



Title	Strongly real 2-blocks and the Frobenius-Schur indicator
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Citation	Osaka Journal of Mathematics. 2006, 43(1), p. 201-213
Version Type	VoR
URL	https://doi.org/10.18910/8874
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STRONGLY REAL 2-BLOCKS AND THE FROBENIUS-SCHUR INDICATOR

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(Received August 17, 2004, revised November 8, 2004)

Abstract

Let G be a finite group, let k be an algebraically closed field of characteristic 2 and let $\Omega := \{g \in G \mid g^2 = 1_G\}$. It is shown that for a block B of kG , the permutation module $k\Omega$ has a B -composition factor if and only if the Frobenius-Schur indicator of the regular character of B is non-zero or equivalently if and only if B is real with a strongly real defect class.

1. Introduction

Let G be a finite group. In this paper we investigate the permutation module of G acting by conjugation on its involutions, over a field of characteristic 2. This develops the main theme of [10] and [8]. In the former paper G.R. Robinson considered the projective components of this module. In the latter paper the author showed that each such component is irreducible and self-dual and belongs to a 2-block of defect zero. Here we investigate which 2-blocks have a composition factor in the involution module. There are two apparently different ways of characterising such blocks. One method is local and uses the *defect classes* of the block. This gives rise to the definition of a *strongly real* 2-block. The other method is global and uses the Frobenius-Schur indicators of the irreducible characters in the block. Our main result is Theorem 2. The proof of this theorem requires Corollaries 4, 15, 18 and 20.

J.A. Green proved a number of results about p -blocks, using the observation that the group algebra of G is a module for the group $G \times G$. Here we shall exploit the additional fact that the group algebra is a module for the wreath product of G with a cyclic group of order 2. This was also an essential tool in [8].

Throughout this paper k will be an algebraically closed field of characteristic 2. There are various reasons why we limit ourselves to characteristic 2. Our wreath product group is an extension of $G \times G$ by a group of order 2. It is thus fairly uninteresting, from the point of view of blocks over a field of characteristic not equal to 2. In addition, the prime 2 is useful for studying the contragradient operator and real blocks, as pairing arguments of various kinds can be employed.

Recall also the following classical result. The Frobenius-Schur indicator $\nu(\chi)$ of a generalized character χ of G is the integer $(1/|G|) \sum_{g \in G} \chi(g^2)$. If χ is absolutely irreducible then $\nu(\chi) = 0$ or $1, -1$, depending on whether χ is not real-valued, or χ is real-valued and a χ -module affords a symmetric, respectively anti-symmetric non-degenerate G -invariant bilinear form. Then Frobenius and Schur proved that $|\Omega| = \sum \nu(\chi) \chi(1_G)$, where $\Omega := \{g \in G \mid g^2 = 1_G\}$.

The reader may be interested to know that in odd characteristic, the geometric type (quadratic or symplectic) of an irreducible self dual module is determined by the Frobenius-Schur indicator of a real valued character which contains the Brauer character of the module with odd multiplicity. It is an open problem as to whether there is an analogous Frobenius-Schur indicator in characteristic 2. See [11] for details.

A *component* of a module is a direct summand of the module that is indecomposable. Following Green, a 2-block of G is a component of kG , considered as a $G \times G$ -module in the usual way. For the rest of the paper we use B to denote a 2-block of G .

A *defect class* of B is a conjugacy class of G whose sum appears with non-zero multiplicity in the block idempotent 1_B , and on which the central character ω_B of B does not vanish. Defect classes are known to exist and to consist of elements of odd order.

The irreducible complex characters, Brauer characters and indecomposable modules of G are partitioned among its 2-blocks. We use $\text{Irr}(B)$, $\text{IBr}(B)$ and $\text{Pic}(B)$ to denote, respectively, the set of irreducible characters, the Brauer characters and the principal indecomposable characters of G that belong to B . We use ψ to indicate the irreducible Brauer character associated to $\Psi \in \text{Pic}(B)$. If M is a G -module, $M \downarrow_H$ denotes the restriction of M to $H \leq G$ and $M \uparrow^K$ denotes the induction of M to $K \geq G$. Identical notation applies to the restriction and induction of characters. See [9] for any additional unexplained notation.

The contragradient map o is defined by $(\sum \alpha_g g)^o = \sum \alpha_g g^{-1}$. It is a k -algebra involutory anti-automorphism of kG . A block B is said to be *real* if $B^o = B$. A conjugacy class C of G is said to be *real* if it coincides with the class C^o of the inverses of its elements. It is one of the main results of [5] that each 2-block has at least one defect class that is real.

A real conjugacy class of G is said to be *strongly real* if it is the trivial class or if its elements are inverted by involutions. This leads to the following key definition:

DEFINITION 1. A *strongly real* 2-block is a real 2-block that has a strongly real defect class.

It turns out that if B is strongly real then each of its real defect classes is strongly real. This was proved by Gow in [4]. Notice that the principal 2-block is strongly real; the identity class is a strongly real defect class.

We use 1_G both for the identity element of G and its trivial character. The set Ω consisting of the involutions in G , together with 1_G , forms a G -set under conjugation.

We denote the kG -module with permutation basis Ω by $k\Omega$. Our main result in this paper is:

Theorem 2. *Let B be a 2-block of G . Then the following are equivalent:*

- (i) $k\Omega$ has a B -composition factor;
- (ii) $\sum_{\chi \in \text{Irr}(B)} \nu(\chi)\chi(1_G) > 0$;
- (iii) B is strongly real.

Let g be an element of G . There is a unique decomposition of $\langle g \rangle$ into a direct product of a cyclic 2-group E and a cyclic $2'$ -group O . So $g = g_2 g_{2'} = g_{2'} g_2$, for some $g_2 \in E$ and $g_{2'} \in O$. We call g_2 the 2-part, and $g_{2'}$ the $2'$ -part, of g . Both are uniquely determined by g .

In our first lemma we compute the multiplicity of an irreducible kG -module as a composition factor of $k\Omega$.

Lemma 3. *Let \overline{P} be an irreducible kG -module, let P be the projective cover of \overline{P} , and let $\Phi \in \text{Pic}(G)$ be principal indecomposable character of P . Then \overline{P} occurs with multiplicity $\nu(\Phi)$ as a composition factor of $k\Omega$. In particular, $\nu(\Phi) \geq 0$.*

Proof. The number of solutions in G to the equation $x^2 = g$, for fixed $g \in G$, is given by $\sum_{\chi \in \text{Irr}(G)} \nu(\chi)\chi(g)$. Let $g \in G$ be 2-regular and let $x \in G$ satisfy $x^2 = g$. As x_2 and $x_{2'}$ commute, we have $g = x_2^2 x_{2'}^2 = x_{2'}^2 x_2^2$. So $x_2^2 = 1_G$ and $x_{2'}^2 = g_{2'}$. It follows that $x_2 \in \Omega(C_G(g))$, while $x_{2'} = g_{2'}^{1/2}$ is uniquely determined. Conversely, given any involution $t \in C_G(g)$, then $t g_{2'}^{1/2}$ is a solution to $x^2 = g$ in G . We conclude that $\sum_{\chi \in \text{Irr}(G)} \nu(\chi)\chi(g) = |\Omega \cap C_G(g)|$ coincides with the Brauer character of $k\Omega$. The lemma follows once we note that the virtual Brauer character of the restriction of the generalized character $\sum_{\chi \in \text{Irr}(G)} \nu(\chi)\chi$ to 2-regular elements is given by $\sum_{\Psi \in \text{Pic}(G)} \nu(\Psi)\Psi$. \square

Our Corollary shows that (i) \iff (ii) in Theorem 2.

Corollary 4. *The dimension of the sum of all submodules of $k\Omega$ that belong to B is given by $\sum_{\chi \in \text{Irr}(B)} \nu(\chi)\chi(1_G)$. In particular $\sum_{\chi \in \text{Irr}(B)} \nu(\chi)\chi(1_G) \geq 0$.*

Proof. Both statements follow from Lemma 3 and the fact that

$$\sum_{\chi \in \text{Irr}(B)} \chi(1_G)\chi = \sum_{\Psi \in \text{Pic}(B)} \psi(1_G)\Psi. \quad \square$$

Let \overline{P} , P and Φ be as in Lemma 3. Suppose that $t \in \Omega$. The Frobenius-Nakayama reciprocity formula [9, 3.1.27] shows that \overline{P} occurs with multiplicity $\langle \Phi_{C_G(t)}, 1_{C_G(t)} \rangle$ as a composition factor of $k_{C_G(t)} \uparrow^G$. Then, using the previous lemma, we get

(c.f. [10, Lemma 1])

$$\nu(\Phi) = \langle \Phi, 1_G \rangle + \sum_t \langle \Phi_{C_G(t)}, 1_{C_G(t)} \rangle,$$

where t ranges over a set of representatives for the conjugacy classes of involutions in G .

We now proceed to the main construction needed for the proof of Theorem 2.

The wreath product group $G \wr \Sigma$ is the semi-direct product of $G \times G$ with the symmetric group Σ on two symbols. Here the conjugation action of the involution $\sigma \in \Sigma$ on $G \times G$ is given by $(g_1, g_2)^\sigma = (g_2, g_1)$, for all $g_1, g_2 \in G$. We shall use the notations

$$\underline{g} := (g, g) \in G \wr \Sigma, \quad \text{for each } g \in G, \quad \text{and} \\ \underline{X} := \{\underline{x} \mid x \in X\} \subseteq G \wr \Sigma, \quad \text{for each } X \subseteq G.$$

We highlight the following crucial fact:

Lemma 5. *The centralizer of σ in $G \wr \Sigma$ is $\underline{G} \times \Sigma$.*

Let R be a commutative ring. Then the group algebra RG is a right $RG \wr \Sigma$ -module. For, RG is a $RG \times G$ -module via $x \cdot (g_1, g_2) = g_1^{-1}xg_2$, for each $x \in RG$ and $g_1, g_2 \in G$. The action of σ on RG is given by the contragradient involution $x \cdot \sigma = x^\sigma$. In more detail we have:

Lemma 6. *The $RG \wr \Sigma$ -module RG is isomorphic to the permutation module $(R_{\underline{G} \times \Sigma})^{\uparrow^{G \wr \Sigma}}$.*

Proof. The elements of G form an $RG \wr \Sigma$ -invariant basis of RG . Moreover if $g_1, g_2 \in G$, then $g_2 = g_1 \cdot (g_1, g_2)$. So G is a transitive $G \wr \Sigma$ -set. The stabilizer of $1_G \in RG$ in $G \wr \Sigma$ is $\underline{G} \times \Sigma$. The lemma follows from these facts. \square

Suppose that E is a block algebra of RG . Set $E^* := E + E^\sigma$. Then E^* is an $RG \wr \Sigma$ -submodule of RG . If $E \neq E^\sigma$, we have $E^* = E^{\uparrow^{G \wr \Sigma}}$. If $E = E^\sigma$, it is still useful to distinguish between the $RG \times G$ -module E and the $RG \wr \Sigma$ -module E^* , even though the underlying R -modules are the same.

Lemma 7. *Let E_1, \dots, E_r be the real blocks and $E_{r+1}, E_{r+1}^\sigma, \dots, E_{r+s}, E_{r+s}^\sigma$ be the nonreal blocks of RG . Then there is a unique indecomposable decomposition of RG as $RG \wr \Sigma$ -module:*

$$RG = E_1^* \oplus \dots \oplus E_{r+s}^*.$$

Proof. This follows from the indecomposable decomposition of RG into a direct sum of its blocks algebras, as $R(G \times G)$ -module. \square

As a particular case, consider when $R = \mathbb{C}$. Let $\chi \in \text{Irr}(G)$ and let M be a $\mathbb{C}G$ -module that affords χ . We use $E(\chi)$ to denote the corresponding Wedderburn component $\text{End}_{\mathbb{C}}(M)$ of $\mathbb{C}G$. Clearly $E(\chi)$ has $G \times G$ -character $\overline{\chi} \otimes \chi: (g_1, g_2) \rightarrow \chi(g_1^{-1})\chi(g_2)$, for $g_1, g_2 \in G$. Suppose now that $\chi = \overline{\chi}$ is real valued. Then $\overline{\chi} \otimes \chi = \chi \otimes \chi$ has two (irreducible) extensions to $G \wr \Sigma$. These will be denoted by χ^{+1} and χ^{-1} . Here if $\varepsilon \in \{\pm 1\}$ then $\chi^{\varepsilon}(g_1, g_2)\sigma = \varepsilon\chi(g_1g_2)$, for all $g_1, g_2 \in G$. When $\overline{\chi} \neq \chi$, the next lemma shows why it is useful to denote the induced $G \wr \Sigma$ -character $(\overline{\chi} \otimes \chi)\uparrow^{G \wr \Sigma}$ by χ^0 .

Lemma 8. *Let χ be an irreducible character of G and let $E(\chi)^*$ be the corresponding $G \wr \Sigma$ -component of $\mathbb{C}G$. Then $E(\chi)^*$ has character $\chi^{v(\chi)}$.*

Proof. This is obvious when $v(\chi) = 0$. So we may assume that $\chi = \overline{\chi}$. Then

$$\langle \chi^{\pm 1} \downarrow_{\underline{G} \times \Sigma}, 1_{\underline{G} \times \Sigma} \rangle = \frac{1}{2|G|} \sum_{g \in G} (\chi(g^{-1})\chi(g) \pm \chi(g^2)) = \frac{1 \pm v(\chi)}{2}.$$

The result now follows from Lemma 6 and Frobenius reciprocity. \square

Recall the following result [7, Theorem 1] of Green. A modern proof is [9, 5.10.8].

Lemma 9. *Let D be a defect group of B . Then B has vertex \underline{D} , as indecomposable $k(G \times G)$ -module.*

We use this to make a preliminary observation about the vertices of the component B^* of the $G \wr \Sigma$ -module kG . This will be refined in Proposition 14.

Lemma 10. *Let D be a defect group of B . If B is not real then \underline{D} is a vertex of B^* ; if B is real then there exists $e \in N_G(D)$, with $e^2 \in D$, such that $\underline{D}\langle e\sigma \rangle$ is a vertex of B^* .*

Proof. Suppose first that B is not real. So $B^* = B\uparrow^{G \wr \Sigma}$. It then follows from Lemma 9 that B^* has vertex \underline{D} .

Suppose then that B is real. Lemma 6 shows that B^* is $\underline{G} \times \Sigma$ -projective. We choose a vertex V of B^* so that $V \leq \underline{G} \times \Sigma$. Now B^* is a quasi-permutation module, $G \times G$ is a normal subgroup of $G \wr \Sigma$, and $B^*\downarrow_{G \times G} = B$ is indecomposable. A variant of Lemma 9.7 of [2] then implies that $V \cap (G \times G) = V \cap \underline{G}$ is a vertex of B . Using Lemma 9, we may choose D so that $V \cap \underline{G} = \underline{D}$. As $G \wr \Sigma / G \times G$ is

a 2-group, and as $B^* \downarrow_{G \times G}$ is indecomposable, Green's indecomposability theorem [6, Theorem 8], implies that $V \not\leq (G \times G)$. The last statement of the lemma follows from this. \square

It is easy to compute the stabilizer subgroup of an element g of the $G \wr \Sigma$ -set G in the group $\underline{G} \times \Sigma$. This subgroup will be denoted $C_{\underline{G} \times \Sigma}(g)$. We hope that the reader will not confuse this group with $C_{\underline{G} \times \Sigma}(\underline{g}) = C_G(g) \times \Sigma$.

Lemma 11. *Let $g \in G$. If g is not G -conjugate to g^{-1} , then $C_{\underline{G} \times \Sigma}(g) = C_G(g)$. If $g^t = g^{-1}$, for $t \in G$, then $C_{\underline{G} \times \Sigma}(g) = C_G(g) \langle \underline{t}\sigma \rangle$.*

Proof. These statements follow from the fact that $g \cdot \sigma = g^{-1}$. \square

For $H \leq G$ and M a kG -module, let M^H denote the sum of all the trivial H -submodules of $M \downarrow_H$. The relative trace map $\text{Tr}_H^G: M^H \rightarrow M^G$ is defined by $\text{Tr}_H^G(m) = \sum m \cdot g$, for $m \in M^H$. Here g ranges over any set of representatives for the right cosets of H in G . We write $\text{Tr}_H^G(M)$, instead of $\text{Tr}_H^G(M^H)$, for the image of the trace map on M^H .

The reader is warned that B^* is generally not a $\underline{G} \times \Sigma$ -algebra, in the sense of Green [7]. In particular, there is no Mackey-type decomposition of a product of the form $\text{Tr}_X^{\underline{G} \times \Sigma}(a) \text{Tr}_Y^{\underline{G} \times \Sigma}(b)$, for $X, Y \leq \underline{G} \times \Sigma$. However, we do have the following useful result.

Lemma 12. *Let A be a k -algebra and a kG -module such that each element of G acts on A as a k -algebra automorphism or as a k -algebra anti-automorphism. Suppose also that A^G is contained in the centre $Z(A)$ of A . Then A^G is a subalgebra of $Z(A)$. Also $\text{Tr}_H^G(A)$ is an ideal of A^G , for each $H \leq G$.*

Proof. Write the G -action on A in exponential form. It is obvious that $\text{Tr}_H^G(A)$ is a k -subspace of A . Let $a \in A$, $z \in A^G$ and $g \in G$. Suppose that g acts as a k -algebra anti-automorphism. Then $a^g z = (z^{g^{-1}} a)^g = (za)^g = (az)^g$. Similarly $a^g z = (az)^g$, if g acts as a k -algebra automorphism. It follows that the map $a \rightarrow az$ is a kG -endomorphism of A . In particular, if $a \in A^H$, then $\text{Tr}_H^G(a)z = \text{Tr}_H^G(az)$. Taking $H = G$, we get that A^G is a subalgebra of $Z(A)$. More generally, we can conclude that $\text{Tr}_H^G(A)$ is an ideal of A^G . \square

We will apply this Lemma to the algebra B^* and the group $\underline{G} \times \Sigma$. Denote by $Z^*(kG)$ the σ -fixed point subalgebra of $Z(kG)$. It has k -basis $\{(C \cup C^o)^+\}$, where C ranges over the conjugacy classes of G . Note that $Z(kG) = kG^{\underline{G}}$ and $Z^*(kG) = kG^{\underline{G} \times \Sigma}$.

Corollary 13. *Let P be a 2-subgroup of G and let $q \in N_G(P)$ with $q^2 \in P$. Then*

- (i) $\text{Tr}_{\underline{P}}^{\underline{G} \times \Sigma}(kG)$ is an ideal of $Z^*(kG)$ with k -basis $\{(X \cup X^o)^+\}$. Here X ranges over the set of non-real conjugacy classes of G such that P contains a Sylow 2-subgroup of $C_G(x)$, for some $x \in X$.
- (ii) $\text{Tr}_{\underline{P}\langle q\sigma \rangle}^{\underline{G} \times \Sigma}(kG)$ is an ideal of $Z^*(kG)$ with k -basis $\{(X \cup X^o)^+\} \cup \{Y^+\}$. Here X has the same meaning as in (i), while Y ranges over the set of real conjugacy classes of G such that P contains a Sylow 2-subgroup of $C_G(y)$, and $y^{pq} = y^{-1}$, for some $y \in Y$ and $p \in P$.

Proof. Lemma 12 implies that both $\text{Tr}_{\underline{P}}^{\underline{G} \times \Sigma}(kG)$ and $\text{Tr}_{\underline{P}\langle q\sigma \rangle}^{\underline{G} \times \Sigma}(kG)$ are ideals of $Z^*(kG)$.

In general, suppose that G is a finite group, H is a subgroup of G , and M is a permutation kG -module. Then it is well known that the k -space $\text{Tr}_H^G(M)$ has basis of the form $\{O^+\}$. Here O ranges over the G -orbits on the permutation basis such that H contains a Sylow 2-subgroup of the stabilizer subgroup of some element of O in G . The Corollary follows by applying this, and Lemma 11, to the group $\underline{G} \times \Sigma$, its subgroups \underline{P} and $\underline{P}\langle q\sigma \rangle$ and the module kG . \square

If g is an element of G , its extended centraliser is the following subgroup of G :

$$C_G^*(g) := \{x \in G \mid g^x = g \text{ or } g^{-1}\}.$$

We can now identify the vertices of B^* .

Proposition 14. *Suppose that B is real. Then B has a real defect class. Let $c \in G$ belong to a real defect class of B , let D be a Sylow 2-subgroup of $C_G(c)$ and let $D\langle e \rangle$ be a Sylow 2-subgroup of $C_G^*(c)$. Then $1_B \in \text{Tr}_E^{\underline{G} \times \Sigma}(kG)$, for $E \leq \underline{G} \times \Sigma$ if and only if $\underline{D}\langle e\sigma \rangle \leq_{\underline{G}} E$. Also $\underline{D}\langle e\sigma \rangle$ is a vertex of B^* .*

Proof. To show that B has a real defect class, we repeat the original argument of Gow, from [4, Lemma 1.2], for the convenience of the reader. Write $1_B = \sum \lambda_K K^+$, where K runs over the conjugacy classes of G and $\lambda_K \in k$, for each class K . Then

$$1_k = \omega_B(1_B) = \sum \lambda_K \omega_B(K^+).$$

Now $\lambda_K = \lambda_{K^o}$ and $\omega_B(K^+) = \omega_B(K^{o+})$, as B is real. It follows that the contribution of a nonreal class K and its inverse class K^o to the above sum is $2\lambda_K \omega_B(K^+) = 0_k$. So there must exist a real class C such that $\lambda_C \omega_B(C^+) \neq 0_k$. Each such C is a real defect class of B .

Now fix a real defect class C of B . So $\lambda_C \neq 0_k$, using the notation of the previous paragraph. Suppose that $1_B \in \text{Tr}_E^{\underline{G} \times \Sigma}(kG)$, where $E \leq \underline{G} \times \Sigma$. Then Corollary 13

implies that there exists $c \in C$ and D a Sylow 2-subgroup of $C_G(c)$ and $D\langle e \rangle$ a Sylow 2-subgroup of $C_G^*(c)$, such that $\underline{D}\langle e\sigma \rangle \leq E$.

Mackey's Theorem implies that $(B^*)^{\underline{G} \times \Sigma} \subseteq \sum \text{Tr}_{P \cap (\underline{G} \times \Sigma)}^{\underline{G} \times \Sigma}(kG)$. Here P ranges over the vertices of B^* . Corollary 13 implies that each subspace $\text{Tr}_{P \cap (\underline{G} \times \Sigma)}^{\underline{G} \times \Sigma}(kG)$ is an ideal of $Z^*(kG)$. But 1_B is a primitive idempotent in $Z^*(kG)$. So by Rosenberg's lemma [9, 5.1.1], there exists a vertex V of B^* such that $1_B \in \text{Tr}_{V \cap \underline{G} \times \Sigma}^{\underline{G} \times \Sigma}(kG)$.

The last two paragraphs imply that $\underline{D}\langle e\sigma \rangle \leq_G V$. But $|\underline{D}\langle e\sigma \rangle| = |V|$, as a consequence of Lemma 10. It follows that $1_B \in \text{Tr}_{\underline{D}\langle e\sigma \rangle}^{\underline{G} \times \Sigma}(kG)$, and also that $\underline{D}\langle e\sigma \rangle$ is a vertex of B^* . \square

Our Corollary shows that (ii) \implies (iii) in Theorem 2.

Corollary 15. *Suppose that there exists $g \in G$ such that $\sum_{\chi \in \text{Irr}(B)} \nu(\chi)\chi(g) > 0$. Then B is real. Let $\underline{D}\langle e\sigma \rangle$ be a vertex of B^* . Then there exists $d \in D$ such that g_2 is G -conjugate to $(de)^2$. In particular, if g can be chosen to be 2-regular, then B is strongly real.*

Proof. When B is the principal 2-block of G , the result is true. So assume otherwise. The hypothesis implies that B is real, as it forces $\nu(\chi) \neq 0$, for some $\chi \in \text{Irr}(B)$.

Let (R, F, k) be a 2-modular system for G . Suppose that \hat{B} is the block algebra of RG such that $B = \hat{B}/J(R)\hat{B}$. Then Lemma 8 shows that the $G \wr \Sigma$ -character of \hat{B}^* is $\chi_B := \sum_{\chi \in \text{Irr}(B)} \chi^{\nu(\chi)}$. Now \hat{B}^* and B^* have the same vertices, as both are trivial source modules [9, 4.8.9]. So $\underline{D}\langle e\sigma \rangle$ is a vertex of \hat{B}^* . As $\chi_B((1_G, g)\sigma) = \sum_{\chi \in \text{Irr}(B)} \nu(\chi)\chi(g)$, the hypothesis is that $\chi_B((1_G, g)\sigma) \neq 0$. It then follows from a theorem of Green [9, 4.7.4] that the 2-part of $(1_G, g)\sigma$ is contained in a vertex of \hat{B}^* . But $((1_G, g)\sigma)_2 = (1_G, g_2)(g_2^{-1/2}, g_2^{1/2})\sigma$ and $((g_2^{-1/2}, g_2^{1/2})\sigma)^{(1_G, g_2')} = \sigma$. In particular $((1_G, g)\sigma)_2$ is $G \wr \Sigma$ -conjugate to $(1_G, g_2)\sigma$. So there exists $g_1, g_2 \in G$ and $d \in D$ such that $(1_G, g_2)\sigma = (\underline{d}e\sigma)^{(g_1, g_2)} = (g_1^{-1}deg_2, g_2^{-1}deg_1)\sigma$. This gives $g_2^{-1} = g_1^{-1}de$, and hence also $g_2 = [(de)^2]^{g_1}$.

Suppose that $g_2 = 1_G$. Then $(de)^2 = 1_G$. So de is an involution that belongs to $D\langle e \rangle \setminus D$. Then, using Proposition 14, we see that each real defect class of B is strongly real, whence B is strongly real. \square

Let K be a field and let τ be a field automorphism of K . Suppose that γ is a K -representation of G . Then we may form the representation γ^τ of G by applying τ to the matrix entries in $\gamma(g)$, for each $g \in G$. If M is the KG -module corresponding to γ , we let M^τ denote the KG -module corresponding to γ^τ . This construction also applies if τ is an automorphism of a subfield K_0 of K , and γ is realisable over K_0 .

We use this to define the Frobenius twist of a module or character. The Frobenius automorphism Fr of k is given by $\lambda \rightarrow \lambda^2$, for $\lambda \in k$. Every \mathbb{C} -representation of G can be realized over $\mathbb{Q}(\zeta)$, where ζ is a primitive $|G|^{th}$ root of unity. There is a Galois

automorphism Fr of $\mathbb{Q}(\zeta)$ given by $\zeta \rightarrow \zeta_2 \zeta_2^2$, for $\zeta \in \mathbb{Q}(\zeta)$. If M is a G -module, with Brauer or ordinary character χ , the Frobenius twist module M^{Fr} has character $\chi^{\text{Fr}}: g \rightarrow \chi(g_2^2 g_2)$, for each g in the domain of definition of χ .

Note that $\{\chi^{\text{Fr}} \mid \chi \in \text{Irr}(B)\}$ is the set of irreducible characters in a 2-block B^{Fr} of G . If M is an indecomposable kG -module, then M belongs to B if and only if M^{Fr} belongs to B^{Fr} .

We identify G and \underline{G} and let \underline{B} be the 2-block of \underline{G} corresponding to B . There is a unique 2-block of $\underline{G} \times \Sigma$ that covers \underline{B} . We denote this block by $\underline{B} \times \Sigma$. Clearly $(\underline{B} \times \Sigma)^{\text{Fr}} = \underline{B}^{\text{Fr}} \times \Sigma$.

Lemma 16. *The Brauer induced block $(\underline{B}^{\text{Fr}} \times \Sigma)^{G \wr \Sigma}$ is defined. It is the unique 2-block of $G \wr \Sigma$ that covers the block $B \otimes B$ of $G \times G$.*

Proof. Let D be a defect group of B . Then $\underline{D} \times \Sigma$ is a defect group of $\underline{B}^{\text{Fr}} \times \Sigma$. Since $C_{G \wr \Sigma}(\underline{D} \times \Sigma) = \underline{C}_G(D) \times \Sigma$ is contained in $\underline{G} \times \Sigma$, the induced block $(\underline{B}^{\text{Fr}} \times \Sigma)^{G \wr \Sigma}$ is defined [9, 5.3.6].

Let $B^{\otimes 2}$ be the unique 2-block of $G \wr \Sigma$ that covers the 2-block $B \otimes B$ of $G \times G$. So χ^{+1} belongs to $B^{\otimes 2}$, whenever $\chi \in \text{Irr}(B)$.

Now $C_{G \wr \Sigma}(\sigma) = \underline{G} \times \Sigma$. Each Brauer character of $\underline{G} \times \Sigma$ can be identified with a Brauer character of \underline{G} . Using Brauer's second main theorem [9, 5.4.2], we have

$$(1) \quad \chi^{+1}(\underline{g}\sigma) = \sum_{\theta \in \text{IBr}(\underline{G})} d_{\chi, \theta}^{\sigma} \theta(g), \quad \text{for all } g \in G \text{ of odd order,}$$

where the $d_{\chi, \theta}^{\sigma}$ are algebraic integers with the property that $d_{\chi, \theta}^{\sigma} = 0$, unless θ belongs to a 2-block B_1 of G such that $(\underline{B}_1 \times \Sigma)^{G \wr \Sigma} = B^{\otimes 2}$. On the other hand, the definition gives

$$(2) \quad \chi^{+1}(\underline{g}\sigma) = \chi(g^2) = \chi^{\text{Fr}}(g) = \sum_{\theta \in \text{IBr}(\underline{G})} d_{\chi^{\text{Fr}}, \theta} \theta(g), \quad \text{for all } g \in G \text{ of odd order.}$$

But the irreducible Brauer characters of \underline{G} are linearly independent on the 2-regular classes of \underline{G} . So (1) and (2) imply that $d_{\chi, \theta}^{\sigma} = d_{\chi^{\text{Fr}}, \theta}$, for all $\theta \in \text{IBr}(\underline{G})$. As $d_{\chi^{\text{Fr}}, \theta} \neq 0$, for some $\theta \in \text{IBr}(B^{\text{Fr}})$, we conclude that $(\underline{B}^{\text{Fr}} \times \Sigma)^{G \wr \Sigma} = B^{\otimes 2}$. \square

The following lemma is a key step in the proof of Theorem 2.

Lemma 17. *Restriction $\downarrow_{\underline{G}}^{G \times \Sigma}$ establishes a multiplicity preserving bijection between the components of $B^* \downarrow_{\underline{G} \times \Sigma}^{G \times \Sigma}$ that have a vertex containing Σ and the components of $k\Omega$ that belong to $\underline{B}^{\text{Fr}}$.*

Proof. Let M be a component of $B^* \downarrow_{\underline{G} \times \Sigma}$ that has a vertex V containing Σ . Then Σ is contained in the kernel of M , as M is a trivial source module. Thus M coincides with the inflation of the indecomposable \underline{G} -module $M \downarrow_{\underline{G}}$ to $\underline{G} \times \Sigma$. In addition, $M \downarrow_{\underline{G}}$ has a vertex V/Σ .

The orbits of $\underline{G} \times \Sigma$ on the $G \wr \Sigma$ -set G are $\{C \cup C^o \mid C \text{ a conjugacy class of } G\}$. Lemma 6 and Mackey's theorem imply that

$$kG \downarrow_{\underline{G} \times \Sigma} = \sum_{C, C^o} k(C \cup C^o).$$

Also $k(C \cup C^o) = k_{C_{\underline{G} \times \Sigma}(c)} \uparrow^{\underline{G} \times \Sigma}$, for each $c \in C \cup C^o$. Now Lemma 11 implies that $\Sigma \leq C_{\underline{G} \times \Sigma}(c)$ if and only if $c \in \Omega$. Then by the Krull-Schmidt theorem M is a component of $k_{C_{\underline{G} \times \Sigma}(t)} \uparrow^{\underline{G} \times \Sigma}$, for some $t \in \Omega$. But $(k_{C_{\underline{G} \times \Sigma}(t)} \uparrow^{\underline{G} \times \Sigma}) \downarrow_{\underline{G}} = k_{\underline{C}_G(t)} \uparrow^{\underline{G}}$. We conclude that $M \downarrow_{\underline{G}}$ is a component of $k\Omega$.

Let B_1 be the 2-block of G such that $B_1^{\text{Fr}} \times \Sigma$ contains M . As $\Sigma \leq V$, Lemma 5 forces $C_{G \wr \Sigma}(V) \leq \underline{G} \times \Sigma$. So by a Theorem of Nagao-Green [9, 5.3.12], the induced block $(B_1^{\text{Fr}} \times \Sigma)^{G \wr \Sigma}$ contains B^* . But $(B_1^{\text{Fr}} \times \Sigma)^{G \wr \Sigma} = B_1^{\otimes 2}$, by Lemma 16. So B^* belongs to $B_1^{\otimes 2}$. This forces the $G \times G$ -module B to belong to $B_1 \otimes B_1$. It follows easily that $B = B_1$.

We can reverse the above argument to show that if N is a B^{Fr} -component of $k\Omega$, then the inflation of N to $\underline{G} \times \Sigma$ is a component of $B^* \downarrow_{\underline{G} \times \Sigma}$ that has a vertex containing Σ . \square

Corollaries 4 and 15 give the implication (i) \implies (iii) of Theorem 2. Our next result gives a direct proof of this and also produces some information on the vertices of the components of $k\Omega$.

Corollary 18. *Suppose that some component of $k\Omega$ belongs to B . Then B is strongly real. More precisely, let $c \in G$ belong to a real defect class of B , let D be a Sylow 2-subgroup of $C_G(c)$, and let E be a Sylow 2-subgroup of $C_G^*(c)$ that contains D . Suppose that N is a component of $k\Omega$ that belongs to B . Then there exists $t \in \Omega \cap (E \setminus D)$ such that N has a vertex $V \leq C_D(t)$.*

Proof. Clearly N^{Fr} is also a component of $k\Omega$. Also \underline{V} is a vertex of N^{Fr} , as \underline{G} -module. Let M be the inflation of N^{Fr} to $\underline{G} \times \Sigma$. Then Lemma 17 implies that M is a component of $B^* \downarrow_{\underline{G} \times \Sigma}$. Now M has vertex $\underline{V} \times \Sigma$. As M is a component of $B^* \downarrow_{\underline{G} \times \Sigma}$, it follows that some vertex of B^* contains $\underline{V} \times \Sigma$.

We established in Proposition 14 that $\underline{D} \langle \underline{e}\sigma \rangle$ is a vertex of B^* . Then by the previous paragraph there exists $(g_1, g_2) \in G \times G$ such that $(\underline{V} \times \Sigma)^{(g_1, g_2)} \leq \underline{D} \langle \underline{e}\sigma \rangle$. In particular $\sigma^{(g_1, g_2)} = (g_1^{-1} g_2, (g_1^{-1} g_2)^{-1}) \sigma$ belongs to $\underline{D} \langle \underline{e}\sigma \rangle$. Choose $d \in D$ such that $de = g_1^{-1} g_2$. Then $t := de$ belongs to Ω . So $E = D \langle t \rangle$ splits over D . In particular B is strongly real. Also $\underline{V}^{(g_1, g_2)} \leq \underline{C}_D(\underline{t}\sigma)$. So $V \leq_G C_D(t)$. \square

An important special case is that of a real 2-block of defect zero. The next result was proved by R. Gow (in unpublished work), unifying results in [4] and [10]. Gow's proof used a pairing argument on L.L. Scott's 'orbital characters' defined with respect to the involution module. Our proof uses Alperin-Scott modules.

Theorem 19. *Suppose that B is a real 2-block of defect 0. Let χ be the unique irreducible character in B . Then $\nu(\chi) = +1$. Let C be a real defect class of B and let $t \in \Omega$. Then $\langle \chi \downarrow_{C_G(t)}, 1_{C_G(t)} \rangle = 1$ or 0, depending on whether or not t inverts an element of C .*

Proof. Note that χ is the unique principal indecomposable character in B . The fact that $\nu(\chi) = +1$ appears in [10] and independently in several other places. Here it is a simple consequence of Lemma 3. It also follows from that lemma that the G -character of $\mathbb{C}\Omega$ contains χ with multiplicity 1.

Let $c \in C$ and let $t \in \Omega$ be such that $c^t = c^{-1}$. The proof will be completed by showing that $\langle \chi \downarrow_{C_G(t)}, 1_{C_G(t)} \rangle \neq 0$.

Now $Z(B^*) = (B^*)^{\underline{G} \times \Sigma}$ is spanned by 1_B . Also B^* has a trivial source. So $B^* \downarrow_{\underline{G} \times \Sigma}$ has a unique Scott component, and this component has socle spanned by 1_B . Proposition 14 implies that $1_B \in \text{Tr}_E^{\underline{G} \times \Sigma}(kG)$, for $E \leq \underline{G} \times \Sigma$ if and only if $\langle \underline{t}\sigma \rangle \leq_{\underline{G}} E$. It follows that the Scott component of $B^* \downarrow_{\underline{G} \times \Sigma}$ has vertex $\langle \underline{t}\sigma \rangle$.

An easy calculation shows that $\underline{C}_G(t) \times \Sigma$ coincides with the normalizer of $\langle \underline{t}\sigma \rangle$ in both $C_G(t) \wr \Sigma$ and $\underline{G} \times \Sigma$. The Green correspondence preserves Scott modules. Then, using a result of D. Burry [9, 4.4.7], and the previous paragraph, the Scott module $S(C_G(t) \wr \Sigma; \langle \underline{t}\sigma \rangle)$ has multiplicity 1 as a component of $B^* \downarrow_{C_G(t) \wr \Sigma}$. The restriction of $S(C_G(t) \wr \Sigma; \langle \underline{t}\sigma \rangle)$ to $C_G(t) \times C_G(t)$ has a projective Scott component. We deduce that $B \downarrow_{C_G(t) \times C_G(t)}$ has a Scott component.

Let \hat{B} be a lift of B to a $G \times G$ -module over a field of characteristic 0. Then \hat{B} has character $\overline{\chi} \otimes \chi$. Thus $\langle (\overline{\chi} \otimes \chi) \downarrow_{C_G(t) \times C_G(t)}, 1_{C_G(t) \times C_G(t)} \rangle$ is the number of Scott components of $B \downarrow_{C_G(t) \times C_G(t)}$. But this inner product is

$$\sum_{c_1, c_2 \in C_G(t)} \chi(c_1^{-1}) \chi(c_2) = \langle \chi \downarrow_{C_G(t)}, 1_{C_G(t)} \rangle^2.$$

It then follows from the previous paragraph that $\langle \chi \downarrow_{C_G(t)}, 1_{C_G(t)} \rangle \neq 0$. □

We can now prove that (iii) \implies (ii) in Theorem 2. This completes the proof of that theorem.

Corollary 20. *Suppose that B is strongly real. Then $\sum_{\chi \in \text{Irr}(B)} \nu(\chi) \chi(1_G) > 0$.*

Proof. Let $c \in G$ belong to a real defect class C of B . Then c is strongly real. Fix a Sylow 2-subgroup D of $C_G(c)$ and a Sylow 2-subgroup E of $C_G^*(c)$ that contains

D. Write $E = D\langle t \rangle$, where $t \in \Omega$.

We need a version of Brauer's first main theorem. Let N be the normalizer of D in G and let b be the Brauer correspondent of B with respect to (G, N, D) . Now b is a real 2-block of N , as $(b^o)^G = (b^G)^o = B$, and b is the unique block of kN such that $b^G = B$. If $1_B = \sum_K \lambda_K K^+$, then $1_b = \sum_K \lambda_K (K \cap C_G(D))^+$. Here K ranges over the classes of G . Also $\omega_B(K^+) = \omega_b((K \cap C_G(D))^+)$, for each conjugacy class K of G . It follows that $C_1 := C \cap C_G(D)$ is a real defect class of b .

Set $\bar{N} := N/D$ and let μ be the natural k -algebra projection $kN \rightarrow k\bar{N}$. Let \bar{C} be the conjugacy class of \bar{N} that contains $\bar{c} = Dc$. Then $\mu(C_1^+) = \bar{C}^+$, by [9, 5.8.9]. It follows that $\mu(1_b) \neq 0$. Write $\mu(1_b) = \sum_{i=1}^s 1_{\beta_i}$, where β_1, \dots, β_s are distinct blocks of \bar{N} . As $1_{b^o} = 1_b$, it follows that there is a permutation τ of $\{1, \dots, s\}$ such that $\beta_i^o = \beta_{i\tau}$, for $i = 1, \dots, s$. If $i \neq i\tau$, an easy argument shows that $1_{\beta_i} + 1_{\beta_{i\tau}}$ is supported on the non-real classes of \bar{N} . But \bar{C} is a real class of \bar{N} whose sum appears with non-zero multiplicity in $\mu(1_b)$. We deduce that there exists i such that β_i is a real 2-block of \bar{N} and \bar{C}^+ appears with non-zero multiplicity in 1_{β_i} . Set $\beta := \beta_i$.

Now β has a trivial defect group, by [9, 5.8.7 (ii)], and $C_{\bar{N}}(\bar{c})$ is odd, by [9, 5.8.9 (ii)]. It follows that \bar{C} is a real defect class of β . Let χ be the unique irreducible character in β . Now \bar{t} is an involution in \bar{N} that inverts an element of \bar{C} . So by Theorem 19 we have $\langle \chi_{C_{\bar{N}}(\bar{t})}, 1_{C_{\bar{N}}(\bar{t})} \rangle = 1$. The preimage of $C_{\bar{N}}(\bar{t})$ in N is $C_N(Dt) := \{n \in N \mid t^n \in Dt\}$. Inflating χ to N , we get $\langle \chi_{C_N(Dt)}, 1_{C_N(Dt)} \rangle = 1$. But $C_N(t) \leq C_N(Dt)$. So $\langle \chi_{C_N(t)}, 1_{C_N(t)} \rangle \neq 0$. Let M be the unique irreducible β -module. Then we have just shown that M is a b -composition factor of $k_{C_N(t)} \uparrow^N$. We deduce from Lemma 17 that $b^* \downarrow_{N \times \Sigma}$ has a component with a vertex that contains Σ .

J.L. Alperin proved in [1] that b is a component of $B \downarrow_{N \times N}$. So b^* is a component of $B^* \downarrow_{N \times N}$. This and the previous paragraph show that $B^* \downarrow_{N \times \Sigma}$ has a component with a vertex that contains Σ . Applying Lemma 17, we deduce that $k\Omega$ has a B -composition factor. We conclude from Corollary 4 that $\sum_{\chi \in \text{Irr}(B)} \nu(\chi) \chi(1_G) > 0$. \square

We conclude our paper with a small application of Theorem 2. R. Gow proved the following result in [3, 5.6]:

Proposition 21. *Let B be a real 2-block, let $c \in G$ belong to a real defect class of B , let D be a Sylow 2-subgroup of $C_G(c)$ and let $D\langle e \rangle$ be a Sylow 2-subgroup of $C_G^*(c)$. Then B contains a real-valued irreducible character of height 0 and Frobenius-Schur indicator -1 if and only if $D\langle e \rangle/D'$ does not split over D/D' .*

It is known that each real 2-block has a real-valued irreducible character of Frobenius-Schur indicator $+1$. So Theorem 2 and Proposition 21 combine to give:

Corollary 22. *Let B , D and $D\langle e \rangle$ be as in Proposition 21. Suppose that $D\langle e \rangle/D'$ splits over D/D' but $D\langle e \rangle$ does not split over D . Then B contains a real-valued irreducible character of height greater than 0 and Frobenius-Schur indicator -1 .*

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