REGULAR SUBRINGS OF A POLYNOMIAL RING, II

Dedicated to Professor Yozō Matsushima on his sixtieth birthday

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Introduction. This is a continuation of the previous work of the author's [7] on a finitely generated, two-dimensional, regular subring contained in a polynomial ring. Let \( k \) be an algebraically closed field of characteristic zero, which we fix as the ground field throughout this article. Let \( X = \text{Spec}(A) \) be a nonsingular affine surface defined over \( k \). An \( \mathbb{A}^1 \)-fibration on \( X \) over a curve \( Y \) is a surjective morphism \( \rho: X \to Y \) from \( X \) to a nonsingular curve \( Y \) whose general fibers are isomorphic to the affine line \( \mathbb{A}^1 \). It is known that every fiber of \( \rho \) is supported by a disjoint union of irreducible components, each of which is isomorphic to \( \mathbb{A}^1 \) (cf. [7]). Let \( F = \rho^*(P) \) be a fiber of \( \rho \) lying over a point \( P \) of \( Y \), and write \( F = \sum_{i=1}^{s} n_i C_i \), where \( C_i \) is isomorphic to \( \mathbb{A}^1 \) and \( n_i > 0 \) for every \( i \). We say that \( F \) is a singular fiber of the first kind (resp. the second kind) if \( s \geq 2 \) and \( n_i = 1 \) for some \( i \) (resp. \( n_i \geq 2 \) for every \( i \)). We also say that \( F \) is a multiple fiber of multiplicity \( \mu \) if \( \mu = \gcd(n_1, \ldots, n_s) > 1 \).

Let \( R := k[u_1, \ldots, u_r] \) be a polynomial ring of dimension \( r \) over \( k \), and let \( A \) be a finitely generated, two-dimensional, regular \( k \)-subalgebra of \( R \). Let \( X := \text{Spec}(A) \), which is a nonsingular affine rational surface. We know that the group \( A^* \) of invertible elements of \( A \) coincides with \( k^* := k^{\times} \) (0), that \( X \) has logarithmic Kodaira dimension \( \kappa(X) = -\infty \), and that \( A \) is isomorphic to a polynomial ring of dimension 2 over \( k \) provided \( A \) is a unique factorization domain (cf. [7]). The condition that \( \kappa(X) = -\infty \) implies that there exists an \( \mathbb{A}^1 \)-fibration \( \rho: X \to Y \) over a nonsingular curve \( Y \) (cf. Miyanishi-Sugie [8], Fujita [2]). In the present case, since \( X \) is dominated by the affine \( r \)-space \( \mathbb{A}^r \) := \text{Spec}(R), \( Y \) is isomorphic to \( \mathbb{A}^1 \) or the projective line \( \mathbb{P}^1 \).

The purpose of this paper is to study the converse: When is a nonsingular affine surface \( X \) with an \( \mathbb{A}^1 \)-fibration \( \rho \) over \( \mathbb{A}^1 \) or \( \mathbb{P}^1 \) dominated by \( \mathbb{A}^r \) (\( r \geq 2 \))? If \( X = \text{Spec}(A) \) has an \( \mathbb{A}^1 \)-fibration over \( \mathbb{A}^1 \), we can give the following criterion (Theorem 3.3):

\( X \) is dominated by \( \mathbb{A}^1 \), that is, \( A \) is contained in \( R \) as a \( k \)-subalgebra, if and only if \( \rho \) has at most one singular fiber of the second kind.
This is done by solving a Diophantine equation in \( k[u_1, \ldots, u_r] \) (Theorem 1.2). Meanwhile, if \( X = \text{Spec}(A) \) has an \( A^1 \)-fibration over \( \mathbb{P}^1 \), the situation becomes very much complicated. Namely, in order to discuss the embeddability of \( A \) into \( k[u_1, \ldots, u_r] \) in full generality, we have to know what the solutions of the following Diophantine equation in \( k[u_1, \ldots, u_r] \) look like:

\[
x_1^{a_1} \cdots x_i^{a_i} + y_1^{b_1} \cdots y_j^{b_j} + z_1^{c_1} \cdots z_s^{c_s} = 0,
\]

where \( a_i \geq 2, b_j \geq 2, c_s \geq 2 \) for every index \( i \) (\( 1 \leq i \leq l \)), \( j \) (\( 1 \leq j \leq m \)), \( s \) (\( 1 \leq s \leq n \)). We only give partial answers to the embeddability problem in terms of multiple fibers of \( \rho \), which are stated as follows:

1. Assume that \( A \) is contained in \( R \) as a \( k \)-subalgebra. Then the fibration \( \rho \) has at most three multiple fibers. If \( \rho \) has three multiple fibers, their multiplicities \( \{\mu_1, \mu_2, \mu_3\} \) are given, up to permutation, by one of the following triplets: \( \{2, 2, n\} \) (\( n \geq 2 \)), \( \{2, 3, 3\}, \{2, 3, 4\}, \{2, 3, 5\} \) (cf. Theorem 3.5).

2. Assume, conversely, that \( \rho \) satisfies the following two conditions:
   i. \( \rho \) has no singular fibers of the second kind except at most three multiple fibers, each of which is supported by a single irreducible component;
   ii. if \( \rho \) has three multiple fibers, the set of multiplicities \( \{\mu_1, \mu_2, \mu_3\} \) is, up to permutation, one of the triplets given in the assertion (1).

Then \( A \) is contained in a polynomial ring as a \( k \)-subalgebra (cf. Theorem 3.7).

In order to obtain these results, we consider an affine hypersurface \( S_{p_1,p_2,p_3} \) in \( \mathbb{A}^3 = \text{Spec}(k[x_1, x_2, x_3]) \) defined by an equation

\[
x_1^{p_1} + x_2^{p_2} + x_3^{p_3} = 0 \quad \text{with} \quad p_1, p_2, p_3 \geq 2,
\]

and also a complete intersection \( \Sigma_{p_1,p_2,p_3} \) in \( \mathbb{A}^4 = \text{Spec}(k[x_1, x_2, x_3, x_4]) \) defined by equations

\[
x_1^{p_1} + x_2^{p_2} + x_3^{p_3} + x_4^{p_4} = 0 \quad \text{and} \quad ax_1^{p_1} + x_2^{p_2} + x_4^{p_4} = 0
\]

with \( p_1, p_2, p_3, p_4 \geq 2 \) and \( a \in k - \{0, 1\} \). Indeed, we have to compute \( \kappa(S_{p_1,p_2,p_3}^*) \), where \( S_{p_1,p_2,p_3}^* = S_{p_1,p_2,p_3} - (0) \), and determine when there exists a dominant morphism from \( \mathbb{A}^3 \) to \( S_{p_1,p_2,p_3}^* \) or \( \Sigma_{p_1,p_2,p_3}^* := \Sigma_{p_1,p_2,p_3} - (0) \) (cf. Theorems 2.8 and 2.15).

The terminology and the notations in this article conform to the use in the previous paper [7] and the general current practice. We shall list up the notations in frequent use.

- \( \mathbb{A}^r \): the affine space of dimension \( r \) defined over \( k \);
- \( \mathbb{P}^r \): the projective space of dimension \( r \) defined over \( k \);
- \( \kappa(X) \): the logarithmic Kodaira dimension of a nonsingular algebraic variety \( X \);
- \( \mathbb{G}_m \): the multiplicative group consisting of the invertible elements of \( A \);
- \( (a_1, \ldots, a_n) \) or \( \text{G.C.D.}(a_1, \ldots, a_n) \): the greatest common divisor of positive integers \( a_1, \ldots, a_n \);
L.C.M. \((a_1, \ldots, a_n)\): the least common multiple of positive integers \(a_1, \ldots, a_n\);
\(\{a_1, \ldots, a_n\}\): an \(n\)-tuple of integers;
\(D \sim D'\): a divisor \(D\) is linearly equivalent to a divisor \(D'\);

For a dominant morphism \(\pi: X \to C\) and a point \(P\) of \(C\), \(\pi^*P\) denotes the (scheme-theoretic) complete inverse image, and \(\pi^{-1}(P)\) denotes the set-theoretic inverse image.

1. A Diophantine equation, I

1.1. Let \(R := k[u_1, \ldots, u_r]\) be a polynomial ring of dimension \(r\) over \(k\). Let us consider a Diophantine equation in \((m+n)\)-variables,
\[
x_1^{a_1} \cdots x_m^{a_m} y_1^{b_1} \cdots y_n^{b_n} = 1,
\]
where \(m, n \geq 1\) and \(a_j's\) and \(b_j's\) are integers larger than 1, and look for its solutions in \(R\). A solution \(\{x_i = f_i, y_j = g_j; 1 \leq i \leq m, 1 \leq j \leq n\}\) is called a constant solution if \(f_i \in k\) and \(g_j \in k\) for every \(i\) and every \(j\). Otherwise, it is called a non-constant solution.

1.2. We shall prove the following

**Theorem.** A non-constant solution of the equation (1) in \(R\) has one of the following forms:

1. \(x_i = 0\) for some \(1 \leq i \leq m\), \(y_j = c_j \in k\) for every \(1 \leq j \leq n\), where \(c_1^1 \cdots c_n^m = -1\);

2. \(y_j = 0\) for some \(1 \leq j \leq n\), and \(x_i = c_i \in k\) for every \(1 \leq i \leq m\), where \(c_1^1 \cdots c_n^m = 1\).

The proof will be given in the paragraph 1.3.

1.3. Let \(\{x_i = f_i, y_j = g_j\}\) be a non-constant solution such that \(f_i \in k\) and \(g_j \in k\) for some \(i\) and \(j\). By reducing the number of variables in the equation (1) if necessary, we may assume that \(f_i \in k\) and \(g_j \in k\) for every \(1 \leq i \leq m\) and every \(1 \leq j \leq n\).

On the other hand, we may assume that \(R\) is a polynomial ring in one variable \(u\). In effect, let \(\gamma_1(u), \ldots, \gamma_r(u)\) be sufficiently general polynomials in \(k[u]\), and let \(\varphi_i := f_i(\gamma_1(u), \ldots, \gamma_r(u))\) and \(\psi_j := g_j(\gamma_1(u), \ldots, \gamma_r(u))\). Then \(\{x_i = \varphi_i, y_j = \psi_j\}\) is a non-constant solution of the equation (1) in \(k[u]\) such that \(\varphi_i \in k\) and \(\psi_j \in k\) for every \(1 \leq i \leq m\) and every \(1 \leq j \leq n\). Such polynomials \(\gamma_1(u), \ldots, \gamma_r(u)\) exist because \(k\) is an infinite field. If we can show the non-existence of such a solution in \(k[u]\), it implies the non-existence of a non-constant solution of (1) in \(R\) such that \(f_i \in k\) and \(g_j \in k\) for some \(i\) and \(j\). Thus, we may assume that \(R = k[u]\).

By replacing again the equation (1) by an equation of the same kind in more unknowns if necessary, we may assume that \(f_i = c_i(u - \alpha_i)\) and \(g_j = d_j(u - \beta_j)\),
where \( c, \alpha_i, \beta_j \in k \), and \( \alpha_i \neq \alpha_i', \beta_j \neq \beta_j' \) whenever \( i \neq i' \) and \( j \neq j' \). Finally, we obtain a relation in a variable \( u \),

\[
c(u-\alpha_1)^{e_1} \cdots (u-\alpha_m)^{e_m} - d(u-\beta_1)^{b_1} \cdots (u-\beta_n)^{b_n} = 1, \quad \cdots \cdots (2)
\]

where \( c, d \in k^* \). We shall show that such identity in \( u \) is impossible.

Note that every \( \alpha_i \) is distinct from \( \beta_1, \ldots, \beta_n \) and every \( \beta_j \) is distinct from \( \alpha_1, \ldots, \alpha_m \). By differentiating both hand sides of the equation (2) in \( u \), we obtain a relation,

\[
c \prod_{i=1}^{m} (u-\alpha_i)^{e_i} \cdot \left\{ \sum_{i=1}^{m} \frac{a_i}{u-\alpha_i} \right\} = d \prod_{j=1}^{n} (u-\beta_j)^{b_j} \cdot \left\{ \sum_{j=1}^{n} \frac{b_j}{u-\beta_j} \right\}. \quad \cdots \cdots (3)
\]

Note that we have

\[
\deg \left( \prod_{i=1}^{m} (u-\alpha_i) \cdot \left\{ \sum_{i=1}^{m} \frac{a_i}{u-\alpha_i} \right\} \right) \leq m-1, \quad \text{and}
\]

\[
\deg \left( \prod_{j=1}^{n} (u-\beta_j) \cdot \left\{ \sum_{j=1}^{n} \frac{b_j}{u-\beta_j} \right\} \right) \leq n-1.
\]

Since \( a_i \geq 2 \) and \( b_j \geq 2 \) by assumption, the relation (3) implies that

\[
\prod_{j=1}^{n} (u-\beta_j) \cdot \left\{ \sum_{j=1}^{n} \frac{b_j}{u-\beta_j} \right\}
\]

is divisible by \( \prod_{i=1}^{m} (u-\alpha_i) \). Hence we obtain \( m \leq n-1 \). Similarly, we have \( n \leq m-1 \). This is a contradiction. Therefore, we have shown that if \( \{x_i=f_i, y_j=g_j\} \) is a non-constant solution of the equation (1), then either \( f_i \in k \) for every \( 1 \leq i \leq m \) or \( g_j \in k \) for every \( 1 \leq j \leq n \).

Suppose that the first case takes place, i.e., \( f_i = c_i \in k \) for every \( 1 \leq i \leq m \). Then \( g_j \in k \) for some \( j \). If \( \prod_{i=1}^{m} c_i \neq 0 \), then \( g_j \) would be a unit in \( R \); this is a contradiction. Hence \( \prod_{i=1}^{m} c_i = 0 \), and \( g_j = 0 \) for some \( j \). The other case can be treated in a similar way. Q.E.D.

2. A Diophantine equation, II

2.1. In this section, we shall consider a Diophantine equation

\[
x_1^p + x_2^p + x_3^p = 0, \quad \cdots \cdots \cdots \cdots (4)
\]

where \( p_1, p_2 \) and \( p_3 \) are integers larger than 1, and look for non-constant solutions in \( R := k[u_1, \ldots, u_n] \). Let \( S_{p_1, p_2, p_3} \) be the affine hypersurface in \( A_3^1 := \text{Spec}(k[x_1, x_2, x_3]) \) defined by the equation (4), and let \( S^*_{p_1, p_2, p_3} := S_{p_1, p_2, p_3} - (0) \), where \( (0) \) is the point \((0, 0, 0)\). When there is no fear of confusion, we denote \( S_{p_1, p_2, p_3} \) and \( S^*_{p_1, p_2, p_3} \) simply by \( S \) and \( S^* \), respectively. It is easy to see that \( S \)
is a normal surface with the unique singular point \((0)\). The resolution of singularity of \(S\) at the point \((0)\) is completely understood (cf. Orlik-Wagreich [10]). We recall some of the results which we need in our subsequent arguments.

2.2. Let \(G_m\) be the multiplicative group scheme defined over \(k\). We need the following:

**Lemma.** Let \(X\) be a nonsingular quasi-projective surface with an effective separated \(G_m\)-action. Assume that \(X\) has no fixed points. Let \(Y := X/G_m\) be the quotient variety and let \(\pi : X \rightarrow Y\) be the canonical projection. Then we have:

1. \(Y\) is a nonsingular curve;
2. \(\pi^{-1}(y) = A^1_m\) for every point \(y \in Y\), where \(A^1_m\) is the affine line \(A^1_\mathbb{A}\) with one point deleted off;
3. \(\pi^*y\) is a multiple fiber with multiplicity \(\mu\) if and only if the stabilizer group \(\sigma_x\) is a cyclic group of order \(\mu\) for a point \(x\) in \(\pi^{-1}(y)\).

Proof. Let \(x\) be a point of \(X\). By virtue of Sumihiro [11; Cor. 2], there exists a \(G_m\)-stable affine open neighborhood \(U := \text{Spec}(A)\) of \(x\). Let \(B\) be the subalgebra of \(G_m\)-invariants in \(A\). Then \(V := \text{Spec}(B)\) is an affine open neighborhood of \(y := \pi(x)\). Since \(A\) is regular, \(B\) is normal. Hence \(Y\) is a nonsingular curve. It is known by the theory of quotient varieties with respect to reductive group actions (e.g., Mumford [9; Chap. 1]) that \(\pi^{-1}(y)\) consists of a single orbit under the stated assumption. Hence the assertion (2) holds.

Consider a \(G_m\)-equivariant completion \(X \rightarrow Z\), where we may assume that \(Z\) is a nonsingular projective surface (cf. Sumihiro [11]). Let \(O(x)\) be the orbit through \(x\), and let \(C\) be the closure of \(O(x)\) in \(Z\). Then \(C\) contains a fixed point \(z\). We can find a system of local coordinates \((u, v)\) at \(z\) such that \(u = 0\) defines a branch of \(C\) through \(z\) and the induced \(G_m\)-action on the tangent space \(T_z Z\) is normalized as \(t(\xi, \eta) = (t^\alpha \xi, t^\beta \eta)\), where \(t \in \mathbb{A}^1\), \(\alpha\) and \(\beta\) are integers and \(\xi = \partial/\partial u\) and \(\eta = \partial/\partial v\). Replacing the \(G_m\)-action \((t, z) \rightarrow t^\alpha z\) on \(Z\) by a \(G_m\)-action \((t, z) \rightarrow t^\beta z\) and interchanging the roles of \(u\) and \(v\) if necessary, we may assume that \(\beta > 0\). Since \(O_x \approx k[[u, v]], \alpha\) and \(\beta\) are prime to each other if \(\alpha = 0\) then \(\beta = 1\). Let \(y := \pi(x)\). Then \(O_{y, z} \approx k[[\delta v^{-\alpha}]]\), and the orbit \(O(x)\) is defined by \(u = 0\) in a neighborhood of \(z\). Hence the multiplicity of \(\pi^*y\) is \(\beta\), and the stabilizer group of a point (hence of the point \(x\)) of the orbit \(O(x)\) is \(Z/\beta Z\). Hence the assertion (3) holds true.

Q.E.D.

2.3. Let \(p_1, p_2\) and \(p_3\) be the same as for the equation (4). Let \(d := \text{L.C.M.}(p_1, p_2, p_3)\) and define the integers \(q_i\) \((1 \leq i \leq 3)\) by \(d = p_i q_i\). The group scheme \(G_m\) acts effectively on \(S^*_{p_1, p_2, p_3}\) by

\[
t(x_1, x_2, x_3) = (t^{q_1}x_1, t^{q_2}x_2, t^{q_3}x_3) .
\]

Then \(S^*_{p_1, p_2, p_3}\) has no fixed points. Let \(C := S^*/G_m\) and let \(\pi : S^* \rightarrow C\) be the
Lemma. (1) The genus $g$ of $C$ is given by
\[ g = \frac{d^2}{2q_2q_2} - \frac{d}{2} \left\{ \frac{(q_1, q_2) + (q_2, q_3) + (q_3, q_4)}{q_2q_3} \right\} + 1. \]

(2) $\pi$ has no multiple fibers but possibly $\frac{d(q_1, q_3)}{q_2q_2}$ fibers with multiplicity $(q_1, q_2)$, $\frac{d(q_2, q_3)}{q_3q_3}$ fibers with multiplicity $(q_2, q_3)$ and $\frac{d(q_3, q_4)}{q_4q_4}$ fibers with multiplicity $(q_3, q_4)$.

Proof. (1) Let $T$ be the hypersurface in $\mathbb{A}_k^3 := \text{Spec}(k[y_1, y_2, y_3])$ defined by $y_1^2 + y_2^2 + y_3^2 = 0$, and let $T^* := T - (0)$. Let $\Phi: T^* \to S^*$ be the morphism defined by $(x_1, x_2, x_3, x_4) \mapsto (y_1, y_2, y_3)$. Let $G_m$ act on $T^*$ via $t(y_1, y_2, y_3) = (ty_1, ty_2, ty_3)$. Then $\Phi$ is a $G_m$-equivariant morphism. Let $D := T^*/G_m$. Then $\Phi$ induces a surjective morphism $\phi: D \to C$ such that $\pi \Phi = \phi \pi'$, where $\pi': T^* \to D$ is the canonical quotient morphism. Then it is easy to show that $\deg \phi = q_1q_2q_3$ and the morphism $\phi$ ramifies at $d$ points (on $D$) with ramification index $q_1(q_2, q_3)$, at $d$ points with ramification index $q_2(q_3, q_4)$ and at $d$ points with ramification index $q_3(q_4, q_1)$. Since $D$ has genus $\frac{1}{2}(d-1)(d-2)$, the genus $g$ of $C$ is obtained by the Riemann-Hurwitz formula applied to $\phi: D \to C$. The assertion (2) can be verified by means of Lemma 2.2. Q.E.D.

2.4. Let $p_i (1 \leq i \leq 4)$ be integers larger than 1. Let $\Sigma := \text{Spec}(k[x_1, x_2, x_3, x_4])$ be the surface defined by equations,
\[ x_1^3 + x_2^3 + x_3^3 = 0 \quad \text{and} \quad ax_1^3 + x_2^3 + x_4^3 = 0, \]
where $a \in k - \{0, 1\}$. Let $\Sigma^* := \Sigma - (0)$; we denote these objects by $\Sigma$ and $\Sigma^*$ if there is no fear of confusion. Then $\Sigma^*$ is a nonsingular surface with an effective action of the group scheme $G_m$ defined by
\[ t(x_1, x_2, x_3, x_4) = (t^i x_1, t^j x_2, t^k x_3, t^l x_4), \]
where the integers $q_i (1 \leq i \leq 4)$ are defined by
\[ d = pq_i (1 \leq i \leq 4) \quad \text{and} \quad d = \text{L.C.M.} (p_1, p_2, p_3, p_4). \]
The $G_m$-action on $\Sigma^*$ given above has no fixed points. Let $C := \Sigma^*/G_m$ and let $\pi: \Sigma^* \to C$ be the canonical quotient morphism. We have the following:

Lemma. (1) The genus $g$ of $C$ is given by the formula:
\[ g = \frac{d^3}{q_1q_2q_4} - \frac{d^2}{2q_1q_2q_4} \left\{ \frac{(q_1, q_2) + (q_2, q_3) + (q_3, q_4)}{q_2q_3} \right\} + 1. \]
(2) \( \pi \) has no multiple fibers but possibly \( \frac{d^2(q_1, q_2, q_3)}{q_2q_3} \) fibers with multiplicity \( (q_1, q_2, q_3) \), \( \frac{d^3(q_1, q_2, q_4)}{q_2q_4} \) fibers with multiplicity \( (q_1, q_2, q_4) \), \( \frac{d^2(q_1, q_3, q_4)}{q_2q_4} \) fibers with multiplicity \( (q_1, q_3, q_4) \) and \( \frac{d^3(q_2, q_3, q_4)}{q_2q_4} \) fibers with multiplicity \( (q_2, q_3, q_4) \).

**Proof.** Similar to the proof of Lemma 2.3.

2.5. As an application of Lemma 2.4, we have the following examples:

<table>
<thead>
<tr>
<th>( {p_1, p_2, p_3, p_4} )</th>
<th>( g(\Sigma*/G_m) )</th>
<th>multiple fibers of ( \pi: \Sigma* \to C := \Sigma*/G_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {2, 2, 2, 2s} )</td>
<td>1</td>
<td>4 fibers with multiplicity ( s )</td>
</tr>
<tr>
<td>( {2, 2, 2, 2s+1} )</td>
<td>0</td>
<td>4 fibers with multiplicity ( 2s+1 )</td>
</tr>
<tr>
<td>( {2, 2, 3, 3} )</td>
<td>2</td>
<td>no multiple fibers</td>
</tr>
<tr>
<td>( {2, 2, 3, 4} )</td>
<td>0</td>
<td>2 fibers with multiplicity 2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4 fibers with multiplicity 3</td>
</tr>
<tr>
<td>( {2, 2, 3, 5} )</td>
<td>0</td>
<td>2 fibers with multiplicity 5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 fibers with multiplicity 3</td>
</tr>
</tbody>
</table>

2.6. From this paragraph on up to 2.14, we shall retain the notations of 2.1. Let \( p_i := p_i/(q_2, q_3), p'_2 := p_2/(q_1, q_3) \) and \( p'_i := p_i/(q_1, q_2); p'_i \) \( (1 \leq i \leq 3) \) are integers because, for example, \( d = p_1q_1 \) and \( (q_1, (q_2, q_3)) = 1 \) imply that \( p_i \) is divisible by \( (q_2, q_3) \). As an easy application of Lemma 2.3, we know that \( g = 0 \) (resp. \( g = 1 \), resp. \( g > 1 \)) if and only if \( \frac{1}{p'_1} + \frac{1}{p'_2} + \frac{1}{p'_3} > 1 \) (resp. \( = 1 \), resp. \( < 1 \)).

2.7. We have the following:

**Lemma.** Assume that \( p_1 \leq p_2 \leq p_3 \) and \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \). Then we have:

(1) \( \{p_1, p_2, p_3\} = \{2, 3, 6\}, \{2, 4, 4\} \text{ or } \{3, 3, 3\}. \)

(2) \( C := S^*/G_m \text{ is a nonsingular elliptic curve, and } \pi: S^* \to C \text{ has no multiple fibers, i.e., } S^*/G_m \text{ is an } \mathbb{A}^1_K \text{ -bundle over } C. \)

(3) Let \( b := d/q_1q_2q_3 \). Then \( b = 1, 2, 3 \) for \( \{p_1, p_2, p_3\} = \{2, 3, 6\}, \{2, 4, 4\} \) and \( \{3, 3, 3\} \), respectively. There exists an invertible sheaf \( \mathcal{L} \) of degree \( b \) over \( C \) such that the ruled surface \( V := \text{Proj}(\mathcal{O}_C \oplus \mathcal{L}) \) over \( C \) with the zero section \( M_0 \) and the infinity section \( M_\infty \) deleted off is isomorphic to \( S^*. \)

(4) \( \kappa(S^*) = 0. \)

**Proof.** (1) follows from a well-known straightforward computation. (2) follows from Lemma 2.3. Since \( S^* \) is an \( \mathbb{A}^1_K \)-bundle over \( C, S^* \) is obtained from a ruled surface in the way as specified in the assertion (3). Then \( (M_0^2) = -b, (M_\infty^2) = b \) and \( (M_0^*M_\infty^*) = 0. \) The number \( b := \text{deg} \mathcal{L} \) is equal to \( d/q_1q_2q_3, \)
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because $M_0$ is the unique exceptional curve which arises from the minimal resolution of singularity of the point $(0, 0, 0)$ of $S$ (cf. Orlik-Wagreich [10]). Note that the canonical divisor $K_\nu$ of $V$ is linearly equivalent to $-M_0-M_\omega$. The boundary divisor of $S^*_\nu$ in $V$ is $D:=M_0+M_\omega$. Hence $D+K_\nu\sim 0$. Therefore, we have $\kappa(S^*_\nu)=0$. Q.E.D.

2.8. We shall prove

**Theorem** (cf. Itaka [4]). $S^*_{q_1,q_2,q_3}$ has the logarithmic Kodaira dimension $\kappa(S^*_{q_1,q_2,q_3})=-\infty$, 0, 1 according as $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}>1$, $=1$, $<1$, respectively.

The proof will be given in the paragraphs 2.9~2.11.

2.9. Let $V$ be a nonsingular projective surface with a surjective morphism $\varphi: V\to C:=S^*/G_m$ satisfying the following conditions:

(i) $V$ contains $S^*_{q_1,q_2,q_3}$ as a dense open set, and $\varphi|_{S^*_{q_1,q_2,q_3}}: S^*_{q_1,q_2,q_3}\to C$;
(ii) $V-S^*$ contains no exceptional curves of the first kind which are contained in fibers of $\varphi$.

It is clear that general fibers of $\varphi$ are isomorphic to $P^1_\nu$. The resolution of singularity of $S^*_{q_1,q_2,q_3}$ at the unique singular point $(0)=(0, 0, 0)$ is described in detail in Orlik-Wagreich [10]. We recall some of the necessary results. The morphism $\pi: S^*\to C$ has multiple fibers if one of $(q_1, q_2)$, $(q_2, q_3)$ and $(q_3, q_1)$ is larger than 1. If $(q_1, q_2)>1$, there are $d(q_1, q_2)/q_1q_2$ fibers of multiplicity $(q_1, q_2)$ (cf. Lemma 2.3). For a multiple fiber $F$ of multiplicity $(q_1, q_2)$, set $\alpha:=(q_1, q_2)$ and determine an integer $\beta$ uniquely by the condition that $q_2\beta\equiv 1$ (mod $\alpha$) and $0<\beta<\alpha$. Define positive integers $b_1, \cdots, b_s\geq 2$ by writing $\alpha/(\alpha-\beta)$ in the form of a continued fraction

$$\frac{\alpha}{\alpha-\beta} = b_1 - \frac{1}{b_2 - \frac{1}{\cdots - \frac{1}{b_s}}}.$$ 

which we write in the form $\alpha/(\alpha-\beta)=[b_1, \cdots, b_s]$. For multiple fibers of multiplicity $(q_1, q_2)$ or $(q_2, q_3)$, we determine the corresponding integers $\alpha, \beta, b_1, \cdots, b_s$ etc. Let $N$ be the number of the multiple fibers of $\pi$. Let

$$b:=\frac{d}{q_1q_2q_3} \sum_{i=1}^{\infty} \frac{\beta_i}{\alpha_i},$$

where $\{\alpha_i, \beta_i\}$ ranges over all pairs of integers which are determined for all multiple fibers of $\pi$ in the above-mentioned fashion. Let $g$ be the genus of $C$. Then the dual graph of the exceptional curves which arise from the resolution
of singularity of the point (0) of $S_{A_1, A_2, A_3}$ has a vertex with weight $-b - N$ (corresponding to a nonsingular curve of genus $g$) and has $N$ branches, each of which is a linear chain of nonsingular rational curves as exhibited in the following figure:

![Diagram of a chain of nonsingular rational curves]

2.10. The fibration $\varphi: V \to C$ has two cross-sections $M'_0$ and $M'_\omega$ and $N$ singular fibers $\Phi_1, \ldots, \Phi_N$ such that:

(i) $M'_0$ and $M'_\omega$ are nonsingular curves of genus $g$; $(M'_0) = -b - N$ and $(M'_\omega) = b$;

(ii) Let $\Phi$ be a singular fiber of $\varphi$, then $\Phi \cap S^* = aF$ with $F \cong \mathbb{A}_2^1$, i.e., a multiple fiber of multiplicity $\alpha > 1$; the component $F$ of $\Phi$ (= the closure of $F$ in $V$) is connected to the cross-section $M'_0$ by $s$ components as exhibited in

$$
\begin{array}{ccccccc}
F & -b_1 & \cdots & -b_s & M'_0
\end{array}
$$

By assumption, $\Phi - F$ contains no exceptional curves of the first kind. Hence $F$ is the unique exceptional curve of the first kind contained in the singular fiber $\Phi$. Then it is easily ascertained that the dual graph of the fiber $\Phi$ is a linear chain. It looks like the one given in Miyanishi [6; p. 95]. To fix the notations, we represent it in the next page. The upper half of the chain between $E_0$ and $E(a, m)$ (with $E(a, m)$ excluded) corresponds to the chain

$$
\begin{array}{ccccccc}
- & - & \cdots & - & -b_1
\end{array}
$$

Hence we have $\frac{\alpha}{\alpha - \beta} = [b_1, \ldots, b_s]$

$$
\begin{align*}
\begin{cases}
[m_1 + 1, 2, \ldots, 2, m_3 + 2, 2, \ldots, 2, m_{s-1} + 2, 2, \ldots, 2] & \text{if } a \text{ is even} \\
\frac{m_2 - 1}{m_2 - 1} & \frac{m_s - 1}{m_s - 1}
\end{cases}
\end{align*}
$$

$$
\begin{align*}
\begin{cases}
[m_1 + 1, 2, \ldots, 2, m_3 + 2, 2, \ldots, 2, m_{s-2} + 2, 2, \ldots, 2, m_s + 1] & \text{if } a \text{ is odd} \\
\frac{m_2 - 1}{m_2 - 1} & \frac{m_s - 1}{m_s - 1} - 1
\end{cases}
\end{align*}
$$
Note that $\alpha$ is the multiplicity of $F$ in the fiber $\Phi$. This is clear because $\Phi \cap S^* = \alpha F$. We can check this fact as follows. The multiplicity $\mu(i,j)$ $(1 \leq i \leq a; 1 \leq j \leq m_i)$ of the component $E(i,j)$ in $\Phi$ is given by the function
\[ \mu(i, j) \] defined inductively by:

\[ \begin{align*}
\mu(0, m) &= 1, \mu(1, j) = j & \text{for } 1 \leq j \leq m, \\
\mu(i, 1) &= \mu(i-1, m_{i-1}) + \mu(i-2, m_{i-2}) & \text{for } 1 < i \leq a, \\
\mu(i, j) &= \mu(i, j-1) + \mu(i-1, m_{i-1}) & \text{for } 1 < j \leq m_i.
\end{align*} \]

On the other hand, the integer \( \alpha \) is regained by the method as indicated in the appendix of [10; p. 76] from the above development of \( \alpha/(\alpha-\beta) \) into a continued fraction.

2.11. Note that \( V-S^* \) consists of nonsingular components crossing normally. It is also easy to see that there exists a unique contraction \( \sigma: V \to V_0 \), where

(i) \( \varphi_0: V_0 \to C \) is a relatively minimal ruled surface;
(ii) Let \( M_0 := \sigma_\# M_0 \) and \( M_\infty := \sigma_\# M_\infty' \); Then \( (M_0) = -(b+N) \) and \( (M_\infty) = b+N \).

The canonical divisor \( K_{V_0} \) is given by

\[ K_{V_0} \sim -M_0 - M_\infty + \varphi_\#(K_C) \]

and \( M_\infty \sim M_0 + \varphi_\#(\delta) \),

where \( K_C \) is the canonical divisor of \( C \) and \( \delta \) is a divisor on \( C \) with \( \deg(\delta) = b+N \). In effect, \( V_0 = \text{Proj}(\mathcal{O}_C \oplus \mathcal{O}_C(\delta)) \), and \( M_0 \) and \( M_\infty \) correspond to the zero section and the infinite section of \( V_0 \), respectively.

Each irreducible component \( E(i, j) \) of the singular fiber has the contribution \( k(i, j) \) in the canonical divisor \( K_V \) determined inductively as follows:

\[ \begin{align*}
k(0, m) &= 0, k(1, j) = j & \text{for } 1 \leq j \leq m, \\
k(i, 1) &= k(i-1, m_{i-1}) + k(i-2, m_{i-2}) + 1 & \text{for } 1 < i \leq a, \\
k(i, j) &= k(i, j-1) + k(i-1, m_{i-1}) + 1 & \text{for } 1 < j \leq m_i.
\end{align*} \]

On the other hand, \( E(i, j) \) has multiplicity \( n(i, j) \) in \( \sigma_*(M_\infty) \), which is determined by

\[ \begin{align*}
n(0, m) &= 0, n(1, j) = 1 & \text{for } 1 \leq j \leq m, \\
n(i, 1) &= n(i-1, m_{i-1}) + n(i-2, m_{i-2}) & \text{for } 1 < i \leq a, \\
n(i, j) &= n(i, j-1) + n(i-1, m_{i-1}) & \text{for } 1 < j \leq m_i.
\end{align*} \]

Let \( D \) be the reduced effective divisor such that \( \text{Supp}(D) = V-S^* \). Then it is straightforward to show that the coefficient \( v(i, j) \) of \( E(i, j) \) in \( D + K_V - \Phi \) is given by

\[ v(i, j) = \begin{cases} 
0 & \text{if } (i, j) \neq (a, m) \\
-1 & \text{if } (i, j) = (a, m).
\end{cases} \]

Therefore we have:
\[ D + K_Y \sim \sum_{i=1}^{\ell} \Phi_i - \sum_{i=1}^{\ell} F_i + \varphi^*(K_C) \]

\[ \geq \sum_{i=1}^{\ell} \left( 1 - \frac{1}{\alpha_i} \right) \Phi_i + \varphi^*(K_C), \]

where \( \alpha_i \) is the multiplicity of \( F_i \) in \( \Phi_i \). Let

\[ A := \left( \sum_{i=1}^{\ell} \left( 1 - \frac{1}{\alpha_i} \right) \Phi_i + \varphi^*(K_C) \cdot M'_0 \right). \]

Note that \( \alpha_i \) has one of the values \((f_1, q_2), (q_2, q_3)\) and \((q_3, q_1)\) (cf. 2.9) and that \( A \) is, in effect, equal to

\[ \left( \sum_{p \in \mathbb{P}} \left( 1 - \frac{1}{\alpha_p} \right) \varphi^*(P) + \varphi^*(K_C) \cdot M'_0 \right), \]

where \( \varphi^*(P) = \alpha_p F_p \) with \( F_p \approx A^1_p \). Then we can calculate \( A \) as follows:

\[ A = \frac{d(q_1, q_2) + d(q_2, q_3) + d(q_3, q_1) - d(q_1, q_2)}{q_1 q_2} - \frac{q_2 q_3}{q_2 q_3} - \frac{q_3 q_1}{q_3 q_1} - \frac{q_1 q_2}{q_1 q_2} \cdot \frac{1}{(q_1, q_2)} \cdot 2g - 2 \]

\[ = \frac{d^2}{q_1 q_2 q_3} \left( 1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3} \right). \]

We have clearly \( \kappa(S^*) = 1 \) if \( A > 0 \), because \( D + K_Y \) is linearly equivalent to a divisor supported by fibers and the components contained in fibers of \( \varphi \). If \( A = 0 \) we have \( \kappa(S^*) = 0 \) (cf. 2.7). If \( A < 0 \), i.e., \( 1 < \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \), we have the following under an additional assumption \( 2 \leq p_1 \leq p_2 \leq p_3 \): \( \{ p_1, p_2, p_3 \} = \{ 2, 2, n \} \) \( (n \geq 2) \), \( \{ 2, 3, 3 \} \), \( \{ 2, 3, 4 \} \) or \( \{ 2, 3, 5 \} \). In each of the above four cases for \( A < 0 \), the foregoing arguments of evaluating \( D + K_Y \) shows that \( \kappa(S^*) = -\infty \); note that if \( A < 0 \) then \( g = 0 \). This completes the proof of Theorem 2.8.

2.12. If \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1 \), the surface \( S_{p_1, p_2, p_3} \) is the quotient variety of \( A^3_k \) with respect to a linear action of a Kleinian subgroup \( G \) of \( GL(2, k) \) (cf. Brieskorn [1]). In effect, \( G \) acts freely on \( A^3_k - (0) \). Hence there exists an étale finite morphism \( \rho: A^3_k - (0) \to S^* \), and \( A^3_k - (0) \) is algebraically simply connected.

Suppose that the ground field \( k \) is the field \( \mathbb{C} \) of complex numbers. Let \( U \) be the universal covering space of \( S^*_{p_1, p_2, p_3} \). Then it is known\(^(*)\) that:

\(^{(*)}\) This was communicated by Dr. A. Fujiki.
\[ U \cong \mathbb{C}^2 \quad (0) \Leftrightarrow 1 < \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \]
\[ U \cong \mathbb{C}^2 \quad \Leftrightarrow 1 = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \]
\[ U \cong \mathbb{C} \times D \quad \Leftrightarrow 1 > \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} , \]

where \( D \) is a unit disc.

2.13. For later use, we shall prove:

**Lemma.** Suppose that \( \varrho(S^*_{p_1, p_2, p_3}) > 0 \) and \( C \cong \mathbb{P}_k^3 \). Then \( \pi: S^* \to C \) has three or more multiple fibers.

**Proof.** We have the inequalities,
\[
\frac{(q_2, q_3) + (q_2, q_1) + (q_1, q_3)}{p_1} > \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} ,
\]
(cf. 2.6 and 2.8). Hence it is impossible that \( (q_2, q_3) = (q_3, q_1) = (q_1, q_2) = 1 \). If \( (q_2, q_3) > 1, (q_2, q_1) > 1 \) and \( (q_1, q_2) > 1 \), \( \pi \) has three or more multiple fibers. We shall consider the cases where one or two of \( (q_2, q_3) \), \( (q_3, q_1) \) and \( (q_1, q_2) \) equal 1. Assume first that \( (q_2, q_3) = 1, (q_3, q_1) > 1 \) and \( (q_1, q_2) > 1 \). Suppose that \( d(q_3, q_1)/q_3q_1 = d(q_1, q_3)/q_1q_3 = 1 \). Then \( q_3 = p_1(q_1, q_3) \) and \( q_2 = p_1(q_1, q_2) \). Hence \( (q_2, q_3) \) is divisible by \( p_1 \). Since \( p_1 > 1 \), this contradicts the assumption that \( (q_2, q_3) = 1 \).

Hence \( d(q_3, q_1) > 1 \) or \( d(q_1, q_2) > 1 \). Thus \( \pi \) has three or more multiple fibers.

Consider next the case where \( (q_2, q_3) = (q_3, q_1) = 1 \) and \( (q_1, q_2) > 1 \). Then the above inequalities imply that \( (q_1, q_2) > p_3 \). Hence \( q_3(q_1, q_2) > d \), and
\[
1 > \frac{(q_1, q_2)}{q_2} > d/q_3 .
\]
However, since \( (q_2, q_3) = 1, d \) is divisible by \( q_2q_3 \). This is a contradiction. Thus this case does not occur. The other cases can be treated in a similar fashion.

**Q.E.D.**

2.14. We shall prove the following:

**Theorem.** (1) If \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \leq 1 \), then there are no non-constant morphisms from \( \mathbb{A}_k^3 \) to \( S^*_{p_1, p_2, p_3} \).

(2) If \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1 \), then there are dominant morphisms from \( \mathbb{A}_k^3 \) to \( S^*_{p_1, p_2, p_3} \).

**Proof.** (1) If \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \), \( S^* \) is an \( \mathbb{A}_k^3 \)-bundle over a nonsingular
elliptic curve $C$. Thus, if $f: \mathbb{A}_k^r \to S^*$ is a non-constant morphism, $f(\mathbb{A}_k)\to S^*$ is contained in a fiber of $\pi$, which is isomorphic to $\mathbb{A}_k^r$. This is impossible. So, we may assume that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1$, i.e., $\kappa(S^*) > 0$. Let $f: \mathbb{A}_k^r \to S^*$ be a non-constant morphism if such a morphism exists at all. If $f$ is dominant, we may assume without loss of generality that $r=2$. Then we have
\[-\infty = \kappa(\mathbb{A}_k^r) \geq \kappa(S^*) = 1,
\]
which is impossible. Hence $f(\mathbb{A}_k^r)$ is a rational curve with at most one place at infinity, and $f(\mathbb{A}_k)$ is not contained in any fiber of $\pi$. Thus we have a dominant morphism
\[\psi: = \pi \cdot f: \mathbb{A}_k^r \to S^* \to C.
\]
Hence $C$ is isomorphic to $\mathbb{P}_k^1$, and $\psi(\mathbb{A}_k^r)$ is isomorphic to $\mathbb{A}_k^1$ or $\mathbb{P}_k^1$. Consider first the case where $\psi(\mathbb{A}_k^r) \cong \mathbb{A}_k^1$. By 2.13, there exist points $P$, $Q$ of $C$ such that $P$, $Q \in \psi(\mathbb{A}_k^r)$ and that $\pi*P$ and $\pi*Q$ are multiple fibers of multiplicity $\mu$ and $\nu$, respectively. Choose an inhomogeneous coordinate $t$ of $\mathbb{A}_k^1$ such that $P$ and $Q$ are defined by $t=0$ and $t=1$, respectively. Then there exist non-constant polynomials $g$ and $h$ in $R: = k[u_1, \ldots, u_r]$ such that $\psi*(t) = g^\mu$ and $\psi*(t-1) = h^\nu$. This implies that \(x^g - y^h = 1\).

This contradicts Theorem 1.2. Consider next the case where $\psi(\mathbb{A}_k^r) \cong \mathbb{P}_k^1$. In order to prove, by reductio ad absurdum, the non-existence of such a non-constant morphism as $\psi$, we may assume, by embedding the ground field $k$ into the field $C$ of complex numbers in a suitable way, that $k=C$. Restricting $\psi$ onto a suitable line $\mathbb{A}_k^r$ in $\mathbb{A}_k^r$, we may assume that $r=1$. Then the Nevanlinna theory (cf. Hayman [3]) implies that
\[\sum_{i=1}^{\nu} \left(1 - \frac{1}{\alpha_i}\right) - 2 \leq 0,
\]
where $N$ is the number of multiple fibers of $\pi$ and $\alpha_i$s are multiplicities. The left-hand side of the above inequality is, in effect, equal to $A$ in 2.11. Hence we have $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 1$. This is a contradiction. Thus there are no non-constant morphisms $f: \mathbb{A}_k^r \to S^*$ provided $\kappa(S^*) \geq 0$.

(2) We may assume that $p_1 \geq p_2 \geq p_3$. Then $\{p_1, p_2, p_3\}$ is one of the following triplets: $\{2, 2, n\}$ $(n \geq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$. Except in the case where $\{p_1, p_2, p_3\} = \{2, 3, 5\}$, one can easily find a solution $\{x_1 = f_1, x_2 = f_2, x_3 = f_3\}$ of the equation
\[x_1^2 + x_2^3 + x_3^5 = 0.
\]
in a polynomial ring \( R := k[u_1, \cdots, u_r] \) such that the subvarieties \( \{f_i = 0\} \) (\( 1 \leq i \leq 3 \)) have no common points in \( A_3^r \) and that trans. \( \text{deg}_k k(f_1, f_2, f_3) = 2 \). Then the assignment \( x_i \mapsto f_i \) (\( 1 \leq i \leq 3 \)) gives rise to a dominant morphism \( f: A_3^r \to S^* \). For example, if \( \{p_1, p_2, p_3\} = \{2, 2, 2\} \), such a solution is given by

\[
\begin{align*}
x_1 &= \frac{\xi^2 + \eta^2}{2}, \\
x_2 &= \frac{\xi^2 - \eta^2}{2\sqrt{-1}}, \\
x_3 &= \sqrt{-1} \cdot \xi \eta,
\end{align*}
\]

where \( \xi, \eta \) are polynomials in \( R \) such that \( \{\xi = 0\} \) and \( \{\eta = 0\} \) have no common points in \( A_3^r \) and that trans. \( \text{deg}_k k(\xi, \eta) = 2 \). The case where \( \{p_1, p_2, p_3\} = \{2, 3, 5\} \) seems more subtle.\(^*\) We look for a dominant morphism \( f: A_3^r \to S^* \). Since \( A_3^r \) is algebraically simply connected, such a morphism \( f \) (if it exists at all) is factored by a dominant morphism \( \tilde{f}: A_3^2 \to A_3^2 \) (0) such that \( f = \rho \cdot \tilde{f} \) (cf. 2.12). Conversely, if a dominant morphism \( \tilde{f} \) is given, \( f := \rho \cdot \tilde{f} \) is a required dominant morphism. Hence we have only to find a dominant morphism \( \tilde{f}: A_3^2 \to A_3^2 \) (0). Such a morphism \( \tilde{f} \) exists because a dominant morphism \( f: A_3^2 \to S^* \) provides one. Note that this argument works also for the other cases. Q.E.D.

2.15. We shall prove:

**Theorem.** Let \( \Sigma^*_1, \Sigma^*_2, \Sigma^*_3, \Sigma^*_4 \) be the nonsingular surface defined in 2.4. Assume that \( \{p_1, p_2, p_3, p_4\} \) is one of the following quadruplets: \( \{2, 2, 2, 2s+1\} \) (\( s \geq 1 \)), \( \{2, 2, 3, 4\} \), \( \{2, 2, 3, 5\} \), i.e., those in the examples in 2.5 with \( g(\Sigma^*/G_m) = 0 \). Then there are no non-constant morphisms from \( A_3^r \) to \( \Sigma^*_1, \Sigma^*_2, \Sigma^*_3, \Sigma^*_4 \).

Proof. We only consider the case where \( \{p_1, p_2, p_3, p_4\} = \{2, 2, 2, 3\} \). The other cases can be treated in a similar fashion. Suppose that \( f: A_3^r \to \Sigma^* \) is a non-constant morphism. With the notations of 2.4, \( C \) is then isomorphic to \( P^1 \). Let \( \psi := \pi \cdot f \). Then \( \psi(A_3^r) \) is isomorphic to \( A_3^1 \) or \( P^1 \). The case where \( \psi(A_3^r) \approx A_3^1 \) is impossible because \( \pi \) has four multiple fibers of multiplicity 3 (cf. 2.5 and the proof of Theorem 2.14). Hence \( \psi(A_3^r) \approx P^1 \). Let \( 3F_i \) (\( 1 \leq i \leq 4 \)) be the multiple fibers of \( \pi \). Then \( f^*(F_i) \) is defined by \( f_i = 0 \) with \( f_i \in R := k[u_1, \cdots, u_r] \). Since \( 3F_1 \sim 3F_2 \sim 3F_3 \), for example, we have a relation

\[
f^3 = f_1^3 + bf_1^3,
\]

where \( b \in k^* \).

Since \( f^*(F_1) \cap f^*(F_2) \cap f^*(F_3) = \phi \), we can define a non-constant morphism

\[
g: A_3^r \to S^* \subset \text{Spec}(k[x_1, x_2, x_3]/(x_1^2 + x_2^3 + x_3^3))
\]

by \( g^*(x_1) = b \cdot f_1 \), \( g^*(x_2) = f_2 \) and \( g^*(x_3) = -f_3 \). This is impossible because \( S^*_1, S^*_3, G_m \) is an elliptic curve. Q.E.D.

\(^*\) For the following argument, the author owes Dr. A. Fujiki.
3. Regular subrings in a polynomial ring

3.1. Let $A$ be a finitely generated, two-dimensional, regular $k$-algebra contained in a polynomial ring $R := k[u_1, \ldots, u_r]$ of dimension $r$. Let $X := \text{Spec}(A)$ and let $A_1 := \text{Spec}(R)$. Then the inclusion $A \hookrightarrow R$ gives rise to a dominant morphism $f: A_1 \rightarrow X$. By restricting $f$ onto a linear plane $L$ in $A_1$ which meets general fibers of $f$ in finitely many points, we have a dominant morphism $f_L: L \cong A_1 \rightarrow X$. This implies that $A$ is a $k$-subalgebra of the two-dimensional polynomial ring. Thus we may assume without loss of generality that $r = 2$.

Since $f: A_1 \rightarrow X$ is generically finite, we have $\aleph(X) = -\infty$, which follows from the inequality of logarithmic Kodaira dimensions.

This implies that $X$ contains a cylinderlike open set $U \cong U_0 \times A_1$, where $U_0$ is an affine curve (cf. Miyanishi-Sugie [8]; Fujita [2]). The projection $p: U \rightarrow U_0$ is induced from a dominant morphism $\rho: X \rightarrow \mathbb{P}_1$, where $U_0$ is an open set of $\mathbb{P}_1$. Then $\rho(X) \cong A_1$ or $\rho(X) = \mathbb{P}_1$. Indeed, if $\mathbb{P}_1 - \rho(X)$ consists of more than one point, we may write $\rho(X) = \text{Spec}(k[t, h(t)^{-1}])$, where $t$ is an inhomogeneous coordinate of $\mathbb{P}_1$ and $h(t) \in k[t] - k$; then $k[t, h(t)^{-1}]$ is a $k$-subalgebra of $A$ (and, hence, of $k[u_1, u_r]$); this contradicts the fact that $A^* = k^*$.

Summing up, we have the following:

**Lemma.** Let $X := \text{Spec}(A)$ be a nonsingular affine surface. Then $A$ is contained in a polynomial ring as a $k$-subalgebra if and only if there exists a dominant morphism $f: A_1 \rightarrow X$. In this case, we have:

1. $A^* = k^*$;
2. There exists an $A^1$-fibration $\rho: X \rightarrow Y$, where $Y \cong A_1$ or $\mathbb{P}_1$;
3. Every fiber of $\rho$ is supported by a disjoint union of irreducible curves, each of which is isomorphic to $A_1$.

For the last assertion, see Miyanishi [7].

3.2. A fiber $\rho^*(P)$ of $\rho$ is a singular fiber if either $\rho^{-1}(P)$ is reducible or $\rho^*(P)$ is irreducible and non-reduced. Write $\rho^*(P) = \sum_{i=1}^{s} n_i C_i$, where $C_i \cong A_1$ and $n_i > 0$. $\rho^*(P)$ is called a singular fiber of the first kind if $s \geq 2$ and $n_i = 1$ for some $i$; $\rho^*(P)$ is called a singular fiber of the second kind if $n_i \geq 2$ for every $i$. Let $\mu := \text{G.C.D.} (n_1, \ldots, n_s)$. If $\mu > 1$, the fiber $\rho^*(P)$ is called a multiple fiber and $\mu$ is called the multiplicity.

3.3. We shall prove:

**Theorem.** Let $X := \text{Spec}(A)$ be a nonsingular surface with an $A^1$-fibration $\rho: X \rightarrow Y$, where $Y \cong A_1$. Then $A$ is contained in a polynomial ring as a $k$-sub-
algebra if and only if $\rho$ has at most one singular fiber of the second kind.

Proof. (I) Let $f: A^3_1 \to X$ be a dominant morphism. Then note that $\rho \cdot f(A^3_1) = Y$. Suppose that $\rho$ has two singular fibers of the second kind $\rho^*(P)$ and $\rho^*(Q)$. Then $f^* \rho^*(P)$ and $f^* \rho^*(Q)$ are defined by the equations

$$g_1^{*1} \cdots g_m^{*n} = 0 \quad \text{and} \quad h_1^{*1} \cdots h_n^{*n} = 0$$

respectively, where $g_1, \ldots, g_m$ and $h_1, \ldots, h_n$ are non-constant polynomials in $k[u_1, u_2]$ and where $a_i \geq 2$ ($1 \leq i \leq m$) and $b_j \geq 2$ ($1 \leq j \leq n$). We may choose an inhomogeneous coordinate $t$ of $Y = \text{Spec}(k[t])$ in such a way that the points $P$ and $Q$ are defined by $t = 0$ and $t = 1$, respectively. Then we have a relation

$$g_1^{*1} \cdots g_m^{*n} - h_1^{*1} \cdots h_n^{*n} = 1.$$

This is impossible by virtue of Theorem 1.2. Therefore $\rho$ has at most one singular fiber of the second kind provided $A$ is contained in a polynomial ring as a $k$-subalgebra.

(II) We shall prove the "if" part of the theorem. Let $\rho^*(P) = \sum_{i=1}^N n_i C_i$ be a singular fiber of the first kind. We shall show that after replacing $X$ by a suitable affine open set with an $A^1$-fibration similar to that for $X$, $\rho^*(P)$ can be assumed to be an irreducible and reduced fiber. For this purpose, embed $X$ into a nonsingular projective surface $V$ as a dense open set. Then $V - X$ consists only of components of codimension 1. Since $X$ is affine, there exists an effective ample divisor $D$ on $V$ such that $\text{Supp}(D) = V - X$. For $\rho^*(P) = \sum_{i=1}^N n_i C_i$, suppose that $n_i = 1$. Then there exists an ample divisor $D'$ on $V$ such that $\text{Supp}(D') = (V - X) \cup \bigcup_{i \in \mathbb{Z}} C_i$, where $C_i$ is the closure of $C_i$ in $V$. Replace $X$ by $X' := X - \text{Supp}(D')$. Then $X'$ is an affine open set of $X$ and $\rho' := \rho|_{X'}: X' \to Y$ is an $A^1$-fibration over $Y$ for which the fiber $\rho^*(P)$ is irreducible and reduced.

Performing this operation to all singular fibers of the first kind of $\rho$, we may assume that $\rho$ has no singular fibers of the first kind. Let $\rho^*(P)$ denote anew a singular fiber of the second kind if such a fiber exists at all. If $\rho^*(P)$ is reducible, we may delete all irreducible components but one by replacing $X$ by a smaller affine open set with an $A^1$-fibration over $Y$ similar to that for $X$. Hence we may assume that $\rho^*(P)$ is an irreducible multiple fiber, i.e., $\rho^*(P) = n C$ with $C \cong A^1$ and $n \geq 2$.

Write $Y = \text{Spec}(k[t])$, and assume that the point $P$ is defined by $t = 0$. Let $Z := \text{Spec}(k[\tau]) \to Y$ be the morphism defined by $t = \tau^*$, which is a finite covering ramifying totally over $P$. Let $W$ be the normalization of $X \times_{Y} Z$. Then $W$ is a nonsingular affine surface, and the canonical surjective morphism
σ: W → Z is an $A^1$-fibration over Z. This can be seen as follows. Let $x$ be a point of X lying over the point P, and find a system of local coordinates ($\xi$, $\eta$) around $x$ such that the curve $C$ is defined by $\xi = 0$. Then we have a relation $\xi^a = at$, where $a$ is a unit in $O_{x,x}$. Then $\xi/\tau$ is regular at every point $x$ of W lying over $x$. Analytically, W around $x$ is defined as a hypersurface $(\xi/\tau)^n = a$ in the $(\xi/\tau, \tau, \eta)$-space. By the Jacobian criterion of smoothness, W is nonsingular at every point $x$ lying over $x$. It is easy to see that W is nonsingular at every point of W lying over $X - P^{-1}(P)$. Hence W is nonsingular. By construction, general fibers of $\sigma$ are isomorphic to $A^1$. Let $\tilde{P}$ be the point of Z lying over P. Every fiber of $\sigma$ except the fiber $\sigma^{*}\tilde{P}$ is irreducible and reduced, while $\sigma^{*}\tilde{P}$ is reduced and reducible with $n$ irreducible components. Let $W'$ be an affine open set of W obtained by deleting all components of $\sigma^{*}\tilde{P}$ except one. Then $\sigma' = \sigma|_{W'}: W' → Z$ is an $A^1$-bundle over Z $\approx A^1$, whence $W'$ is isomorphic to $A^1$ (cf. Kambayashi-Miyanishi [5]). Let $f$ be the composite of the natural morphisms

$$f: A^1 \cong W' \hookrightarrow W \to X \times Z \to X.$$ 

Since $f$ is apparently a dominant morphism, $A$ is contained in a polynomial ring as a $k$-subalgebra.

3.4. Corollary. Let $X$ be a nonsingular affine surface which satisfies the condition in Theorem 3.3. Then the torsion part $Pic(X)_{tor}$ of the Picard group of $X$ is a cyclic group.

Proof. Let $\rho: X → Y$ be the $A^1$-fibration as in Theorem 3.3. Let $\rho^{*}P_i$ ($0 ≤ i ≤ m$) exhaust all singular fibers of $\rho$; if there exists a singular fiber of the second kind, we let $\rho^{*}P_0$ denote it. Write $\rho^{*}P_i = \sum_{i \leq j \leq \mu} n_{ij} C_{ij}$, where $C_{ij} \cong A^1$ and $n_{ij} > 0$. Then, since $Y \cong A^1$, the Picard group $Pic(X)$ of $X$ is an abelian group with the following generators and relations:

$$\{\xi_{ij} | 0 ≤ i ≤ m, 1 ≤ j ≤ s_i\} \quad \text{and} \quad \sum_{i \leq j \leq \mu} n_{ij} \xi_{ij} = 0 \quad \text{for} \quad 0 ≤ i ≤ m.$$ 

It is then clear that $Pic(X) \cong \prod_{i=0}^{m} G_i$, where $G_i$ is an abelian group with generators and relations given as above with $i$ fixed and with $1 ≤ j ≤ s_i$. Since $(n_{i1}, \cdots, n_{im}) = 1$ for $i ≥ 1$ by assumption, we have $G_i \cong \mathbb{Z}^{\oplus (s_i - 1)}$. Let $\mu = (n_{i1}, \cdots, n_{im})$. Then $G_0 \cong \mathbb{Z}_{\mu} \cong \mathbb{Z}^{\oplus (c_0 - 1)}$. Hence we have $Pic(X)_{tor} \cong \mathbb{Z}_{\mu} \mathbb{Z}$. Q.E.D.

3.5. We shall prove:

Theorem. Let $X = Spec(A)$ be a nonsingular affine surface with an $A^1$-fibration $\rho: X → Y$, where $Y \cong P^1$. Assume that $A$ is contained in a polynomial ring as a $k$-subalgebra. Then the fibration $\rho$ has at most three multiple fibers. If
\( \rho \) has three multiple fibers, their multiplicities \( \{\mu_1, \mu_2, \mu_3\} \) are given, up to permutation, by one of the following triplets: \( \{2, 2, n\} \) \((n \geq 2)\), \( \{2, 3, 3\} \), \( \{2, 3, 4\} \) and \( \{2, 3, 5\} \).

Proof. Suppose that \( \rho \) has three or more multiple fibers. Let \( \rho \ast P_i := \mu_i F_i \) \((1 \leq i \leq 3)\) be a multiple fiber of multiplicity \( \mu_i > 1 \). Let \( f: A^3_i := \text{Spec}(k[u_i, u_2]) \rightarrow X \) be a dominant morphism as in 3.1. Then \( \rho \cdot f(A^3_i) \cong A^3_i \) or \( \rho \cdot f(A^3_i) = Y \). If \( \rho \cdot f(A^3_i) \cong A^3_i \), we may assume that \( P_1, P_2 \subseteq \rho \cdot f(A^3_i) \). However, this assumption leads to a contradiction by the argument in the step (I) of the proof of Theorem 3.3. Hence \( \rho \cdot f(A^3_i) = Y \). Then \( f^* F_i \) \((1 \leq i \leq 3)\) is defined by an equation \( f_i = 0 \), where \( f_i \) is a non-constant polynomial in \( k[u_i, u_2] \). Since \( \mu_1 f^* F_1 \sim \mu_2 f^* F_2 \sim \mu_3 f^* F_3 \) (linear equivalence), we have

\[
\frac{f_1^a}{f_1^b} = a \frac{f_2^a}{f_2^b} + b,
\]

where \( a, b \in k^* \). Without loss of generality, we may assume that \( a = b = -1 \). Namely, we have a relation

\[
f_1^a f_2^a + f_3^a = 0.
\]

Note that \( f^*(F_i) \cap f^*(F_j) = \emptyset \) whenever \( i \neq j \). The assignment \( x_i \mapsto f_i \) defines a non-constant morphism

\[
\psi: A^3_i \rightarrow S^* *= \text{Spec}(k[x_1, x_2, x_3]/((x_1^a + x_2^a + x_3^a)).
\]

Hence \( \{\mu_1, \mu_2, \mu_3\} \) is, up to permutation, one of the following triplets: \( \{2, 2, n\} \) \((n \geq 2)\), \( \{2, 3, 3\} \), \( \{2, 3, 4\} \) and \( \{2, 3, 5\} \) (cf. 2.14).

Suppose that \( \rho \) has four multiple fibers \( \rho \ast P_i = \mu_i F_i \) with multiplicity \( \mu_i \) \((1 \leq i \leq 4)\). Let \( f^* F_i \) be defined by \( f_i = 0 \), where \( f_i \) is a non-constant polynomial in \( k[u_i, u_2] \). Then we obtain relations of the following form:

\[
f_1^a f_2^a + f_3^a = 0
\]

\[
a f_1^a f_2^a + f_3^a = 0,
\]

where \( a \in k \setminus \{0, 1\} \). In view of the above observations on possible multiplicities of three multiple fibers of \( \rho \), we know that \( \{\mu_1, \mu_2, \mu_3, \mu_4\} \) is, up to permutation, one of the following quadruplets: \( \{2, 2, 2, n\} \) \((n \geq 2)\), \( \{2, 2, 3, 3\} \), \( \{2, 2, 3, 4\} \) and \( \{2, 2, 3, 5\} \). The induced relations provide a non-constant morphism

\[
\psi: A^3_i \rightarrow \sum_{\mu_1, \mu_2, \mu_3, \mu_4}
\]

This is impossible by 2.5 and 2.15.

Q.E.D.

3.6 Corollary. Let \( X \) be the same surface as in 3.5. Then \( \text{Pic}(X)_{\text{tor}} \) has at most two cyclic components. If \( \text{Pic}(X)_{\text{tor}} \) has two cyclic components, it is of the form:
3.7. We shall prove:

**Theorem.** Let $X := \text{Spec}(A)$ be a nonsingular affine surface with an $A^1$-fibration $\rho: X \to Y$, where $Y \cong \mathbb{P}^1$. Assume that $\rho$ satisfies the following conditions:

1. $\rho$ has no singular fibers of the second kind but at most three multiple fibers with a single irreducible component;

2. if $\rho$ has three multiple fibers, the set of multiplicities $\{\mu_1, \mu_2, \mu_3\}$ is one of the following triplets: $\{2, 2, n\}$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ and $\{2, 3, 5\}$.

Then $A$ is contained in a polynomial ring as a $k$-subalgebra.

Proof. (I) By performing the same operation as we did in the second step of the proof of Theorem 3.3, we may assume that $\rho$ has no singular fibers of the first kind. Suppose that $\rho$ has at most two multiple fibers. Let $P$ be a point of $Y$ such that $\rho^*P$ is a multiple fiber (if such a fiber exists at all), and let $X' := X - \rho^{-1}(P)$. Then the nonsingular affine surface $X'$ with an $A^1$-fibration $\rho' := \rho|_{X'}$ over $Y' := Y - \{P\}$ has at most one singular fiber of the second kind. By Theorem 3.3, there exist a dominant morphism $A_i \to X'$, and hence a dominant morphism $A_i \to X$. Therefore $A$ is contained in a polynomial ring as a $k$-subalgebra.

(II) Suppose that $\rho$ has three multiple fibers $\rho^*P_i = \mu_iF_i$ ($1 \leq i \leq 3$) with multiplicity $\mu_i$. We consider first the case where $\{\mu_1, \mu_2, \mu_3\} = \{2, 2, n\}$ ($n \geq 2$). Let $Y'' \to Y$ be a double covering of $Y$ which ramifies over the points $P_1$ and $P_2$; then $Y'' \cong \mathbb{P}^1$. Let $X'$ be the normalization of $X \times_Y Y''$ and let $\rho': X' \to Y'$ be the natural projection. Then $X'$ is a nonsingular affine surface and $\rho'$ is an $A^1$-fibration over $Y'$ (cf. the proof of Theorem 3.3). Moreover, $\rho'^*P'_i$ ($i=1, 2$) is a reduced singular fiber with two irreducible components, $P'_i$ being the unique point of $Y'$ lying over $P_i$, and $\rho'^*Q_i$ ($i=1, 2$) is a multiple fiber of multiplicity $n$ with single irreducible component, $Q_i$ and $Q_2$ being two points of $Y'$ lying over $P_3$. Replacing $X'$ by an affine open set, we may assume that $\rho'$ has no singular fibers of the first kind. Let $Y'''' \to Y'$ be an $n$-ple covering which ramifies totally over $Q_1$ and $Q_2$, let $X''''$ be the normalization of $X' \times_{Y'} Y''''$, and let $\rho'': X'''' \to Y''''$ be the natural projection. Then $X''''$ is a nonsingular affine surface and $\rho''$ is an $A^1$-fibration over $Y'''' \cong \mathbb{P}^1$. The fibration $\rho''$ has two reduced singular fibers $\rho''^*Q'_i$ ($i=1, 2$) with $n$ irreducible components, where $Q'_i$ ($i=1, 2$) is the unique point of $Y''''$ lying over $Q_i$. Then, by virtue of the step (I), there exist a dominant morphism $A_i \to X''''$, and hence a dominant morphism $A_i \to X$. Therefore, $A$ is contained in a polynomial ring as a $k$-subalgebra.
(III) The other cases except the last one can be treated in a similar fashion, that is, by choosing suitable multiple coverings $P_1 \rightarrow P_1$ and then taking the normalizations of the fiber products with respect to such multiple coverings. The following diagram will indicate roughly the necessary steps:

\[
\begin{array}{ccc}
\{2, 3, 3\} & \overset{\text{triple covering}}{\longrightarrow} & \{2, 2, 2\} \\
\{2, 2, 4\} & \overset{\text{double covering}}{\longrightarrow} & \{2, 3, 3\}
\end{array}
\]

The former case.

(IV) In the case where $\{\mu_1, \mu_2, \mu_3\} = \{2, 3, 5\}$, we know by the theory of Kleinian singularities that there exists a ramified covering $\tau: Y' \rightarrow Y$ of degree 60 with 30 points over $P_1$ with ramification index 2, 20 points over $P_2$ with ramification index 3 and 12 points over $P_3$ with ramification index 5, where $Y' \cong P_1$. Let $X'$ be the normalization of $X \times Y'$ and $\rho': X' \rightarrow Y'$ be the natural $A^1$-fibration. Then $\rho'$ has no multiple fibers of the second kind. So, we are done. Q.E.D.

References


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