\circ}_{\nu+1}\Lambda^{-\overline{m}'_{\nu+1}} \quad \text{for} \quad \kappa=(k_1,\cdots,k_\nu)\neq\kappa^0_\nu.$$

Set for $\iota<\iota'$

$$Q_{\nu+1,(\kappa);\iota,\iota'} = P'_\iota\square^{\varepsilon}_{k_\iota}P'_{\iota+1}\square^{\varepsilon}_{k_{\iota+1}}\cdots\square^{\varepsilon}_{k_{\iota'-1}}P'_{\iota'} \qquad (P'_{\nu+1}=P^{\circ}_{\nu+1}\Lambda^{-\overline{m}'_\nu+1}).$$

For j with $j<\nu$ and $k_j=1$ we set $\theta\equiv\theta_\kappa(j)=\max\{j'; 0\leqq j'<j, k_j=1\}$ $(k_0=1)$ and consider the part $Q_{\nu+1,(\kappa);\theta+1,j+1}$. Using (1.59) we write it in the form

$$Q_{\nu+1,(\kappa);\theta+1,j+1} = (P''_{\theta+1}\odot\Lambda^{\overline{\overline{m}}'_{\theta+1}})\square^{\varepsilon}_0(\Lambda^{-\overline{\overline{m}}'_{\theta+1}}\odot P''_{\theta+2}\odot\Lambda^{\overline{\overline{m}}'_{\theta+2}})\square^{\varepsilon}_0\cdots \cdot\square^{\varepsilon}_0(\Lambda^{-\overline{\overline{m}}'_{j-1}}\odot P''_j\odot\Lambda^{\overline{\overline{m}}'_j})\square^{\varepsilon}_1P'_{j+1},$$

where $\overline{\overline{m}}'_\iota\equiv\overline{\overline{m}}'_{\theta+1,\iota}=m'_{\theta+1}+\cdots+m'_\iota$ $(\theta+1\leqq\iota\leqq j)$. In the case of $\overline{\overline{m}}'_j\equiv\overline{\overline{m}}'_{\theta+1,j}\leqq 0$, applying (1.51) and (1.52), we have

$$(1.64) \qquad |q_{\nu+1,(\kappa);\theta+1,j+1}|^{(m_{\theta+1},\cdots,m_j,m_{j+1}+m'_{j+1})}_{0,l_0} \leqq(1+\varepsilon)^{M'(j-\theta-1)}\prod_{\iota=\theta+1}^{j}\|p''_\iota\|^{(m_\iota)}_{0,l_0,l_0;n+1}\|p'_{j+1}\|^{(m_{j+1}+m'_{j+1})}_{0,l_0,l_0;n+1} \quad (\overline{\overline{m}}'_{\theta+1,j}\leqq 0).$$

In the case of $\overline{\overline{m}}'_j>0$ we write by using (1.50)

$$Q_{\nu+1,(\kappa);\theta+1,j+1} = (P''_{\theta+1}\odot\Lambda^{\overline{\overline{m}}'_{\theta+1}})\square^{\varepsilon}_0(\Lambda^{-\overline{\overline{m}}'_{\theta+1}}\odot P''_{\theta+2}\odot\Lambda^{m'_{\theta+2}})\square^{\varepsilon}_0\cdots \cdot\square^{\varepsilon}_0(\Lambda^{-\overline{\overline{m}}'_{j-1}}\odot P''_j\odot\Lambda^{\overline{\overline{m}}'_{j-1}})\square_{1,\mu^0}P'_{j+1}$$

with $\mu^0=[M'/(1-\delta)]^*$. Then, from (1.51) and (1.53) we have

$$(1.65) \qquad |q_{\nu+1,(\kappa);\theta+1,j+1}|^{(m_{\theta+1},\cdots,m_j,m_{j+1}+m'_{j+1})}_{0,l_0} \leqq C_{l_0,\mu^0,\varepsilon}(1+\varepsilon)^{M'(j-\theta-1)}\prod_{\iota=\theta+1}^{j-1}\|p''_\iota\|^{(m_\iota)}_{0,l_0,l_0;n+1} \times\|p''_j\|^{(m_j)}_{0,l_0,l_0+\mu^0;n+1}\|p'_{j+1}\|^{(m_{j+1}+m'_{j+1})}_{0,l_0+\mu^0,l,;n+1} \qquad (\overline{\overline{m}}'_{\theta+1,j}>0).$$

For $j^0=\max\{j; k_j=1\}$ we write $Q_{\nu+1(\kappa);j^0,\nu+1}$ in the form

$$Q_{\nu+1,(\kappa);j^0,\nu+1} = (P''_{j^0}\odot\Lambda^{m'_{j^0}})\square^{\varepsilon}_{1,\mu^0}(\Lambda^{-\overline{m}'_{j^0}}\odot P''_{j^0+1}\odot\Lambda^{\overline{m}'_{j^0+1}})\square^{\varepsilon}_0\cdots \cdot\square^{\varepsilon}_0(\Lambda^{-\overline{m}'_{\nu-1}}\odot P''_\nu\odot\Lambda^{\overline{m}'_\nu})\square^{\varepsilon}_0(\Lambda^{-\overline{m}'_\nu}\odot P''_{\nu+1})$$
$$(=(P''_\nu\odot\Lambda^{m'_\nu})\square^{\varepsilon}_{1,\mu^0}(\Lambda^{\overline{m}'_\nu}\odot P''_{\nu+1}) \qquad \text{if} \quad j^0=\nu).$$

Then, the multiple symbol $q_{\nu+1,(\kappa);j^0,\nu+1}(x^{j^0-1}, \xi^{j^0}, \cdots, x^\nu, \xi^{\nu+1})$ of $Q_{\nu+1,(\kappa);j^0,\nu+1}$ satisfies

$$(1.66)\qquad \begin{cases} |q_{\nu+1,(\kappa);j^0,\nu+1}|_{0,l_0}^{(m_{j^0},\cdots,m_{\nu+1})} \\ \quad \leqq C_{l_0,\mu^0,\varepsilon}(1+\varepsilon)^{M'(\nu-j^0)}\|p''_{j^0}\|_{0,l_0,l_0+\mu^0;n+1}^{(m_{j^0})} \\ \qquad \times\|p''_{j^0+1}\|_{0,l_0+\mu^0,l_0;n+1}^{(m_{j^0+1})}\prod_{\iota=j^0+2}^{\nu}\|p''_\iota\|_{0,l_0,l_0;n+1}^{(m_\iota)}|p''_{\nu+1}|_{0,l_0}^{(m_{\nu+1})} \quad \text{if } j^0<\nu, \\ |q_{\nu+1,(\kappa);\nu,\nu+1}|_{0,l_0}^{(m_\nu,m_{\nu+1})} \\ \quad \leqq C_{l_0,\mu^0,\varepsilon}\|p''_\nu\|_{0,l_0,l_0+\mu^0;n+1}^{(m_\nu)}|p''_{\nu+1}|_{0,l_0+\mu^0}^{(m_{\nu+1})} \qquad (j^0=\nu)\,. \end{cases}$$

Now, we set

$$\Gamma(\kappa)=\{j;\ 1\leqq j<\nu,\ k_j=1,\ \sum_{\iota=\theta_\kappa(j)}^{j} m'_\iota>0\}\cup\{j^0\}\,.$$

Then, from the definition of N^0 and the relation

$$\Gamma(\kappa)\subset\{j;\ k_j=1,\ m'_j>0\}\cup\{j;\ k_j=1,\ k_\theta=\cdots=k_{j-1}=0\,, \\ m'_{\theta+1}>0 \text{ for some } \theta<j\}\cup\{j^0\}$$

the number l of the elements in $\Gamma(\kappa)$ does not exceed N^0+1. Set

$$\Gamma(\kappa)=\{j_1, j_2, \cdots, j_l\} \qquad (j_1<j_2<\cdots<j_l=j^0)$$

and write the multi-product $Q_{\nu+1,(\kappa)}$ of (1.63) in the form

$$Q_{\nu+1,(\kappa)}=P'_1\square^\varepsilon_{k_1}P'_2\square^\varepsilon_{k_2}\cdots\square^\varepsilon_{k_{j_1-1}}P'_{j_1}\square^\varepsilon_{1,\mu^0}P'_{j_1+1}\square^\varepsilon_{k_{j_1+1}}\cdots \\ \cdot\square^\varepsilon_{k_{j_l-1}}P'_{j_l-1}\square^\varepsilon_{1,\mu^0}P'_{j_l}\square^\varepsilon_0\cdots\square^\varepsilon_0 P^\circ_{\nu+1}\Lambda^{-\bar{m}'_{\nu+1}}\,.$$

Then, using the discussions in the preceding paragraph and Proposition 1.8–ii), we get the following: There exists a constant C_ε depending on M', N^0 and ε (but independent of ν) such that the multiple symbol $q_{\nu+1,(\kappa)}(x^0, \tilde{x}, \tilde{\xi}^{\nu+1})$ of $Q_{\nu+1,(\kappa)}$ satisfies (1.32) with B replaced by

$$(1.67)\qquad B_{\nu+1,(\kappa)}=C_\varepsilon(1+\varepsilon)^{M'\nu}\|p''_1\|_{0,0,l_0+s_1\mu^0;n+1}^{(m_1)} \\ \times\prod_{j=2}^{\nu}\|p''_j\|_{0,l_0+s_{j-1}\mu^0,l_0+s_j\mu^0;n+1}^{(m_j)}|p''_{\nu+1}|_{0,l_0+s_\nu\mu^0}^{(m_{\nu+1})}\,.$$

Here, $s_j=0$ for $j\notin\Gamma(\kappa)$ and $s_j=1$ for $j\in\Gamma(\kappa)$. Hence, by Proposition 1.7 we get

$$(1.68)\qquad |(q_{\nu+1,(\kappa)})_L(x, \xi)|\leqq A_0{}^\nu B_{\nu+1,(\kappa)}\langle\xi\rangle^{\bar{m}_{\nu+1}}\,.$$

IV) From (1.57) we have

$$q^\circ_{\nu+1}(x, \xi)=\sum_{\kappa\in K_\nu}(q_{\nu+1,(\kappa)})_L(x, \xi)\langle\xi\rangle^{\bar{m}'_{\nu+1}}\,.$$

Hence, we obtain from (1.62) and (1.68)

$$(1.69)\qquad |q^\circ_{\nu+1}(x,\xi)|\leqq(2A_0)^\nu \max_{\kappa\in K_\nu} B_{\nu+1,(\kappa)}\langle\xi\rangle^{\bar m_{\nu+1}+\bar m'_{\nu+1}}.$$

On the other hand, (1.59) and (1.55) imply

$$(1.70)\qquad \begin{cases} \|p_1''\|^{(m_1)}_{0,0,l_0+\mu^0;\,n+1}\leqq \tilde C|p^\circ_1|^{(m_1+m'_1)}_{n+1,0}, \\ \begin{cases} \|p_j''\|^{(m_j)}_{0,l_0,l_0;\,n+1}\leqq A_4|p^\circ_j|^{(m_j+m'_j)}_{n+1,l_0}, \\ \|p_j''\|^{(m_j)}_{0,l_0,l_0+\mu^0;\,n+1}\leqq \tilde CA_4|p^\circ_j|^{(m_j+m'_j)}_{n+1,l_0}, \\ \|p_j''\|^{(m_j)}_{0,l_0+\mu^0,l_0+\mu^0;\,n+1}\leqq \tilde CA_4|p^\circ_j|^{(m_j+m'_j)}_{n+1,l_0+\mu^0} \qquad (2\leqq j\leqq\nu), \end{cases} \\ |p''_{\nu+1}|^{(m_{\nu+1})}_{0,l_0+\mu^0}\leqq \tilde C|p^\circ_{\nu+1}|^{(m_{\nu+1}+m'_{\nu+1})}_{0,l_0+\mu^0} \end{cases}$$

for a constant A_4 independent of M' and ν and a constant $\tilde C$ independent of ν. Hence, from (1.60) and (1.67) we have for any $\kappa\in K_\nu$

$$B_{\nu+1,(\kappa)}\leqq C_\varepsilon \tilde C^{2(N^0+1)}A_4^\nu(1+\varepsilon)^{M'\nu}|p^\circ_1|^{(m_1+m'_1)}_{n+1,0} \times\prod_{j\notin\Gamma(\kappa)}|p^\circ_{j+1}|^{(m_{j+1}+m'_{j+1})}_{n+1,l_0}\prod_{j\in\Gamma(\kappa)}|p^\circ_{j+1}|^{(m_{j+1}+m'_{j+1})}_{n+1,l_0+\mu^0} \qquad (\Gamma(\kappa^0_\nu)=\phi).$$

For any fixed $\sigma>1$ we take $\varepsilon=\varepsilon_{M'}\in(0,1]$ satisfying

$$(1+\varepsilon)^{M'}\leqq\sigma.$$

Then, there exists a constant C° independent of ν such that

$$(1.71)\qquad B_{\nu+1,(\kappa)}\leqq C^\circ(A_4\sigma)^\nu \max_{\substack{k_j=0,1\\ k_2+\cdots+k_{\nu+1}\leqq N^0+1}}\{|p^\circ_1|^{(m_1+m'_1)}_{n+1,0}\prod_{j=2}^{\nu+1}|p^\circ_j|^{(m_j+m'_j)}_{n+1,l_0+k_j\mu^0}\}.$$

Consequently, setting $\tilde A_0=2\sigma A_4A_0$, we get (1.15) from (1.69) and (1.71). This concludes the proof of Proposition 1.2.

2. Multi-products of Fourier integral operators. Throughout this section we denote by I_ϕ the Fourier integral operator with phase function $\phi(x,\xi)\in\mathscr{P}_\rho(\tau)$ and symbol 1. Following [7] we define for $\phi(x,\xi)\in\mathscr{P}_\rho(\tau)$ the conjugate Fourier integral operator I_{ϕ^*} (with symbol 1) by

$$(2.1)\qquad (I_{\phi^*}u)(x)=O_s-\iint e^{i(x\cdot\xi-\phi(x',\xi))}u(x')dx'đ\xi \qquad \textit{for}\quad u\in\mathcal{S}.$$

Set for $\phi(x,\xi)\in\mathscr{P}_\rho(\tau)$

$$(2.2)\qquad \begin{cases} \tilde\nabla_x\phi(x,\xi,x')=\int_0^1\nabla_x\phi(x'+\theta(x-x'),\xi)d\theta, \\ \tilde\nabla_\xi\phi(\xi,x',\xi')=\int_0^1\nabla_\xi\phi(x',\xi'+\theta(\xi-\xi'))d\theta. \end{cases}$$

We employ the following lemma, which is a slightly different version of Proposition 1.5 of Chap. 10 in [8], but it can be proved by a similar way.

Lemma 2.1. *Let $\phi(x, \xi)$ belong to $\mathscr{P}_\rho(\tau)$, $0\leqq\tau<1$, $1/2\leqq\rho\leqq1$. Then, we have the following:*

i) *The equation $\eta=\tilde{\nabla}_x\phi(x, \xi, x')$ has the unique solution $\xi=\tilde{\nabla}_x\phi^{-1}(x, \eta, x')$ and it satisfies*

$$(2.3)\quad \begin{cases} \text{a)} & |\xi-\eta|\leqq\tau\langle\eta\rangle \quad \textit{with} \quad \xi=\tilde{\nabla}_x\phi^{-1}(x, \eta, x')\,, \\ \text{b)} & C^{-1}\langle\eta\rangle\leqq\langle\tilde{\nabla}_x\phi^{-1}(x, \eta, x')\rangle\leqq C\langle\eta\rangle\,, \\ \text{c)} & |\partial_\eta^\alpha D_x^\beta D_{x'}^{\beta'}\tilde{\nabla}_x\phi^{-1}(x, \eta, x')|\leqq C_{\alpha,\beta,\beta'}\langle\eta\rangle^{1-|\alpha|+(1-\rho)(|\alpha+\beta+\beta'|-1)} \\ & \qquad (|\alpha+\beta+\beta'|\geqq1)\,. \end{cases}$$

ii) *The equation $y'=\tilde{\nabla}_\xi\phi(\xi, x', \xi')$ has the unique solution $x'=\tilde{\nabla}_\xi\phi^{-1}(\xi, y', \xi')$ and it satisfies*

$$(2.4)\quad \begin{cases} \text{a)} & |\tilde{\nabla}_\xi\phi^{-1}(\xi, y', \xi')-y'|\leqq C\,, \\ \text{b)} & |\partial_\xi^\alpha\partial_{\xi'}^{\alpha'}D_{y'}^{\beta'}\tilde{\nabla}_\xi\phi^{-1}(\xi, y', \xi')| \\ & \quad \leqq C_{\alpha,\alpha',\beta'}\langle\xi;\,\xi'\rangle^{(1-\rho)(|\alpha+\alpha'+\beta'|-1)} \qquad (|\alpha+\alpha'+\beta'|\geqq1)\,, \\ \text{c)} & |\partial_\xi^\alpha\partial_{\xi'}^{\alpha'}D_{y'}^{\beta'}\{\chi((\xi-\xi')/\langle\xi'\rangle)\tilde{\nabla}_\xi\phi^{-1}(\xi, y', \xi')\}| \\ & \quad \leqq C_{\alpha,\alpha',\beta'}\langle\xi'\rangle^{-|\alpha+\alpha'|+(1-\rho)(|\alpha+\alpha'+\beta'|-1)} \quad (|\alpha+\alpha'+\beta'|\geqq1)\,, \end{cases}$$

where $\langle\xi;\,\xi'\rangle=\langle\xi\rangle+\langle\xi'\rangle$ $(=\langle\xi;\,\xi'\rangle_1)$ and χ is a C^∞-function satisfying

$$(2.5)\qquad 0\leqq\chi\leqq1\,,\quad \chi=1\ (|\xi|\leqq2/5)\,,\quad \chi=0\ (|\xi|\geqq1/2)\,.$$

Moreover, if $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is bounded in $\mathscr{P}_\rho(\tau)$, we can take the constants C, $C_{\alpha,\beta,\beta'}$ and $C_{\alpha,\alpha',\beta'}$ in (2.3) *and* (2.4) *independent of $\gamma\in\Gamma$.*

REMARK 1. In the lemma and in what follows, we say that for $\phi_\gamma\in\mathscr{P}_\rho(\tau)$ [resp. $\phi_\gamma\in\mathscr{P}_\rho(\tau, l)$] the set $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is bounded in $\mathscr{P}_\rho(\tau)$ [resp. $\mathscr{P}_\rho(\tau, l)$] if the corresponding set $\{\|J_\gamma\|_{l'}\}_{\gamma\in\Gamma}$ of semi-norms $\|J_\gamma\|_{l'}$ of (6) in Introduction is bounded for any $l'=0, 1, 2, \cdots$.

REMARK 2. Throughout this section we denote by $\chi(\xi)$ a C^∞-function satisfying (2.5).

Now, we show the existence of pseudo-differential operators R and R' satisfying (8).

Proposition 2.2. *There exist a constant $\tilde{\tau}$ (<1) and an integer $\tilde{l}_0$ such that for a phase function $\phi(x, \xi)$ in $\mathscr{P}_\rho(\tilde{\tau}, \tilde{l}_0)$ we can find pseudo-differential operators R and R' in S_ρ^0 satisfying*

$$(2.6)\qquad I_{\phi^*}I_\phi R=RI_{\phi^*}I_\phi=I\,,$$

$$(2.7)\qquad I_\phi I_{\phi^*} R' = R' I_\phi I_{\phi^*} = I$$

and

$$(2.8)\qquad \begin{cases} \text{i)} & I_\phi R I_{\phi^*} = I\,, \\ \text{ii)} & I_{\phi^*} R' I_\phi = I\,. \end{cases}$$

If the set $\{\phi_\gamma\}_{\gamma\in\Gamma}$ *is bounded in* $\mathcal{P}_\rho(\tilde{\tau}, \tilde{l}_0)$, *the corresponding sets* $\{\sigma(R_\gamma)\}_{\gamma\in\Gamma}$ *and* $\{\sigma(R'_\gamma)\}_{\gamma\in\Gamma}$ *are bounded in* S^0_ρ.

Proof. The property (2.8) follows immediately from (2.6) and (2.7). The existence of R' satisfying (2.7) is proved in Theorem 6.1 of Chap. 10 in [8]. So, it remains to prove the existence of R satisfying (2.6).

Set $P = I_{\phi^*} I_\phi$. Then, we have

$$(2.9)\qquad p(x, \xi')\ (=\sigma(P)) = O_s\text{-}\iint e^{-i\psi} dx' đ\xi$$

with $\psi = x\cdot\xi - \phi(x', \xi) + \phi(x', \xi') - x\cdot\xi'$. Set

$$(2.10)\qquad \tilde{p}(\xi, x', \xi') = \Big\{ |\det \frac{\partial}{\partial x} \tilde{\nabla}_\xi \phi(\xi, w, \xi')|^{-1} \Big\}_{w=\tilde{\nabla}_\xi \phi^{-1}(\xi, x', \xi')},$$

where for a vector $f = {}^t(f_1, \cdots, f_n)$ of functions $f_j(x, \xi)$ $\frac{\partial}{\partial x} f$ is $\Big(\partial f_j/\partial x_k \begin{smallmatrix} j\downarrow 1, \cdots, n \\ k\rightarrow 1, \cdots, n \end{smallmatrix}\Big)$. In what follows we also use $\frac{\partial}{\partial \xi} f = \Big(\partial f_j/\partial \xi_k \begin{smallmatrix} j\downarrow 1, \cdots, n \\ k\rightarrow 1, \cdots, n \end{smallmatrix}\Big)$. Since ψ is written as $\psi = (x - \tilde{\nabla}_\xi \phi(\xi, x', \xi'))\cdot(\xi - \xi')$, by a change of the variables $y = \tilde{\nabla}_\xi \phi(\xi, x', \xi') - x$, $\eta = \xi - \xi'$, we have from (2.9)–(2.10)

$$(2.11)\qquad p(x, \xi') = O_s\text{-}\iint e^{-iy\cdot\eta} \tilde{p}(\xi' + \eta, x + y, \xi') dy đ\eta\,.$$

Here, the oscillatory integral in (2.11) is well-defined because of

$$|\partial^\alpha_\eta D^\beta_y \tilde{p}(\xi' + \eta, x + y, \xi')| \leqq C_{\alpha,\beta} \langle \eta \rangle^{\delta|\alpha+\beta|} \qquad \textit{for any fixed } x \textit{ and } \xi'$$

with $\delta = 1 - \rho$ (<1). Set

$$(2.12)\qquad \tilde{q}(\xi, x', \xi') = \tilde{p}(\xi, x', \xi') - 1\,.$$

Since $\frac{\partial}{\partial x} \tilde{\nabla}_\xi \phi = E + \frac{\partial}{\partial x} \tilde{\nabla}_\xi J$ (E is an identity matrix), $\tilde{q}(\xi, x', \xi')$ has the form

$$(2.13)\qquad \tilde{q}(\xi, x', \xi') = [\{1 - \det(E + \frac{\partial}{\partial x} \tilde{\nabla}_\xi J(\xi, w, \xi'))\} / \det(E + \frac{\partial}{\partial x} \tilde{\nabla}_\xi J(\xi, w, \xi'))]_{w=\tilde{\nabla}_\xi \phi^{-1}(\xi, x, \xi')}\,.$$

Fix a constant $\tilde{\tau}'$ satisfying $0 \leqq \tilde{\tau}' < 1$. Then, if $\phi(x, \xi)$ belongs to $\mathcal{P}_\rho(\tilde{\tau}', l)$, we can prove by applying Lemma 2.1–ii) to $\tilde{\nabla}_\xi \phi^{-1}(\xi, x', \xi')$ that the symbol $\tilde{q}(\xi, x', \xi')$ satisfies

$$(2.14)\quad \begin{cases} \text{i)} & |\partial_\xi^\alpha \partial_{\xi'}^{\alpha'} D_{x'}^{\beta'} \tilde{q}(\xi, x', \xi')| \\ & \quad \leqq C_{\tilde{\tau}', \alpha, \alpha', \beta'} \|J\|_l \langle \xi ; \xi' \rangle^{(1-\rho)|\alpha+\alpha'+\beta'|} \quad (|\alpha+\alpha'+\beta'| \leqq l), \\ \text{ii)} & |\partial_\xi^\alpha \partial_{\xi'}^{\alpha'} D_{x'}^{\beta'} \{\chi((\xi-\xi')/\langle \xi' \rangle) \tilde{q}(\xi, x', \xi')\}| \\ & \quad \leqq C_{\tilde{\tau}', \alpha, \alpha', \beta'} \|J\|_l \langle \xi' \rangle^{-|\alpha+\alpha'|+(1-\rho)|\alpha+\alpha'+\beta'|} \quad (|\alpha+\alpha'+\beta'| \leqq l) \end{cases}$$

with a constant $C_{\tilde{\tau}', \alpha, \alpha', \beta'}$ depending on $\tilde{\tau}'$. Write the simplified symbol $q(x, \xi)$ of $\tilde{q}(\xi, x', \xi')$ as

$$(2.15)\quad q(x, \xi') = O_s\text{-}\iint e^{-iy\cdot\eta} \chi(\eta/\langle \xi' \rangle) \tilde{q}(\xi'+\eta, x+y, \xi') dy\, d\!\!\bar{}\,\eta$$
$$+ O_s\text{-}\iint e^{-iy\cdot\eta} (1-\chi(\eta/\langle \xi' \rangle)) \tilde{q}(\xi'+\eta, x+y, \xi') dy\, d\!\!\bar{}\,\eta$$

and use (2.14)–ii) to the first term of (2.15) and (2.14)–i) to the second term of (2.15). Then, we can find a constant A_5 (depending on $\tilde{\tau}'$) and an integer $\tilde{l}_0$ such that we have for $l_0 = [n/\rho + 1]$

$$(2.16)\quad |q|_{n+1, l_0}^{(0)} \leqq A_5 \|J\|_{\tilde{l}_0}$$

if $\phi(x, \xi)$ belongs to $\mathcal{P}_\rho(\tilde{\tau}', \tilde{l}_0)$. Take a constant $\tilde{\tau} (\leqq \tilde{\tau}')$ satisfying

$$(2.17)\quad \tilde{\tau} < 1/(AA_5)$$

with a constant A in Theorem 2. Then, $q(x, \xi)$ satisfies (14) and by means of Theorem 3 the inverse R of the operator $p(X, D_x) = I + q(X, D_x)$ is obtained with the form $R = r(X, D_x)$ for a symbol $r(x, \xi)$ in S_ρ^0. This R satisfies (2.6). Finally, from the above discussions we obtain the last statement of the proposition. Q.E.D.

From now on, for $p(x, \xi) \in S_\rho^m$ we shall use the semi-norms

$$(2.18)\quad |p|_l^{(m)} = \max_{|\alpha+\beta| \leqq l} \sup_{x, \xi} \{|p_{(\beta)}^{(\alpha)}(x, \xi)| \langle \xi \rangle^{-(m-|\alpha|+(1-\rho)|\alpha+\beta|)}\}$$

instead of using (5).

Proposition 2.3. *Let $\phi(x, \xi)$ belong to $\mathcal{P}_\rho(\tilde{\tau}, \tilde{l}_0)$ for the constant $\tilde{\tau}$ and the integer $\tilde{l}_0$ in Proposition 2.2. Let $p(x, \xi)$ belong to S_ρ^m $(-\infty < m < \infty, 1/2 \leqq \rho \leqq 1)$. Then, we have the following:*

i) *There exist pseudo-differential operators $P_j = p_j(X, D_x)$, $j = 1, 2$, in S_ρ^m such that*

$$(2.19)\quad P_\phi = P_1 I_\phi,$$

$$(2.20)\quad P_\phi = I_\phi P_2$$

and estimates

$$(2.21)\qquad |p_j|_l^{(m)} \leqq C_l |p|_{l'}^{(m)} \qquad (j=1, 2)$$

hold for any l, *where* C_l *is a constant depending only on* m, ρ, l *and* $\|J\|_{l''}$ (*for some* l'') *and* l' *is an integer depending only on* m, ρ *and* l.

ii) *There exist pseudo-differential operators* $P_j = p_j(X, D_x)$, $j=1, 2, 3, 4$, *in* S_ρ^m *such that we have*

$$(2.22)\qquad PI_\phi = I_\phi P_1, \quad I_\phi P = P_2 I_\phi,$$

$$(2.23)\qquad PI_{\phi^*} = I_{\phi^*} P_3, \quad I_{\phi^*} P = P_4 I_{\phi^*}$$

and the symbols $p_j(x, \xi)$, $j=1, 2, 3, 4$, *have the semi-norm estimates similar to* (2.21).

Proof. i) Set

$$\begin{cases} P_1 = P_\phi I_{\phi^*} R', \\ P_2 = R I_{\phi^*} P_\phi \end{cases}$$

with pseudo-differential operators R and R' constructed in Proposition 2.2. Note that from Theorem 1.6 and Theorem 1.7 of Chap. 10 in [8] the operators P_1 and P_2 are pseudo-differential operators in S_ρ^m. From (2.8) they satisfy (2.19) and (2.20). If we go over the proof carefully once again, we obtain (2.21).

ii) Set

$$\begin{cases} P_1 = R I_{\phi^*} P I_\phi, \quad P_2 = I_\phi P I_{\phi^*} R', \\ P_3 = R' I_\phi P I_{\phi^*}, \quad P_4 = I_{\phi^*} P I_\phi R. \end{cases}$$

Then, as in i) we see that P_j, $j=1, 2, 3, 4$, are pseudo-differential operators in S_ρ^m and they satisfy (2.22), (2.23), and the last statement of ii). Q.E.D.

In order to study products of Fourier integral operators, we shall review some results of multi-products of phase functions. Proofs are found in Section 1 of [11] or Section 5 of Chap. 10 in [8]. First, we introduce

DEFINITION 2.4. For $-\infty < m < \infty$, $1/2 \leqq \rho \leqq 1$ and an integer k we define a class $S_\rho^m((k))$ by the set of symbols $p(x, \xi) \in S_\rho^m$ satisfying

$$(2.24)\qquad p_{(\beta)}^{(\alpha)}(x, \xi) \in S_\rho^{m-|\alpha|} \qquad \text{for} \quad |\alpha|+|\beta| \leqq k.$$

The class $S_\rho^m((k))$ is a Fréchet space with semi-norms

$$(2.25)\qquad |p|_l^{(m)} = \max_{|\alpha+\beta| \leqq l} \sup_{x,\xi} \{|p_{(\beta)}^{(\alpha)}(x, \xi)| \langle\xi\rangle^{-(m-|\alpha|+(1-\rho)(|\alpha+\beta|-k)_+)}\}.$$

Now, we begin with

Proposition 2.5. i) *Let τ_o' be a constant satisfying $0 \leqq \tau_o' < 1/3$. Let $\phi_j(x, \xi)$ belong to $\mathcal{P}_\rho(\tau_j), j=1, 2, \cdots, \nu+1, \cdots$, and suppose that $\sum_{j=1}^{\infty} \tau_j \leqq \tau_o'$. Then, the equation*

$$(2.26) \quad \begin{cases} x^j = \nabla_\xi \phi_j(x^{j-1}, \xi^j), \\ \xi^j = \nabla_x \phi_{j+1}(x^j, \xi^{j+1}), \qquad j = 1, \cdots, \nu \quad (x^0 = x, \xi^{\nu+1} = \xi) \end{cases}$$

has the unique C^∞-solution $\{X_\nu^j, \Xi_\nu^j\}_{j=1}(x, \xi)$.

ii) *Setting $J_j(x, \xi) = \phi_j(x, \xi) - x\cdot\xi$, we assume, furthermore, that the set $\{J_j/\tau_j\}$ is bounded in $S_\rho^1((k+2))$ with some $k \geqq 0$. Then, the sets $\{X_\nu^j\}_{j,\nu}$ and $\{\Xi_\nu^j\}_{j,\nu}$ are bounded in $S_\rho^0((k+1))$ and $S_\rho^1((k+1))$, respectively.*

REMARK. In [11] and [8] only the case $k=0$ is considered, but we can prove the proposition similarly for the case $k \geqq 1$.

For any fixed ν we define

$$(2.27) \quad \Phi_{\nu+1}(x, \xi) = \sum_{j=1}^{\nu} (\phi_j(X_\nu^{j-1}, \Xi_\nu^j) - X_\nu^j \cdot \Xi_\nu^j) + \phi_{\nu+1}(X_\nu^\nu, \xi) \quad (X_\nu^0 = x).$$

Then, if $\phi_j \in \mathcal{P}_\rho(\tau_j, l)$, setting $\tilde{J}_{\nu+1}(x, \xi) = \Phi_{\nu+1}(x, \xi) - x\cdot\xi$ it follows that

$$(2.28) \quad \|\tilde{J}_{\nu+1}\|_l \leqq c_{o,l} \bar{\tau}_{\nu+1} \qquad (\bar{\tau}_{\nu+1} = \tau_1 + \tau_2 + \cdots + \tau_{\nu+1})$$

for a constant $c_{o,l}$ determined only by n, ρ, τ_o' and l. Taking account of this we have

Proposition 2.6. *Let $\phi_j(x, \xi)$ belong to $\mathcal{P}_\rho(\tau_j, l)$ and assume that the set $\{J_j/\tau_j\}$ $(J_j(x, \xi) = \phi_j(x, \xi) - x\cdot\xi)$ is bounded in $S_\rho^1((k+2))$ $(k \geqq 0)$. Then, if $c_{o,l}\bar{\tau}_{\nu+1} < 1$, the function $\Phi_{\nu+1}(x, \xi)$ of (2.27) is a phase function in $\mathcal{P}_\rho(c_{o,l}\bar{\tau}_{\nu+1}, l)$ and the set $\{\tilde{J}_{\nu+1}/\bar{\tau}_{\nu+1}\}$ is bounded in $S_\rho^1((k+2))$, where $\tilde{J}_{\nu+1} = \Phi_{\nu+1} - x\cdot\xi$.*

Setting $c_o = c_{o,0}$, we take a constant τ_o satisfying $0 \leqq \tau_o \leqq \tau_o'$ and $c_o\tau_o < 1$. Then, applying the above proposition with $l=0$, the following is justified.

DEFINITION 2.7. Let τ_o be the constant above. For phase functions $\phi_j(x, \xi) \in \mathcal{P}_\rho(\tau_j)$, $j = 1, 2, \cdots, \nu+1$, with $\sum_{j=1}^{\nu+1} \tau_j \leqq \tau_o$ we define the *multi(-#)-product* $\Phi_{\nu+1}(x, \xi) = \phi_1 \# \phi_2 \# \cdots \# \phi_{\nu+1}(x, \xi)$ $(\in \mathcal{P}_\rho(c_o\bar{\tau}_{\nu+1}))$ of $\phi_1(x, \xi)$, $\phi_2(x, \xi)$, $\cdots$, $\phi_{\nu+1}(x, \xi)$ by (2.27).

We return to products of Fourier integral operators.

Proposition 2.8. i) *Let $\phi_j(x, \xi)$ belong to $\mathcal{P}_\rho(\tau_j)$, $j=1, 2$, $\tau_1 + \tau_2 \leqq \tau_o$ and let $\{X, \Xi\} = \{X_1^1, \Xi_1^1\}(x, \xi)$ be the solution of (2.26) with $\nu=1$. Set*

$$(2.29) \quad \Phi(x, \xi) \equiv \phi_1 \# \phi_2(x, \xi) = \phi_1(x, \Xi) - X\cdot\Xi + \phi_2(X, \xi)$$

and

$$(2.30)\quad p(x, \xi') = O_s-\iint e^{i\psi}dx'd\xi$$

with

$$\psi \equiv \psi(x, x'; \xi, \xi') = \phi_1(x, \xi)-x'\cdot\xi+\phi_2(x', \xi')-\Phi(x, \xi').$$

Then, we have $p(x, \xi)\in S_\rho^0$ *and*

$$(2.31)\quad I_{\phi_1}I_{\phi_2} = P_\Phi .$$

ii) *Let* $\{\phi_{1,\gamma}\}_{\gamma\in\Gamma}$ *and* $\{\phi_{2,\gamma}\}_{\gamma\in\Gamma}$ *be bounded sets in* $\mathcal{P}_\rho(\tau_o)$ *and assume that for any* $\gamma\in\Gamma$ *the pair* $\{\phi_{1,\gamma}, \phi_{2,\gamma}\}$ *satisfies the condition in* i). *Then, for the symbol* $p_\gamma(x, \xi)$ *defined from the pair* $\{\phi_{1,\gamma}, \phi_{2,\gamma}\}$ *the set* $\{p_\gamma\}_{\gamma\in\Gamma}$ *is bounded in* S_ρ^0.

REMARK. In [4] Hörmander gave this proposition in the generalized form. Here, we shall give the simplified version of the proof studied in [10].

Proof. We devide the proof into two steps.

I) From the definition of I_{ϕ_1} and I_{ϕ_2} we have

$$(2.32)\quad I_{\phi_1}I_{\phi_2}u = O_s-\iiint e^{i\{\phi_1(x,\xi)-x'\cdot\xi+\phi_2(x',\xi')\}}\hat{u}(\xi')d\xi'\cdot dx'd\xi$$

Substituting (2.30) into (2.32),

$$I_{\phi_1}I_{\phi_2}u = \int e^{i\Phi(x,\xi')}p(x, \xi')\hat{u}(\xi')d\xi'$$

holds. This is nothing but (2.31).

Now, we set

$$(2.33)\quad \chi_\infty(\xi, \xi') = 1-\chi((\xi-\xi')/\langle\xi'\rangle)$$

and consider

$$(2.34)\quad p_\infty(x, \xi') = O_s-\iint e^{i\psi}\chi_\infty(\xi, \xi')dx'd\xi$$
$$= O_s-\iint e^{-ix'\cdot\xi}\tilde{p}_\infty(x', \xi; x, \xi')dx'd\xi ,$$

where

$$\tilde{p}_\infty(x', \xi; x, \xi') = e^{i\{\phi_1(x,\xi)+\phi_2(x',\xi')-\Phi(x,\xi')\}}\chi_\infty(\xi, \xi') .$$

Considering x and ξ' as parameters, the symbol $\partial_\xi^\alpha D_x^\beta\tilde{p}_\infty(x', \xi; x, \xi')$ belongs to $\mathcal{A}_{0,|\alpha|}^{|\beta|}$ defined in §6 of Chap. 1 of [8]. Hence, applying Theorem 6.6 of Chap. 1 of [8] we obtain

$$(2.35)\quad p_{\infty(\beta)}^{(\alpha)}(x, \xi') = O_s-\iint e^{-ix'\cdot\xi}\partial_{\xi'}^\alpha D_x^\beta\tilde{p}_\infty(x', \xi; x, \xi')dx'd\xi .$$

Set

$$\tilde{p}_{\infty,((\alpha,\beta))}(x',\xi;x,\xi')=e^{-i\{\phi_1(x,\xi)+\phi_2(x',\xi')-\Phi(x,\xi')\}}\partial_{\xi'}^{\alpha}D_x^{\beta}\tilde{p}_{\infty}(x',\xi;x,\xi').$$

Then, we have

$$p_{\infty(\beta)}^{(\alpha)}(x,\xi')=O_s-\iint e^{i\psi}\tilde{p}_{\infty,((\alpha,\beta))}(x',\xi;x,\xi')dx'd\xi. \tag{2.36}$$

From $|\nabla_\xi\{\phi_1(x,\xi)+\phi_2(x',\xi')-\Phi(x,\xi')\}|\leqq C\langle x-x'\rangle$ and $|\nabla_x\{\phi_1(x,\xi)+\phi_2(x',\xi')-\Phi(x,\xi')\}|\leqq C\langle\xi;\xi'\rangle$ we have

$$|\partial_{\xi}^{\alpha'}D_{x'}^{\beta'}\tilde{p}_{\infty,((\alpha,\beta))}(x',\xi;x,\xi')|\leqq C_{\alpha,\beta,\alpha',\beta'}\langle x-x'\rangle^{|\alpha|}\langle\xi;\xi'\rangle^{|\beta|}\langle\xi'\rangle^{(1-\rho)|\beta'|}. \tag{2.37}$$

Since we have $|\xi-\xi'|\geqq(2/5)\langle\xi'\rangle$ on supp $\chi_\infty(\xi,\xi')$, we obtain on supp $\tilde{p}_{\infty,((\alpha,\beta))}$

$$\begin{aligned}|\nabla_{x'}\psi|&=|-\xi+\nabla_x\phi_2(x',\xi')|\\&\geqq|\xi-\xi'|-\tau_2\langle\xi'\rangle\\&\geqq\frac{1}{6}|\xi-\xi'|\geqq\frac{1}{15}\langle\xi'\rangle.\end{aligned}$$

Moreover, we can prove

$$1+|\nabla_\xi\psi|\geqq C\langle x-x'\rangle$$

with some positive constant C. Set

$$\begin{cases}L_1=-i|\nabla_{x'}\psi|^{-2}\nabla_{x'}\psi\cdot\nabla_{x'},\\ L_2=(1+|\nabla_\xi\psi|^2)^{-1}(1-i\nabla_\xi\psi\cdot\nabla_\xi)\end{cases}$$

and write

$$p_{\infty(\beta)}^{(\alpha)}(x,\xi')=\iint e^{i\psi}(L_1{}^t)^{l_1}(L_2{}^t)^{l_2}\tilde{p}_{\infty,((\alpha,\beta))}dx'd\xi$$

for a fixed $l_2>n+|\alpha|$ and large l_1. Then, we get for any N

$$|p_{\infty(\beta)}^{(\alpha)}(x,\xi')|\leqq C_{N,\alpha,\beta}\langle\xi'\rangle^{-N}, \tag{2.38}$$

that is, we have

$$p_\infty(x,\xi)\in S^{-\infty}. \tag{2.39}$$

II) For $\chi_O(\xi,\xi')=\chi((\xi-\xi')/\langle\xi'\rangle)$ we consider

$$p_O(x,\xi')=O_s-\iint e^{i\psi}\chi_O(\xi,\xi')dx'd\xi. \tag{2.40}$$

Using a change of the variables: $x'=X(x,\xi')+y$, $\xi=\Xi(x,\xi')+\eta$, we write

$$p_O(x,\xi')=O_s-\iint e^{-i\tilde{\psi}(y,\eta;x,\xi')}\tilde{\chi}_O(\eta;x,\xi')dyd\eta. \tag{2.41}$$

Here,

$$\tilde{\chi}_O(\eta;\, x,\, \xi) = \chi((\Xi(x,\, \xi)+\eta-\xi)/\langle\xi\rangle) \tag{2.42}$$

and

$$\begin{aligned}\tilde{\psi} \equiv \tilde{\psi}(y,\, \eta;\, x,\, \xi) &= -\psi(x,\, X(x,\, \xi)+y;\, \Xi(x,\, \xi)+\eta,\, \xi)\\ &= y\cdot\eta-\{\phi_1(x,\, \Xi+\eta)-X\cdot\eta-\phi_1(x,\, \Xi)\}\\ &\qquad -\{\phi_2(X+y,\, \xi)-y\cdot\Xi-\phi_2(X,\, \xi)\}\,.\end{aligned} \tag{2.43}$$

Since $\{X, \Xi\}$ is the solution of (2.26) with $\nu=1$, we have

$$|X-x|\leqq\tau_1\leqq\frac{1}{3} \qquad |\Xi-\xi|\leqq\tau_2\langle\xi\rangle\leqq\frac{1}{3}\langle\xi\rangle\,.$$

Hence, we have from the definition of $\tilde{\chi}_O=\tilde{\chi}_O(\eta;\, x,\, \xi)$

$$\begin{cases} |\Xi+\theta\eta-\xi|\leqq\theta|\Xi+\eta-\xi|+(1-\theta)|\Xi-\xi|\leqq\dfrac{1}{2}\langle\xi\rangle\,, \\ \dfrac{1}{2}\langle\xi\rangle\leqq\langle\Xi+\theta\eta\rangle\leqq2\langle\xi\rangle \qquad (0\leqq\theta\leqq1) \quad \textit{on} \quad \text{supp}\,\tilde{\chi}_O\,. \end{cases} \tag{2.44}$$

Taking account of (2.26) for $\nu=1$ we have

$$\begin{cases} \nabla_\eta\tilde{\psi} = y-\Big(\displaystyle\int_0^1\frac{\partial}{\partial\xi}\nabla_\xi J_1(x,\, \Xi+\theta\eta)d\theta\Big)\eta\,, \\ \nabla_y\tilde{\psi} = \eta-\Big(\displaystyle\int_0^1\frac{\partial}{\partial x}\nabla_x J_2(X+\theta y,\, \xi)d\theta\Big)y\,. \end{cases}$$

Then, from (2.44) we have

$$\begin{cases} |\nabla_\eta\tilde{\psi}|\geqq|y|-2\tau_1\langle\xi\rangle^{-1}|\eta|\geqq|y|-\dfrac{2}{3}\langle\xi\rangle^{-1}|\eta|\,, \\ |\nabla_y\tilde{\psi}|\geqq|\eta|-\tau_2\langle\xi\rangle|y|\geqq|\eta|-\dfrac{1}{3}\langle\xi\rangle|y| \qquad \textit{on} \quad \text{supp}\,\tilde{\chi}_O \end{cases}$$

and get

$$\langle\xi\rangle^2|\nabla_\eta\tilde{\psi}|^2+|\nabla_y\tilde{\psi}|^2\geqq\frac{1}{2}(\langle\xi\rangle|\nabla_\eta\tilde{\psi}|+|\nabla_y\tilde{\psi}|)^2\geqq\frac{1}{18}(\langle\xi\rangle|y|+|\eta|)^2 \quad \textit{on} \quad \text{supp}\,\tilde{\chi}_O\,. \tag{2.45}$$

On the other hand, we rewrite $\tilde{\psi}$ in the form

$$\tilde{\psi} = y\cdot\eta-B\eta\cdot\eta-B'y\cdot y \tag{2.46}$$

with

$$(2.47)\quad \begin{cases} B = B(\eta;\, x,\, \xi) = \int_0^1 (1-\theta)\dfrac{\partial}{\partial \xi}\nabla_\xi J_1(x,\, \Xi+\theta\eta)d\theta\,, \\ B' = B'(y;\, x,\, \xi) = \int_0^1 (1-\theta)\dfrac{\partial}{\partial x}\nabla_x J_2(X+\theta y,\, \xi)d\theta\,. \end{cases}$$

Then, as in the first step, we have

$$(2.48)\quad p_{O(\beta)}^{(\alpha)}(x,\, \xi) = O_s\text{-}\iint e^{-iy\cdot\eta}\partial_\xi^\alpha D_x^\beta\{e^{i(B\eta\cdot\eta+B'y\cdot y)}\tilde{\chi}_O\}\, dy\, đ\eta$$
$$= O_s\text{-}\iint e^{-i\tilde{\psi}}\tilde{p}_{O,((\alpha,\beta))}(y,\, \eta;\, x,\, \xi)dy\, đ\eta$$

with

$$(2.49)\quad \tilde{p}_{O,((\alpha,\beta))}(y,\, \eta;\, x,\, \xi) = e^{-i(B\eta\cdot\eta+B'y\cdot y)}\partial_\xi^\alpha D_x^\beta(e^{i(B\eta\cdot\eta+B'y\cdot y)}\tilde{\chi}_O)$$
$$= \sum_{k=0}^{|\alpha+\beta|}\ \sum_{\substack{\alpha'+\alpha^1+\cdots+\alpha^k=\alpha \\ \beta^1+\cdots+\beta^k=\beta}} C_{k,\alpha',\tilde{\alpha}^k,\tilde{\beta}^k}$$
$$\times \prod_{j=1}^{k} \{(\partial_\xi^{\alpha^j}D_x^{\beta^j}B)\eta\cdot\eta+(\partial_\xi^{\alpha^j}D_x^{\beta^j}B')y\cdot y\}\,\partial_\xi^{\alpha'}\tilde{\chi}_O\,.$$

This expression (2.49) yields

$$(2.50)\quad |\partial_\eta^{\alpha'}D_y^{\beta'}\tilde{p}_{O,((\alpha,\beta))}(y,\, \eta;\, x,\, \xi)|$$
$$\leqq C_{\alpha,\alpha',\beta,\beta'}\langle\xi\rangle^{-|\alpha|+(1-\rho)|\alpha+\beta|+\frac{1}{2}(|\beta'|-|\alpha'|)}\{1+\langle\xi\rangle^{-1/2}(\langle\xi\rangle|y|+|\eta|)\}^{2|\alpha+\beta|}$$

in view of the fact that the symbols B and B' in (2.47) have the orders -1 and 1, respectively, with respect to ξ. We set

$$L_3 = (1+\langle\xi\rangle^{-1}(\langle\xi\rangle^2|\nabla_\eta\tilde{\psi}|^2+|\nabla_y\tilde{\psi}|^2))^{-1}$$
$$\times(1+i\langle\xi\rangle^{-1}(\langle\xi\rangle^2\nabla_\eta\tilde{\psi}\cdot\nabla_\eta+\nabla_y\tilde{\psi}\cdot\nabla_y))$$

and write (2.48) in the form

$$p_{O(\beta)}^{(\alpha)}(x,\, \xi) = \iint e^{-i\tilde{\psi}}(L_3{}^t)^l\tilde{p}_{O,((\alpha,\beta))}(y,\, \eta;\, x,\, \xi)dy\, đ\eta$$

for $l=2n+1+2(|\alpha+\beta|)$. Then, we have

$$(2.51)\quad |p_{O(\beta)}^{(\alpha)}(x,\xi)| \leqq C_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|+(1-\rho)|\alpha+\beta|}\,.$$

Consequently, we have

$$(2.52)\quad p_O\in S_\rho^0$$

and combining this with (2.39) we obtain

$$(2.53)\quad p(x,\, \xi)\in S_\rho^0$$

for $p(x, \xi)$ in (2.30). Finally, we get ii) if we go over the proof carefully once again. Q.E.D.

Now, we apply Proposition 2.3-i) to the Fourier integral operator P_Φ in (2.31). Then, we have

Corollary 2.9. *Let $\phi_j(x, \xi)$ belong to $\mathcal{P}_\rho(\tau_j)$, $j=1, 2$, $\tau_1+\tau_2\leqq\tau_o$, and assume that the phase function $\Phi(x, \xi)$ defined by* (2.29) *belongs to $\mathcal{P}_\rho(\tilde{\tau}, \tilde{l}_o)$ for the constant $\tilde{\tau}$ and the integer $\tilde{l}_o$ in Proposition* 2.2. *Then, there exist symbols $p_j(x, \xi)$, $j=1, 2$, in S^0_ρ such that for $P_j=p_j(X, D_x)$*

$$(2.54) \qquad I_{\phi_1}I_{\phi_2} = P_1 I_\Phi = I_\Phi P_2$$

holds.

REMARK 1. Let $c_{o,\tilde{l}_o}$ be the constant defined in Proposition 2.6 with $l=\tilde{l}_o$. Then, if $\phi_j(x, \xi)$ in Corollary 2.9 belongs to $\mathcal{P}_\rho(\tau_j, \tilde{l}_o)$, $j=1, 2$, and $\tau_1+\tau_2\leqq\tilde{\tau}/c_{o,\tilde{l}_o}$ holds, the phase function $\Phi(x, \xi)$ of (2.29) belongs to $\mathcal{P}_\rho(\tilde{\tau}, \tilde{l}_o)$.

REMARK 2. Let $\{\phi_{1,\gamma}\}_{\gamma\in\Gamma}$ and $\{\phi_{2,\gamma}\}_{\gamma\in\Gamma}$ be bounded sets in $\mathcal{P}_\rho(\tau_o)$ and assume that for any $\gamma\in\Gamma$ the pair $\{\phi_{1,\gamma}, \phi_{2,\gamma}\}$ satisfies the condition in the corollary. Then, for the symbols $p_{1,\gamma}(x, \xi)$ and $p_{2,\gamma}(x, \xi)$ defined from the pair $\{\phi_{1,\gamma}, \phi_{2,\gamma}\}$ the sets $\{p_{1,\gamma}\}_{\gamma\in\Gamma}$ and $\{p_{2,\gamma}\}_{\gamma\in\Gamma}$ are bounded in S^0_ρ.

Lemma 2.10. *Let $\phi_j(x, \xi)$ belong to $\mathcal{P}_\rho(\tau_j, \tilde{l}_o)$, $j=1, 2$, with $\tau_1+\tau_2\leqq\tau_o$, $\tau_j\leqq\tilde{\tau}$ satisfying $\Phi(x, \xi)\equiv\phi_1\#\phi_2(x, \xi)\in\mathcal{P}_\rho(\tilde{\tau}, \tilde{l}_o)$ for the constant $\tilde{\tau}$ and the integer $\tilde{l}_o$ in Proposition* 2.2, *and let $p(x, \xi)$ belong to S^m_ρ. Then, there exist pseudo-differential operators P' and P'' in S^m_ρ such that*

$$(2.55) \qquad I_{\phi_1}P_{\phi_2} = P' I_\Phi$$

and

$$(2.56) \qquad P_{\phi_1}I_{\phi_2} = I_\Phi P''\,.$$

Moreover, estimates

$$(2.57) \qquad \begin{cases} |p'|_l^{(m)}\leqq C_l|p|_{l'}^{(m)}\,, \\ |p''|_l^{(m)}\leqq C_l|p|_{l'}^{(m)} \end{cases}$$

hold for a constant C_l depending only on m, ρ, l and $\{\|J_j\|_{l''}\}_{j=1,2}$ (for some l'') and an integer l' depending only on m, ρ and l.

Proof. We prove (2.55). Then, we can prove (2.56) similarly. From Proposition 2.3-i) there exists a pseudo-differential operator P_1 in S^m_ρ satisfying

$$P_{\phi_2} = P_1 I_{\phi_2}\,.$$

Next, we apply Proposition 2.3-ii) to find a pseudo-differential operator P_2 in S^m_ρ satisfying

$$I_{\phi_1}P_1 = P_2 I_{\phi_1}\,.$$

Then, we have

$$I_{\phi_1}P_{\phi_2} = P_2 I_{\phi_1} I_{\phi_2}\,.$$

Use Corollary 2.9 to find a pseudo-differential operator R° in S^0_ρ satisfying

$$I_{\phi_1} I_{\phi_2} = R^\circ I_\Phi\,.$$

Then, setting $P'=P_2R^\circ$, we get (2.55). If we go over the proof once again, we can prove the last statement. Q.E.D.

Now, we *prove Theorem* 1. We take the integer $\tilde{l}_o$ in Proposition 2.2 as the one in Theorem 1. Define

$$\tau^0 = \min(\tau_o/c_o,\ \tilde{\tau}/c_{o,\tilde{l}_o}) \tag{2.58}$$

with the constants τ_o, $c_o\ (=c_{o,0})$, $\tilde{\tau}$ and $c_{o,\tilde{l}_o}$ introduced in Definition 2.7, Proposition 2.2 and Proposition 2.6. Then, if phase functions $\phi_j(x, \xi)\in\mathscr{P}_\rho(\tau_j, \tilde{l}_o)$ satisfy (*) in Introduction, we have for multi-products $\Phi_j=\phi_1\#\phi_2\#\cdots\#\phi_j$

$$\Phi_j\in\mathscr{P}_\rho(\tau_o)\,,\quad \Phi_{j+1} = \Phi_j\#\phi_{j+1}\in\mathscr{P}_\rho(\tilde{\tau}, \tilde{l}_o) \qquad (\Phi_1=\phi_1)$$

from (1.30) of [11] and Proposition 2.6. Using this we prove (9)-i) for the multi-product

$$\tilde{Q}_{\nu+1} = P_{1,\phi_1}P_{2,\phi_2}\cdots P_{\nu+1,\phi_{\nu+1}}$$

with $P_{j,\phi_j}=p_{j,\phi_j}(X, D_x)$ for $p_j(x, \xi)\in S^{m_j}_\rho$. First, we apply Proposition 2.3-i). Then, there exists a pseudo-differential operator P'_1 in $S^{m_1}_\rho$ such that

$$P_{1,\phi_1} = P'_1 I_{\phi_1}\,. \tag{2.59}$$

For j with $j\geqq 2$ we apply Lemma 2.10. Then, there exists a pseudo-differential operator P'_j in $S^{m_j}_\rho$ such that

$$I_{\Phi_{j-1}}P_{j,\phi_j} = P'_j I_{\Phi_j} \qquad (\Phi_1 = \phi_1)\,. \tag{2.60}$$

Combining (2.59) and (2.60), we get

$$\begin{aligned} \tilde{Q}_{\nu+1} &= P'_1(I_{\phi_1}P_{2,\phi_2})P_{3,\phi_3}\cdots P_{\nu+1,\phi_{\nu+1}} \\ &= P'_1P'_2(I_{\Phi_2}P_{3,\phi_3})P_{4,\phi_4}\cdots P_{\nu+1,\phi_{\nu+1}} \\ &= \cdots\cdots\cdots \\ &= P'_1P'_2\cdots P'_\nu(I_{\Phi_\nu}P_{\nu+1,\phi_{\nu+1}}) \\ &= P'_1P'_2\cdots P'_\nu P'_{\nu+1} I_{\Phi_{\nu+1}}\,. \end{aligned} \tag{2.61}$$

This proves (9)-i). Similarly, we can prove (9)-ii).

From the above discussion the boundedness of the sequence $\{m_j\}$ implies

$$(2.62)\qquad \begin{cases} |\sigma(P_j')|_l^{(m_j)} \leqq C_l |p_j|_{l'}^{(m_j)}, \\ |\sigma(P_j'')|_l^{(m_j)} \leqq C_l |p_j|_{l'}^{(m_j)} \end{cases}$$

with a constant C_l and an integer l' independent of j. This comes from the fact that P_j' and P_j'' are determined only by P_{j,ϕ_j} and $\{\phi_k\}_{k=1}^{\nu+1}$. Combining (9) and (2.62) with Theorem 2 we get Theorem 1. This concludes the proof of Theorem 1.

The asymptotic expansion for the symbol $q_{\nu+1}(x, \xi)$ of multi-products (3) was discussed in the proof of Theorem 2.4 in [10]. Here, we give its well-arranged form, which is not used in the following but which is derived directly from the discussions of the proof of Theorem 1.

Theorem 2.11. *Let $\phi_j(x, \xi)$ belong to $\mathscr{P}_\rho(\tau_j)$ and $p_j(x, \xi)$ belong to $S_\rho^{m_j}$ for $1/2<\rho\leqq 1$ verifying the assumptions in Theorem 1. Let $\{X_\nu^j, \Xi_\nu^j\}_{j=1}^\nu$ be the solution of (2.26). Then, the symbol $q_{\nu+1}(x, \xi)$ of the multi-product (3) of Fourier integral operators $P_{j,\phi_j}=p_{j,\phi_j}(X, D_x)$ satisfies*

$$(2.63)\qquad q_{\nu+1}(x, \xi) \sim \sum_{k=0}^{\infty} \sum_{|\tilde{\alpha}^\nu|+|\tilde{\beta}^\nu|\leqq 2k} r_{k,\tilde{\alpha}^\nu,\tilde{\beta}^\nu}^{\nu+1}(x, \xi) p_1^{(\alpha^1)}(x, \Xi_\nu^1) p_{2(\beta^1)}^{(\alpha^2)}(X_\nu^1, \Xi_\nu^2) \cdots \\ \times p_{\nu(\beta^{\nu-1})}^{(\alpha^\nu)}(X_\nu^{\nu-1}, \Xi_\nu^\nu) p_{\nu+1(\beta^\nu)}(X_\nu^\nu, \xi)$$

in the sense of Definition 1.6 of Chap. 2 in [8], where $r_{k,\tilde{\alpha}^\nu,\tilde{\beta}^\nu}^{\nu+1}(x,\xi)$ belong to $S_\rho^{m(k,\tilde{\alpha}^\nu,\tilde{\beta}^\nu)}$ with $m(k, \tilde{\alpha}^\nu, \tilde{\beta}^\nu)=-(2\rho-1)k+|\tilde{\alpha}^\nu|-(1-\rho)(|\tilde{\alpha}^\nu|+|\tilde{\beta}^\nu|)$ and $|\tilde{\alpha}^\nu|=|\alpha^1|+\cdots+|\alpha^\nu|$, $|\tilde{\beta}^\nu|=|\beta^1|+\cdots+|\beta^\nu|$ for $\tilde{\alpha}^\nu=(\alpha^1, \cdots, \alpha^\nu)$, $\tilde{\beta}^\nu=(\beta^1, \cdots, \beta^\nu)$.

Proof. From the proof of Theorem 1 the pseudo-differential operators P_j' $(j=1, 2, \cdots, \nu)$ in (2.59)–(2.60) have the forms

$$(2.64)\qquad \begin{cases} P_1' = P_{1,\phi_1} I_{\phi^*} R_1', \\ P_j' = I_{\Phi_{j-1}}(P_{j,\phi_j} I_{\phi_{j-1}^*} R_j') I_{\Phi_{j-1}^*} R_j'' \qquad (2\leqq j\leqq \nu) \end{cases}$$

with some pseudo-differential operators R_j' and R_j'' in S_ρ^0, where $\Phi_1=\phi_1$, $\Phi_1=\phi_1 \# \phi_2 \# \cdots \# \phi_j$ $(j\geqq 2)$. For $j=\nu+1$, applying Proposition 2.3-i), we write

$$(2.65)\qquad P_{\nu+1,\phi_{\nu+1}} = I_{\phi_{\nu+1}} P_{\nu+1}''$$

with

$$P_{\nu+1}'' = R_{\nu+1} I_{\phi_{\nu+1}^*} P_{\nu+1,\phi_{\nu+1}} \qquad (\sigma(R_{\nu+1})\in S_\rho^0).$$

Then, with the aid of $I_{\Phi_\nu} I_{\phi_{\nu+1}}=R_O I_{\Phi_{\nu+1}}$ by Corollary 2.9 the multi-product $\tilde{Q}_{\nu+1}$ of (3) has the form

(2.66) $\tilde{Q}_{\nu+1} = P_1'P_2'\cdots P_\nu' R_O I_{\Phi_{\nu+1}} P_{\nu+1}''$.

Here, R_O is a pseudo-differential operator in S_ρ^0.

Denote for a phase function $\phi(x, \xi) \in \mathcal{P}_\rho(\tau)$ the inverses of $\xi = \nabla_x\phi(x, \eta)$ and $x = \nabla_\xi\phi(y, \xi)$ by $\eta = \nabla_x\phi^{-1}(x, \xi)$ and $y = \nabla_\xi\phi^{-1}(x, \xi)$, respectively. We note that $\nabla_x\phi^{-1}(x, \xi) = \tilde{\nabla}_x\phi^{-1}(x, \xi, x)$ and $\nabla_\xi\phi^{-1}(x, \xi) = \tilde{\nabla}_\xi\phi^{-1}(\xi, x, \xi)$ hold. Using Theorem 1.6, Theorem 1.7 and Theorem 2.1 of Chap. 10 in [8] we have from (2.64) and (2.65)

(2.67) $$p_1'(x, \xi) \sim \sum_{k=0}^{\infty} \sum_{|\alpha| \leq 2k} r_{1,k,\alpha}(x, \xi) p_1^{(\alpha)}(x, \nabla_x\phi^{-1}(x, \xi)),$$

(2.68) $$p_j'(x, \xi) \sim \sum_{k=0}^{\infty} \sum_{|\alpha+\beta| \leq 2k} r_{j,k,\alpha,\beta}(x, \xi) p_{j(\beta)}^{(\alpha)}(\nabla_\xi\Phi_{j-1}(x, \eta), \nabla_x\phi_j^{-1}(\nabla_\xi\Phi_{j-1}(x, \eta), \eta))|_{\eta=\nabla_x\Phi_{j-1}^{-1}(\xi,x)} \quad (2 \leq j \leq \nu),$$

(2.69) $$p_{\nu+1}''(x, \xi) \sim \sum_{k=0}^{\infty} \sum_{|\beta| \leq 2k} r_{\nu+1,k,\beta}(x, \xi) p_{\nu+1(\beta)}(\nabla_\xi\phi_{\nu+1}^{-1}(x, \xi), \xi)$$

with symbols

(2.70) $$\begin{cases} r_{1,k,\alpha} \in S_\rho^{-(2\rho-1)k+\rho|\alpha|}, \\ r_{j,k,\alpha,\beta} \in S_\rho^{-(2\rho-1)k+\rho|\alpha|-(1-\rho)|\beta|} \quad (2 \leq j \leq \nu), \\ r_{\nu+1,k,\beta} \in S_\rho^{-(2\rho-1)k-(1-\rho)|\beta|}. \end{cases}$$

On the other hand, we can prove by the same method as the discussions in Section 1 of [11]

(2.71) $$\nabla_x\phi_1^{-1}(x, \nabla_x\Phi_{\nu+1}(x, \xi)) = \Xi_\nu^1(x, \xi),$$

(2.72) $$\begin{cases} \nabla_x\Phi_{j-1}^{-1}(x, \nabla_x\Phi_{\nu+1}(x, \xi)) = \Xi_\nu^{j-1}(x, \xi), \\ \nabla_\xi\Phi_{j-1}(x, \Xi_\nu^{j-1}) = X_\nu^{j-1}(x, \xi), \\ \nabla_x\phi_j^{-1}(X_\nu^{j-1}, \Xi_\nu^{j-1}) = \Xi_\nu^j(x, \xi) \quad (2 \leq i \leq \nu), \end{cases}$$

(2.73) $$\nabla_\xi\phi_{\nu+1}^{-1}(\nabla_\xi\Phi_{\nu+1}(x, \xi), \xi) = X_\nu^\nu(x, \xi).$$

Consequently, applying Theorem 2.5 of Chap. 7 in [8], we can derive (2.63) from (2.66)–(2.69), (2.71)–(2.73) and (2.4-i) and (2.23-i) of Chap. 10 in [8]. Q.E.D.

3. Commutative law for #-products of phase functions. Let $\phi_j(t, s; x, \xi)$ $(j=1, 2)$ be the phase function defined by an eikonal equation

(3.1) $$\begin{cases} \partial\phi/\partial t - \lambda(t, x, \nabla_x\phi) = 0 \quad \text{on} \quad [0, T], \\ \phi|_{t=s} = x \cdot \xi \end{cases}$$

for $\lambda(t, x, \xi) = \lambda_j(t, x, \xi)$ (real symbol of order one), and let $I_{\phi_j}(t, s)$ be the Fourier integral operator with phase function $\phi_j(t, s; x, \xi)$ and symbol 1. What we want to study is the following problem: When do $I_{\phi_1}(t, s)$ and $I_{\phi_2}(t, s)$ commute, or in a wider sense, when is the product $I_{\phi_2}(t, \theta)I_{\phi_1}(\theta, s)$ is equal to $I_{\phi_1}(t, \omega)I_{\phi_2}(\omega, s)R$ for an appropriate constant ω and a pseudo-differential operator R? The positive answer of this problem suggests the possibility of the reduction of the infinite sum expression (2) for the fundamental solution to the finite sum expression (4), that is, the possibility of the proof of Theorem 4. If the Poisson bracket

$$\{\tau-\lambda_1, \tau-\lambda_2\} = \partial\lambda_1/\partial t-\partial\lambda_2/\partial t+\nabla_\xi\lambda_1\cdot\nabla_x\lambda_2-\nabla_x\lambda_1\cdot\nabla_\xi\lambda_2$$

of $\tau-\lambda_1$ and $\tau-\lambda_2$ (τ is the dual variable of t) is identically zero, Kumano-go-Taniguchi-Tozaki [11] proved

$$(3.2) \qquad (\phi_2(t, \theta)\#\phi_1(\theta, s))(x, \xi) = (\phi_1(t, t-\theta+s)\#\phi_2(t-\theta+s, s))(x, \xi),$$

which implies

$$(3.3) \qquad I_{\phi_2}(t, \theta)I_{\phi_1}(\theta, s) = I_{\phi_1}(t, t-\theta+s)I_{\phi_2}(t-\theta+s, s)R$$

on account of (2.54). In this way the above problem is reduced to the problem of the commutative law for phase functions. In the present paper, we shall show their commutative law under the condition

$$(3.4) \qquad \{\tau-\lambda_1, \tau-\lambda_2\} = a(t, x, \xi)\,(\lambda_1-\lambda_2)+a'(t, x, \xi)$$

where $a(t, x, \xi)$ and $a'(t, x, \xi)$ are real symbols of order zero.

For the further study, we shall review the properties with additional results for the phase function $\phi(t, s; x, \xi)$ defined by an eikonal equation (3.1). We note that (3.1) corresponds to a hyperbolic operator

$$(3.5) \qquad \mathcal{L}_0 = D_t-\lambda(t, X, D_x) \quad \text{on} \quad [0, T],$$

where $D_t=-i\partial_t$, $\partial_t=\partial/\partial t$. To begin with, we introduce the following definition.

DEFINITION 3.1. Let Z be a subset of Euclidean space $R^{\tilde{n}}_{\tilde{t}}$ and let $F(\subset S^{\infty}_{\rho,\delta})$ be a Fréchet space of symbol class of pseudo-differential operators (for example, $F=S^m_{\rho,\delta}$, S^m_ρ or $S^m_\rho((k))$). We say that a C^l-function $p(\tilde{t}, x, \xi)$ in $Z\times R^n_x\times R^n_\xi$ belongs to a class $M^l(Z; F)$ when $\partial^\alpha_\xi D^\beta_x p(\tilde{t}, x, \xi)$ is a C^l-function for any α, β, $p(\tilde{t}, x, \xi)$ belongs to F for any $\tilde{t}\in Z$ and the set $\{(\partial/\partial\tilde{t})^{\tilde{\gamma}}p(\tilde{t}, x, \xi)\}_{\tilde{t}\in Z}$ is a bounded set in F for any $\tilde{\gamma}$ with $|\tilde{\gamma}|\leqq l$. We set $M(Z; F)=\bigcap_{l=0}^{\infty} M^l(Z; F)$ and use the expression "$\{p_\theta(\tilde{t}, x, \xi)\}_{\theta\in\Theta}$ is bounded in $M^l(Z; F)$ [resp. in $M(Z; F)$]" if the set $\{(\partial/\partial\tilde{t})^{\tilde{\gamma}}p_\theta(\tilde{t}, x, \xi)\}_{\tilde{t}\in Z,\theta\in\Theta}$ is a bounded set in F for any $\tilde{\gamma}$ satisfying $|\tilde{\gamma}|\leqq l$ [resp. $|\tilde{\gamma}|<\infty$]. For an integer k and $\rho\in[1/2, 1]$ we also set

$$\bar{M}(Z; S^m_\rho; k) = \bigcap_{l=0}^{k} M^l(Z; S^m_\rho((k-l))) \cap \bigcap_{l=k+1}^{\infty} M^l(Z; S^{m+(1-\rho)(l-k)}_\rho).$$

We consider the Hamilton equation corresponding to (3.1):

$$(3.6) \qquad \begin{cases} \dfrac{dq}{dt} = -\nabla_\xi \lambda(t, q, p), \quad \dfrac{dp}{dt} = \nabla_x \lambda(t, q, p), \\ \{q, p\}_{|t=s} = \{y, \eta\}. \end{cases}$$

Then, we have

Lemma 3.2. i) *Let* $\lambda(t, x, \xi)$ *belong to* $M^0([0, T]; S^1_\rho((k+2)))$ $(k \geqq 0)$. *Then, the solution* $\{q, p\}(t, s; y, \eta)$ *of* (3.6) *satisfies for a small* $T_1(\leqq T)$

$$(3.7) \qquad \begin{cases} \{(q-y)/|t-s|\} & \textit{is bounded in} \quad S^0_\rho((k+1)), \\ \{(p-\eta)/|t-s|\} & \textit{is bounded in} \quad S^1_\rho((k+1)) \\ & \qquad (0 \leqq s, \ t \leqq T_1, \ s \neq t), \end{cases}$$

and

$$(3.8) \qquad \begin{cases} q-y \in M^0(Z(T_1); S^0_\rho((k+1))) \cap M^1(Z(T_1); S^0_\rho((k))), \\ p \in M^0(Z(T_1); S^1_\rho((k+1))) \cap M^1(Z(T_1); S^1_\rho((k))), \end{cases}$$

where $Z(T) = \{(t, s); 0 \leqq t, s \leqq T\}$.

ii) *We assume, furthermore, that* $\lambda(t, x, \xi)$ *belongs to* $M([0, T]; S^1_\rho((k+2)))$. *Then,* $q(t, s; y, \eta) - y$ *belongs to* $\bar{M}(Z(T_1); S^0_\rho; k+1)$ *and* $p(t, s; y, \eta)$ *belongs to* $\bar{M}(Z(T_1); S^1_\rho; k+1)$.

Proof. By the similar way as in the proof of Lemma 3.1 in [7] we can prove (3.7) and

$$(3.8)' \qquad \begin{cases} q-y \in M^0(Z(T_1); S^0_\rho((k+1))), \\ p \in M^0(Z(T_1); S^1_\rho((k+1))) \end{cases}$$

for a small T_1 $(\leqq T)$. Consider the equation (3.6) and

$$(3.9) \qquad \begin{bmatrix} \partial_s q(t, s; y, \eta) \\ \partial_s p(t, s; y, \eta) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial y} q(t, s; y, \eta) & \dfrac{\partial}{\partial \eta} q(t, s; y, \eta) \\ \dfrac{\partial}{\partial y} p(t, s; y, \eta) & \dfrac{\partial}{\partial \eta} p(t, s; y, \eta) \end{bmatrix} \begin{bmatrix} \nabla_\xi \lambda(s, y, \eta) \\ -\nabla_x \lambda(s, y, \eta) \end{bmatrix}$$

Then, from (3.8)' we get (3.8). For the proof of ii) we differentiate the equations in (3.6) and (3.9) with respect to t and s. Then, using (3.8) we get ii) inductively. Q.E.D.

Let ε_1 be $0 < \varepsilon_1 \leqq 1$. Then, from (3.7) we can find a constant T_2 $(\leqq T_1)$ such that

$$(3.10)\qquad \left\|\frac{\partial}{\partial y}q-E\right\|\leqq 1-\varepsilon_1 \quad \textit{for}\quad 0\leqq s,\, t\leqq T_2$$

holds, where E is the identity matrix and $||W||$ is a matrix norm $\sum_{j,k}|w_{jk}|$ of a matrix $W=(w_{jk})$. We fix such a T_2. Then, we have

Lemma 3.3. *Let $\lambda(t, x, \xi)$ belong to $M^0([0, T]; S^1_\rho((k+2)))$. Then, for the above $q(t, s; y, \eta)$ the equation $x=q(t, s; y, \xi)$ has the unique solution $y=Y(t, s; x, \xi)$ satisfying*

$$(3.11)\qquad \begin{cases} Y(t, s; x, \xi)-x\in M^0(Z(T_2); S^0_\rho((k+1)))\cap M^1(Z(T_2); S^0_\rho((k))), \\ \{(Y-x)/|t-s|\} \quad \textit{is bounded in}\quad S^0_\rho((k+1)) \qquad (0\leqq s,\, t\leqq T_2,\, s\neq t). \end{cases}$$

Furthermore, if we assume $\lambda(t, x, \xi)\in M([0, T]; S^1_\rho((k+2)))$, $Y(t, s; x, \xi)-x$ belongs to $\bar{M}(Z(T_2); S^0_\rho; k+1)$.

We can prove this lemma by the similar way as the one in Lemma 3.2 of [7].

Proposition 3.4. *Let $\lambda(t, x, \xi)$ belong to $M^0([0, T]; S^1_\rho((k+2)))$ and let $\{q, p\}(t, s; y, \eta)$ and $Y(t, s; x, \xi)$ be the symbols constructed in Lemma 3.2 and Lemma 3.3. We put*

$$(3.12)\qquad u(t, s; y, \eta)=y\cdot\eta+\int_s^t \{\lambda-\xi\cdot\nabla_\xi\lambda\}\,(\sigma, q(\sigma, s; y, \eta), p(\sigma, s; y, \eta))d\sigma$$

and define

$$(3.13)\qquad \phi(t, s; x, \xi)=u(t, s;\, Y(t, s; x, \xi), \xi)\,.$$

Then, $\phi(t, s; x, \xi)$ is a solution of (3.1) and satisfies

$$(3.14)\qquad \nabla_\xi\phi(t, s; x, \xi)= Y(t, s; x, \xi)\,,$$

$$(3.15)\qquad \nabla_x\phi(t, s; x, \xi)=p(t, s;\, Y(t, s; x, \xi), \xi)\,,$$

$$(3.16)\qquad \partial_s\phi(t, s; x, \xi)= -\lambda(s, \nabla_\xi\phi(t, s; x, \xi), \xi)\,.$$

For any $l(\geqq 0)$ there exists a constant $\tilde{c}_{o,l}$ such that, if $\tilde{c}_{o,l}T_2<1$, $\phi(t, s; x, \xi)$ belongs to $\mathcal{P}_\rho(\tilde{c}_{o,l}|t-s|, l)$ and $\{J(t, s)/|t-s|\}$ is bounded in $S^1_\rho((k+2))$ for $0\leqq t$, $s\leqq T_2$, $t\neq s$, where $J(t, s; x, \xi)=\phi(t, s; x, \xi)-x\cdot\xi$. We assume, furthermore, that $\lambda(t, x, \xi)$ belongs to $M([0, T]; S^1_\rho((k+2)))$. Then, $J(t, s; x, \xi)$ belongs to $\bar{M}(Z(T_2); S^1_\rho; k+2)$.

If we follow the proofs of Theorem 3.1 in [7] and Proposition 2.2 in [11], we obtain the above proposition.

Take $\lambda_j(t, x, \xi)$, $j=1, 2, \cdots, \nu+1, \cdots$, as $\lambda(t, x, \xi)$ of (3.5) and let $\phi_j(t, s)\equiv\phi_j(t, s; x, \xi)$ be the solution of (3.1) corresponding to λ_j. Assume that $\{\lambda_j(t, x, \xi)\}_{j=1}^\infty$ is bounded in $M^0([0, T]; S^1_\rho((2)))$. Then, by Proposition 3.4

there exists a constant $\tilde{c}$ independent of j such that

$$\phi_j(t, s; x, \xi) \in \mathscr{P}_\rho(\tilde{c}|t-s|)\,. \tag{3.17}$$

Take a constant T_0 satisfying $T_0 \leqq \tau_0/\tilde{c}$ for the constant τ_0 in Definition 2.7. Then, the multi-product

$$\Phi_{\nu+1}(t_0, \tilde{t}^{\nu+1}; x, \xi) = (\phi_1(t_0, t_1) \# \phi_2(t_1, t_2) \# \cdots \# \phi_{\nu+1}(t_\nu, t_{\nu+1}))(x, \xi) \qquad (\tilde{t}^{\nu+1} = (t_1, t_2, \cdots, t_{\nu+1})) \tag{3.18}$$

is well-defined for $(t_0, \tilde{t}^{\nu+1}) \in \bar{\Delta}_{\nu+1}(T_0) \equiv \{(t_0, \tilde{t}^{\nu+1});\, 0 \leqq t_{\nu+1} \leqq t_\nu \leqq \cdots \leqq t_1 \leqq t_0 \leqq T_0\}$. In the following, we denote (3.18) simply by $\Phi_{\nu+1}(t_0, \tilde{t}^{\nu+1})$ or $\Phi_{\nu+1}$ unless otherwise specified. Corresponding to (3.18) we denote by $\{X_\nu^j, \Xi_\nu^j\}_{j=1}^{\nu}(t_0, \tilde{t}^{\nu+1}; x, \xi)$ for $(t_0, \tilde{t}^{\nu+1}) \in \bar{\Delta}_{\nu+1}(T_0)$ the solution of

$$\begin{cases} x^j = \nabla_\xi \phi_j(t_{j-1}, t_j; x^{j-1}, \xi^j)\,, \\ \xi^j = \nabla_x \phi_{j+1}(t_j, t_{j+1}, x^j, \xi^{j+1})\,, \qquad j = 1, \cdots, \nu \qquad (x^0 = x,\ \xi^{\nu+1} = \xi) \end{cases} \tag{3.19}$$

and we also write $X_\nu^j(t_0, \tilde{t}^{\nu+1}; x, \xi)$ and $\Xi_\nu^j(t_0, \tilde{t}^{\nu+1}; x, \xi)$ simply by X_ν^j and Ξ_ν^j.

Concerning the multi-products (3.18) the following is obtained by Kumano-go–Taniguchi–Tozaki [11].

Proposition 3.5. i) *$\Phi_{\nu+1} = \Phi_{\nu+1}(t_0, \tilde{t}^{\nu+1}; x, \xi)$ satisfies*

$$\begin{cases} \partial_{t_0} \Phi_{\nu+1} = \lambda_1(t_0, x, \nabla_x \Phi_{\nu+1})\,, \\ \partial_{t_j} \Phi_{\nu+1} = \lambda_{j+1}(t_j, X_\nu^j, \Xi_\nu^j) - \lambda_j(t_j, X_\nu^j, \Xi_\nu^j)\,, \qquad j = 1, \cdots, \nu\,, \\ \partial_{t_{\nu+1}} \Phi_{\nu+1} = -\lambda_{\nu+1}(t_{\nu+1}, \nabla_\xi \Phi_{\nu+1}, \xi)\,. \end{cases} \tag{3.20}$$

ii) *The following holds.*

$$\begin{cases} \phi_1(t, s) \# \phi_2(s, s) = \phi_1(t, s)\,, \\ \phi_1(t, t) \# \phi_2(t, s) = \phi_2(t, s)\,. \end{cases} \tag{3.21}$$

Proposition 3.6. *Assume that the set $\{\lambda_j\}_{j=1}^\infty$ is bounded in $M([0, T]; S_\rho^1((k+2)))$. Then, we have the following:*

i) *For the solution $\{X_\nu^j, \Xi_\nu^j\}(t_0, \tilde{t}^{\nu+1}; x, \xi)$ of (3.19) we have*

$$\begin{cases} \{\partial_{(t_0, \tilde{t}^{\nu+1})}^{\tilde{\gamma}^{\nu+1}} X_\nu^j\}_{j,\nu} \text{ is bounded in } S_\rho^0((k+1-l)) & \text{for } |\tilde{\gamma}^{\nu+1}| = l \leqq k+1\,, \\ \{\partial_{(t_0, \tilde{t}^{\nu+1})}^{\tilde{\gamma}^{\nu+1}} X_\nu^j\}_{j,\nu} \text{ is bounded in } S_\rho^{(1-\rho)(l-k-1)} & \text{for } |\tilde{\gamma}^{\nu+1}| = l \geqq k+2\,, \\ \{\partial_{(t_0, \tilde{t}^{\nu+1})}^{\tilde{\gamma}^{\nu+1}} \Xi_\nu^j\}_{j,\nu} \text{ is bounded in } S_\rho^1((k+1-l)) & \text{for } |\tilde{\gamma}^{\nu+1}| = l \leqq k+1\,, \\ \{\partial_{(t_0, \tilde{t}^{\nu+1})}^{\tilde{\gamma}^{\nu+1}} \Xi_\nu^j\}_{j,\nu} \text{ is bounded in } S_\rho^{1+(1-\rho)(l-k-1)} & \text{for } |\tilde{\gamma}^{\nu+1}| = l \geqq k+2\,, \end{cases} \tag{3.22}$$

where $\partial_{(t_0, \tilde{t}^{\nu+1})}^{\tilde{\gamma}^{\nu+1}} = \partial_{t_0}^{\gamma_0} \partial_{t_1}^{\gamma_1} \cdots \partial_{t_\nu}^{\gamma_\nu} \partial_{t_{\nu+1}}^{\gamma_{\nu+1}}$ and $|\tilde{\gamma}^{\nu+1}| = \gamma_0 + \cdots + \gamma_{\nu+1}$ for $(\nu+2)$-tuple

$\tilde{\gamma}^{\nu+1}=(\gamma_0, \gamma_1, \cdots, \gamma_{\nu+1})$.

ii) *Set*

$$\tilde{J}_{\nu+1}(t_0, \tilde{t}^{\nu+1}; x, \xi) = \Phi_{\nu+1}(t_0, \tilde{t}^{\nu+1}; x, \xi)-x\cdot\xi .$$

Then, we have

$$(3.23)\quad \begin{cases} \{\partial_{(t_0,\tilde{t}^{\nu+1})}^{\tilde{\gamma}^{\nu+1}}\tilde{J}_{\nu+1}\} \text{ is bounded in } S_\rho^1((k+2-l)) & \text{for } |\tilde{\gamma}^{\nu+1}|=l\leqq k+2 ,\\ \{\partial_{(t_0,\tilde{t}^{\nu+1})}^{\tilde{\gamma}^{\nu+1}}\tilde{J}_{\nu+1}\} \text{ is bounded in } S_\rho^{1+(1-\rho)(l-k-2)} & \text{for } |\tilde{\gamma}^{\nu+1}|=l\geqq k+3 .\end{cases}$$

Proof. Since $\{\lambda_j\}$ is bounded in $M([0, T];S_\rho^1((k+2)))$, the following holds by virtue of Proposition 3.4:

$$\begin{cases} \{J_j(t, s; x, \xi)/|t-s|\}_{0\leqq s<t\leqq T_2} \text{ is bounded in } S_\rho^1(k+2)) ,\\ \{\partial_t^l\partial_s^{l'}J_j(t, s; x, \xi)\}_{0\leqq s\leqq t\leqq T_2} \text{ is bounded in} \\ \qquad\qquad S_\rho^{1+(1-\rho)(l+l'-k-2)_+}((k+2-l-l')_+)) ,\end{cases}$$

where $J_j(t, s; x, \xi)=\phi_j(t, s; x, \xi)-x\cdot\xi$. Hence, we obtain (3.22) and (3.23) with $\tilde{\gamma}^{\nu+1}=0$ by Proposition 2.6. Concerning the derivatives of X_ν^j and Ξ_ν^j with respect to $(t_0, \tilde{t}^{\nu+1})$ we follow the proof of Theorem 1.7′ of [11]. Then, we get (3.22) for any $\tilde{\gamma}^{\nu+1}$. Using this and (3.20) we get (3.23) from the boundedness of $\{\lambda_j\}$. Q.E.D.

For the above $\lambda_1, \lambda_2, \cdots$, we consider the solution $\{q^j, p^j\}(t, s; y, \eta)$ of the Hamilton equation (3.6) corresponding to λ_j, and define for the point $(y, \eta)\in R_y^n\times R_\eta^n$ and $(t_0, \tilde{t}^{\nu+1})\in\tilde{\Delta}_{\nu+1}(T_2)$ the trajectory $\{\tilde{q}_{1,\cdots,j}, \tilde{p}_{1,\cdots,j}\}(t_0, t_1, \cdots, t_{j-1}, \sigma; y, \eta)$ $(t_j\leqq\sigma\leqq t_{j-1})$ by

$$(3.24)\quad \begin{cases} \{\tilde{q}_1, \tilde{p}_1\}(t_0, \sigma; y, \eta) = \{q^1, p^1\}(\sigma, t_0; y, \eta) \qquad (t_1\leqq\sigma\leqq t_0)\\ \{\tilde{q}_{1,\cdots,j}, \tilde{p}_{1,\cdots,j}\}(t_0, t_1, \cdots, t_{j-1}, \sigma; y, \eta) \\ \qquad = \{q^j, p^j\}(\sigma, t_{j-1}; \{\tilde{q}_{1,\cdots,j-1}, \tilde{p}_{1,\cdots,j-1}\}(t_0, t_1, \cdots, t_{j-1}; y, \eta)) \\ \qquad\qquad (t_j\leqq\sigma\leqq t_{j-1}),\ j\geqq 2 .\end{cases}$$

Proposition 3.7. *Let* $\{X_\nu^j, \Xi_\nu^j\}_{j=1}^\nu(t_0, \tilde{t}^{\nu+1}; x, \xi)$ *be the solution of* (3.19). *Then, we have*

$$(3.25)\quad \begin{cases} q^1(t_1, t_0; x, \nabla_x\Phi_{\nu+1}(t_0, \tilde{t}^{\nu+1}; x, \xi)) = X_\nu^1(t_0, \tilde{t}^{\nu+1}; x, \xi) ,\\ p^1(t_1, t_0; x, \nabla_x\Phi_{\nu+1}(t_0, \tilde{t}^{\nu+1}; x, \xi)) = \Xi_\nu^1(t_0, \tilde{t}^{\nu+1}; x, \xi) .\end{cases}$$

$$(3.26)\quad \begin{cases} q^j(t_j, t_{j-1}; X_\nu^{j-1}, \Xi_\nu^{j-1}) = X_\nu^j ,\\ p^j(t_j, t_{j-1}; X_\nu^{j-1}, \Xi_\nu^{j-1}) = \Xi_\nu^j \qquad (2\leqq j\leqq\nu)\end{cases}$$

and for any $j\leqq\nu$

$$(3.27)_j \quad \{\tilde{q}_{1,\cdots,j}, \tilde{p}_{1,\cdots,j}\}(t_0, t_1, \cdots, t_j; x, \nabla_x \Phi_{\nu+1}(t_0, \tilde{t}^{\nu+1}; x, \xi)) = \{X_\nu^j, \Xi_\nu^j\}(t_0, \tilde{t}^{\nu+1}; x, \xi).$$

Proof. From Lemma 3.3, (3.14) and (3.15) we get for any j

$$\begin{cases} q^j(t, s; \nabla_\xi \phi_j(t, s; x, \xi), \xi) = x, \\ p^j(t, s; \nabla_\xi \phi_j(t, s; x, \xi), \xi) = \nabla_x \phi_j(t, s; x, \xi). \end{cases}$$

Hence, by the uniqueness of the initial value problem (3.6) for $\lambda = \lambda_j$ we get

$$(3.28)_j \quad \begin{cases} q^j(s, t; x, \nabla_x \phi_j(t, s; x, \xi)) = \nabla_\xi \phi_j(t, s; x, \xi), \\ p^j(s, t; x, \nabla_x \phi_j(t, s; x, \xi)) = \xi. \end{cases}$$

From (1.25) of [11] we have $\nabla_x \Phi_{\nu+1} = \nabla_x \phi_1(t_0, t_1; x, \Xi_\nu^1)$. Using this with $(3.28)_1$ and (3.19) we obtain

$$\begin{cases} q^1(t_1, t_0; x, \nabla_x \Phi_{\nu+1}) = q^1(t_1, t_0; x, \nabla_x \phi_1(t_0, t_1; x, \Xi_\nu^1)) \\ \qquad = \nabla_\xi \phi_1(t_0, t_1, x, \Xi_\nu^1) = X_\nu^1, \\ p^1(t_1, t_0; x, \nabla_x \Phi_{\nu+1}) = p^1(t_1, t_0; x, \nabla_x \phi_1(t_0, t_1; x, \Xi_\nu^1)) = \Xi_\nu^1. \end{cases}$$

Hence, we get (3.25). Next, we use

$$\nabla_x \phi_j(t_{j-1}, t_j; X_\nu^{j-1}, \Xi_\nu^j) = \Xi_\nu^{j-1}$$

in (3.19). Then, we get from $(3.28)_j$ and (3.19)

$$\begin{cases} q^j(t_j, t_{j-1}; X_\nu^{j-1}, \Xi_\nu^{j-1}) = q^j(t_j, t_{j-1}; X_\nu^{j-1}, \nabla_x \phi_j(t_{j-1}, t_j; X_\nu^{j-1}, \Xi_\nu^j)) \\ \qquad = \nabla_\xi \phi_j(t_{j-1}, t_j; X_\nu^{j-1}, \Xi_\nu^j) = X_\nu^j, \\ p^j(t_j, t_{j-1}; X_\nu^{j-1}, \Xi_\nu^{j-1}) = p^j(t_j, t_{j-1}; X_\nu^{j-1}, \nabla_x \phi_j(t_{j-1}, t_j; X_\nu^{j-1}, \Xi_\nu^j)) \\ \qquad = \Xi_\nu^j. \end{cases}$$

Hence, we get (3.26).

We prove $(3.27)_j$ by the induction. Since $(3.27)_1$ is (3.25), we suppose $(3.27)_j$ and prove $(3.27)_{j+1}$. From $(3.27)_j$ and (3.26) we have

$$\begin{aligned} &\{\tilde{q}_{1,\cdots,j+1}, \tilde{p}_{1,\cdots,j+1}\}(t_0, t_1, \cdots, t_j, t_{j+1}; x, \nabla_x \Phi_{\nu+1}) \\ &= \{q^{j+1}, p^{j+1}\}(t_{j+1}, t_j; \{\tilde{q}_{1,\cdots,j}, \tilde{p}_{1,\cdots,j}\}(t_0, t_1, \cdots, t_j; x, \nabla_x \Phi_{\nu+1})) \\ &= \{q^{j+1}, p^{j+1}\}(t_{j+1}, t_j; X_\nu^j, \Xi_\nu^j) \\ &= \{X_\nu^{j+1}, \Xi_\nu^{j+1}\}. \end{aligned}$$

Hence, we obtain $(3.27)_{j+1}$. Q.E.D.

From Proposition 3.6 and Proposition 3.7 we get the following proposition.

Proposition 3.8. *Assume that the set* $\{\lambda_j\}$ *is bounded in* $M([0, T]$;

$S^1_\rho((k+2)))$. *Then, we have for the trajectory* $\{\tilde{q}_{1,\cdots,j}, \tilde{p}_{1,\cdots,j}\}(t_0, \tilde{t}^{j-1}, \sigma; y, \eta)$ *defined by* (3.24)

$$(3.29)\quad \begin{cases} \{\partial^{\tilde{\gamma}^j}_{(t_0,\tilde{t}^{j-1},\sigma)}\tilde{q}_{1,\cdots,j}\}_{j,(t_0,\tilde{t}^{j-1},\sigma)} \\ \qquad\qquad \text{is bounded in } S^0_\rho((k+1-l)) \text{ for } |\tilde{\gamma}^j| = l \leqq k+1\,, \\ \{\partial^{\tilde{\gamma}^j}_{(t_0,\tilde{t}^{j-1},\sigma)}\tilde{q}_{1,\cdots,j}\}_{j,(t_0,\tilde{t}^{j-1},\sigma)} \\ \qquad\qquad \text{is bounded in } S^{(1-\rho)(l-k-1)}_\rho \text{ for } |\tilde{\gamma}^j| = l \geqq k+2\,, \\ \{\partial^{\tilde{\gamma}^j}_{(t_0,\tilde{t}^{j-1},\sigma)}\tilde{p}_{1,\cdots,j}\}_{j,(t_0,\tilde{t}^{j-1},\sigma)} \\ \qquad\qquad \text{is bounded in } S^1_\rho((k+1-l)) \text{ for } |\tilde{\gamma}^j| = l \leqq k+1\,, \\ \{\partial^{\tilde{\gamma}^j}_{(t_0,\tilde{t}^{j-1},\sigma)}\tilde{p}_{1,\cdots,j}\}_{j,(t_0,\tilde{t}^{j-1},\sigma)} \\ \qquad\qquad \text{is bounded in } S^{1+(1-\rho)(l-k-1)}_\rho \text{ for } |\tilde{\gamma}^j| = l \geqq k+2\,. \end{cases}$$

Now, we turn to study the commutative law for $\#$-products of phase functions. Let $\{\lambda_j\}_{j=1}^\infty$ be a bounded set of real symbols $\lambda_j(t, x, \xi)$ in $M([0, T]; S^1_\rho((3)))$ and let $\phi_j(t, s; x, \xi) \in \mathcal{P}_\rho(\tilde{c}|t-s|)$ be the phase function corresponding to $\lambda_j(t, x, \xi)$. For the multi-product (3.18) we commute ϕ_j and ϕ_{j+1} and denote

$$(3.30)\quad \begin{aligned} \Phi_{\nu+1;j}(t_0, \tilde{t}^{\nu+1}; x, \xi) &= (\phi_1(t_0, t_1)\#\cdots\#\phi_{j-1}(t_{j-2}, t_{j-1}) \\ &\#\phi_{j+1}(t_{j-1}, t_j)\#\phi_j(t_j, t_{j+1})\#\phi_{j+2}(t_{j+1}, t_{j+2})\#\cdots\#\phi_{\nu+1}(t_\nu, t_{\nu+1}))(x, \xi) \\ &\text{for}\quad (t_0, \tilde{t}^{\nu+1}) \in \bar{\Delta}_{\nu+1}(T_O)\,. \end{aligned}$$

We put an assumption: There exist real symbols $a_j(t, x, \xi)$ in $M([0, T]; S^0_\rho((1)))$ and $a'_j(t, x, \xi)$ in $M([0, T]; S^0_\rho)$ such that

$$(3.31)\quad \{\tau-\lambda_j, \tau-\lambda_{j+1}\} = a_j(t, x, \xi)(\lambda_j-\lambda_{j+1})+a'_j(t, x, \xi)\,.$$

Then, we have

Theorem 3.9[2]**.** *Let* $\{\lambda_j(t, x, \xi)\}_{j=1}^\infty$ *be a bounded set in* $M([0, T]; S^1_\rho((3)))$ *and let* $\phi_j(t, s; x, \xi) \in \mathcal{P}_\rho(\tilde{c}|t-s|)$ *(with some* $\tilde{c}$*) be the phase function corresponding to* λ_j. *We assume that* (3.31) *holds and that the sets* $\{a_j\}$ *and* $\{a'_j\}$ *are bounded in* $M([0, T]; S^0_\rho((1)))$ *and* $M([0, T]; S^0_\rho)$, *respectively. For any* ν, j $(\leqq \nu)$ *and* $(t_0, t^{\nu+1}) \in \bar{\Delta}_{\nu+1}(T_O)$ *(for some* T_O*) we consider the multi-products* $\Phi_{\nu+1}(t_0, \tilde{t}^{\nu+1})$ *and* $\Phi_{\nu+1;j}(t_0, \tilde{t}^{\nu+1})$ *of* (3.18) *and* (3.30). *Then, there exists a constant* T'_O *independent of* ν *such that the following hold:*

I) *We can find for any* ν *and* j $(\leqq \nu)$ *a symbol* $\Omega_{\nu,j}(t_0, \tilde{t}^{\nu+1}; x, \xi)$ *in* $\bar{M}(\bar{\Delta}_{\nu+1}(T'_O); S^0_\rho; 1)$ *such that it satisfies*

---

2) The idea of the proof is found in Section 1 of [10], where the theorem is proved for the case of $a_{m,k} \equiv a'_{m,k} \equiv 0$. In [13] Morimoto proved this theorem in the case of $a_{m,k}(t, x, \xi) \equiv a_{m,k}(t)$ and $a'_{m,k}(t, x, \xi) \equiv 0$.

(3.32) $t_{j+1} \leqq \Omega_{\nu,j}(t_0, \tilde{t}^{\nu+1}; x, \xi) \leqq t_{j-1}$,

(3.33) $\Omega_{\nu,j|t_j=t_{j-1}} = t_{j+1}$, $\Omega_{\nu,j|t_j=t_{j+1}} = t_{j-1}$,

and

(3.34)
$$\begin{aligned}\Phi_{\nu+1;j}(t_0, t_1, \cdots, t_{j-1}, t_j, t_{j+1}, \cdots, t_{\nu+1}; x, \xi) \\ = \Phi_{\nu+1}(t_0, t_1, \cdots, t_{j-1}, \Omega_{\nu,j}(t_0, \tilde{t}^{\nu+1}; x, \xi), t_{j+1}, \cdots, t_{\nu+1}; x, \xi) \\ +\psi_{\nu,j}(t_0, \tilde{t}^{\nu+1}; x, \xi)\end{aligned}$$

with some $\psi_{\nu,j}(t_0, \tilde{t}^{\nu+1}, x, \xi)$ *satisfying*

(3.35) $\psi_{\nu,j}(t_0, \tilde{t}^{\nu+1}; x, \xi) \in \bar{M}(\bar{\Delta}_{\nu+1}(T'_0); S^0_\rho; 0)$

and

(3.36) $\psi_{\nu,j} \equiv 0$ if $a'_j = 0$.

II) *It holds that*

(3.37)
$$\begin{cases} \{\Omega_{\nu,j}\}_{\nu,j,(t_0,\tilde{t}^{\nu+1})} & \text{is bounded in } S^0_\rho((1)), \\ \{\partial^{\tilde{\gamma}^{\nu+1}}_{(t_0,\tilde{t}^{\nu+1})}\Omega_{\nu,j}\}_{\nu,j,(t_0,\tilde{t}^{\nu+1})} & \text{is bounded in } S^{(1-\rho)(l-1)}_\rho \\ & \quad \text{for} \quad |\tilde{\gamma}^{\nu+1}| = l \geqq 1, \\ \{\partial^{\tilde{\gamma}^{\nu+1}}_{(t_0,\tilde{t}^{\nu+1})}\psi_{\nu,j}\}_{\nu,j,(t_0,\tilde{t}^{\nu+1})} & \text{is bounded in } S^{(1-\rho)l}_\rho \\ & \quad \text{for} \quad |\tilde{\gamma}^{\nu+1}| = l. \end{cases}$$

We can find a constant A_6 *independent of* ν *such that we have*

(3.38) $|\partial_{t_j}\Omega_{\nu,j}+1| \leqq A_6(t_0-t_{\nu+1})$.

REMARK. In [10] and [13] the commutative law for multi-#-products follows from the commutative law for #-products between two phase functions, since $\{\Omega_{\nu,j}\}$ are determined only by $(t, \tilde{t}^{\nu+1})$. In our case we emphasize that we cannot apply the above method because $\{\Omega_{\nu,j}\}$ depend also on x and ξ.

We begin the proof with finding $\Omega_{\nu,j}(t_0, \tilde{t}^{\nu+1}; x, \xi)$ satisfying (3.32)–(3.34). To simplify the notation below, we use (t, θ, s) or (t, ω, s) instead of (t_{j-1}, t_j, t_{j+1}) and write

$$\begin{cases} \Phi_{\nu+1}(t, \omega, s) \equiv \Phi_{\nu+1}(t, \omega, s; \tilde{t}^{0,\nu+1}_j, x, \xi) \\ \qquad = \Phi_{\nu+1}(t_0, \cdots, t_{j-2}, t, \omega, s, t_{j+2}, \cdots, t_{\nu+1}; x, \xi), \\ \Phi_{\nu+1;j}(t, \theta, s) \equiv \Phi_{\nu+1;j}(t, \theta, s; \tilde{t}^{0,\nu+1}_j, x, \xi) \\ \qquad = \Phi_{\nu+1;j}(t_0, \cdots, t_{j-2}, t, \theta, s, t_{j+2}, \cdots, t_{\nu+1}; x, \xi), \end{cases}$$

where $\tilde{t}^{0,\nu+1}_j = (t_0, \cdots, t_{j-2}, t_{j+2}, \cdots, t_{\nu+1})$ when $\nu \geqq 2$. Now, we set

$$(3.39)\qquad \psi \equiv \psi_{\nu,j}(t, \theta, s) = \Phi_{\nu+1\,;\,j}(t, \theta, s) - \Phi_{\nu+1}(t, \Omega, s)$$

and seek the symbol $\Omega = \Omega(t, \theta, s; \tilde{t}_j^{0,\nu+1}, x, \xi)$ $(=\Omega_{\nu,j}(t_0, \cdots, t_{j-2}, t, \theta, s, t_{j+2}, \cdots, t_{\nu+1}; x, \xi))$ such that ψ belongs to $\bar{M}(S^0_\rho; 0)$ and Ω satisfies

$$(3.32)'\qquad s \leqq \Omega(t, \theta, s) \leqq t\,,$$

$$(3.33)'\qquad \Omega(t, t, s) = s\,, \quad \Omega(t, s, s) = t\,.$$

Here, we suppress the domain of $(t_0, \cdots, t_{j-2}, t, \theta, s, t_{j+2}, \cdots, t_{\nu+1})$ and write $\bar{M}(S^0_\rho; 0)$ instead of writing $\bar{M}(\{0 \leqq t_{\nu+1} \leqq \cdots \leqq t_{j+2} \leqq s \leqq \theta \leqq t \leqq t_{j-2} \leqq \cdots \leqq t_0 \leqq T_O\}; S^0_\rho; 0)$. In the following we also suppress domains of $t_0, \cdots, t_{j-2}, t, \theta, \omega, s, t_{j+2}, \cdots, t_{\nu+1}$ and use the notation $\bar{M}(S^m_\rho; k)$ if no confusion occurs.

Let $\{X^k_\nu, \Xi^k_\nu\}^\nu_{k=1} \equiv \{X^k_\nu, \Xi^k_\nu\}^\nu_{k=1}(t, \omega, s; \tilde{t}_j^{0,\nu+1}, x, \xi)$ be the solution of

$$(3.40)\qquad \begin{cases} x^k = \nabla_\xi \phi_k(t_{k-1}, t_k; x^{k-1}, \xi^k)\,, \\ \xi^k = \nabla_x \phi_{k+1}(t_k, t_{k+1}; x^k, \xi^{k+1})\,, \qquad k = 1, \cdots, \nu \\ \qquad\qquad (x^0 = x,\ \xi^{\nu+1} = \xi;\ t_{j-1} = t,\ t_j = \omega\,,\ t_{j+1} = s) \end{cases}$$

and let $\{\tilde{X}^k_\nu, \tilde{\Xi}^k_\nu\}^\nu_{k=1} \equiv \{\tilde{X}^k_\nu, \tilde{\Xi}^k_\nu\}^\nu_{k=1}(t, \theta, s; \tilde{t}_j^{0,\nu+1}, x, \xi)$ be the solution of

$$(3.41)\qquad \begin{cases} \begin{cases} x^k = \nabla_\xi \phi_k(t_{k-1}, t_k; x^{k-1}, \xi^k) & (1 \leqq k \leqq \nu,\ k \neq j,\ j+1)\,, \\ x^j = \nabla_\xi \phi_{j+1}(t_{j-1}, t_j; x^{j-1}, \xi^j)\,, \\ x^{j+1} = \nabla_\xi \phi_j(t_j, t_{j+1}; x^j, \xi^{j+1})\,, \end{cases} \\ \begin{cases} \xi^k = \nabla_x \phi_{k+1}(t_k, t_{k+1}; x^k, \xi^{k+1}) & (1 \leqq k \leqq \nu,\ k \neq j-1, j)\,, \\ \xi^{j-1} = \nabla_x \phi_{j+1}(t_{j-1}, t_j; x^{j-1}, \xi^j)\,, \\ \xi^j = \nabla_x \phi_j(t_j, t_{j+1}, x^j, \xi^{j+1}) \end{cases} \\ \qquad\qquad (x^0 = x,\ \xi^{\nu+1} = \xi;\ t_{j-1} = t,\ t_j = \theta,\ t_{j+1} = s)\,. \end{cases}$$

For convenience, we set

$$\begin{cases} \lambda_0(t, x, \xi) = 0\,, \\ X^0_\nu = \tilde{X}^0_\nu = x\,, \quad \Xi^0_\nu = \nabla_x \Phi_{\nu+1}(t, \omega, s)\,, \quad \tilde{\Xi}^0_\nu = \nabla_x \Phi_{\nu+1\,;\,j}(t, \theta, s)\,. \end{cases}$$

Then, we have from (3.20)

$$(3.42)\qquad \begin{aligned} \partial_t \psi &= (\partial_{t_{j-1}} \Phi_{\nu+1\,;\,j})(t, \theta, s) - (\partial_{t_{j-1}} \Phi_{\nu+1})(t, \Omega, s) - \partial_\omega \Phi_{\nu+1}(t, \Omega, s) \partial_t \Omega \\ &= \lambda_{j+1}(t, \tilde{X}^{j-1}_\nu, \tilde{\Xi}^{j-1}_\nu) - \lambda_{j-1}(t, \tilde{X}^{j-1}_\nu, \tilde{\Xi}^{j-1}_\nu) \\ &\qquad - \{\lambda_j(t, X^{j-1}_\nu, \Xi^{j-1}_\nu) - \lambda_{j-1}(t, X^{j-1}_\nu, \Xi^{j-1}_\nu)\}_{|\omega=\Omega} \\ &\qquad - \partial_\omega \Phi_{\nu+1}(t, \Omega, s) \partial_t \Omega\,. \end{aligned}$$

When $j \geqq 2$, we use the trajectory $\{\tilde{q}_{1,\cdots,j-1}, \tilde{p}_{1,\cdots,j-1}\}(t_0, t_1, \cdots, t_{j-2}, t; y, \eta)$ defined by (3.24). Then, we have from Proposition 3.7

$$(3.43)\qquad \begin{cases} \{\tilde{q}_{1,\cdots,j-1}, \tilde{p}_{1,\cdots,j-1}\}(t_0, \tilde{t}^{j-2}, t; x, \nabla_x \Phi_{\nu+1}) = \{X^{j-1}_\nu, \Xi^{j-1}_\nu\}\,, \\ \{\tilde{q}_{1,\cdots,j-1}, \tilde{p}_{1,\cdots,j-1}\}(t_0, \tilde{t}^{j-2}, t; x, \nabla_x \Phi_{\nu+1\,;\,j}) = \{\tilde{X}^{j-1}_\nu, \tilde{\Xi}^{j-1}_\nu\}\,. \end{cases}$$

Hence, if we set

$$\begin{cases} \tilde{\lambda}_1(t; z, \zeta) = \lambda_2(t, z, \zeta), \\ \tilde{\lambda}_j(t; t_0, \tilde{t}^{j-2}, z, \zeta) = \{\lambda_{j+1}-\lambda_{j-1}\}\ (t, \{\tilde{q}_{1,\cdots,j-1}, \tilde{p}_{1,\cdots,j-1}\}(t_0, \\ \qquad \tilde{t}^{j-2}, t; z, \zeta)) \qquad (j \geqq 2) \end{cases}$$

$\tilde{\lambda}_j$ belongs to $\bar{M}(S^1_\rho; 2)$ from Proposition 3.8 and satisfies

$$(3.44)\quad \begin{cases} \tilde{\lambda}_j(t; t_0, \tilde{t}^{j-2}, x, \nabla_x \Phi_{\nu+1}) = \{\lambda_{j+1}-\lambda_{j-1}\}\ (t, X^{j-1}_\nu, \Xi^{j-1}_\nu), \\ \tilde{\lambda}_j(t; t_0, \tilde{t}^{j-2}, x, \nabla_x \Phi_{\nu+1\,;\,j}) = \{\lambda_{j+1}-\lambda_{j-1}\}\ (t, \tilde{X}^{j-1}_\nu, \tilde{\Xi}^{j-1}_\nu) \\ \qquad ((t; t_0, \tilde{t}^{j-2}) = t \qquad \text{for } j=1). \end{cases}$$

Define the symbol $\Lambda_j(\omega) \equiv \Lambda_j(\omega; t, \theta, s, \tilde{t}^{0,\nu+1}_j, x, \xi)$ in $\bar{M}(S^0_\rho; 1)$ by

$$(3.45)\quad \Lambda_j(\omega) = \int_0^1 \nabla_\xi \tilde{\lambda}_j(t; t_0, \tilde{t}^{j-2}, x, \sigma \nabla_x \Phi_{\nu+1\,;\,j}(t, \theta, s; \tilde{t}^{0,\nu+1}_j, x, \xi) \\ +(1-\sigma)\nabla_x \Phi_{\nu+1}(t, \omega, s; \tilde{t}^{0,\nu+1}_j, x, \xi))d\sigma .$$

Then, we can write

$$(3.46)\quad \tilde{\lambda}_j(t; t_0, \tilde{t}^{j-2}, x, \nabla_x \Phi_{\nu+1\,;\,j}(t, \theta, s)) = \tilde{\lambda}_j(t; t_0, \tilde{t}^{j-2}, x, \nabla_x \Phi_{\nu+1}(t, \omega, s)) \\ +\Lambda_j(\omega)\cdot(\nabla_x \Phi_{\nu+1\,;\,j}(t, \theta, s) - \nabla_x \Phi_{\nu+1}(t, \omega, s)) .$$

From (3.39) we have

$$(3.47)\quad \nabla_x \psi = \nabla_x \Phi_{\nu+1\,;\,j}(t, \theta, s) - \nabla_x \Phi_{\nu+1}(t, \Omega, s) - \partial_\omega \Phi_{\nu+1}(t, \Omega, s)\nabla_x \Omega .$$

Hence, from (3.42), (3.44), (3.46) and (3.47) we have

$$(3.48)\quad \partial_t \psi = \tilde{\lambda}_j(t; t_0, \tilde{t}^{j-2}, x, \nabla_x \Phi_{\nu+1\,;\,j}(t, \theta, s)) - \tilde{\lambda}_j(t; t_0, \tilde{t}^{j-2}, x, \nabla_x \Phi_{\nu+1}(t, \Omega, s)) \\ -\partial_\omega \Phi_{\nu+1}(t, \Omega, s)\partial_t \Omega - \{\lambda_j - \lambda_{j+1}\}\ (t, X^{j-1}_\nu, \Xi^{j-1}_\nu)_{|\omega=\Omega} \\ = \Lambda_j(\Omega)\cdot\nabla_x \psi - \partial_\omega \Phi_{\nu+1}(t, \Omega, s)\,[\partial_t \Omega - \Lambda_j(\Omega)\cdot\nabla_x \Omega] \\ -\{\lambda_j - \lambda_{j+1}\}\ (t, X^{j-1}_\nu, \Xi^{j-1}_\nu)_{|\omega=\Omega} .$$

Let $\{q^j, p^j\}\ (t, s; y, \eta)$ be the solution of (3.6) with λ replaced by λ_j and set $\lambda^\circ(\sigma, t; y, \eta)\ (=\lambda^\circ_j(\sigma, t; y, \eta)) = \{\lambda_j - \lambda_{j+1}\}\ (\sigma, \{q^j, p^j\}\ (\sigma, t; y, \eta))$. Then, as the proof of Corollary of Theorem 2.3 in [11] we get from (3.31)

$$(3.49)\quad \frac{d\lambda^\circ}{d\sigma} = \{\tau-\lambda_j, \tau-\lambda_{j+1}\}\ (\sigma, \{q^j, p^j\}\ (\sigma, t; y, \eta)) \\ = a_j(\sigma, \{q^j, p^j\}\ (\sigma, t; y, \eta))\lambda^\circ(\sigma, t; y, \eta) + a'_j(\sigma, (\{q^j, p^j\}\ (\sigma, t; y, \eta))$$

and the solution $\lambda^\circ(\sigma, t; y, \eta)$ of (3.49) has the form

$$(3.50)\quad \lambda^\circ(\sigma, t; y, \eta) \\ = \lambda^\circ(\omega, t; y, \eta)\exp\int_\omega^\sigma a_j(\sigma', \{q^j, p^j\}\ (\sigma', t; y, \eta))d\sigma'$$

$$+\int_{\omega}^{\sigma}(\exp\int_{\sigma'}^{\sigma}a_j(\sigma'', \{q^j, p^j\}(\sigma'', t; y, \eta))d\sigma'')$$
$$\times a_j'(\sigma', \{q^j, p^j\}(\sigma', t; y, \eta))d\sigma' .$$

From (3.25)–(3.26) and (3.20) we get

$$\begin{cases} q^j(t_j, t_{j-1}; X_\nu^{j-1}, \Xi_\nu^{j-1}) = X_\nu^j , \\ p^j(t_j, t_{j-1}; X_\nu^{j-1}, \Xi_\nu^{j-1}) = \Xi_\nu^j \end{cases}$$

and

$$\lambda^\circ(t_j, t_{j-1}; X_\nu^{j-1}, \Xi_\nu^{j-1}) = \{\lambda_j - \lambda_{j+1}\}(t_j, X_\nu^j, \Xi_\nu^j) = -\partial_\omega\Phi_{\nu+1} \tag{3.51}$$

with $t_j=\omega$ and $t_{j-1}=t$. Set

$$\begin{aligned} \alpha_j(\omega) &\equiv \alpha_j(\omega; t, s, \tilde{t}_j^{0,\nu+1}, x, \xi) \\ &= \exp\int_{\omega}^{t}a_j(\sigma', \{q^j, p^j\}(\sigma', t; X_\nu^{j-1}, \Xi_\nu^{j-1}))d\sigma' \qquad (\in \bar{M}(S_\rho^0; 1)) \end{aligned} \tag{3.52}$$

and

$$\begin{aligned} \alpha_j'(\omega) &\equiv \alpha_j'(\omega; t, s, \tilde{t}_j^{0,\nu+1}, x, \xi) \\ &= \int_{\omega}^{t}(\exp\int_{\sigma'}^{t}a_j(\sigma'', \{q^j, p^j\}(\sigma'', t; X_\nu^{j-1}, \Xi_\nu^{j-1}))d\sigma'') \\ &\quad \times a_j'(\sigma', \{q^j, p^j\}(\sigma', t; X_\nu^{j-1}, \Xi_\nu^{j-1}))d\sigma' \qquad (\in \bar{M}(S_\rho^0; 0)) . \end{aligned} \tag{3.53}$$

Then, we obtain from (3.48), (3.51) and (3.50) with $\sigma=t$, $y=X_\nu^{j-1}$ and $\eta=\Xi_\nu^{j-1}$

$$\begin{aligned} \partial_t\psi &= \Lambda_j(\Omega)\cdot\nabla_x\psi - \alpha_j'(\Omega) \\ &\quad -\partial_\omega\Phi_{\nu+1}(t, \Omega, s)[\partial_t\Omega - \Lambda_j(\Omega)\cdot\nabla_x\Omega - \alpha_j(\Omega)] . \end{aligned} \tag{3.54}$$

Consider the equation with respect to Ω:

$$\partial_t\Omega - \Lambda_j(\Omega)\cdot\nabla_x\Omega - \alpha_j(\Omega) = 0 \tag{3.55}$$

with the initial condition

$$\Omega_{|t=\theta} = s . \tag{3.56}$$

Since (3.55) is a quasi-linear equation, we may solve the ordinary differential equation

$$\begin{cases} \dfrac{d\tilde{r}}{dt} = -\Lambda_j(\tilde{z}; t, \theta, s, \tilde{t}_j^{0,\nu+1}, \tilde{r}, \xi) , \\ \dfrac{d\tilde{z}}{dt} = \alpha_j(\tilde{z}; t, s, \tilde{t}_j^{0,\nu+1}, \tilde{r}, \xi) , \\ \tilde{r}_{|t=\theta} = y, \ \tilde{z}_{|t=\theta} = s . \end{cases} \tag{3.57}$$

Lemma 3.10. *There exists a constant* T_0'' *such that the equation* (3.57) *has a solution* $\{\tilde{r}, \tilde{z}\}$ $(t, \theta, s; \tilde{t}_j^{0,\nu+1}, x, \xi)$ *in* $\bar{M}(S_\rho^0; 1)$ *for* $(t, \theta, s; \tilde{t}_j^{0,\nu+1})$ *with* $0\leqq s\leqq\theta\leqq t\leqq T_0''$ *(and* $t_0\leqq T_0''$*) and* $\{\tilde{r}, \tilde{z}\}$ *satisfy*

$$(3.58)\qquad \begin{cases} s\leqq\tilde{z}(t, \theta, s; \tilde{t}_j^{0,\nu+1}, y, \xi)\leqq t\,, \\ \tilde{z}(t, s, s) = t \end{cases}$$

and

$$(3.59)\qquad \left\|\frac{\partial}{\partial y}\tilde{r}-E\right\|\leqq A_7(t-s)$$

with a constant A_7 *independent of* ν.

Admitting this lemma for a moment, we continue the proof of the theorem. Take $T_0'(\leqq T_0'')$ such that $T_0'A_7<1$. Then, from (3.59) the equation

$$(3.60)\qquad \tilde{r}(t, \theta, s; \tilde{t}_j^{0,\nu+1}, \tilde{Y}, \xi) = x$$

has a solution $\tilde{Y}(t, \theta, s; t_j^{0,\nu+1}, x, \xi)$ satisfying

$$(3.61)\qquad \tilde{Y}(t, \theta, s; \tilde{t}_j^{0,\nu+1}, x, \xi)-x\in\bar{M}(S_\rho^0; 1)$$

when $0\leqq s\leqq\theta\leqq t\leqq T_0'$. In the following the inequality $0\leqq s\leqq\theta\leqq t\leqq T_0'$ always holds. Set

$$(3.62)\qquad \Omega(t, \theta, s; \tilde{t}_j^{0,\nu+1}, x, \xi) = \tilde{z}(t, \theta, s; \tilde{t}_j^{0,\nu+1}, \tilde{Y}(t, \theta, s; \tilde{t}_j^{0,\nu+1}, x, \xi), \xi)\,.$$

Then, $\Omega(t, \theta, s)$ is a solution of (3.55)–(3.56). From (3.56) and (3.58) $\Omega(t, \theta, s)$ satisfies (3.32)′–(3.33)′.

For the solution $\Omega(t, \theta, s)$ of (3.55)–(3.56) the equation (3.54) is reduced to the equation

$$(3.63)\qquad \partial_t\psi = \Lambda_j(\Omega)\cdot\nabla_x\psi-\alpha_j'(\Omega)\,.$$

On the other hand, the equation

$$(3.64)\qquad \psi_{|t=\theta} = 0$$

holds, since we have from (3.33)′ and (3.21)

$$\begin{aligned} \psi(\theta, \theta, s) &= \Phi_{\nu+1;j}(\theta, \theta, s)-\Phi_{\nu+1}(\theta, s, s) \\ &= \Phi_{\nu+1;j}(t_0, \cdots, t_{j-2}, \theta, \theta, s, t_{j+2}, \cdots, t_{\nu+1}) \\ &\quad -\Phi_{\nu+1}(t_0, \cdots, t_{j-2}, \theta, s, s, t_{j+2}, \cdots, t_{\nu+1}) \\ &= \phi_1(t_0, t_1)\#\cdots\#\phi_{j-1}(t_{j-2}, \theta)\#\{\phi_{j+1}(\theta, \theta)\#\phi_j(\theta, s)\} \\ &\qquad \#\phi_{j+2}(s, t_{j+2})\#\cdots\#\phi_{\nu+1}(t_\nu, t_{\nu+1}) \\ &\quad -\phi_1(t_0, t_1)\#\cdots\#\phi_{j-1}(t_{j-2}, \theta)\#\{\phi_j(\theta, s)\#\phi_{j+1}(s, s)\} \\ &\qquad \#\phi_{j+2}(s, t_{j+2})\#\cdots\#\phi_{\nu+1}(t_\nu, t_{\nu+1}) \\ &= 0\,. \end{aligned}$$

Hence, if we set $\beta_j(t, \theta, s; \tilde{t}_j^{0,\nu+1}, x, \xi)=\alpha_j'(\Omega(t, \theta, s; \tilde{t}_j^{0,\nu+1}, x, \xi); t, s, \tilde{t}_j^{0,\nu+1}, x, \xi)$, the symbol ψ can be written in the form

$$(3.65) \qquad \psi = -\int_\theta^t \beta_j(\sigma, \theta, s; \tilde{t}_j^{0,\nu+1}, \tilde{r}(\sigma, \theta, s; \tilde{t}_j^{0,\nu+1}, \tilde{Y}(t, \theta, s; \tilde{t}_j^{0,\nu+1}, x, \xi), \xi), \xi) d\sigma ,$$

where $\tilde{r}$ and $\tilde{Y}$ are defined by Lemma 3.10 and (3.60). Hence, ψ belongs to $\bar{M}(S_\rho^0, 0)$ and is identically zero when $a_j' \equiv 0$. Consequently, we have proved I) in the theorem. From the above discussions we also get (3.37).

For the proof of (3.38) we set $\Omega_\theta(t, \theta, s)=\partial_\theta \Omega(t, \theta, s; \tilde{t}_j^{0,\nu+1}, x, \xi)$. Then, from (3.55)

$$(3.66) \qquad \partial_t \Omega_\theta(t, \theta, s) = \Lambda_j(\Omega)\cdot \nabla_x \Omega_\theta + \beta_j'(t, \theta, s)$$

holds with

$$\beta_j'(t, \theta, s) \equiv \beta_j'(t, \theta, s; \tilde{t}_j^{0,\nu+1}, x, \xi) = \partial_\theta(\Lambda_j(\Omega))\cdot \nabla_x \Omega + \partial_\theta(\alpha_j(\Omega)) .$$

On the other hand, writing

$$(3.67) \qquad \Omega(t, \theta, s) = s + \int_\theta^t \{\Lambda_j(\Omega(\sigma, \theta, s); \sigma, \theta, s, \tilde{t}_j^{0,\nu+1}, x, \xi)\cdot \nabla_x \Omega(\sigma, \theta, s) + \alpha_j(\Omega(\sigma, \theta, s); \sigma, s, \tilde{t}_j^{0,\nu+1}, x, \xi)\} d\sigma ,$$

we have

$$(3.68) \qquad \Omega_\theta(\theta, \theta, s) = -\alpha_j(s; \theta, s, \tilde{t}_j^{0,\nu+1}, x, \xi) = -\exp \int_s^\theta a_j d\sigma ,$$

since $\nabla_x \Omega(\theta, \theta, s)=0$ from (3.67). Hence, as in (3.65) we can write

$$(3.69) \qquad \Omega_\theta(t, \theta, s) = \Omega_\theta(\theta, \theta, s) + \int_\theta^t \beta_j'(\sigma, \theta, s; \tilde{t}_j^{0,\nu+1}, \tilde{r}(\sigma, \theta, s; \tilde{t}_j^{0,\nu+1}, \tilde{Y}(t, \theta, s; \tilde{t}_j^{0,\nu+1}, x, \xi), \xi), \xi) d\sigma$$

and get (3.38) from (3.68)–(3.69). This completes the proof of Theorem 3.9.

REMARK. If $a_j(t, x, \xi)$ is identically zero, the solution Ω of (3.55)–(3.56) is

$$\Omega = t - \theta + s .$$

This corresponds to the result in Theorem 1.10 of [10].

Proof of Lemma 3.10. We solve (3.57) by the Picard's method of successive approximation. For simplicity we suppress the dependence of $\tilde{t}_j^{0,\nu+1}$ and j and write $\Lambda(\omega; t, \theta, s, x, \xi)$ and $\alpha(\omega; t, s, x, \xi)$ instead of writing $\Lambda_j(\omega; t, \theta, s, \tilde{t}_j^{0,\nu+1}, x, \xi)$ and $\alpha_j(\omega; t, s, \tilde{t}_j^{0,\nu+1}, x, \xi)$. Define $\{\tilde{r}^{(N)}, \tilde{z}^{(N)}\}(t) \equiv \{\tilde{r}^{(N)}, \tilde{z}^{(N)}\}(t, \theta, s; x, \xi)$, $N=0, 1, 2, \cdots$, by

$$(3.70) \qquad \tilde{r}^{(0)}(t) = y, \quad \tilde{z}^{(0)}(t) = t - \theta + s ,$$

$$(3.71)\qquad \begin{cases} \tilde{r}^{(N+1)}(t) = y - \int_\theta^t \Lambda(\tilde{z}^{(N)}(\sigma);\, \sigma,\, \theta,\, s,\, \tilde{r}^{(N)}(\sigma),\, \xi)\, d\sigma\,, \\ \tilde{z}^{(N+1)}(t) = s + \int_\theta^t \alpha(\tilde{z}^{(N)}(\sigma);\, \sigma,\, s,\, \tilde{r}^{(N)}(\sigma),\, \xi)\, d\sigma\,. \end{cases}$$

In order for $\{\tilde{r}^{(N+1)}, \tilde{z}^{(N+1)}\}(t)$ to be well-defined, we must prove

$$(3.72)\qquad s \leqq \tilde{z}^{(N)}(t,\, \theta,\, s;\, x,\, \xi) \leqq t\,.$$

But, (3.72) is derived from $\tilde{z}^{(N)}(t, t, s) = s$,

$$(3.73)_N\qquad -2 \leqq \partial_\theta \tilde{z}^{(N)} \leqq 0\,,$$

and

$$(3.74)_N\qquad \tilde{z}^{(N)}(t,\, s,\, s) = t\,.$$

Hence, we shall prove $(3.73)_N$, $(3.74)_N$ and

$(3.75)_N$ $\quad |\partial_\theta \tilde{r}^{(N)}| \leqq A_8$ *with some* $A_8 > 0$ (*independent of* ν)

by the induction. From (3.52) $\alpha(t;\, t,\, s,\, x,\, \xi) = 1$ holds. Hence, using $(3.74)_{N-1}$ we get $(3.74)_N$. In fact, we have

$$\begin{aligned} \tilde{z}^{(N)}(t,\, s,\, s) &= s + \int_s^t \alpha(\tilde{z}^{(N-1)}(\sigma,\, s,\, s);\, \sigma,\, s,\, \tilde{r}^{(N-1)}(\sigma),\, \xi)\, d\sigma \\ &= s + \int_s^t d\sigma = t\,. \end{aligned}$$

Now, we prove $(3.73)_N$. Since Λ, $\alpha \in \bar{M}(S_\rho^0;\, 1)$ and $|\alpha(\omega;\, t,\, s,\, x,\, \xi) - 1| \leqq C_1 \times (t-\omega)$, we have

$$\begin{aligned} &|\partial_\theta \tilde{z}^{(N)} + 1| \\ &= |-\{\alpha(s;\theta,s,y,\xi)-1\} + \int_\theta^t \partial_\theta \{\alpha(\tilde{z}^{(N-1)}(\sigma,\theta,s);\sigma,s,\tilde{r}^{(N-1)}(\sigma,\theta,s),\xi)\}\, d\sigma| \\ &\leqq C_1(\theta-s) + C_2(t-\theta) \end{aligned}$$

by using $(3.73)_{N-1}$ and $(3.75)_{N-1}$. Hence, if T_0'' is small enough, we obtain

$$|\partial_\theta \tilde{z}^{(N)} + 1| \leqq 1$$

and $(3.73)_N$ when $0 \leqq s \leqq \theta \leqq t \leqq T_0''$. Similarly, we can prove $(3.75)_N$ by using $(3.73)_{N-1}$ and $(3.75)_{N-1}$. Consequently, by the induction the functions $\{\tilde{r}^{(N)}, \tilde{z}^{(N)}\}(t,\, \theta,\, s;\, y,\, \xi)$ are well-defined and satisfy (3.72) and $(3.73)_N$–$(3.75)_N$ for $0 \leqq s \leqq \theta \leqq t \leqq T_0''$ if T_0'' is small enough.

As usual we can prove

$$(3.76)\qquad \begin{cases} |\tilde{z}^{(N+1)} - \tilde{z}^{(N)}| \leqq C^N (t-\theta)^N / N!\,, \\ |\tilde{r}^{(N+1)} - \tilde{z}^{(N)}| \leqq C^N (t-\theta)^N / N! \end{cases}$$

with a constant C independent of N. Hence, we obtain the desired symbols $\tilde{z}(t, \theta, s; y, \xi)$ and $\tilde{r}(t, \theta, s; y, \xi)$ as limits of $\{\tilde{z}^{(N)}\}$ and $\{\tilde{r}^{(N)}\}$. From (3.72) and $(3.74)_N$ we get (3.58). Moreover, we can easily prove $\tilde{z}\in\bar{M}(S^0_\rho; 1)$, $\tilde{r}\in\bar{M}(S^0_\rho; 1)$ and (3.59). Q.E.D.

REMARK. If $a_j(t, x, \xi)$ in (3.31) are functions of only t, we can relax the conditions in Theorem 3.9 as the following: Assume $\lambda_j(t, x, \xi)\in M^0([0, T]; S^1_\rho((2)))\cap C^1([0, T]\times R^{2n})$ and

$$\{\tau-\lambda_j, \tau-\lambda_{j+1}\} = a_j(t)(\lambda_j-\lambda_{j+1})+a'_j(t, x, \xi)$$

with C^0-functions $a_j(t)$ and symbols $a'_j(t, x, \xi)$ in $M^0([0, T]; S^0_\rho)$. Then, for the function $\Omega_{\nu,j}$ determined by

$$\Omega_{\nu,j} = a_j^{-1}(a_j(t_{j-1})-a_j(t_j)+a_j(t_{j+1})) \quad \text{(c.f. (2.20) of [13])}$$

with $a_j(t)=\int(\exp\int a_j(t)dt)dt$ the results (3.34)–(3.35) holds. In fact, we first prove (3.34)–(3.35) for $\nu=1$ by the method of proving Theorem 3.9. Then, the result (3.34)–(3.35) for any ν will be derived by the method of proving Theorem 1.7′ and Theorem 1.8′ in [11]. It seems to us that we cannot prove (3.35) directly from (3.63)–(3.64) when $\nu\geqq 2$ and $\lambda_j\in M^0([0, T]; S^1_\rho((2)))$, since $\Lambda_j(\omega)$ of (3.45) may not belong to $\bar{M}(S^0_\rho; 1)$ when $\nu\geqq 2$.

4. Fundamental solutions for hyperbolic systems. In this section we prove Theorem 4 by using Theorem 1 and Theorem 3.9. First, we construct the fundamental solution $\boldsymbol{E}(t, s)$ of the Cauchy problem

$$(4.1) \qquad \begin{cases} \mathcal{L}U(t)=0 & \text{on } [0, T], \\ U(0)=U_o \end{cases}$$

for the hyperbolic operator $\mathcal{L}$ of (1). Let $\phi_m(t, s)\equiv\phi_m(t, s; x, \xi)$ be the phase function corresponding to $\lambda_m(t, x, \xi)$. Set $M_\nu=\{\mu=(m_1, \cdots, m_\nu); m_j=1, \cdots, l\}$ $(\nu=1, 2, \cdots)$ and denote

$$(4.2) \qquad \begin{aligned} &\Phi_{\nu,(\mu)}(t, t_1, \cdots, t_{\nu-1}, s; x, \xi) \\ &\quad = (\phi_{m_1}(t, t_1)\#\phi_{m_2}(t_1, t_2)\#\cdots\#\phi_{m_{\nu-1}}(t_{\nu-2}, t_{\nu-1})\#\phi_{m_\nu}(t_{\nu-1}, s))(x, \xi) \end{aligned}$$

for $\mu=(m_1, \cdots, m_\nu)\in M_\nu$ when $\nu\geqq 2$. Set

$$(4.3) \qquad \boldsymbol{I}_\phi(t, s) = \begin{bmatrix} I_{\phi_1}(t, s) & & 0 \\ & \ddots & \\ 0 & & I_{\phi_l}(t, s) \end{bmatrix} \qquad (\sigma(I_{\phi_m}(t, s))=1).$$

Then, we have

Proposition 4.1. *Let $1/2 \leqq \rho \leqq 1$. Assume that $\lambda_m(t, x, \xi)$ in* (1) *belong to $M^0([0, T]; S^1_\rho((2)))$ and $b_{mk}(t, x, \xi)$ in* (1) *belong to $M^0([0, T]; S^0_\rho)$. Then, the fundamental solution $\boldsymbol{E}(t, s)$ of the Cauchy problem* (4.1) *for the hyperbolic system* (1) *can be represented in the form*

$$\boldsymbol{E}(t, s) = \boldsymbol{I}_\phi(t, s) + \int_s^t \boldsymbol{I}_\phi(t, \theta)\{\sum_{m=1}^l W_{m,\phi_m}(\theta, s) + \sum_{\nu=2}^\infty \sum_{\mu\in M_\nu} \int_s^\theta \int_s^{t_1} \cdots \int_s^{t_{\nu-2}} W_{\nu,(\mu),\Phi_{\nu,(\mu)}}(\theta, t_1, \cdots, t_{\nu-1}, s)\, dt_{\nu-1}, \cdots, dt_1\}\, d\theta \tag{4.4}$$

$$(t_0 = \theta;\ 0 \leqq s \leqq t \leqq T_o)$$

for some T_o, and $w_m(t, s; x, \xi) = \sigma(W_{m,\phi_m}(t, s))$ and $w_{\nu,(\mu)}(t, \tilde{t}^{\nu-1}, s; x, \xi) = \sigma(W_{\nu,(\mu),\Phi_{\nu,(\mu)}}(t, \tilde{t}^{\nu-1}, s))$ $(\tilde{t}^{\nu-1} = (t_1, \cdots, t_{\nu-1}))$ satisfy the following: There exists a constant C_O independent of ν such that the set $\{w_m\} \cup \{C_O^{-\nu} w_{\nu,(\mu)}\}$ is bounded in S^0_ρ. Moreover, if $\lambda_m(t, x, \xi)$ belong to $M([0, T]; S^1_\rho((2)))$ and $b_{mk}(t, x, \xi)$ belong to $M([0, T]; S^0_\rho)$, then, setting $\bar{M}(Z; S^0_\rho) = \bigcap_{k=0}^\infty M^k(Z; S^{(1-\rho)k}_\rho)$ $(= \bar{M}(Z; S^0_\rho; 0))$, the symbols $w_m(t, s; x, \xi)$ and $w_{\nu,(\mu)}(t, \tilde{t}^{\nu-1}, s; x, \xi)$ $(\nu \geqq 2, \mu \in M_\nu)$ in (4.4) *belong to $\bar{M}(\Delta_0(T_o); S^0_\rho)$ and $\bar{M}(\Delta_{\nu-1}(T_o); S^0_\rho)$, respectively, and there exists a constant C_O independent of ν such that for any $\tilde{\gamma}^\nu = (\gamma_0, \gamma_1, \cdots, \gamma_\nu)$ with $\gamma_0 + \cdots + \gamma_\nu = k$ the set $\{C_O^{-\nu} \partial^{\tilde{\gamma}^\nu}_{(t, \tilde{t}^{\nu-1}, s)} w_{\nu,(\mu)}\}$ is bounded in $S^{(1-\rho)k}_\rho$. Here, $\Delta_0(T_o) = \{(t, s); 0 \leqq s \leqq t \leqq T_o\}$ and $\Delta_\nu(T_o) = \{(t, \tilde{t}^\nu, s);\ 0 \leqq s \leqq t_\nu \leqq \cdots \leqq t_1 \leqq t \leqq T_o\}$ $(\nu \geqq 1)$.*

REMARK. By virtue of Theorem 1 we can take smoothing operators in Sobolev spaces away from the expression (3.17) in [10].

Proof. We fix a constant T_o such that $T_o \leqq T_2$ and $T_o \leqq \tau^0/\tilde{c}_{o,\tilde{l}_o}$. Here, T_2 is the constant in Section 3, τ^0 is the constant defined by (2.58) and $\tilde{c}_{o,\tilde{l}_o}$ is the constant in Proposition 3.4 for $l = \tilde{l}_o$ (the integer defined in Proposition 2.2). Operate $\mathcal{L}$ to (4.3). Then, we have

$$\mathcal{L}\boldsymbol{I}_\phi(t, s) = \boldsymbol{R}_\phi(t, s) \tag{4.5}$$

for

$$\boldsymbol{R}_\phi(t, s) = \sum_{m=1}^l R_{m,\phi_m}(t, s)\,, \tag{4.6}$$

where $R_{m,\phi_m}(t, s)$ is a matrix of Fourier integral operators with phase function $\phi_m(t, s; x, \xi)$ and its symbol $r_m(t, s; x, \xi)$ belongs to $M^0(\Delta_0(T_o); S^0_\rho)$ (c.f. Theorem 2.2 of Chap. 10 in [8]). From (4.5) we see that the fundamental solution $\boldsymbol{E}(t, s)$ for $\mathcal{L}$, as the continuous operator from the Sobolev space H_σ into itself for any fixed real σ, is constructed in the form

$$\boldsymbol{E}(t, s) = \boldsymbol{I}_\phi(t, s) + \int_s^t \boldsymbol{I}_\phi(t, \theta) \sum_{\nu=1}^\infty \boldsymbol{W}_\nu(\theta, s)\, d\theta\,. \tag{4.7}$$

Here, $\{\boldsymbol{W}_\nu(t, s)\}_{\nu=1}^\infty$ are defined by

$$(4.8) \qquad \begin{cases} \boldsymbol{W}_1(t, s) = -i\boldsymbol{R}_\phi(t, s), \\ \boldsymbol{W}_{\nu+1}(t, s) = \int_s^t \boldsymbol{W}_1(t, \theta)\boldsymbol{W}_\nu(\theta, s)d\theta, \quad \nu = 1, 2, \cdots. \end{cases}$$

Set

$$w_m(t, s; x, \xi) = -ir_m(t, s; x, \xi) \quad (m = 1, \cdots, l).$$

Then, $\boldsymbol{W}_\nu(t, s)$ for $\nu \geqq 2$ can be written in the form

$$(4.9) \qquad \begin{cases} \boldsymbol{W}_\nu(t, s) = \int_s^t\int_s^{t_1}\cdots\int_s^{t_{\nu-2}} \boldsymbol{W}^{(\nu)}(t, t_1, \cdots, t_{\nu-1}, s)dt_{\nu-1}\cdots dt_1 \qquad (t_0 = t), \\ \boldsymbol{W}^{(\nu)}(t, t_1, \cdots, t_{\nu-1}, s) = \sum_{\mu_\nu \in M_\nu} W_{m_1,\phi_{m_1}}(t, t_1)W_{m_2,\phi_{m_2}}(t_1, t_2)\cdots W_{m_\nu,\phi_{m_\nu}}(t_{\nu-1}, s) \end{cases}$$

with $W_{m,\phi_m}(t, s) = w_{m,\phi_m}(t, s; X, D_x)$. Consequently, we get the first assertion of the theorem by applying Theorem 1.

Next, we assume $\lambda_m \in M([0, T]; S^1_\rho((2)))$ and $b_{mk} \in M([0, T]; S^0_\rho)$. Then, the symbols $r_m(t, s; x, \xi)$ in (4.6) belong to $\bar{M}(\Delta_0(T_o); S^0_\rho)$. Consequently, we get the second assertion of the theorem, when we use the expression (4.9), Proposition 3.6 and the discussions in Section 2. Q.E.D.

Now, we *prove Theorem* 4. First, we assume the condition (I). Since the expression (4.4) holds including the case $\rho = 1/2$, by the method of proving Theorem 3 in [13] we can prove Theorem 4 under the condition (I) not only in the case of $1/2 < \rho \leqq 1$ but also in the case of $\rho = 1/2$. We note that in Theorem 3 of [13] only the case $\rho = 1$ was treated. Next, we consider Theorem 4 under the condition (II). For the proof we prepare the following three propositions.

First, we shall restate Theorem 3.9 in our use. For $\mu = (m_1, \cdots, m_j, m_{j+1}, \cdots, m_\nu) \in M_\nu$ we change the order of m_j and m_{j+1} and set $\mu(j) = (m_1, \cdots, m_{j-1}, m_{j+1}, m_j, m_{j+2}, \cdots, m_\nu)$. We note that $\Phi_{\nu,(\mu)}(t, \tilde{t}^{\nu-1}, s) \equiv \Phi_{\nu,(\mu)}(t, \tilde{t}^{\nu-1}, s; x, \xi)$ and $\Phi_{\nu,(\mu(j))}(t, \tilde{t}^{\nu-1}, s) \equiv \Phi_{\nu,\mu(j))}(t, \tilde{t}^{\nu-1}, s; x, \xi)$ have the forms $\Phi_{\nu,(\mu)}(t, \tilde{t}^{\nu-1}, s) = \phi_{m_1}(t, t_1) \# \cdots \# \phi_{m_j}(t_{j-1}, t_j) \# \phi_{m_{j+1}}(t_j, t_{j+1}) \# \cdots \# \phi_{m_\nu}(t_{\nu-1}, s)$ and $\Phi_{\nu,(\mu(j))}(t, \tilde{t}^{\nu-1}, s) = \phi_{m_1}(t, t_1) \# \cdots \# \phi_{m_{j+1}}(t_{j-1}, t_j) \# \phi_{m_j}(t_j, t_{j+1}) \# \cdots \# \phi_{m_\nu}(t_{\nu-1}, s)$. Then, from Theorem 3.9 we obtain

Proposition 4.2. *Let $\lambda_m(t, x, \xi)$, $m = 1, \cdots, l$, belong to $M([0, T]; S^1_\rho((3)))$. Assume for any m and k the equation* (18) *holds with $a_{m,k}(t, x, \xi)$ and $a'_{m,k}(t, x, \xi)$ in* (17). *Then, if we take a sufficiently small constant T'_0 (independent of ν), there exist symbols $\Omega_{\nu,(\mu),j}(t, \tilde{t}^{\nu-1}, s; x, \xi)$ in $\bar{M}(\Delta_{\nu-1}(T'_0); S^0_\rho; 1)$ and $\psi_{\nu,(\mu),j}(t, \tilde{t}^{\nu-1}, s; x, \xi)$ in $\bar{M}(\Delta_{\nu-1}(T'_0); S^0_\rho)$ for any ν $(\geqq 2)$, j $(\leqq \nu - 1)$ and $\mu \in M_\nu$ such that they satisfy*

$$(4.10) \qquad t_{j+1} \leqq \Omega_{\nu,(\mu),j}(t, \tilde{t}^{\nu-1}, s; x, \xi) \leqq t_{j-1} \quad (t_0 = t, t_\nu = s),$$

(4.11) $\Omega_{\nu,(\mu),j|t_j=t_{j-1}}=t_{j+1},\ \Omega_{\nu,(\mu),j|t_j=t_{j+1}}=t_{j-1}\,,$

(4.12)
$$\begin{aligned}&\Phi_{\nu,(\mu(j))}(t, t_1, \cdots, t_{j-1}, t_j, t_{j+1}, \cdots, t_{\nu-1}, s; x, \xi)\\&=\Phi_{\nu,(\mu)}(t, t_1, \cdots, t_{j-1}, \Omega_{\nu,(\mu),j}(t, \tilde{t}^{\nu-1}, s; x, \xi), t_{j+1}, \cdots, t_{\nu-1}, s; x, \xi)\\&\quad+\psi_{\nu,(\mu),j}(t, \tilde{t}^{\nu-1}, s; x, \xi)\,,\end{aligned}$$

(4.13)
$$\begin{cases}\{\Omega_{\nu,(\mu),j}\} \text{ is bounded in } S^0_\rho\,,\\ \{\partial^{\tilde{\gamma}^\nu}_{(t,\tilde{t}^{\nu-1},s)}\Omega_{\nu,(\mu),j}\} \text{ is bounded in } S^{(1-\rho)(k-1)}_\rho \quad \text{for } |\tilde{\gamma}^\nu|=k\geqq 1\,,\\ \{\partial^{\tilde{\gamma}^\nu}_{(t,\tilde{t}^{\nu-1},s)}\psi_{\nu,(\mu),j}\} \text{ is bounded in } S^{(1-\rho)k}_\rho \quad \text{for } |\tilde{\gamma}^\nu|=k\end{cases}$$

and

(4.14) $|\partial_{t_j}\Omega_{\nu,(\mu),j}+1|\leqq A_6(t-s)$

for a constant A_6 independent of ν.

Proposition 4.3. *In Proposition 4.2 we assume, furthermore, that the constant T'_0 satisfies $A_6T'_0\leqq 1/2$. Then, the equation*

(4.15) $\omega=\Omega_{\nu,(\mu),j}(t, t_1, \cdots, t_{j-1}, \theta, t_{j+1}, \cdots, t_{\nu-1}, s; x, \xi)\quad (1\leqq j\leqq \nu-1)$

has the inverse $\theta=\Theta_{\nu,(\mu),j}(t, t_1, \cdots, t_{j-1}, \omega, t_{j+1}, \cdots, t_{\nu-1}, s; x, \xi)$ satisfying $t_{j+1}\leqq \Theta_{\nu,(\mu),j}\leqq t_{j-1}$ and

(4.16)
$$\begin{cases}\{\Theta_{\nu,(\mu),j}\} \text{ is bounded in } S^0_\rho((1))\,,\\ \{\partial^{\tilde{\gamma}^\nu}_{(t,\tilde{t}^{\nu-1},s)}\Theta_{\nu,(\mu),j}\} \text{ is bounded in } S^{(1-\rho)(k-1)}_\rho \quad \text{for } |\tilde{\gamma}^\nu|=k\geqq 1\,.\end{cases}$$

Proof. Set

$$\begin{aligned}\mathcal{A}=\{&\Theta(t, t_1, \cdots, t_{j-1}, \omega, t_{j+1}, \cdots, t_{\nu-1}, s; x, \xi)\in C^\infty;\\ &t_{j+1}\leqq\Theta\leqq t_{j-1},\ \Theta_{|\omega=t_{j-1}}=t_{j+1},\ \Theta_{|\omega=t_{j+1}}=t_{j-1},\ -2\leqq\partial_\omega\Theta\leqq 0\}\,,\end{aligned}$$

and consider a mapping $\Gamma\equiv\Gamma_{\nu,(\mu),j}\colon \mathcal{A}\ni\Theta\to G=\Gamma(\Theta)\in\mathcal{A}$ defined by

(4.17)
$$\begin{aligned}G&\equiv G(t, t_1, \cdots, t_{j-1}, \omega, t_{j+1}, \cdots, t_{\nu-1}, s; x, \xi)\\&=-\omega+\Omega_{\nu,(\mu),j}(t, t_1, \cdots, t_{j-1}, \Theta, t_{j+1}, \cdots, t_{\nu-1}, s; x, \xi)+\Theta\\&\qquad\qquad (t_0=t,\ \ t_\nu=s)\,.\end{aligned}$$

Since $t_{j+1}\leqq\Theta\leqq t_{j-1}$ for $\Theta\in\mathcal{A}$ the mapping Γ is well-defined. From (4.11), (4.14) and $A_6T'_0\leqq 1/2$ we get for $G=\Gamma(\Theta)$ with $\Theta\in\mathcal{A}$

$$\begin{aligned}G_{|\omega=t_{j-1}}&=-t_{j-1}+\Omega_{\nu,(\mu),j}(t, \cdots, t_{j-1}, \Theta_{|\omega=t_{j-1}}, t_{j+1}, \cdots, s)+\Theta_{|\omega=t_{j-1}}\\&=-t_{j-1}+\Omega_{\nu,(\mu),j}(t, \cdots, t_{j-1}, t_{j+1}, t_{j+1}, \cdots, s)+t_{j+1}\\&=t_{j+1}\,,\end{aligned}$$

$$
(4.18)\quad \begin{cases} G_{|\omega=t_{j+1}} = -t_{j+1}+\Omega_{\nu,(\mu),j}(t, \cdots, t_{j-1}, \Theta_{|\omega=t_{j+1}}, t_{j+1}, \cdots, s)+\Theta_{|\omega=t_{j+1}} \\ \qquad = -t_{j+1}+\Omega_{\nu,(\mu),j}(t, \cdots, t_{j-1}, t_{j-1}, t_{j+1}, \cdots, s)+t_{j-1} \\ \qquad = t_{j-1}, \\ |\partial_\omega G+1| = |\{\partial_{t_j}\Omega_{\nu,(\mu),j}+1\}\partial_\omega\Theta| \\ \qquad \leqq A_6 T'_0\cdot 2 \leqq \frac{1}{2}\cdot 2 = 1 \end{cases}
$$

and

$$
(4.19)\quad t_{j+1} = G_{|\omega=t_{j-1}} \leqq G \leqq G_{|\omega=t_{j+1}} = t_{j-1}
$$

by $\partial_\omega G \leqq 0$. This shows that the mapping $\Gamma: \mathcal{A}\to\mathcal{A}$ is into. We define a sequence $\{\Theta^{(N)}\}_{N=0}^\infty$ in $\mathcal{A}$ by

$$
\begin{cases} \Theta^{(0)} = t_{j-1}-\omega+t_{j+1}, \\ \Theta^{(N+1)} = \Gamma(\Theta^{(N)}). \end{cases}
$$

Then, from (4.14) and $A_6 T'_0 \leqq 1/2$ we get for some constant C independent of N

$$
|\Theta^{(N+1)}-\Theta^{(N)}| \leqq C 2^{-N}.
$$

Consequently, we can find the desired solution $\Theta=\Theta_{\nu,(\mu),j}$ of (4.15) as the limit of the sequence $\{\Theta^{(N)}\}_{N=0}^\infty$. Consider the equation (4.15). Then, we get (4.16) by the usual method. Q.E.D.

Proposition 4.4. *Let $p(t, \tilde{t}^{\nu-1}, s; x, \xi)$ belong to $\bar{M}(\Delta_{\nu-1}(T); S^0_\rho)$ and let $\{\Theta_N(t, \tilde{t}^{\nu-1}, s; x, \xi)\}_{N=1}^\infty$ and $\{g_N(t, \tilde{t}^{\nu-1}, s; x, \xi)\}_{N=1}^\infty$ be sequences in $\bar{M}(\Delta_{\nu-1}(T); S^0_\rho; 1)$ and $\bar{M}(\Delta_{\nu-1}(T); S^0_\rho)$, respectively. Assume*

$$
(4.20)\quad \begin{cases} \{\Theta_N\} \text{ is bounded in } S^0_\rho((1)), \\ \{\partial^{\tilde{\gamma}^\nu}_{(t,\tilde{t}^{\nu-1},s)}\Theta_N\} \text{ is bounded in } S^{(1-\rho)(k-1)}_\rho \quad \text{for } |\tilde{\gamma}^\nu| = k \geqq 1, \\ \{\partial^{\tilde{\gamma}^\nu}_{(t,\tilde{t}^{\nu-1},s)} g_N\} \text{ is bounded in } S^{(1-\rho)k}_\rho \quad \text{for } |\tilde{\gamma}^\nu| = k. \end{cases}
$$

For a fixed sequence $\{j_N\}_{N=1}^\infty$ $(1 \leqq j_N \leqq \nu-1)$, we set inductively

$$
(4.21)\quad \begin{aligned} & p_N(t, \tilde{t}^{\nu-1}, s; x, \xi) \\ & = p_{N-1}(t, t_1, \cdots, t_{j_N-1}, \Theta_N(t, \tilde{t}^{\nu-1}, s; x, \xi), t_{j_N+1}, \cdots, t_{\nu-1}, \\ & \qquad s; x, \xi) g_N(t, \tilde{t}^{\nu-1}, s; x, \xi) \quad (p_0 = p;\ t_0 = t,\ t_\nu = s). \end{aligned}
$$

Then, for any k there exists a constant C_k independent of N and ν such that

$$
(4.22)\quad ||p_N||^{(0)}_k \leqq C_k^N ||p||^{(0)}_k,
$$

where $||p||^{(0)}_k = \max\limits_{0\leqq k'\leqq k} \max\limits_{|\tilde{\gamma}^\nu|=k'} |\partial^{\tilde{\gamma}^\nu}_{(t,\tilde{t}^{\nu-1},s)} p|^{((1-\rho)k')}_{k-k'}$.

We can prove this proposition by the induction.

Using these propositions and the discussions for the proof of Theorem 3 in [13] we can reduce (4.4) to the finite sum expression

$$(4.23)\qquad E(t, s) = \sum_{m=1}^{l} W^0_{1,m,\Phi_m}(t, s) + \sum_{\nu=2}^{l} \sum_{\mu \in M^\circ_\nu} \int_s^t \int_s^{t_1} \cdots \int_s^{t_{\nu-2}} W^0_{\nu,(\mu),\Phi_{\nu,(\mu)}}(t, \tilde{t}^{\nu-1}, s)\, dt_{\nu-1} \cdots dt_1 \quad (t_0 = t,\ 0 \leqq s \leqq t \leqq T_o)$$

with some T_o ($\leqq T'_o$) and symbols $\sigma(W_{\nu,(\mu),\Phi_{\nu,(\mu)}})$ in $\bar{M}(\Delta_{\nu-1}(T_o); S^0_\rho)$ ($1 \leqq \nu \leqq l$), where $M^\circ_\nu = \{\mu = (m_1, \cdots, m_\nu) \in M_\nu;\ m_1 < m_2 < \cdots < m_\nu\}$ ($2 \leqq \nu \leqq l$). This proves Theorem 4.

REMARK. When we use the remark at the end of Section 3, we can prove Theorem 4 under the condition (I) with (16) replaced by

$$(16)'\qquad \{\tau - \lambda_m, \tau - \lambda_k\} = a_{k,j}(t)\,(\lambda_m - \lambda_k) + a'_{m,k}(t, x, \xi)\,.$$

Here, $a_{m,k}(t)$ are continuous functions of t and $a'_{m,k}(t, x, \xi)$ are symbols in $M^0([0, T]; S^0_\rho)$. This result contains the one studied in [5]. In [5] Ichinose proved this when $a_{m,k}(t)=0$ and $l=2$. But, he did not discuss the convergence of the symbols $\sigma(W_{\nu,(\mu),\Phi_{\nu,(\mu)}})$ derived from the successive approximation.

As a corollary of Theorem 4 we get immediately from the expression (4.23) and Theorem 3.14 of Chap. 10 of [8]

Corollary 4.5. *In Theorem* 4 *we assume, furthermore, that the symbols* $\lambda_m(t, x, \xi)$ *are homogeneous for large* $|\xi|$. *Then, for the solution* $U(t)$ *of the Cauchy problem* (4.1) *we have*

$$(4.24)\qquad WF(U(t)) \subset \textit{Conic hull of } \Gamma_t$$

for the wave front set $WF(U(t)) = \bigcup_{m=1}^{l} WF(u_m(t))$ *of* $U(t) = {}^t(u_1(t), \cdots, u_l(t))$, *which is defined in* [4].

In (4.24) Γ_t is defined by

$$(4.25)\qquad \Gamma_t = \{\{q_{m_1,\cdots,m_\nu}, p_{m_1,\cdots,m_\nu}\}(t, t_1, \cdots, t_{\nu-1}; y, \eta);\ (m_1, \cdots, m_\nu) \in M^\circ_\nu,\ \nu = 1, \cdots, l,\ \ 0 \leqq t_{\nu-1} \leqq \cdots \leqq t_1 \leqq t,\ (y, \eta) \in WF(U_o) \ \textit{for large } |\eta|\} \quad (t_0 = t)$$

for the trajectory $\{q_{m_1,\cdots,m_\nu}, p_{m_1,\cdots,m_\nu}\}(t, t_1, \cdots, t_{\nu-1}; y, \eta)$ ($(m_1, \cdots, m_\nu) \in M^0_\nu$) determined by the following: Let $\{q_{m_\nu}, p_{m_\nu}\}(t; y, \eta)$ be the solution of

$$(4.26)\qquad \frac{dq}{dt}=-\nabla_\xi\lambda_{m_\nu}(t, q, p),\ \frac{dp}{dt}=\nabla_x\lambda_{m_\nu}(t, q, p),\ \{q, p\}_{|t=0}=\{y, \eta\}\,.$$

Then, $\{q_{m_k,\cdots,m_\nu}, p_{m_k,\cdots,m_\nu}\}\ (t, t_k, \cdots, t_{\nu-1}; y, \eta)$ is defined as the solution of

$$(4.27)\qquad \begin{cases} \dfrac{dq}{dt}=-\nabla_\xi\lambda_{m_k}(t, q, p),\ \dfrac{dp}{dt}=\nabla_x\lambda_{m_k}(t, q, p), \\ \{q, p\}_{|t=t_k}=\{q_{m_{k+1},\cdots,m_\nu}, p_{m_{k+1},\cdots,m_\nu}\}\ (t_k, \cdots, t_{\nu-1}; y, \eta) \\ \qquad\qquad\qquad (1\leqq k\leqq \nu-1)\,. \end{cases}$$

In [13] Morimoto has obtained Corollary 4.5 by a different method. In the condition of Corollary 4.5 the symbols $\lambda_m(t, x, \xi)$ belong to $M([0, T]; S^1_{1,0})$. But using the discussions in Section 3 of [5] we can prove the property: Assume that there exist continuous functions $\lambda^\circ_m(t, x, \xi)$, $m=1, \cdots, l$, which have Lipschitz continuous derivatives with respect to x and ξ for $|\xi|\geqq 1$, are homogeneous of order 1 with respect to ξ and satisfy for some $\kappa<1$

$$(4.28)\qquad |\partial_\xi^\alpha D_x^\beta(\lambda^\circ_m-\lambda_m)|\leqq C\langle\xi\rangle^{\kappa-|\alpha|}\quad (|\alpha+\beta|\leqq 1,\ |\xi|\geqq 1)\,.$$

Then, the property (4.25) holds with $\lambda_m(t, x, \xi)$ replaced by $\lambda^\circ_m(t, x, \xi)$ in (4.26)–(4.27). Here, we need not assume the homogeneity of $\lambda_m(t, x, \xi)$.

Finally, we shall study examples in Introduction. First, we consider (19). The characteristic roots are $\lambda_\pm(x, \xi)=\pm\sqrt{a_k(x)}\,|\xi|$, which are C^{k-1}-class with Lipschitz continuous derivatives of $(k-1)$-st order for $|\xi|\geqq 1$. We approximate $\lambda_+(x, \xi)$ and $\lambda_-(x, \xi)$ by $\lambda_1(x, \xi)=(a_k(x)|\xi|^2+1)^{1/2}$ and $\lambda_2(x, \xi)=-(a_k(x)|\xi|^2+1)^{1/2}$, respectively. Then, setting $\rho=1-1/k$, $\lambda_1(x, \xi)$ and $\lambda_2(x, \xi)$ belong to $S^1_\rho((k))$ (c.f. §4 of [3]) and we can find pseudo-differential operators B and B' in S^0_ρ such that (19) has the form

$$(4.29)\qquad L_1=(D_t-\lambda_2(X, D_x)+B)\,(D_t-\lambda_1(X, D_x)-B)+B'\,.$$

Hence, the study for the operator (19) is reduced to the study for the system $\mathcal{L}_O$ of the form

$$(4.30)\qquad \mathcal{L}_O=D_t-\begin{bmatrix}\lambda_1(X, D_x) & 0\\ 0 & \lambda_2(X, D_x)\end{bmatrix}+\begin{bmatrix}-B & -1\\ B' & B\end{bmatrix}.$$

Since the system (4.30) is involutive, that is,

$$(4.31)\qquad \{\tau-\lambda_1, \tau-\lambda_2\}=0$$

holds, the fundamental solution of (19) is constructed in the form

$$(4.32)\qquad W^0_{1,\phi_1}(t, s)+W^0_{2,\phi_2}(t, s)+\int_s^t W^0_{\phi_1\#\phi_2}(t, \theta, s)d\theta\,.$$

We note that in the case of $k\geqq 3$ we can apply the approximation theory in

[9] in order to reduce (19) to (4.30) with $\rho=(k-1)/(k+1)$.

Next, we consider (20). The characteristic roots are $\lambda_\pm(x, \xi)=\pm a(x_1) \times(\xi_1^2+a(x_1)^2\xi_2^2)^{1/2}$, which are of C^1-class with Lipschitz continuous derivatives. Set

$$\mu(x, \xi) = \{\xi_1^2+a(x_1)^2\xi_2^2+\langle\xi\rangle\}^{1/2} \tag{4.33}$$

and approximate $\lambda_+(x, \xi)$ and $\lambda_-(x, \xi)$ by $\lambda_1(x, \xi)=a(x_1)\mu(x, \xi)$ and $\lambda_2(x, \xi)= -a(x_1)\mu(x, \xi)$. Then, we can prove by the method of [15] that $\lambda_m(x, \xi)$, $m=1, 2$, belong to $S^1_{1/2}((2))$ and that (20) can be reduced to the system (4.30) with appropriate pseudo-differential operators B and B' in $S^0_{1/2}$. In this case we also have (4.31) and the fundamental solution of (20) can be constructed in the form (4.32). For the operator (21) its characteristic roots are $\lambda_\pm(x, \xi)=a(x_1)^2\xi_1 \pm a(x_1)^2(\xi_1^2+a(x_1)^2\xi_2^2)^{1/2}$ of class C^2. Hence, if we approximate $\lambda_+(x, \xi)$ and $\lambda_-(x, \xi)$ by $\lambda_1(x, \xi)=a(x_1)^2\xi_1+a(x_1)^2\mu(x, \xi)$ and $\lambda_2(x, \xi)=a(x_1)^2\xi_1-a(x_1)^2\mu(x, \xi)$ with the aid of $\mu(x, \xi)$ in (4.33), the symbols $\lambda_m(x, \xi)$ belong to $S^1_{1/2}((3))$ and satisfy

$$\{\tau-\lambda_1, \tau-\lambda_2\} = a_{1,2}(x, \xi)\,(\lambda_1-\lambda_2) \tag{4.34}$$

with a symbol $a_{1,2}(x, \xi)$ in $S^0_{1/2}((1))$. The operator (21) can also be reduced to a system (4.30) with pseudo-differential operators B and B' in $S^0_{1/2}$. Hence, the fundamental solution of (21) can be obtained in the form (4.32).

In three examples (19), (20) and (21) the characteristic roots $\lambda_\pm(x, \xi)$ and the corresponding approximated symbols $\lambda_m(x, \xi)$, $m=1, 2$, satisfy (4.28) with $\lambda_1^\circ=\lambda_+$, $\lambda_2^\circ=\lambda_-$ and $\kappa=1/2$. Hence, from the statement after Corollary 4.5 we get

$$WF(u(t))\subset \textit{Conic hull of } \Gamma_t \tag{4.35}$$

for the solution $u(t)$ of

$$\begin{cases} L_j u(t) = 0\,, \\ u(0) = u_0, \quad \partial_t u(0) = u_1\,, \end{cases}$$

where $\Gamma(t)=\{\{Q, P\}\,(t, s; y, \eta);\, 0\leqq s\leqq t,\, (y, \eta)\in WF(u_0)\cup WF(u_1)\ \textit{for large}\ |\eta|\}$ for the trajectory $\{Q, P\}\,(t, s; y, \eta)$ defined by the following: Let $\{q, p\}\,(t; y, \eta)$ be the solution of

$$\begin{cases} \dfrac{dq}{dt} = -\nabla_\xi\lambda_-(t, q, p), \quad \dfrac{dp}{dt} = \nabla_x\lambda_-(t, q, p)\,, \\ \{q, p\}_{|t=0} = \{y, \eta\}\,. \end{cases}$$

Then, $\{Q, P\}\,(t, s; y, \eta)$ is defined as the solution of

$$\begin{cases} \dfrac{dQ}{dt} = -\nabla_\xi\lambda_+(t, Q, P), \quad \dfrac{dP}{dt} = \nabla_x\lambda_+(t, Q, P)\,, \\ \{Q, P\}_{|t=s} = \{q, p\}\,(s, y, \eta)\,. \end{cases}$$

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