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EXPONENTIAL ATTRACTOR FOR AN ADSORBATE-INDUCED PHASE TRANSITION MODEL IN NON SMOOTH DOMAIN

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Abstract

We improve our preceding result obtained in Tsujikawa and Yagi [10]. We construct the similar exponential attractors to the same adsorbate-induced phase transition model as in [10] but in a convex domain by using the compact smoothing property of corresponding nonlinear semigroup. In [10], the domain has been assumed to have $\mathcal{C}^3$ regularity to ensure the squeezing property of semigroup.

1. Introduction

We continue a study ([10]) of the Cauchy problem of the following system

$$
(1.1)\qquad
\begin{cases}
\dfrac{\partial u}{\partial t} = a\Delta u + c\nabla\cdot\{u(1-u)\nabla\chi(\rho)\} - fe^{\alpha\chi(\rho)}u \\
\qquad - gu + h(1-u) & \text{in } \Omega\times(0,\infty),\\
\dfrac{\partial \rho}{\partial t} = b\Delta\rho + d\rho(\rho+u-1)(1-\rho) & \text{in } \Omega\times(0,\infty),\\
\dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty),\\
u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x) & \text{in } \Omega.
\end{cases}
$$

This model has been presented by Hildebrand et al. [5] to describe the process of pattern formation by a specific kind of molecules adsorbated on a catalytic surface under the influence of adsorbate-induced phase transition. Here, Ω denotes a two-dimensional bounded metallic surface, $u(x,t)$ denotes the adsorbate coverage rate of the surface by the molecules at a position $x\in\Omega$ and time $t\geq 0$, and $\rho(x,t)$ denotes the structural state of the surface at $x\in\Omega$ and $t\geq 0$. In addition, a and b are diffusion constants, $\chi(\rho)$ is a chemical potential, $\rho(\rho+u-1)(1-\rho)$ is a phase transition function, and

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g and h are the desorption rate and the adsorption rate of the molecules respectively. For the modeling, we refer to [5, 6] (cf. [10, Introduction]).

In this paper we assume that Ω is a bounded, convex or $\mathcal{C}^2$ domain in $\mathbb{R}^2$. The function $\chi(\rho)$ is a cubic function of the form

$$\chi(\rho) = -\rho^2(3 - 2\rho) \tag{1.2}$$

(as suggested in [5]). The constants a, b, c, d, f, g, h and α are all given and positive. The Neumann boundary conditions are imposed on u and ρ on the boundary $\partial\Omega$, where $n(x)$ denotes (if it is uniquely determined) the outer normal vector at a boundary point $x \in \partial\Omega$. The initial functions are assumed to satisfy naturally the conditions $0 \le u_0 \le 1$ and $0 \le \rho_0 \le 1$ which will in turn implies $0 \le u \le 1$ and $0 \le \rho \le 1$ for every time $t > 0$.

We are interested not only in constructing a unique global solution of (1.1) for each pair of initial functions u_0, ρ_0 but also in investigating asymptotic behavior of the solution by constructing exponential attractors for the dynamical system determined from our Cauchy problem. Exponential attractor the notion of which has been introduced by Eden et al. [13] is one of very important limit sets in the theory of infinite-dimensional dynamical systems (see [12, 19]). The exponential attractor is a compact set with finite fractal dimension which contains the global attractor interiorly and attracts every trajectory in an exponential rate. If one exponential attractor exists, then there exists a family of them in such a way that in any neighborhood of the global attractor one can find an exponential attractor. The content of exponential attractor is richer than that of the global attractor since the global attractor consists of states in final stages only and since the pattern formation may often be performed in the process approaching to a final stage. The exponential attractor is very robust, in fact as shown in [1], it attracts not only every trajectory but also attracts every approximate solution to its neighborhood in an exponential rate and continues to trap it in the neighborhood forever. In a certain specific sense the exponential attractor depends on a parameter continuously, see [4], although the global attractor merely depends upper semi continuously in general. The finiteness of fractal dimension shows good correspondence to the slaving principle in the theory of self-organizations (see Haken [15]), that is in the process of a self-organization only a finite number of modes are active and the degree of freedom of the system is reduced to those.

In the preceding paper Tsujikawa and Yagi [10], we considered the case when Ω is sufficiently smooth. Indeed, under the assumption that Ω is a bounded $\mathcal{C}^3$ domain in $\mathbb{R}^2$, we have constructed an exponential attractor following the method due to [13]. In [10, Section 5] we could show that the nonlinear semigroup $S(t)$ determined from the Cauchy problem (1.1) enjoys a crucial condition called the squeezing property. The $\mathcal{C}^3$ regularity was needed to use the shift property that $\Delta\rho \in H^1(\Omega)$ together with $\partial\rho/\partial n = 0$ on $\partial\Omega$ yields $\rho \in H^3(\Omega)$ in establishing energy estimates of ρ (for example [10, p.329]). Therefore these methods are no longer available to the present case. Instead of using the squeezing property we will use the compact smoothing property of $S(t)$ ((3.2) below) which has recently been introduced by Efendiev et al. [3] to construct the exponential attractors for a wide class of nonlinear diffusion equations or systems. Following the general strategy devised in the paper Aida et al. [2], we will construct an exponential attractor as before using only a weak shift property that $\Delta\rho \in L^2(\Omega)$ together with $\partial\rho/\partial n = 0$ yields that $\rho \in H^2(\Omega)$. According to Grisvard [14], such a shift property is true in convex or $\mathcal{C}^2$ domains.

The nature of pattern is expected to depend seriously on the shape of domain Ω. As a matter of fact, we can observe various interesting pattern solutions by numerical computations in the case of square Ω (see [9]).

This paper is organized as follows. In Section 3 we review the general methods for constructing exponential attractors for abstract parabolic evolution equations studied in [2]. Section 4 is devoted to constructing local solutions to our problem (1.1) and Section 5 to establishing a priori and absorbing estimates for any local solution. Finally in Section 6 we shall construct an exponential attractor.

Notation. Let X be a Banach space with norm $\|\cdot\|_X$. Let $\mathcal{X}$ be a subset of X, then $\mathcal{X}$ is a metric space with the induced distance $d(U, V) = \|U - V\|_X$ $(U, V \in \mathcal{X})$. For $U \in \mathcal{X}$ and a set $B \subset \mathcal{X}$, $d(U, B)$ is defined by $d(U, B) = \inf_{V\in B} d(U, V)$. For two sets $B_1, B_2 \subset \mathcal{X}$, their distance $d(B_1, B_2)$ is defined by $d(B_1, B_2) = \max\{h(B_1, B_2), h(B_2, B_1)\}$, where $h(B_1, B_2)$ denotes the Hausdorff pseudodistance given by

$$h(B_1, B_2) = \sup_{U\in B_1} d(U, B_2) = \sup_{U\in B_1} \inf_{V\in B_2} d(U, V). \tag{1.3}$$

For two Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators from X into Y which is equipped with the uniform operator norm $\|\cdot\|_{\mathcal{L}(X,Y)}$. When $X = Y$, $\mathcal{L}(X, X)$ is abbreviated as $\mathcal{L}(X)$.

Let X be a Banach space and let I be an interval. $\mathcal{C}(I; X)$, $\mathcal{C}^\theta(I; X)$ $(0 < \theta < 1)$ and $\mathcal{C}^1(I; X)$ denote the space of X-valued continuous functions, Hölder continuous functions with exponent θ, and continuously differentiable functions equipped with the usual function norms, respectively.

We shall use a universal notation C to denote varying positive constants which are determined by the initial constants a, b, c, d, f, g, h, and α and by the domain Ω and the function $\chi(\rho)$. It may therefore change from occurrence to occurrence. If C depends also on some parameter, say r, it will be denoted by C_r.

2. Preliminary

Throughout this section, Ω is a bounded, $\mathcal{C}^2$ or convex domain in the plane. As well known, a bounded convex domain has a Lipschitz boundary (cf. [14, Corollary 1.2.2.3]).

For $0 \le s \le 2$, $H^s(\Omega)$ denotes the Sobolev space, its norm being denoted by

$\|\cdot\|_{H^s}$ (see [14, Chap. 1] and [20]). For $0 \le s_0 \le s \le s_1 \le 2$, $H^s(\Omega)$ coincides with the complex interpolation space $[H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta$, where $s = (1-\theta)s_0 + \theta s_1$, and the estimate

$$\|\cdot\|_{H^s} \le C\|\cdot\|_{H^{s_0}}^{1-\theta}\|\cdot\|_{H^{s_1}}^{\theta} \tag{2.1}$$

holds. When $0 \le s < 1$, $H^s(\Omega) \subset L^p(\Omega)$, where $1/p = (1-s)/2$, with continuous embedding

$$\|\cdot\|_{L^p} \le C\|\cdot\|_{H^s}. \tag{2.2}$$

When $s = 1$, $H^1(\Omega) \subset L^q(\Omega)$ for any finite $2 < q < \infty$ with the estimate

$$\|\cdot\|_{L^q} \le C_{p,q}\|\cdot\|_{H^1}^{1-(p/q)}\|\cdot\|_{L^p}^{p/q}, \tag{2.3}$$

where $1 \le p < q < \infty$. When $s > 1$, $H^s(\Omega) \subset \mathcal{C}\left(\overline{\Omega}\right)$ with continuous embedding

$$\|\cdot\|_{\mathcal{C}} \le C\|\cdot\|_{H^s}. \tag{2.4}$$

These results are well known when Ω is the whole plane $\mathbb{R}^2$ or a bounded smooth domain (see [20]). The results can then be generalized to the convex domain with the aid of the extension theorem due to Stein [17, Chap. VI, Theorem 5].

For any $0 < \theta \le 1$, it holds that

$$\|uv\|_{H^{1+\theta}} \le C_\theta\|u\|_{H^{1+\theta}}\|v\|_{H^{1+\theta}}, \quad u, v \in H^{1+\theta}(\Omega). \tag{2.5}$$

This is verified directly by remembering the definition of Sobolev norm $\|\cdot\|_{H^{1+\theta}}$ (cf. [14, Chap. 1]) and by noting (2.4).

We next consider a sesquilinear form given by

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla\overline{v}\, dx + \int_\Omega u\overline{v}\, dx, \quad u, v \in H^1(\Omega)$$

on the product space $H^1(\Omega) \times H^1(\Omega)$. From this form we can define realization A of the Laplace operator $-\Delta + 1$ in $L^2(\Omega)$ under the Neumann boundary conditions on the boundary $\partial\Omega$, see Lions and Magenes [16, Chap. 2, No. 9]. The realization A is a positive definite self-adjoint operator of $L^2(\Omega)$. Its domain is characterized by

$$\mathcal{D}(A) = H_N^2(\Omega) = \left\{u \in H^2(\Omega);\, \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\}, \tag{2.6}$$

and the following estimate

$$\|u\|_{H^2} \le C\|Au\|_{L^2}, \quad u \in \mathcal{D}(A) \tag{2.7}$$

holds. These characterization and estimate are verified by using the a priori estimates for elliptic operators when Ω is a bounded $\mathcal{C}^2$ domain. For the convex domain, these are then shown by the fact that the constant appearing in (2.7) depends essentially only on the negative part of the curvature of $\partial\Omega$, see [14, Theorem 3.2.1.3].

For $\theta \geq 0$, we consider the fractional power A^θ of A. As shown in [7, Sec. 2], we can characterize for $0 < \theta < 1$, $\theta \neq 3/4$, its domain in the form

$$\mathcal{D}(A^\theta) = \begin{cases} H^{2\theta}(\Omega), & \text{when } 0 \leq \theta < \dfrac{3}{4} \\ H_N^{2\theta}(\Omega) = \left\{u \in H^{2\theta}(\Omega)\,;\, \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\}, & \text{when } \dfrac{3}{4} < \theta \leq 1. \end{cases} \tag{2.8}$$

For $1 < \theta \leq 2$, $\theta \neq 7/4$, since $u \in \mathcal{D}(A^\theta)$ if and only if $u \in \mathcal{D}(A)$ with $Au \in \mathcal{D}(A^{\theta-1})$, its domain is given by

$$\mathcal{D}(A^\theta) = \mathcal{H}_N^{2\theta}(\Omega) = \begin{cases} \left\{u \in H_N^2(\Omega)\,;\, \Delta u \in H^{2(\theta-1)}(\Omega)\right\}, & \text{when } 1 \leq \theta < \dfrac{7}{4} \\ \left\{u \in H_N^2(\Omega)\,;\, \Delta u \in H_N^{2(\theta-1)}(\Omega)\right\}, & \text{when } \dfrac{7}{4} < \theta \leq 2. \end{cases} \tag{2.9}$$

3. General strategy

Let X be a Banach space with norm $\|\cdot\|_X$. Let $\mathcal{X}$ be a subset of X, $\mathcal{X}$ being a metric space with the distance $d(\,\cdot\,,\,\cdot\,)$ induced from $\|\cdot\|_X$. A family of nonlinear operators $S(t)$, $0 \leq t < \infty$, from $\mathcal{X}$ into itself is called a continuous semigroup on $\mathcal{X}$ if $S(0) = 1$ (identity on $\mathcal{X}$), $S(t+s) = S(t)S(s)$ for $0 \leq t,\ s < \infty$, and if

$$G(t, U_0) = S(t)U_0 \quad \text{is a continuous mapping from} \quad [0,\infty) \times \mathcal{X} \quad \text{into} \quad \mathcal{X}. \tag{3.1}$$

Let $S(t)$ be a continuous semigroup on $\mathcal{X}$. Then the set of all $\mathcal{X}$-valued continuous functions $S(\cdot)U_0$, $U_0 \in \mathcal{X}$, on $[0,\infty)$ is called a dynamical system determined by the semigroup $S(t)$ on the phase space $\mathcal{X}$ in the universal space X. The system is denoted by $(S(t), \mathcal{X}, X)$.

We are concerned with the case where $\mathcal{X}$ is a compact set of X. From the compactness, it is immediately seen that the set

$$\mathcal{A} = \bigcap_{0 \leq t < \infty} S(t)\mathcal{X}$$

is a global attractor of $(S(t), \mathcal{X}, X)$. That is, $\mathcal{A}$ is a compact set of X, $\mathcal{A}$ is an invariant set of $S(t)$ (this means that $S(t)\mathcal{A} = \mathcal{A}$ for every $t \geq 0$), and $\mathcal{A}$ attracts $\mathcal{X}$ in the sense that $h(S(t)\mathcal{X}, \mathcal{A})$ converges to 0 as $t \to \infty$, where $h(\,\cdot\,,\,\cdot\,)$ is the Hausdorff pseudodistance defined by (1.3).

The exponential attractor is then defined as follows (see Eden et al. [13]). A subset $\mathcal{M} \subset \mathcal{X}$ is called an exponential attractor of $(S(t), \mathcal{X}, X)$ if

(1) $\mathcal{M}$ contains the global attractor $\mathcal{A}$;
(2) $\mathcal{M}$ is a compact subset of X with finite fractal dimension;
(3) $\mathcal{M}$ is a positively invariant set of $S(t)$, namely $S(t)\mathcal{M} \subset \mathcal{M}$ for every $t \geq 0$;
(4) $\mathcal{M}$ attracts the whole space $\mathcal{X}$ exponentially in the sense that

$$h(S(t)\mathcal{X}, \mathcal{M}) \leq Ce^{-kt}, \quad 0 \leq t < \infty$$

with some exponent $k > 0$ and a constant $C > 0$.

Concerning construction of exponential attractors we present a method due to Efendiev et al. [3]. We assume the following two conditions. There exists another Banach space $Z \subset X$ with a compact embedding such that the operator $S(t^*)$ with some fixed $t^* > 0$ satisfies a compact Lipschitz condition of the form

$$\|S(t^*)U_0 - S(t^*)V_0\|_Z \leq L_1\|U_0 - V_0\|_X, \quad U_0, V_0 \in \mathcal{X} \tag{3.2}$$

with a constant $L_1 > 0$. The mapping $G(t, U_0) = S(t)U_0$ from $[0, t^*] \times \mathcal{X}$ into $\mathcal{X}$ satisfies the usual Lipschitz condition

$$\|G(t, U_0) - G(s, V_0)\|_X \leq L_2\{|t - s| + \|U_0 - V_0\|_X\}, \quad t, s \in [0, t^*], \ U_0, V_0 \in \mathcal{X}. \tag{3.3}$$

According to [3, Proposition 1] and to [13, Theorem 3.1], we have the following result.

Theorem 3.1. *Let $S(t^*)$ satisfy* (3.2) *with some Banach space Z embedded compactly in X and let G satisfy* (3.3). *Then, an exponential attractor $\mathcal{M}$ is constructed for the dynamical system $(S(t), \mathcal{X}, X)$.*

We also observe that if there exists one exponential attractor for $(S(t), \mathcal{X}, X)$, then there exists a family of exponential attractors in such a way that in any neighborhood of the global attractor $\mathcal{A}$ there exists an exponential attractor. In fact we can easily verify the following theorem from the definition of exponential attractor.

Theorem 3.2. *Let G satisfy* (3.3). *If $\mathcal{M}$ is an exponential attractor, then so is $S(t)\mathcal{M}$ for every time $t > 0$ and, as $t \to \infty$, $d(S(t)\mathcal{M}, \mathcal{A})$ converges to* 0, *where $d(\cdot, \cdot)$ is the distance of sets.*

In the second half of this section we shall review a general strategy for applying Theorem 3.1 to the dynamical systems which are determined from the Cauchy problems of abstract parabolic evolution equations. This has been obtained essentially in our previous paper [2].

Let X be a Banach space. We consider the Cauchy problem for an abstract parabolic evolution equation

$$\begin{cases} \dfrac{dU}{dt} + A(U)U = F(U), \quad 0 < t < \infty, \\ U(0) = U_0 \end{cases} \tag{3.4}$$

in X. For each $U \in Z$, $A(U)$ is a densely defined closed linear operator in X with a constant domain $\mathcal{D}(A(U)) \equiv \mathcal{D}$, where $Z \subset X$ is a second Banach space with continuous embedding. The operator F is a nonlinear operator from W into X, where $W \subset Z$ with continuous embedding.

Basic assumptions. For $U = 0$, we assume that $A(0)$ is a sectorial operator of X with angle less than $\pi/2$, i.e.

$$\sigma(A(0)) \subset \Sigma_\omega = \{\lambda \in \mathbb{C}\,;\, |\arg \lambda| < \omega\}$$

with $0 < \omega < \pi/2$, and

$$\left\|(\lambda - A(0))^{-1}\right\|_{\mathcal{L}(X)} \leq M\big/|\lambda|, \quad \lambda \notin \Sigma_\omega,$$

with some constant $M \geq 1$. The space $\mathcal{D}$ is then equipped with the graph norm $\|\cdot\|_{\mathcal{D}} = \|A(0)\cdot\|_X$.

We assume also that, for every $0 < R < \infty$, there exists a number $k_R > 0$ such that the operators $A_R(U) = A(U) + k_R$ are sectorial operators of X for all $U \in K_R = \{U \in Z\,;\, \|U\|_Z < R\}$. More precisely, the spectral set $\sigma(A_R(U))$ is contained in an open sectorial domain

$$\sigma(A_R(U)) \subset \Sigma_\omega = \{\lambda \in \mathbb{C}\,;\, |\arg \lambda| < \omega\}, \quad U \in K_R$$

with angle $0 < \omega < \pi/2$, and the resolvent satisfies

$$\left\|(\lambda - A_R(U))^{-1}\right\|_{\mathcal{L}(X)} \leq M_R\big/|\lambda|, \quad \lambda \notin \Sigma_\omega,\ U \in K_R \tag{3.5}$$

with some constant $M_R \geq 1$ dependent on R. We assume also that $A_R(U)$ satisfies the Lipschitz condition

$$\left\|A_R(U)\left\{A_R(U)^{-1} - A_R(V)^{-1}\right\}\right\|_{\mathcal{L}(X)} \leq N_R\|U - V\|_Y, \quad U, V \in K_R \tag{3.6}$$

with some constant $N_R > 0$ dependent on R, where Y is another Banach space such that $Z \subset Y \subset X$ with continuous embeddings.

For the nonlinear operator $F\colon W \to X$, we assume the Lipschitz condition

$$\begin{aligned} \|F(U) - F(V)\|_X &\leq \varphi(\|U\|_Z + \|V\|_Z) \\ &\quad \times \left\{\|U - V\|_W + (\|U\|_W + \|V\|_W)\|U - V\|_Z\right\}, \quad U, V \in W \end{aligned} \tag{3.7}$$

with some increasing continuous function $\varphi(\,\cdot\,)$.

For the spaces $W \subset Z \subset Y \subset X$, we assume that there exist exponents $0 \le \alpha_0 < \beta_0 < \eta_0 < 1$ such that $\mathcal{D}(A(0)^{\alpha_0}) \subset Y$, $\mathcal{D}(A(0)^{\beta_0}) \subset Z$, $\mathcal{D}(A(0)^{\eta_0}) \subset W$ with continuous embedding

$$\begin{cases} \|\widetilde{U}\|_Y \le D_1 \|A(0)^{\alpha_0}\widetilde{U}\|_X, & \widetilde{U} \in \mathcal{D}(A(0)^{\alpha_0}), \\ \|\widetilde{U}\|_Z \le D_2 \|A(0)^{\beta_0}\widetilde{U}\|_X, & \widetilde{U} \in \mathcal{D}(A(0)^{\beta_0}), \\ \|\widetilde{U}\|_W \le D_3 \|A(0)^{\eta_0}\widetilde{U}\|_X, & \widetilde{U} \in \mathcal{D}(A(0)^{\eta_0}) \end{cases} \tag{3.8}$$

respectively.

Let γ_0 be any exponent such that $\beta_0 < \gamma_0 \le 1$. Then the domain

$$\mathcal{D}_{\gamma_0} \equiv \mathcal{D}(A(0)^{\gamma_0}) \tag{3.9}$$

can be taken as a space of initial values. The space $\mathcal{D}_{\gamma_0}$ is equipped with the graph norm $\|\cdot\|_{\mathcal{D}_{\gamma_0}} = \|A(0)^{\gamma_0}\cdot\|_X$.

Local solutions. By [2, Theorem 1 and Corollary 1] (see [11] also), we can construct local solutions to (3.4).

Choose new exponents $\alpha, \beta, \gamma, \eta$ in such a way that $\alpha_0 < \alpha < \beta_0 < \beta < \eta_0 < \eta < 1$ and $\beta < \gamma < \gamma_0 \le 1$. Then, for each $0 < R < \infty$, the family of linear operators $A_R(U)$, $U \in K_R$, is readily observed to satisfy all the structural assumptions [2, (3.2), (3.3) and (3.5)]. In particular, the continuous embedding [2, (3.5)] is verified from (3.8) as follows. Since

$$D_R^{-1}\|A(0)\widetilde{U}\|_X \le \|A_R(U)\widetilde{U}\|_X \le D_R\|A(0)\widetilde{U}\|_X, \quad \widetilde{U} \in \mathcal{D},\ U \in K_R$$

with a suitable constant $D_R > 0$ dependent on R due to (3.6), we have

$$\begin{cases} \|A(0)^{\alpha_0}\widetilde{U}\|_X \le \widetilde{D}_{1,R}\|A_R(U)^{\alpha}\widetilde{U}\|_X, & \widetilde{U} \in \mathcal{D}(A_R(U)^{\alpha}),\ U \in K_R, \\ \|A(0)^{\beta_0}\widetilde{U}\|_X \le \widetilde{D}_{2,R}\|A_R(U)^{\beta}\widetilde{U}\|_X, & \widetilde{U} \in \mathcal{D}(A_R(U)^{\beta}),\ U \in K_R, \\ \|A(0)^{\eta_0}\widetilde{U}\|_X \le \widetilde{D}_{3,R}\|A_R(U)^{\eta}\widetilde{U}\|_X, & \widetilde{U} \in \mathcal{D}(A_R(U)^{\eta}),\ U \in K_R \end{cases}$$

with suitable constants $\widetilde{D}_{i,R} > 0$ $(i = 1, 2, 3)$. In the present case the nonlinear operator F satisfies only a very weaker Lipschitz condition (3.7) than [2, (3.4)]. But by (3.8), we verify that

$$\begin{aligned} \|F(U) - F(V)\|_X \le \varphi\left(\|A(0)^{\beta_0}U\|_X + \|A(0)^{\beta_0}V\|_X\right) \\ \times \Big\{\|A(0)^{\eta_0}(U-V)\|_X \\ + (\|A(0)^{\eta_0}U\|_X + \|A(0)^{\eta_0}V\|_X)\,\|A(0)^{\beta_0}(U-V)\|_X\Big\}, \\ U, V \in \mathcal{D}(A(0)^{\eta_0}). \end{aligned}$$

This then shows that we can use similar techniques devised in the proof of [8, Theorem 3.1] for handling the semilinear term in (3.4).

For each $0 < r < \infty$, consider the set of initial values

$$B_r = \{U_0 \in \mathcal{D}_{\gamma_0}\,;\, \|U_0\|_{\mathcal{D}_{\gamma_0}} \le r\}.$$

For a suitable $0 < R < \infty$, we have $B_r \subset K_R$. And $U_0 \in B_r$ implies $U_0 \in \mathcal{D}(A(0)^{\gamma_0}) \subset \mathcal{D}(A_R(U_0)^\gamma)$ with the estimate

$$\|A_R(U_0)^\gamma U_0\|_X \le D_r \|A(0)^{\gamma_0} U_0\|_X \le D_r r,$$

where $D_r > 0$ is some constant depending only on r. This shows that if $U_0 \in B_r$, then U_0 satisfies the compatibility condition [2, (3.20)] (and also [8, (In)]).

Therefore, by [2, Theorem 1 and Corollary 1] and [8, Theorem 3.1], we obtain the following result.

Theorem 3.3. *Under* (3.5), (3.6), (3.7) *and* (3.8), *for any initial value* $U_0 \in B_r$, $0 < r < \infty$, *there exists a unique local solution to* (3.4) *in the function space*

$$\begin{cases} U \in \mathcal{C}^{\gamma-\beta}([0, T_r]; Z) \cap \mathcal{C}^{\gamma-\alpha}([0, T_r]; Y) \cap \mathcal{C}^1((0, T_r]; X), \\ t^{1-\gamma} U \in \mathcal{C}([0, T_r]; \mathcal{D}), \quad A(U)^\gamma U \in \mathcal{C}([0, T_r]; X), \end{cases}$$

here $T_r > 0$ *depends only on* r *and is uniform in* $U_0 \in B_r$.

Dynamical system. In order to obtain the global solutions to (3.4), we have to establish a priori estimates for all local solutions. We here assume that there exists an increasing continuous function $p(\,\cdot\,) > 0$ such that for any initial values $U_0 \in \mathcal{D}_{\gamma_0}$, the estimate

$$\|U(t)\|_{\mathcal{D}_{\gamma_0}} \le p\left(\|U_0\|_{\mathcal{D}_{\gamma_0}}\right), \quad 0 \le t \le T_U \tag{3.10}$$

holds for all local solutions with the initial condition $U(0) = U_0$. Then by Theorem 3.3, any local solution can be extended as local solution over a fixed time length $T_{p\left(\|U_0\|_{\mathcal{D}_{\gamma_0}}\right)} > 0$ depending only on the initial value. Hence, (3.4) possess a unique global solution $U(t; U_0)$ on the interval $[0, \infty)$.

For each $0 < r < \infty$, let B_r be the closed ball of $\mathcal{D}_{\gamma_0}$ defined above. We are then led to define a phase space $\mathcal{X}_r$ and a nonlinear semigroup $S(t)$ by setting

$$\mathcal{X}_r = \bigcup_{U_0 \in B_r} \{U(t; U_0)\,;\, 0 \le t < \infty\}$$

and

$$S(t)U_0 = U(t; U_0), \quad U_0 \in \mathcal{X}_r,$$

where $U(t; U_0)$ denotes the global solution of (3.4) with $U_0 \in \mathcal{X}_r$. Then $S(t)$ is a nonlinear semigroup acting on $\mathcal{X}_R$.

In view of (3.10), we verify that $\mathcal{X}_R$ is a bounded set of $\mathcal{D}_{\gamma_0}$. Indeed, $\mathcal{X}_r \subset \overline{B}^{\mathcal{D}_{\gamma_0}}(0; p(r))$. We can then appeal to [2, Theorem 2] in order to verify that $S(t)$ is Lipschitz continuous from $(\mathcal{X}_R, d_X)$, the metric space equipped with the distance induced from X, into X. More precisely, there is a number $\tau > 0$ such that

$$\|S(t)U_0 - S(t)V_0\|_X \le L^n \|U_0 - V_0\|_X, \quad U_0, V_0 \in \mathcal{X}_r, \ t \in [(n-1)\tau, n\tau] \tag{3.11}$$

holds for any integer $n \ge 1$ with some constant $L > 0$. Indeed, when $n = 1$, this is verified directly from [2, Theorem 2]. Assume that (3.11) is true for n. For $n\tau \le t \le (n+1)\tau$, we can write $S(t) = S(t - n\tau)S(n\tau)$; then (3.11) for $n + 1$ is readily verified (note that $S(n\tau)\mathcal{X}_R \subset \mathcal{X}_R$). Moreover from this we can conclude that the mapping $G(t, U_0) = S(t)U_0$ is continuous from $[0, \infty) \times (\mathcal{X}_r, d_X)$ into X.

In this way, under (3.10), we have constructed a dynamical system $(S(t), \mathcal{X}_r, X)$ determined from the problem (3.4). Since $\mathcal{X}_r \supset \mathcal{B}_r$, it is obvious that $\mathcal{D}_{\gamma_0} = \bigcup_{0<r<\infty} \mathcal{X}_r$.

Absorbing and positively invariant compact set. In this part, we add the fundamental assumptions

$$\begin{cases} X & \text{is a reflexive Banach space,} \\ Z & \text{is compactly embedded in } X. \end{cases} \tag{3.12}$$

Then, since $A(0)^{\gamma_0}$ is an isomorphism from $\mathcal{D}_{\gamma_0}$ onto X, $\mathcal{D}_{\gamma_0}$ is also reflexive. Similarly, since $\mathcal{D}_{\gamma_0} = \mathcal{D}(A(0)^{\gamma_0}) \subset \mathcal{D}(A(0)^{\beta_0}) \subset Z$ continuously, $\mathcal{D}_{\gamma_0}$ is also embedded compactly in X. Therefore, (3.12) yields that any bounded ball of $\mathcal{D}_{\gamma_0}$ is a compact set of X.

We assume also that there exists a number $\widetilde{C} > 0$ for which the following assertion is valid. For any bounded set B of $\mathcal{D}_{\gamma_0}$, there is a time t_B such that

$$\sup_{U_0 \in B} \sup_{t \ge t_B} \|S(t)U_0\|_{\mathcal{D}_{\gamma_0}} \le \widetilde{C}. \tag{3.13}$$

Using this constant $\widetilde{C}$, we define the closed ball

$$\mathcal{B} = \overline{B}^{\mathcal{D}_{\gamma_0}}\left(0; \widetilde{C}\right).$$

In terms of dynamical systems, $\mathcal{B}$ is an absorbing set, namely for any bounded set B of $\mathcal{D}_{\gamma_0}$ there exists a time t_B such that $S(t)B \subset \mathcal{B}$ for every $t \ge t_B$. As noticed above, under (3.12), $\mathcal{B}$ is a compact set of X.

Finally we set

$$\mathcal{X} = \bigcup_{0 \le t < \infty} S(t)\mathcal{B} = \bigcup_{0 \le t \le t_{\mathcal{B}}} S(t)\mathcal{B}. \tag{3.14}$$

Note that, as $\mathcal{B}$ is one of bounded sets of $\mathcal{D}_{\gamma_0}$, $S(t)\mathcal{B}$ itself is absorbed by $\mathcal{B}$ for every $t \ge t_{\mathcal{B}}$. Furthermore, since $\mathcal{X}$ is an image of $[0, t_{\mathcal{B}}] \times \mathcal{B}$ by the continuous mapping $G(t, U) = S(t)U$, and since $[0, t_{\mathcal{B}}] \times \mathcal{B}$ is a compact set of $[0, \infty) \times X$, $\mathcal{X}$ is also a compact set of X. By definition, $\mathcal{X}$ is a positively invariant set of $S(t)$. Since $\mathcal{X} \supset \mathcal{B}$, $\mathcal{X}$ is also an absorbing set. Therefore, $\mathcal{X}$ is an absorbing and positively invariant compact set.

Under (3.12) and (3.13) we have thus verified that the set $\mathcal{X}$ defined by (3.14) is an absorbing and positively invariant compact set. In this sense, the asymptotic behavior of any dynamical system $(S(t), \mathcal{X}_r, X)$ is reduced to that of a new dynamical system $(S(t), \mathcal{X}, X)$ with the compact phase space $\mathcal{X}$.

Exponential attractors. We next construct exponential attractors for the dynamical system $(S(t), \mathcal{X}, X)$. But it is not possible to apply Theorem 3.1 directly, because $S(t)$ may not satisfy the Lipschitz condition (3.3). So we need to introduce an auxiliary phase space.

Since $\mathcal{X}$ is a bounded set of $\mathcal{D}_{\gamma_0}$, $\mathcal{X} \subset B_R$ with a suitable $0 < R < \infty$. We can then apply Theorem 3.3 in order to obtain that

$$\|A_R(S(t)U_0)S(t)U_0\|_X \le C_{\mathcal{X}} t^{\gamma-1}, \quad 0 < t \le T_{\mathcal{X}},\ U_0 \in \mathcal{X}$$

with a sufficiently small time $T_{\mathcal{X}} > 0$ and a suitable constant $C_{\mathcal{X}} > 0$. We now consider the set

$$\widetilde{\mathcal{X}} = S(T_{\mathcal{X}})\mathcal{X} \subset \mathcal{X}.$$

By the similar arguments as above, this set is seen to be an absorbing and positively invariant compact set. In addition, $\widetilde{\mathcal{X}} \subset \mathcal{D}$ with the estimate

$$\begin{aligned} \|U\|_{\mathcal{D}} &= \|S(T_{\mathcal{X}})U_0\|_{\mathcal{D}} \\ &\le \left\|A(0)A_R(S(T_{\mathcal{X}})U_0)^{-1}\right\|_{\mathcal{L}(X)} \|A_R(S(T_{\mathcal{X}})U_0)S(T_{\mathcal{X}})U_0\|_X \\ &\le C_{\mathcal{X}} T_{\mathcal{X}}^{\gamma-1}, \quad U = S(T_{\mathcal{X}})U_0 \in \widetilde{\mathcal{X}},\ U_0 \in \mathcal{X}. \end{aligned}$$

Therefore, $(S(t), \mathcal{X}, X)$ is reduced to the new system $\left(S(t), \widetilde{\mathcal{X}}, X\right)$ in which the phase space $\widetilde{\mathcal{X}}$ is a compact set of X and is a bounded set of $\mathcal{D}$.

We are now ready to apply Theorem 3.1 to the system $(S(t), \widetilde{\mathcal{X}}, X)$. We use again [2, Theorem 2] to conclude that, for a sufficiently small time $t^* > 0$, the operator $S(t^*)$ from $\widetilde{\mathcal{X}}$ into Z fulfills the compact Lipschitz condition (3.2). Similarly, by (3.11),

$$\|S(t)U_0 - S(t)V_0\|_X \le C_{\widetilde{\mathcal{X}}}\|U_0 - V_0\|_X, \quad 0 \le t \le t^*,\ U_0, V_0 \in \widetilde{\mathcal{X}}$$

with some constant $C_{\widetilde{\mathcal{X}}} > 0$. In the meantime, we have

$$\begin{aligned}\|S(t)V_0 - S(s)V_0\|_X &= \left\|\int_s^t \{F(S(\tau)V_0) - A(S(\tau)V_0)S(\tau)V_0\}\,d\tau\right\|_X \\ &\le C_{\widetilde{\mathcal{X}}}(t-s)\sup_{0\le\tau\le t^*}\|S(\tau)V_0\|_{\mathcal{D}} \le C_{\widetilde{\mathcal{X}}}(t-s), \\ &\qquad 0 \le s < t \le t^*,\ \ V_0 \in \widetilde{\mathcal{X}}.\end{aligned}$$

Hence the condition (3.3) is also fulfilled. Thus by Theorems 3.1 and 3.2, a family of exponential attractors $\mathcal{M}$ are constructed for $\left(S(t), \widetilde{\mathcal{X}}, X\right)$. It is immediate to verify that any exponential attractor of $\left(S(t), \widetilde{\mathcal{X}}, X\right)$ is that of $(S(t), \mathcal{X}, X)$.

4. Local solutions

To formulate the phase transition system (1.1) as an abstract equation of the form (3.4), we set the underlying space as

$$X = \left\{\begin{pmatrix} u \\ \rho \end{pmatrix};\ u \in L^2(\Omega) \text{ and } \rho \in H^2_N(\Omega)\right\}, \tag{4.1}$$

where $H^2_N(\Omega)$ is the space given by (2.6). In addition, Y is set as $Y = X$ and Z as

$$Z = \left\{\begin{pmatrix} u \\ \rho \end{pmatrix};\ u \in H^{1/2}(\Omega) \text{ and } \rho \in \mathcal{H}^{5/2}_N(\Omega)\right\}, \tag{4.2}$$

where $\mathcal{H}^{5/2}_N(\Omega)$ is the space given by (2.9).

For each $U = \binom{u}{\rho} \in Z$, a linear operator $A(U)$ is defined by

$$A(U)\widetilde{U} = \begin{pmatrix} A_1 & 0 \\ B_{21}(U) & A_2 \end{pmatrix}\begin{pmatrix} \widetilde{u} \\ \widetilde{\rho} \end{pmatrix}, \quad \widetilde{U} = \begin{pmatrix} \widetilde{u} \\ \widetilde{\rho} \end{pmatrix} \in \mathcal{D}(A(U)). \tag{4.3}$$

Here, $A_1 = -a\Delta + g$ is a self-adjoint operator of $L^2(\Omega)$ with $\mathcal{D}(A_1) = H^2_N(\Omega)$ and $A_2 = -b\Delta + d$ is also a self-adjoint operator of $L^2(\Omega)$ with $\mathcal{D}(A_2) = H^2_N(\Omega)$, but A_2 is at present considered as an operator from $\mathcal{H}^4_N(\Omega) = \mathcal{D}\left(A_2^2\right)$ to $H^2_N(\Omega) = \mathcal{D}(A_2)$. And for each $U = \binom{u}{\rho} \in Z$, $B_{21}(U)$ is a bounded linear operator from $H^2_N(\Omega)$ into itself defined by

$$B_{21}(U)\widetilde{u} = d\rho(\rho - 1)\widetilde{u}, \quad \widetilde{u} \in H^2_N(\Omega).$$

Due to (2.5), $B_{21}(U)$ satisfies the estimate

$$\|B_{21}(U)\widetilde{u}\|_{H^2} \le C\|\rho\|_{H^2}(\|\rho\|_{H^2} + 1)\,\|\widetilde{u}\|_{H^2} \le C\|U\|_X(\|U\|_X + 1)\,\|\widetilde{u}\|_{H^2}. \tag{4.4}$$

Thus, $A(U)$'s are all linear operators of X with constant domain

$$\mathcal{D}(A(U)) \equiv \mathcal{D} = \left\{ \begin{pmatrix} \widetilde{u} \\ \widetilde{\rho} \end{pmatrix} ; \widetilde{u} \in H^2_N(\Omega) \text{ and } \widetilde{\rho} \in \mathcal{H}^4_N(\Omega) \right\}, \quad U \in Z.$$

Obviously,

$$A(0) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

is a positive definite self-adjoint operator of X.

The nonlinear operator F in (3.4) is given by

$$F(U) = \begin{pmatrix} c\nabla \cdot \{u(1-u)\nabla\chi(\rho)\} - fe^{\alpha\chi(\rho)}u + h(1-u) \\ d\rho^2(2-\rho) \end{pmatrix}, \quad U = \begin{pmatrix} u \\ \rho \end{pmatrix} \in W. \tag{4.5}$$

Here, W is a fourth space which is defined by

$$W = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix} ; u \in H^{2-\varepsilon}_N(\Omega) \text{ and } \rho \in \mathcal{H}^{4-\varepsilon}_N(\Omega) \right\} \tag{4.6}$$

with an arbitrarily fixed exponent $0 < \varepsilon < 1/2$.

Then the problem (1.1) is rewritten in the form (3.4) in the underlying Banach space X.

It is not difficult to verify that all the structural assumptions (3.5)–(3.8) are fulfilled by $A_R(U)$, $U \in K_R$ and F.

In fact, for $\lambda \notin (0, \infty)$, we see that

$$(\lambda - A(U))\widetilde{U} = \widetilde{F}, \quad U = \begin{pmatrix} u \\ \rho \end{pmatrix} \in Z, \quad \widetilde{U} = \begin{pmatrix} \widetilde{u} \\ \widetilde{\rho} \end{pmatrix} \in \mathcal{D}, \quad \widetilde{F} = \begin{pmatrix} \widetilde{f} \\ \widetilde{\mu} \end{pmatrix} \in X$$

if and only if

$$\begin{cases} \widetilde{u} = (\lambda - A_1)^{-1}\widetilde{f}, \\ \widetilde{\rho} = (\lambda - A_2)^{-1}\left\{\widetilde{\mu} + B_{21}(U)(\lambda - A_1)^{-1}\widetilde{f}\right\}. \end{cases}$$

In view of (4.4), we conclude that (3.5) holds for each $0 < R < \infty$ with an arbitrarily fixed $0 < \omega < \pi/2$ if $k_R > 0$ is chosen suitably.

Let next $U, V \in K_R$ with $U = \left(\begin{smallmatrix} u \\ \rho \end{smallmatrix}\right)$ and $V = \left(\begin{smallmatrix} v \\ \varsigma \end{smallmatrix}\right)$. Then, since

$$\begin{aligned} A_R(U)\{A_R(U)^{-1} - A_R(V)^{-1}\}\widetilde{F} &= \{A_R(V) - A_R(U)\}A_R(V)^{-1}\widetilde{F} \\ &= \begin{pmatrix} 0 & 0 \\ B_{21}(U) - B_{21}(V) & 0 \end{pmatrix} A_R(V)^{-1}\widetilde{F}, \quad \widetilde{F} \in X, \end{aligned}$$

(3.6) is reduced to the condition

$$\|\{\rho(\rho-1)-\zeta(\zeta-1)\}\widetilde{u}\|_{H^2} \le C_R\|\rho-\zeta\|_{H^2}\,\|\widetilde{u}\|_{H^2}, \quad \widetilde{\rho} \in H^2_N(\Omega).$$

But this is also verified immediately by (2.5).

Let us now verify the condition (3.7). For $U = \binom{u}{\rho} \in W$, we see that

$$\nabla\cdot\{u(1-u)\nabla\chi(\rho)\} = \nabla\{u(1-u)\}\cdot\nabla\chi(\rho) + u(1-u)\Delta\chi(\rho),$$

and

$$\nabla\chi(\rho) = \chi'(\rho)\nabla\rho \quad \text{and} \quad \Delta\chi(\rho) = \chi''(\rho)|\nabla\rho|^2 + \chi'(\rho)\Delta\rho. \tag{4.7}$$

From (1.2) it then follows that

$$\begin{aligned}&\nabla\cdot\{u(1-u)\nabla\chi(\rho)\}\\ &= 6(1-2u)\rho(1-\rho)\nabla u\cdot\nabla\rho + 6u(1-u)(2\rho-1)|\nabla\rho|^2 + 6u(1-u)\rho(\rho-1)\Delta\rho.\end{aligned}$$

While, by (2.2) and (2.4),

$$\begin{aligned}&\|(1-2u)\rho(1-\rho)\nabla u\cdot\nabla\rho\|_{L^2}\\ &\le C(\|u\|_{L^4}+1)\|\rho\|_{L^\infty}(\|\rho\|_{L^\infty}+1)\|\nabla u\|_{L^{2/\varepsilon}}\|\nabla\rho\|_{L^{4/(1-2\varepsilon)}}\\ &\le C(\|u\|_{H^{1/2}}+1)\|\rho\|_{H^{1+\varepsilon}}(\|\rho\|_{H^{1+\varepsilon}}+1)\|u\|_{H^{2-\varepsilon}}\|\rho\|_{H^{(3+2\varepsilon)/2}} \le C\|U\|_Z^2(\|U\|_Z+1)^2\|U\|_W.\end{aligned}$$

Similarly,

$$\begin{aligned}&\left\|u(1-u)(2\rho-1)|\nabla\rho|^2\right\|_{L^2}\\ &\le C\|u\|_{L^\infty}(\|u\|_{L^4}+1)(\|\rho\|_{L^\infty}+1)\|\nabla\rho\|_{L^8}^2\\ &\le C\|u\|_{H^{1+\varepsilon}}(\|u\|_{H^{1/2}}+1)(\|\rho\|_{H^{1+\varepsilon}}+1)\|\rho\|_{H^{7/4}}^2 \le C\|U\|_Z^2(\|U\|_Z+1)^2\|U\|_W.\end{aligned}$$

Finally,

$$\begin{aligned}\|u(1-u)\rho(\rho-1)\Delta\rho\|_{L^2} &\le C\|u\|_{L^4}(\|u\|_{L^4}+1)\|\rho\|_{L^\infty}(\|\rho\|_{L^\infty}+1)\|\Delta\rho\|_{L^\infty}\\ &\le C\|u\|_{H^{1/2}}(\|u\|_{H^{1/2}}+1)\|\rho\|_{H^{1+\varepsilon}}(\|\rho\|_{H^{1+\varepsilon}}+1)\|\rho\|_{\mathcal{H}^{3+\varepsilon}}\\ &\le C\|U\|_Z^2(\|U\|_Z+1)^2\|U\|_W.\end{aligned}$$

From these calculations we conclude that

$$\begin{aligned}&\|\nabla\cdot\{u(1-u)\nabla\chi(\rho)\}-\nabla\cdot\{v(1-v)\nabla\chi(\zeta)\}\|_{L^2}\\ &\le C\left(\|U\|_Z^4+\|V\|_Z^4+1\right)\\ &\quad\times\{\|U-V\|_W+(\|U\|_W+\|V\|_W)\|U-V\|_Z\}, \quad U=\begin{pmatrix}u\\ \rho\end{pmatrix},\ V=\begin{pmatrix}v\\ \zeta\end{pmatrix}\in W.\end{aligned}$$

In addition, by a direct calculation,

$$
\begin{aligned}
&\|e^{\alpha\chi(\rho)}u - e^{\alpha\chi(\zeta)}v\|_{L^2} \\
&\le e^{C\left(\|\rho\|_{L^\infty}^3+\|\zeta\|_{L^\infty}^3+1\right)}(\|u\|_{L^2}+\|v\|_{L^2}+1)\{\|u-v\|_{L^2}+\|\rho-\zeta\|_{L^\infty}\} \\
&\le e^{C\left(\|U\|_Z^3+\|V\|_Z^3+1\right)}(\|U\|_Z+\|V\|_Z+1)\|U-V\|_Z, \quad U=\begin{pmatrix} u \\ \rho \end{pmatrix}, \ V=\begin{pmatrix} v \\ \zeta \end{pmatrix} \in W.
\end{aligned}
$$

And, by (2.5),

$$
\begin{aligned}
\left\|\rho^2(2-\rho)-\zeta^2(2-\zeta)\right\|_{H^2} &\le C\left(\|\rho\|_{H^2}^2+\|\zeta\|_{H^2}^2+1\right)\|\rho-\zeta\|_{H^2} \\
&\le C\left(\|U\|_Z^2+\|V\|_Z^2+1\right)\|U-V\|_Z, \\
&\qquad U=\begin{pmatrix} u \\ \rho \end{pmatrix}, \ V=\begin{pmatrix} v \\ \zeta \end{pmatrix} \in W.
\end{aligned}
$$

In this way we have verified that F fulfils the condition (3.7).

As for exponents $0 \le \alpha_0 < \beta_0 < \gamma_0 \le 1$ and $\beta_0 < \eta_0 < 1$, we take $\alpha_0 = 0$, $\beta_0 = 1/4$, $\gamma_0 = 1/2$ and $\eta_0 = 1-(\varepsilon/2)$. Then $Z = \mathcal{D}(A(0)^{\beta_0})$ and $W = \mathcal{D}(A(0)^{\eta_0})$ with norm equivalence. The space of initial values is then given by

$$
\mathcal{D}_{\gamma_0} = \mathcal{D}\left(A(0)^{1/2}\right) = \left\{\begin{pmatrix} \widetilde{u} \\ \widetilde{\rho} \end{pmatrix}; \widetilde{u} \in H^1(\Omega) \text{ and } \widetilde{\rho} \in \mathcal{H}_N^3(\Omega)\right\}. \tag{4.8}
$$

We take also exponents $0 < \alpha < \beta < \gamma < 1$ and $\beta < \eta < 1$ in such a way that $0 < \alpha < 1/4 < \beta < \gamma < 1/2$ and $1-(\varepsilon/2) < \eta < 1$.

Then, by virtue of Theorem 3.3, for any initial function $U_0 = \binom{u_0}{\rho_0}$, where $u_0 \in H^1(\Omega)$ and $\rho_0 \in \mathcal{H}_N^3(\Omega)$, there exists a unique local solution to (1.1) in the function space:

(4.9)

$$
\begin{cases}
u \in \mathcal{C}^{\gamma-\beta}([0,T_{U_0}];H^{1/2}(\Omega)) \cap \mathcal{C}^1((0,T_{U_0}];L^2(\Omega)), & t^{1-\gamma}u \in \mathcal{C}([0,T_{U_0}];H_N^2(\Omega)), \\
\rho \in \mathcal{C}^{\gamma-\beta}([0,T_{U_0}];\mathcal{H}_N^{5/2}(\Omega)) \cap \mathcal{C}^1((0,T_{U_0}];H_N^2(\Omega)), & t^{1-\gamma}\rho \in \mathcal{C}([0,T_{U_0}];\mathcal{H}_N^4(\Omega)).
\end{cases}
$$

Here, $T_{U_0} > 0$ is determined by the norm $\|U_0\|_{\mathcal{D}_{1/2}} = \|u_0\|_{H^1} + \|\rho_0\|_{\mathcal{H}_N^3}$ alone.

We shall conclude this section with showing that $0 \le u_0 \le 1$ and $0 \le \rho_0 \le 1$ imply $0 \le u \le 1$ and $0 \le \rho \le 1$ for any local solution in the function space (4.9). So let $u_0 \in H^1(\Omega)$ and $\rho_0 \in \mathcal{H}_N^3(\Omega)$ be initial functions with the conditions $0 \le u_0 \le 1$ and $0 \le \rho_0 \le 1$. Let u, ρ be any local solution to (1.1) such that

(4.10)

$$
\begin{cases}
u \in \mathcal{C}^{\gamma-\beta}\left([0,T_{u,\rho}];H^{1/2}(\Omega)\right) \cap \mathcal{C}^1\left((0,T_{u,\rho}];L^2(\Omega)\right), & t^{1-\gamma}u \in \mathcal{C}\left([0,T_{u,\rho}];H_N^2(\Omega)\right), \\
\rho \in \mathcal{C}^{\gamma-\beta}\left([0,T_{u,\rho}];\mathcal{H}_N^{5/2}(\Omega)\right) \cap \mathcal{C}^1\left((0,T_{u,\rho}];H_N^2(\Omega)\right), & t^{1-\gamma}\rho \in \mathcal{C}\left([0,T_{u,\rho}];\mathcal{H}_N^4(\Omega)\right).
\end{cases}
$$

Since the complex conjugate $\overline{u}, \overline{\rho}$ of u, ρ is also a solution to (1.1), the uniqueness of solution yields that $u = \overline{u}$ and $\rho = \overline{\rho}$, namely u and ρ are real valued functions.

Let us regard u as a solution to the linear problem

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = a\Delta u - gu + \boldsymbol{\chi}(x,t)\cdot\nabla u + F(x,t)u + h & \text{in } \Omega\times(0,T_{u,\rho}),\\
\dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega\times(0,T_{u,\rho}),\\
u(0) = u_0 & \text{in } \Omega
\end{cases}
\tag{4.11}
$$

in $L^2(\Omega)$, where

$$
\begin{aligned}
\boldsymbol{\chi}(x,t) &= c(1-2u)\nabla\chi(\rho) = 6c\rho(\rho-1)(1-2u)\nabla\rho,\\
F(x,t) &= c(1-u)\Delta\chi(\rho) - fe^{\alpha\chi(\rho)} - h.
\end{aligned}
$$

We here observe from (4.10) that

$$
\begin{aligned}
\|\boldsymbol{\chi}(x,t)\cdot\nabla\widetilde{u}\|_{L^2} &\le C\|\rho\|_{L^\infty}(\|\rho\|_{L^\infty}+1)(\|u\|_{L^4}+1)\|\nabla\rho\|_{L^{4/(1-2\varepsilon)}}\|\nabla\widetilde{u}\|_{L^{2/\varepsilon}}\\
&\le C\|\rho\|_{H^2}^2(\|\rho\|_{H^2}+1)(\|u\|_{H^{1/2}}+1)\|\nabla\widetilde{u}\|_{H^{1-\varepsilon}}\\
&\le C_{u,\rho}\left\|A_1^{1-\varepsilon/2}\widetilde{u}\right\|_{L^2}, \quad \widetilde{u}\in\mathcal{D}\left(A_1^{1-(\varepsilon/2)}\right).
\end{aligned}
$$

In addition, from (4.10),

$$
\|\{\boldsymbol{\chi}(x,t)-\boldsymbol{\chi}(x,s)\}\cdot\nabla\widetilde{u}\|_{L^2} \le C_{u,\rho}(t-s)^{\gamma-\beta}\left\|A_1^{1-(\varepsilon/2)}\widetilde{u}\right\|_{L^2}, \quad \widetilde{u}\in\mathcal{D}\left(A_1^{1-(\varepsilon/2)}\right).
$$

Similarly we observe from (4.7) and (4.10) that

$$
\begin{aligned}
\|F(x,t)\widetilde{u}\|_{L^2} \le C\Big[&\big(\|u\|_{L^4}+1\big)\Big\{(\|\rho\|_{L^\infty}+1)\|\nabla\rho\|_{L^4}^2\\
&+\big(\|\rho\|_{L^\infty}^2+1\big)\|\Delta\rho\|_{L^4}\Big\} + e^{\alpha\|\rho\|_{L^\infty}}+1\Big]\|\widetilde{u}\|_{L^\infty}\\
\le C&\left\{(\|u\|_{H^{1/2}}+1)\left(\|\rho\|_{H^2}^2+1\right)\|\rho\|_{\mathcal{H}^{5/2}} + e^{\alpha\|\rho\|_{H^2}}+1\right\}\|\widetilde{u}\|_{H^{1+\varepsilon}}\\
\le C&_{u,\rho}\left\|A_1^{1-(\varepsilon/2)}\widetilde{u}\right\|_{L^2}, \quad \widetilde{u}\in\mathcal{D}\left(A_1^{1-(\varepsilon/2)}\right),
\end{aligned}
$$

and

$$
\|\{F(x,t)-F(x,s)\}\widetilde{u}\|_{L^2} \le C_{u,\rho}(t-s)^{\gamma-\beta}\left\|A_1^{1-(\varepsilon/2)}\widetilde{u}\right\|_{L^2}, \quad \widetilde{u}\in\mathcal{D}\left(A_1^{1-(\varepsilon/2)}\right).
$$

We can then appeal to the theory of linear abstract evolution equations. According to [18, Chap. 5, Theorem 2.3], there exists a unique solution of the problem (4.11) such that

$$
u\in\mathcal{C}\left((0,T_{u,\rho}];H_N^2(\Omega)\right)\cap\mathcal{C}^1\left((0,T_{u,\rho}];L^2(\Omega)\right)\cap\mathcal{C}\left([0,T_{u,\rho}];L^2(\Omega)\right).
$$

We further consider approximate linear problems

$$\begin{cases} \dfrac{\partial u_k}{\partial t} = a\Delta u_k - g u_k + \boldsymbol{\chi}_k(x,t)\cdot\nabla u_k + F_k(x,t)u_k + h & \text{in } \Omega\times(0,T_{u,\rho}),\\ \dfrac{\partial u_k}{\partial n} = 0 & \text{on } \partial\Omega\times(0,T_{u,\rho}),\\ u_k(0) = u_0 & \text{in } \Omega \end{cases}$$

for $k = 1, 2, 3, \dots$. Here,

$$\boldsymbol{\chi}_k(x,t) = \boldsymbol{\Phi}_k(\boldsymbol{\chi}(x,t)),$$
$$F_k(x,t) = \Phi_k(F(x,t)),$$

and $\boldsymbol{\Phi}_k(\boldsymbol{\chi})$, $\boldsymbol{\chi} \in \mathbb{R}^2$, and $\Phi_k(F)$, $F \in \mathbb{R}$, are both cutoff functions such that $\boldsymbol{\Phi}_k(\boldsymbol{\chi}) = \mathbf{0}$ for $\|\boldsymbol{\chi}\| \geq k$ and $\Phi_k(F) = 0$ for $|F| \geq k$. Then the solution u_k converges to the solution u of (4.11) in $\mathcal{C}([0,T_{u,\rho}];L^2(\Omega))$ as $k \to \infty$. In the meantime we can easily verify by the truncation method (see [10, Theorem 3.3]) that $u_0 \geq 0$ implies $u_k(t) \geq 0$ for every k. As a consequence we conclude that $u(t) \geq 0$.

In a similar way we conclude also that $\rho_0 \geq 0$ implies $\rho(t) \geq 0$.

The estimates $u(t) \leq 1$ and $\rho(t) \leq 1$ are also verified in an analogous way. In fact it is sufficient to notice that $w = 1 - u$ and $\xi = 1 - \rho$ satisfy the following linear equations

$$\begin{cases} \dfrac{\partial w}{\partial t} = a\Delta w - hw - c\nabla\cdot\{wu\nabla\chi(\rho)\} + fe^{\alpha\chi(\rho)}u + gu,\\ \dfrac{\partial \xi}{\partial t} = b\Delta\xi - d\rho(\rho + u - 1)\xi \end{cases}$$

respectively. From $w(0) = 1 - u_0 \geq 0$ and $\xi(0) = 1 - \rho_0 \geq 0$ it follows that $w(t) \geq 0$ and $\xi(t) \geq 0$. Note that $fe^{\alpha\chi(\rho)}u + gu \geq 0$ is already known. We have hence verified that $u(t) \leq 1$ and $\rho(t) \leq 1$.

Thus, if $U_0 = \binom{u_0}{\rho_0} \in \mathcal{D}_{1/2}$ with $0 \leq u_0 \leq 1$ and $0 \leq \rho_0 \leq 1$, then any local solution $U = \binom{u}{\rho}$ in the space (4.10) satisfies

$$0 \leq u(t) \leq 1 \quad \text{and} \quad 0 \leq \rho(t) \leq 1, \quad 0 \leq t \leq T_{u,\rho}. \tag{4.12}$$

5. Global solutions

Let the initial functions satisfy

$$u_0 \in H^1(\Omega) \quad \text{and} \quad \rho_0 \in \mathcal{H}^3_N(\Omega) \quad \text{with} \quad 0 \leq u_0 \leq 1 \quad \text{and} \quad 0 \leq \rho_0 \leq 1.$$

And let u, ρ be any local solution to (1.1) in the function space (4.10).

We can then verify the following a priori estimate

$$\|u(t)\|_{H^1} + \|\rho(t)\|_{\mathcal{H}^3} \le p\left(\|u_0\|_{H^1} + \|\rho_0\|_{\mathcal{H}^3}\right), \quad 0 \le t \le T_{u,\rho} \tag{5.1}$$

with some continuous increasing function $p(\cdot)$ which is independent of u, ρ. This estimate will be established by several steps. Throughout the proof, we shall use universal notations δ and $p(\,\cdot\,)$ to denote varying positive exponents and varying continuous increasing functions which are determined by the constants $a, b, c, d, f, g, h,$ and α and by Ω and $\chi(\rho)$. They may therefore change from occurrence to occurrence.

We write

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u - gu + c\nabla\cdot\{u(1-u)\nabla\chi(\rho)\} + P(u,\rho), \\ \dfrac{\partial \rho}{\partial t} = b\Delta\rho - d\rho + Q(u,\rho), \end{cases} \tag{5.2}$$

where

$$P(u,\rho) = -fe^{\alpha\chi(\rho)} + h(1-u),$$
$$Q(u,\rho) = d\rho(u+\rho-1)(1-\rho) + d\rho.$$

As shown by (4.12), we already know that $0 \le u(t) \le 1$ and $0 \le \rho(t) \le 1$ for every $0 \le t \le T_{u,\rho}$.

STEP 1. Multiply the second equation of (5.2) by $\Delta\rho$ and integrate the product in Ω. Then, noting (4.12), we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\rho|^2\,dx + b\int_\Omega |\Delta\rho|^2 dx + d\int_\Omega |\nabla\rho|^2\,dx$$
$$= -\int_\Omega Q(u,\rho)\Delta\rho\,dx \le \frac{b}{2}\|\Delta\rho\|_{L^2}^2 + C.$$

Therefore,

$$\frac{d}{dt}\int_\Omega |\nabla\rho|^2\,dx + b\int_\Omega |\Delta\rho|^2\,dx + 2d\int_\Omega |\nabla\rho|^2\,dx \le C.$$

Solving this in $\int_\Omega |\nabla\rho|^2 dx$, we obtain that

$$\int_\Omega |\nabla\rho(t)|^2\,dx \le e^{-2dt}\|\rho_0\|_{H^1}^2 + C.$$

Hence, in view of (4.12),

$$\|\rho(t)\|_{H^1}^2 \le C\left[e^{-2dt}\|\rho_0\|_{H^1}^2 + 1\right], \quad 0 \le t \le T_{u,\rho}. \tag{5.3}$$

STEP 2. Multiply the first equation of (5.2) by u and integrate the product in Ω. Then,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + a\int_\Omega |\nabla u|^2\,dx + g\int_\Omega u^2\,dx\\&= -c\int_\Omega u(1-u)\nabla\chi(\rho)\cdot\nabla u\,dx + \int_\Omega P(u,\rho)u\,dx\\&\le C\int_\Omega (|\nabla u|\,|\nabla\rho| + 1)\,dx \le \frac{a}{2}\|\nabla u\|_{L^2}^2 + C\left\{\|\nabla\rho\|_{L^2}^2 + 1\right\}.\end{aligned}$$

By using (5.3), we have

$$\frac{d}{dt}\int_\Omega u^2\,dx + a\int_\Omega |\nabla u|^2\,dx + 2g\int_\Omega u^2\,dx \le C\left\{\|\rho_0\|_{H^1}^2 + 1\right\}.$$

In the meantime, multiply the second equation of (5.2) by $\Delta^2\rho$ and integrate the product in Ω. Then,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta\rho|^2\,dx + b\int_\Omega |\nabla\Delta\rho|^2\,dx + d\int_\Omega |\Delta\rho|^2\,dx\\&= -\int_\Omega \nabla Q(u,\rho)\cdot\nabla\Delta\rho\,dx \le \frac{b}{2}\|\nabla\Delta\rho\|_{L^2}^2 + C\left\{\|\nabla u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2\right\}.\end{aligned}$$

By (5.3) it follows that

$$\frac{d}{dt}\int_\Omega |\Delta\rho|^2\,dx + b\int_\Omega |\nabla\Delta\rho|^2\,dx + 2d\int_\Omega |\Delta\rho|^2\,dx \le C\|\nabla u\|_{L^2}^2 + C\left\{\|\rho_0\|_{H^1}^2 + 1\right\}.$$

We join this with the inequality obtained above to obtain that

$$\frac{d}{dt}\int_\Omega \left(u^2 + |\Delta\rho|^2\right)dx + \delta\int_\Omega \left(u^2 + |\Delta\rho|^2\right)dx \le C\left\{\|\rho_0\|_{H^1}^2 + 1\right\}.$$

Solving this, we conclude that

$$\begin{aligned}x\|u(t)\|_{L^2}^2 + \|\rho(t)\|_{H^2}^2 &\le C\left[e^{-\delta t}\{\|u_0\|_{L^2}^2 + \|\rho_0\|_{H^2}^2\} + \|\rho_0\|_{H^1}^2 + 1\right]\\&\le C\left[e^{-\delta t}\|\rho_0\|_{H^2}^2 + \|\rho_0\|_{H^1}^2 + 1\right], \quad 0 \le t \le T_{u,\rho}.\end{aligned} \tag{5.4}$$

STEP 3. Multiply the first equation of (5.2) by Δu and integrate the product in Ω. Then, after some estimation,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + a\int_\Omega |\Delta u|^2\,dx + g\int_\Omega |\nabla u|^2\,dx\\&\le \frac{a}{2}\int_\Omega |\Delta u|^2\,dx + C\int_\Omega \left[|\nabla\cdot\{u(1-u)\nabla\chi(\rho)\}|^2 + P(u,\rho)^2\right]dx.\end{aligned}$$

By a direct calculation (see (4.7)), we verify that

$$|\nabla \cdot \{u(1-u)\nabla\chi(\rho)\}|^2 \leq C\left(|\Delta\rho|^2 + |\nabla\rho|^4 + |\nabla u|^2|\nabla\rho|^2\right).$$

Moreover, by (2.3) and (5.4),

$$\int_\Omega \left(|\Delta\rho|^2 + |\nabla\rho|^4\right) dx \leq C\left(\|\rho\|_{H^2}^2 + \|\nabla\rho\|_{H^1}^4\right) \leq p\left(\|\rho_0\|_{H^2}\right), \tag{5.5}$$

and by (2.7),

$$\begin{aligned}\int_\Omega |\nabla u|^2|\nabla\rho|^2\,dx &\leq C\|\nabla u\|_{L^3}^2\|\nabla\rho\|_{L^6}^2 \leq C\|u\|_{H^2}^{4/3}\|u\|_{L^2}^{2/3}\|\nabla\rho\|_{H^1}^2 \\ &\leq \frac{a}{4}\|\Delta u\|_{L^2}^2 + C\{\|\rho_0\|_{H^2}^6 + 1\}.\end{aligned} \tag{5.6}$$

Therefore we obtain that

$$\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + \frac{a}{2}\int_\Omega |\Delta u|^2\,dx + 2g\int_\Omega |\nabla u|^2\,dx \leq p(\|\rho_0\|_{H^2}).$$

In the meantime, we observe in view of (4.10) that

$$\frac{d}{dt}\int_\Omega |\nabla\Delta\rho|^2\,dx = -2\int_\Omega \left(\frac{\partial}{\partial t}\Delta\rho\right)\Delta^2\rho\,dx.$$

Take the product of the second equation of (5.2) operated by Δ and $\Delta^2\rho$, and integrate the product in Ω. Then,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\Delta\rho|^2\,dx + b\int_\Omega |\Delta^2\rho|^2\,dx + d\int_\Omega |\nabla\Delta\rho|^2\,dx \\ &= -\int_\Omega \Delta Q(u,\rho)\Delta^2\rho\,dx \leq \frac{b}{2}\int_\Omega |\Delta^2\rho|^2\,dx + C\int_\Omega |\Delta Q(u,\rho)|^2\,dx.\end{aligned}$$

By a direct calculation, we verify that

$$|\Delta Q(u,\rho)|^2 \leq C\left(|\Delta u|^2 + |\Delta\rho|^2 + |\nabla u|^2|\nabla\rho|^2 + |\nabla\rho|^4\right).$$

Therefore, by using (5.5) and (5.6) again, we obtain that

$$\frac{d}{dt}\int_\Omega |\nabla\Delta\rho|^2\,dx + b\int_\Omega |\Delta^2\rho|^2\,dx + 2d\int_\Omega |\nabla\Delta\rho|^2\,dx \leq C\|\Delta u\|_{L^2}^2 + p(\|\rho_0\|_{H^2}).$$

This together with the inequality obtained above then yields that

$$\frac{d}{dt}\int_\Omega \left(|\nabla u|^2 + |\nabla\Delta\rho|^2\right) dx + \delta\int_\Omega \left(|\nabla u|^2 + |\nabla\Delta\rho|^2\right) dx \leq p(\|\rho_0\|_{H^2}).$$

As a consequence, it follows that

$$\|u(t)\|_{H^1}^2 + \|\rho(t)\|_{\mathcal{H}^3}^2 \le Ce^{-\delta t}\left\{\|u_0\|_{H^1}^2 + \|\rho_0\|_{\mathcal{H}^3}^2\right\} + p(\|\rho_0\|_{H^2}), \quad 0 \le t \le T_{u,\rho}. \tag{5.7}$$

We have thus established the desired a priori estimate (5.1), namely (3.10).

As a consequence we conclude that for any pair of initial functions in the space

$$\begin{aligned} \mathcal{D}_{1/2;[0,1]} = \Bigg\{ \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} ;\ & u_0 \in H^1(\Omega) \text{ and } \rho_0 \in \mathcal{H}_N^3(\Omega) \\ & \text{with } 0 \le u_0 \le 1 \text{ and } 0 \le \rho_0 \le 1 \Bigg\}, \end{aligned} \tag{5.8}$$

the problem (1.1) possesses a unique global solution in the function space (4.10) in which $T_{u,\rho}$ can be any finite time, and $0 \le u(t) \le 1$ and $0 \le \rho(t) \le 1$ hold for every $0 < t < \infty$.

6. Exponential attractors

Following the strategy described in Section 3 we shall define a dynamical system from our problem (1.1) and shall construct exponential attractors.

As the global solution of (1.1) is constructed for every pair of initial functions in the space $\mathcal{D}_{1/2;[0,1]}$ (given by (5.8)) and the solution takes its values in the space, we can define a nonlinear semigroup $S(t)$ acting on $\mathcal{D}_{1/2;[0,1]}$. Furthermore we can verify the absorbing estimate (3.13). In fact, let $r > 0$ be any number. Let u_0 and ρ_0 be in $\mathcal{D}_{1/2;[0,1]}$ with $\|u_0\|_{H^1} + \|\rho_0\|_{\mathcal{H}^3} \le r$. Then from (5.3) there exits a time $t_r > 0$ such that $\|\rho(t)\|_{H^1} \le \widetilde{C} + 1$ for every $t \ge t_r$, where $\widetilde{C}$ is the constant appearing in (5.3). Then, applying (5.4) with initial time t_r and with initial functions $u(t_r)$ and $\rho(t_r)$, we observe that there exists a time $t_r' > t_r$ such that $\|u(t)\|_{L^2} + \|\rho(t)\|_{H^2} \le \widetilde{C} + 2$ for every $t \ge t_r'$. By the similar argument we observe from (5.7) that there exists a time $t_r'' > t_r'$ such that $\|u(t)\|_{H^1} + \|\rho(t)\|_{\mathcal{H}^3} \le \widetilde{C} + 3$ for every $t \ge t_r''$. We have therefore verified that

$$\begin{cases} \displaystyle\sup_{U_0 \in \mathcal{D}_{1/2;[0,1]}} \sup_{t \ge t_r''} \le \widetilde{C} + 3. \\ \|U_0\|_{\mathcal{D}_{1/2}} \le r \end{cases}$$

This then shows that the set

$$\mathcal{B} = \left\{ \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in \mathcal{D}_{1/2;[0,1]} ;\ \|u_0\|_{H^1} + \|\rho_0\|_{\mathcal{H}^3} \le \widetilde{C} + 3 \right\}$$

is an absorbing set of the semigroup $S(t)$.

The rest of arguments is quite the same as in Section 3. As $\mathcal{B}$ itself is absorbed by $\mathcal{B}$, there exists a time $t_{\mathcal{B}} > 0$ such that $S(t)\mathcal{B} \subset \mathcal{B}$ for every $t \ge t_{\mathcal{B}}$. Then $\mathcal{X} =$

$\bigcup_{0\le t\le t_{\mathcal{B}}} S(t)\mathcal{B}$ is an absorbing and positively invariant set of $S(t)$ which is also a compact set of the underlying space X (given by (4.1)). Hence, $(S(t), \mathcal{X}, X)$ is a dynamical system determined from (1.1) with compact phase space. In addition the compact perturbation property (3.1) and the Lipschitz condition (3.2) are verified if the phase space $\mathcal{X}$ is replaced by a new one of the form $\widetilde{\mathcal{X}} = S(T_{\mathcal{X}})$ with a suitable $T_{\mathcal{X}} > 0$. Thus we have accomplished construction of exponential attractors for $\big(S(t), \widetilde{\mathcal{X}}, X\big)$ and therefore for $(S(t), \mathcal{X}, X)$.

Theorem 6.1. *Let Ω be a two-dimensional bounded, convex or $\mathcal{C}^2$ domain. Then the dynamical system $(S(t), \mathcal{X}, X)$ defined from* (1.1) *possesses a family of exponential attractors.*

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