<table>
<thead>
<tr>
<th>Title</th>
<th>Regular subrings of a polynomial ring. II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Miyanishi, Masayoshi</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 19(4) P.901-P.921</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1982</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/8875">https://doi.org/10.18910/8875</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/8875</td>
</tr>
<tr>
<td>rights</td>
<td></td>
</tr>
</tbody>
</table>

Osaka University Knowledge Archive : OUKA
https://ir.library.osaka-u.ac.jp/repo/ouka/all/

Osaka University
REGULAR SUBRINGS OF A POLYNOMIAL RING, II

Dedicated to Professor Yozō Matsushima on his sixtieth birthday

MASAYOSHI MIYANISHI

(Received December 26, 1980)

Introduction. This is a continuation of the previous work of the author's [7] on a finitely generated, two-dimensional, regular subring contained in a polynomial ring. Let \( k \) be an algebraically closed field of characteristic zero, which we fix as the ground field throughout this article. Let \( X = \text{Spec}(A) \) be a nonsingular affine surface defined over \( k \). An \( A^1 \)-fibration on \( X \) over a curve \( Y \) is a surjective morphism \( \rho: X \to Y \) from \( X \) to a nonsingular curve \( Y \) whose general fibers are isomorphic to the affine line \( A_1 \). It is known that every fiber of \( \rho \) is supported by a disjoint union of irreducible components, each of which is isomorphic to \( A_1 \) (cf. [7]). Let \( F = \rho^*(P) \) be a fiber of \( \rho \) lying over a point \( P \) of \( Y \), and write \( F = \sum n_i C_i \), where \( C_i \) is isomorphic to \( A_1 \) and \( n_i > 0 \) for every \( i \).

We say that \( F \) is a singular fiber of the first kind (resp. the second kind) if \( s^2 \) and \( n_i \) for some \( i \) (resp. \( n_i \geq 2 \) for every \( i \)). We also say that \( F \) is a multiple fiber of multiplicity \( \mu \) if \( \mu := \text{G.C.D.} (n_1, \ldots, n_s) \).

Let \( R := k[u_1, \ldots, u_r] \) be a polynomial ring of dimension \( r \) over \( k \), and let \( A \) be a finitely generated, two-dimensional, regular \( k \)-subalgebra of \( R \). Let \( X = \text{Spec}(A) \), which is a nonsingular affine rational surface. We know that the group \( A^* \) of invertible elements of \( A \) coincides with \( k^* := k - (0) \), that \( X \) has logarithmic Kodaira dimension \( \kappa(X) = -\infty \), and that \( A \) is isomorphic to a polynomial ring of dimension 2 over \( k \) provided \( A \) is a unique factorization domain (cf. [7]). The condition that \( \kappa(X) = -\infty \) implies that there exists an \( A^1 \)-fibration \( \rho: X \to Y \) over a nonsingular curve \( Y \) (cf. Miyanishi-Sugie [8], Fujita [2]).

In the present case, since \( X \) is dominated by the affine \( r \)-space \( A_r^1 = \text{Spec}(R) \), \( Y \) is isomorphic to \( A_r \) or the projective line \( P^1_r \).

The purpose of this paper is to study the converse: When is a nonsingular affine surface \( X \) with an \( A^1 \)-fibration \( \rho \) over \( A_r^1 \) or \( P^1_r \) dominated by \( A_r^1 \) (\( r \geq 2 \))? If \( X = \text{Spec}(A) \) has an \( A^1 \)-fibration over \( A_r \), we can give the following criterion (Theorem 3.3):

\( X \) is dominated by \( A_r^1 \), that is, \( A \) is contained in \( R \) as a \( k \)-subalgebra, if and only if \( \rho \) has at most one singular fiber of the second kind.
This is done by solving a Diophantine equation in $k[u_1, \ldots, u_r]$ (Theorem 1.2). Meanwhile, if $X = \text{Spec}(A)$ has an $A^1$-fibration over $P^1$, the situation becomes very much complicated. Namely, in order to discuss the embeddability of $A$ into $k[u_1, \ldots, u_r]$ in full generality, we have to know what the solutions of the following Diophantine equation in $k[u_1, \ldots, u_r]$ look like:

$$x_1^{a_1} \cdots x_l^{a_l} + y_1^{b_1} \cdots y_w^{b_w} + z_1^{c_1} \cdots z_r^{c_r} = 0,$$

where $a_i \geq 2$, $b_j \geq 2$, $c_s \geq 2$ for every index $i$ ($1 \leq i \leq l$), $j$ ($1 \leq j \leq m$), $s$ ($1 \leq s \leq n$).

We only give partial answers to the embeddability problem in terms of multiple fibers of $ho$, which are stated as follows:

1. Assume that $A$ is contained in $R$ as a $k$-subalgebra. Then the fibration $\rho$ has at most three multiple fibers. If $\rho$ has three multiple fibers, their multiplicities $\{\mu_1, \mu_2, \mu_3\}$ are given, up to permutation, by one of the following triplets: \{2, 2, n\} ($n \geq 2$), \{2, 3, 3\}, \{2, 3, 4\} and \{2, 3, 5\} (cf. Theorem 3.5).

2. Assume, conversely, that $\rho$ satisfies the following two conditions:
   (i) $\rho$ has no singular fibers of the second kind except at most three multiple fibers, each of which is supported by a single irreducible component;
   (ii) if $\rho$ has three multiple fibers, the set of multiplicities $\{\mu_1, \mu_2, \mu_3\}$ is, up to permutation, one of the triplets given in the assertion (1).

Then $A$ is contained in a polynomial ring as a $k$-subalgebra (cf. Theorem 3.7).

In order to obtain these results, we consider an affine hypersurface $S_{p_1, p_2, p_3}$ in $A_1^3 = \text{Spec}(k[x_1, x_2, x_3])$ defined by an equation

$$x_1^{p_1} + x_2^{p_2} + x_3^{p_3} = 0 \quad \text{with} \quad p_1, p_2, p_3 \geq 2,$$

and also a complete intersection $\Sigma_{p_1, p_2, p_3}$ in $A_4^5 = \text{Spec}(k[x_1, x_2, x_3, x_4])$ defined by equations

$$x_1^{p_1} + x_2^{p_2} + x_3^{p_3} = 0 \quad \text{and} \quad ax_1^{p_1} + bx_2^{p_2} + cx_4^{p_4} = 0$$

with $p_1, p_2, p_3, p_4 \geq 2$ and $a \in k - \{0, 1\}$. Indeed, we have to compute $\kappa(S_{p_1, p_2, p_3}^*)$, where $S_{p_1, p_2, p_3}^* = S_{p_1, p_2, p_3}^* - (0)$, and determine when there exists a dominant morphism from $A_1^3$ to $S_{p_1, p_2, p_3}^*$ or $\Sigma_{p_1, p_2, p_3}^* := \Sigma_{p_1, p_2, p_3}^* - (0)$ (cf. Theorems 2.8 and 2.15).

The terminology and the notations in this article conform to the use in the previous paper [7] and the general current practice. We shall list up the notations in frequent use.

- $A_1^r$: the affine space of dimension $r$ defined over $k$;
- $P_1^r$: the projective space of dimension $r$ defined over $k$;
- $\kappa(X)$: the logarithmic Kodaira dimension of a nonsingular algebraic variety $X$;
- $A^*$: the multiplicative group consisting of the invertible elements of $A$;
- $(a_1, \ldots, a_n)$ or $G.C.D. (a_1, \ldots, a_n)$: the greatest common divisor of positive integers $a_1, \ldots, a_n$;
Let \( R := k[u_1, \ldots, u_n] \) be a polynomial ring of dimension \( r \) over \( k \). Let us consider a Diophantine equation in \((m+n)\)-variables,

\[ x_1^{a_1} \cdots x_m^{a_m} - y_1^{b_1} \cdots y_n^{b_n} = 1, \]  

(1)

where \( m, n \geq 1 \) and \( a_j's \) and \( b_j's \) are integers larger than 1, and look for its solutions in \( R \). A solution \( \{x_i = f_i, y_j = g_j; 1 \leq i \leq m, 1 \leq j \leq n\} \) is called a constant solution if \( f_i \in k \) and \( g_j \in k \) for every \( i \) and every \( j \). Otherwise, it is called a non-constant solution.

1.2. We shall prove the following

**Theorem.** A non-constant solution of the equation (1) in \( R \) has one of the following forms:

1. \( x_i = 0 \) for some \( 1 \leq i \leq m \), \( y_j = c_j \in k \) for every \( 1 \leq j \leq n \), where \( c_1^1 \cdots c_n^n = -1 \);

2. \( y_j = 0 \) for some \( 1 \leq j \leq n \), and \( x_i = c_i \in k \) for every \( 1 \leq i \leq m \), where \( c_1^1 \cdots c_n^n = 1 \).

The proof will be given in the paragraph 1.3.

1.3. Let \( \{x_i = f_i, y_j = g_j\} \) be a non-constant solution such that \( f_i \in k \) and \( g_j \in k \) for some \( i \) and \( j \). By reducing the number of variables in the equation (1) if necessary, we may assume that \( f_i \in k \) and \( g_j \in k \) for every \( 1 \leq i \leq m \) and every \( 1 \leq j \leq n \).

On the other hand, we may assume that \( R \) is a polynomial ring in one variable \( u \). In effect, let \( \gamma_i(u), \ldots, \gamma_r(u) \) be sufficiently general polynomials in \( k[u] \), and let \( \varphi_i := f_i(\gamma_i(u), \ldots, \gamma_r(u)) \) and \( \psi_j := g_j(\gamma_i(u), \ldots, \gamma_r(u)) \). Then \( \{x_i = \varphi_i, y_j = \psi_j\} \) is a non-constant solution of the equation (1) in \( k[u] \) such that \( \varphi_i \in k \) and \( \psi_j \in k \) for every \( 1 \leq i \leq m \) and every \( 1 \leq j \leq n \). Such polynomials \( \gamma_i(u), \ldots, \gamma_r(u) \) exist because \( k \) is an infinite field. If we can show the non-existence of such a solution in \( k[u] \), it implies the non-existence of a non-constant solution of (1) in \( R \) such that \( f_i \in k \) and \( g_j \in k \) for some \( i \) and \( j \). Thus, we may assume that \( R = k[u] \).

By replacing again the equation (1) by an equation of the same kind in more unknowns if necessary, we may assume that \( f_i = c_i(u - \alpha_i) \) and \( g_j = d_j(u_j - \beta_j) \),
where $c_i, \alpha_i, d_j, \beta_j \in k$, and $\alpha_i \neq \alpha'_i$, $\beta_j \neq \beta'_j$ whenever $i \neq i'$ and $j \neq j'$. Finally, we obtain a relation in a variable $u$,

$$c(u-\alpha_1)^{a_1} \cdots (u-\alpha_m)^{a_m} - d(u-\beta_1)^{b_1} \cdots (u-\beta_n)^{b_n} = 1, \quad \cdots \cdots \cdots (2)$$

where $c, d \in k^*$. We shall show that such identity in $u$ is impossible.

Note that every $\alpha_i$ is distinct from $\beta_1, \ldots, \beta_n$ and every $\beta_j$ is distinct from $\alpha_1, \ldots, \alpha_m$. By differentiating both hand sides of the equation (2) in $u$, we obtain a relation,

$$c \prod_{i=1}^m (u-\alpha_i)^{a_i} \cdot \left\{ \sum_{i=1}^m \frac{a_i}{u-\alpha_i} \right\} = d \prod_{j=1}^n (u-\beta_j)^{b_j} \cdot \left\{ \sum_{j=1}^n \frac{b_j}{u-\beta_j} \right\}. \quad \cdots \cdots \cdots (3)$$

Note that we have

$$\deg\left( \prod_{i=1}^m (u-\alpha_i) \cdot \left\{ \sum_{i=1}^m \frac{a_i}{u-\alpha_i} \right\} \right) \leq m-1, \text{ and}$$

$$\deg\left( \prod_{j=1}^n (u-\beta_j) \cdot \left\{ \sum_{j=1}^n \frac{b_j}{u-\beta_j} \right\} \right) \leq n-1.$$ 

Since $a_i \geq 2$ and $b_j \geq 2$ by assumption, the relation (3) implies that

$$\prod_{i=1}^m (u-\alpha_i) \cdot \left\{ \sum_{j=1}^n \frac{b_j}{u-\beta_j} \right\}$$

is divisible by $\prod_{i=1}^m (u-\alpha_i)$. Hence we obtain $m \leq n-1$. Similarly, we have $n \leq m-1$. This is a contradiction. Therefore, we have shown that if $\{x_i=f_i, y_j=g_j\}$ is a non-constant solution of the equation (1), then either $f_i \in k$ for every $1 \leq i \leq m$ or $g_j \in k$ for every $1 \leq j \leq n$.

Suppose that the first case takes place, i.e., $f_i = c_i \in k$ for every $1 \leq i \leq m$. Then $g_j \not\in k$ for some $j$. If $\prod_{i=1}^m c_i \neq 1$, then $g_j$ would be a unit in $R$; this is a contradiction. Hence $\prod_{i=1}^m c_i = 1$, and $g_j = 0$ for some $j$. The other case can be treated in a similar way. Q.E.D.

2. A Diophantine equation, II

2.1. In this section, we shall consider a Diophantine equation

$$x_1^p + x_2^p + x_3^p = 0, \quad \cdots \cdots \cdots \cdots (4)$$

where $p_1, p_2$ and $p_3$ are integers larger than 1, and look for non-constant solutions in $R := k[u_1, \ldots, u_n]$. Let $S_{p_1, p_2, p_3}$ be the affine hypersurface in $A_3^1 := \text{Spec}(k[x_1, x_2, x_3])$ defined by the equation (4), and let $S_{p_1, p_2, p_3}^* := S_{p_1, p_2, p_3} \setminus \{0\}$, where $0$ is the point $(0, 0, 0)$. When there is no fear of confusion, we denote $S_{p_1, p_2, p_3}$ and $S_{p_1, p_2, p_3}^*$ simply by $S$ and $S^*$, respectively. It is easy to see that $S$
is a normal surface with the unique singular point (0). The resolution of singularity of $S$ at the point (0) is completely understood (cf. Orlik-Wagreich [10]). We recall some of the results which we need in our subsequent arguments.

2.2. Let $G_m$ be the multiplicative group scheme defined over $k$. We need the following:

**Lemma.** Let $X$ be a nonsingular quasi-projective surface with an effective separated $G_m$-action. Assume that $X$ has no fixed points. Let $Y:=X/G_m$ be the quotient variety and let $\pi: X\to Y$ be the canonical projection. Then we have:

1. $Y$ is a nonsingular curve;
2. $\pi^{-1}(y)\cong \mathbb{A}^1_*$ for every point $y\in Y$, where $\mathbb{A}^1_*$ is the affine line $\mathbb{A}^1$ with one point deleted off;
3. $\pi^*y$ is a multiple fiber with multiplicity $\mu$ if and only if the stabilizer group $\sigma_x$ is a cyclic group of order $\mu$ for a point $x$ in $\pi^{-1}(y)$.

**Proof.** Let $x$ be a point of $X$. By virtue of Sumihiro [11; Cor. 2], there exists a $G_m$-stable affine open neighborhood $U:=\text{Spec}(A)$ of $x$. Let $B$ be the subalgebra of $G_m$-invariants in $A$. Then $\bar{U}: =\text{Spec}(B)$ is an affine open neighborhood of $y:=\pi(x)$. Since $A$ is regular, $B$ is normal. Hence $Y$ is a nonsingular curve. It is known by the theory of quotient varieties with respect to reductive group actions (e.g., Mumford [9; Chap. 1]) that $\pi^{-1}(y)$ consists of a single orbit under the stated assumption. Hence the assertion (2) holds.

Consider a $G_m$-equivariant completion $X\to Z$, where we may assume that $Z$ is a nonsingular projective surface (cf. Sumihiro [11]). Let $O(x)$ be the orbit through $x$, and let $C$ be the closure of $O(x)$ in $Z$. Then $C$ contains a fixed point $z$. We can find a system of local coordinates $(u, v)$ at $z$ such that $u=0$ defines a branch of $C$ through $z$ and the induced $G_m$-action on the tangent space $T_{z,z}$ is normalized as $t(ξ, η)= (t^xξ, t^yη)$, where $t\in k^*$, $ξ$ and $η$ are integers and $ξ=\partial/\partial u$ and $η=\partial/\partial v$. Replacing the $G_m$-action $(t, z)\mapsto t^z z$ on $Z$ by a $G_m$-action $(t, z)\mapsto t^{-1} z$ and interchanging the roles of $u$ and $v$ if necessary, we may assume that $β>0$.

Since $\hat{O}_{z}\cong k[[u, v]]$, $α$ and $β$ are prime to each other; if $α=0$ then $β=1$. Let $y:=\pi(x)$. Then $\hat{O}_{y}\cong k[[u^β v^β]]$, and the orbit $O(x)$ is defined by $u=0$ in a neighborhood of $z$. Hence the multiplicity of $\pi^*y$ is $β$, and the stabilizer group of a point (hence of the point $x$) of the orbit $O(x)$ is $Z/βZ$. Hence the assertion (3) holds true.

Q.E.D.

2.3. Let $p_1, p_2$ and $p_3$ be the same as for the equation (4). Let $d:= L.C.M.(p_1, p_2, p_3)$ and define the integers $q_i$ ($1\leq i\leq 3$) by $d=p_i q_i$. The group scheme $G_m$ acts effectively on $S^{\ast}_{p_1, p_2, p_3}$ by

$$t(x_1, x_2, x_3) = (t^{q_1} x_1, t^{q_2} x_2, t^{q_3} x_3).$$

Then $S^{\ast}_{p_1, p_2, p_3}$ has no fixed points. Let $C:=S^{\ast}/G_m$ and let $\pi: S^{\ast}\to C$ be the
096 M. MIYANISHI

canonical projection. Then we have:

**Lemma.** (1) The genus \( g \) of \( C \) is given by

\[
g = \frac{d^2}{2q_1q_2q_3} - \frac{d}{2} \left( \frac{(q_1, q_2)}{q_3} + \frac{(q_2, q_3)}{q_1} + \frac{(q_3, q_1)}{q_2} \right) + 1.
\]

(2) \( \pi \) has no multiple fibers but possibly \( \frac{d(q_1, q_2)}{q_3q_2} \) fibers with multiplicity \((q_1, q_2)\), \( \frac{d(q_2, q_3)}{q_1q_3} \) fibers with multiplicity \((q_2, q_3)\) and \( \frac{d(q_3, q_1)}{q_2q_1} \) fibers with multiplicity \((q_3, q_1)\).

**Proof.** (1) Let \( T \) be the hypersurface in \( \mathbb{A}^3 := \text{Spec}(k[y_1, y_2, y_3]) \) defined by \( y_1^2 + y_2^2 + y_3^2 = 0 \), and let \( T^* := T - \{0\} \). Let \( \Phi : T^* \to \mathbb{P}^2 \) be the morphism defined by \( (x_1, x_2, x_3) \mapsto (y_1^2, y_2^2, y_3^2) \). Let \( G_m \) act on \( T^* \) via \( t(y_1, y_2, y_3) = (ty_1, ty_2, ty_3) \). Then \( \Phi \) is a \( G_m \)-equivariant morphism. Let \( D := T^*/G_m \). Then \( \Phi \) induces a surjective morphism \( \varphi : D \to C \) such that \( \pi \cdot \Phi = \varphi \cdot \pi' \), where \( \pi' : T^* \to D \) is the canonical quotient morphism. Then it is easy to show that \( \deg \varphi = d \cdot q_3 \) and the morphism \( \varphi \) ramifies at \( d \) points with ramification index \( q_3(q_1, q_2) \), at \( d \) points with ramification index \( q_2(q_3, q_1) \) and at \( d \) points with ramification index \( q_1(q_2, q_3) \). Since \( D \) has genus \( \frac{1}{2}(d-1)(d-2) \), the genus \( g \) of \( C \) is obtained by the Riemann-Hurwitz formula applied to \( \varphi : D \to C \). The assertion (2) can be verified by means of Lemma 2.2. Q.E.D.

2.4. Let \( p_i (1 \leq i \leq 4) \) be integers larger than 1. Let \( \Sigma^* := \text{Spec}(k[x_1, x_2, x_3, x_4]) \) be the surface in \( \mathbb{A}^4 := \text{Spec}(k[y_1, y_2, y_3]) \) defined by equations,

\[ x_1^p_1 + x_2^p_2 + x_3^p_3 = 0 \quad \text{and} \quad ax_1^p_1 + x_2^p_2 + x_4^p_4 = 0, \]

where \( a \in k \setminus \{0, 1\} \). Let \( \Sigma^* \) be the surface in \( \mathbb{A}^4 := \text{Spec}(k[x_1, x_2, x_3, x_4]) \) defined by equations,

\[ x_1^p_1 + x_2^p_2 + x_3^p_3 = 0 \quad \text{and} \quad ax_1^p_1 + x_2^p_2 + x_4^p_4 = 0, \]

where \( a \in k \setminus \{0, 1\} \). Let \( \pi : \Sigma^* \to C \) be the canonical quotient morphism. We have the following:

**Lemma.** (1) The genus \( g \) of \( C \) is given by the formula:

\[
g = \frac{d^3}{q_1q_2q_3q_4} - \frac{d^2}{2} \left( \frac{(q_1, q_2)}{q_3} + \frac{(q_2, q_3)}{q_1} + \frac{(q_3, q_1)}{q_2} \right) + 1.
\]
(2) \( \pi \) has no multiple fibers but possibly \( \frac{d^2(q_1, q_2, q_3)}{q_1 q_2 q_3} \) fibers with multiplicity \( (q_1, q_2, q_3) \), \( \frac{d^2(q_1, q_2, q_4)}{q_1 q_2 q_4} \) fibers with multiplicity \( (q_1, q_2, q_4) \), \( \frac{d^2(q_1, q_3, q_4)}{q_1 q_3 q_4} \) fibers with multiplicity \( (q_1, q_3, q_4) \) and \( \frac{d^2(q_2, q_3, q_4)}{q_2 q_3 q_4} \) fibers with multiplicity \( (q_2, q_3, q_4) \).

Proof. Similar to the proof of Lemma 2.3.

2.5. As an application of Lemma 2.4, we have the following examples:

<table>
<thead>
<tr>
<th>{p_1, p_2, p_3, p_4}</th>
<th>( g(S^*/G_m) )</th>
<th>multiple fibers of ( \pi: S* \to C := S^*/G_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>{2, 2, 2, 2s}</td>
<td>1</td>
<td>4 fibers with multiplicity ( s )</td>
</tr>
<tr>
<td>{2, 2, 2, 2s+1}</td>
<td>0</td>
<td>4 fibers with multiplicity ( 2s+1 )</td>
</tr>
<tr>
<td>{2, 2, 3, 3}</td>
<td>2</td>
<td>no multiple fibers</td>
</tr>
<tr>
<td>{2, 2, 3, 4}</td>
<td>0</td>
<td>2 fibers with multiplicity ( 2 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4 fibers with multiplicity ( 3 )</td>
</tr>
<tr>
<td>{2, 2, 3, 5}</td>
<td>0</td>
<td>2 fibers with multiplicity ( 5 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 fibers with multiplicity ( 3 )</td>
</tr>
</tbody>
</table>

2.6. From this paragraph on up to 2.14, we shall retain the notations of 2.1. Let \( p_i := p_i/(q_2, q_3) \), \( p_i' := p_i/(q_1, q_3) \) and \( p_i' := p_i/(q_1, q_3) \); \( p_i' \) (\( 1 \leq i \leq 3 \)) are integers because, for example, \( d = p_1 q_1 \) and \( (q_1, (q_2, q_3)) = 1 \) imply that \( p_i \) is divisible by \( (q_2, q_3) \). As an easy application of Lemma 2.3, we know that \( g = 0 \) (resp. \( g = 1 \), resp. \( g > 1 \)) if and only if \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1 \) (resp. \( 1 \), resp. \( < 1 \)).

2.7. We have the following:

**Lemma.** Assume that \( p_1 \leq p_2 \leq p_3 \) and \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \). Then we have:

1. \( \{p_1, p_2, p_3\} = \{2, 3, 6\}, \{2, 4, 4\} \) or \( \{3, 3, 3\} \).
2. \( C := S^*/G_m \) is a nonsingular elliptic curve, and \( \pi: S^* \to C \) has no multiple fibers, i.e., \( S^* \) is an \( \mathbb{A}^1 \)-bundle over \( C \).
3. Let \( b := d/q_1 q_2 q_3 \). Then \( b = 1, 2, 3 \) for \( \{p_1, p_2, p_3\} = \{2, 3, 6\}, \{2, 4, 4\} \) and \( \{3, 3, 3\} \), respectively. There exists an invertible sheaf \( \mathcal{L} \) of degree \( b \) over \( C \) such that the ruled surface \( V := \text{Proj}(\mathcal{O}_C \oplus \mathcal{L}) \) over \( C \) with the zero section \( M_0 \) and the infinity section \( M_\infty \) deleted off is isomorphic to \( S^* \).
4. \( \kappa(S^*) = 0 \).

Proof. (1) follows from a well-known straightforward computation. (2) follows from Lemma 2.3. Since \( S^* \) is an \( \mathbb{A}^1 \)-bundle over \( C \), \( S^* \) is obtained from a ruled surface in the way as specified in the assertion (3). Then \( (M_0^2) = -b \), \( (M_\infty^2) = b \) and \( (M_z, M_\infty) = 0 \). The number \( b := \deg \mathcal{L} \) is equal to \( d/q_1 q_2 q_3 \),
because $M_0$ is the unique exceptional curve which arises from the minimal resolution of singularity of the point $(0,0,0)$ of $S$ (cf. Orlik-Wagreich [10]). Note that the canonical divisor $K_v$ of $V$ is linearly equivalent to $-M_0-M_*$. The boundary divisor of $S^*$ in $V$ is $D:=M_0+M_*$. Hence $D+K_V\sim 0$. Therefore, we have $g(S^*)=0$. Q.E.D.

2.8. We shall prove

**Theorem** (cf. Itaka [4]). $S^*_{p_1,p_2,p_3}$ has the logarithmic Kodaira dimension $g(S^*_{p_1,p_2,p_3})=-\infty$, 0, 1 according as $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}>1$, $=1$, $<1$, respectively.

The proof will be given in the paragraphs 2.9~2.11.

2.9. Let $V$ be a nonsingular projective surface with a surjective morphism $\varphi: V\to C:=S^*/G_m$ satisfying the following conditions:

(i) $V$ contains $S^*_{p_1,p_2,p_3}$ as a dense open set, and $\varphi|_{S^*}: S^*\to C$;

(ii) $V-S^*$ contains no exceptional curves of the first kind which are contained in fibers of $\varphi$.

It is clear that general fibers of $\varphi$ are isomorphic to $\mathbb{P}^1$. The resolution of singularity of $S^*_{p_1,p_2,p_3}$ at the unique singular point $(0)=(0,0,0)$ is described in detail in Orlik-Wagreich [10]. We recall some of the necessary results. The morphism $\pi: S^*\to C$ has multiple fibers if one of $(q_1, q_2)$, $(q_2, q_3)$ and $(q_3, q_1)$ is larger than 1. If $(q_1, q_2)>1$, there are $d(q_1, q_2)/q_1q_2$ fibers of multiplicity $(q_1, q_2)$ (cf. Lemma 2.3). For a multiple fiber $F$ of multiplicity $(q_1, q_2)$, set $\alpha:=(q_1, q_2)$ and determine an integer $\beta$ uniquely by the condition that $q_3\beta\equiv 1 \pmod{\alpha}$ and $0<\beta<\alpha$. Define positive integers $b_1, \ldots, b_s\geq 2$ by writing $\alpha/(\alpha-\beta)$ in the form of a continued fraction

$$\frac{\alpha}{\alpha-\beta} = b_1 - \frac{1}{b_2 - \frac{1}{\ldots - \frac{1}{b_s}}}$$

which we write in the form $\alpha/(\alpha-\beta)=[b_1, \ldots, b_s]$. For multiple fibers of multiplicity $(q_2, q_3)$ or $(q_1, q_3)$, we determine the corresponding integers $\alpha, \beta, b_1, \ldots, b_s$ etc. Let $N$ be the number of the multiple fibers of $\pi$. Let

$$b: = \frac{d}{q_1q_2q_3} \sum_{i=1}^{\infty} \frac{\beta_i}{\alpha_i},$$

where $\{\alpha_i, \beta_i\}$ ranges over all pairs of integers which are determined for all multiple fibers of $\pi$ in the above-mentioned fashion. Let $g$ be the genus of $C$. Then the dual graph of the exceptional curves which arise from the resolution
of singularity of the point (0) of $S_{\theta_1, \theta_2, \theta_3}$ has a vertex with weight $-b-N$ (corresponding to a nonsingular curve of genus $g$) and has $N$ branches, each of which is a linear chain of nonsingular rational curves as exhibited in the following figure:

![Diagram](image)

2.10. The fibration $\varphi: V \to C$ has two cross-sections $M_0'$ and $M'_\omega$ and $N$ singular fibers $\Phi_1, \ldots, \Phi_N$ such that:

(i) $M_0'$ and $M'_\omega$ are nonsingular curves of genus $g$; $(M_0'^s) = -b-N$ and $(M'_\omega^s) = b$;

(ii) Let $\Phi$ be a singular fiber of $\varphi$; then $\Phi \cap S^* = a F$ with $F \cong A^1$, i.e., a multiple fiber of multiplicity $\alpha > 1$; the component $F$ of $\Phi$ (=the closure of $F$ in $V$) is connected to the cross-section $M_0'$ by $s$ components as exhibited in

$$F \rightarrow \cdots \rightarrow -b_s \rightarrow -b_{s-1} \rightarrow -b_1 \rightarrow M_0'.$$

By assumption, $\Phi - F$ contains no exceptional curves of the first kind. Hence $F$ is the unique exceptional curve of the first kind contained in the singular fiber $\Phi$. Then it is easily ascertained that the dual graph of the fiber $\Phi$ is a linear chain. It looks like the one given in Miyanishi [6; p. 95]. To fix the notations, we represent it in the next page. The upper half of the chain between $E_0$ and $E(a, m_a)$ (with $E(a, m_a)$ excluded) corresponds to the chain

$$-b_s \rightarrow \cdots \rightarrow -b_1.$$

Hence we have

$$\frac{\alpha}{\alpha - \beta} = [b_1, \ldots, b_s].$$

$$\begin{cases} [m_1+1, \underbrace{2, \ldots, 2, m_3+2, 2, \ldots, 2, m_{s-1}+2, 2, \ldots, 2}] & \text{if } a \text{ is even} \\ m_2-1 & m_a-1 \\ \end{cases}$$

$$\begin{cases} [m_1+1, \underbrace{2, \ldots, 2, m_3+2, 2, \ldots, 2, m_a+2+2, 2, \ldots, 2, m_{s-1}+1}] & \text{if } a \text{ is odd} \\ m_2-1 & m_a-1-1 \\ \end{cases}$$
Note that α is the multiplicity of $F$ in the fiber $Φ$. This is clear because $Φ \cap S^* = αF$. We can check this fact as follows. The multiplicity $μ(i,j)$ ($1 ≤ i ≤ a; 1 ≤ j ≤ m_i$) of the component $E(i,j)$ in $Φ$ is given by the function

<table>
<thead>
<tr>
<th>$a$: even</th>
<th>$a$: odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-(m_a+1)$</td>
<td>$-(m_a+1)$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$E(2,1)$</td>
<td>$E(2,1)$</td>
</tr>
<tr>
<td>$m_a-1$</td>
<td>$m_a-1$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$E(2,m_a-1)$</td>
<td>$E(2,m_a-1)$</td>
</tr>
<tr>
<td>$-(m_a+2)$</td>
<td>$-(m_a+2)$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$E(2,m_a)$</td>
<td>$E(2,m_a)$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$E(4,1)$</td>
<td>$E(4,1)$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$E(a-2,m_a-2)$</td>
<td>$E(a-3,m_a-3)$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$E(a,1)$</td>
<td>$E(a-1,1)$</td>
</tr>
<tr>
<td>$m_a-1$</td>
<td>$m_a-1$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$E(a,m_a-1)$</td>
<td>$E(a-1,m_a-1)$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$E(a,m_a):=F$</td>
<td>$E(a-1,m_a):=F$</td>
</tr>
<tr>
<td>$-(m_a+1)$</td>
<td>$-(m_a+1)$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$E(a-1,m_a-1)$</td>
<td>$E(a,m_a-1)$</td>
</tr>
<tr>
<td>$m_a-1$</td>
<td>$m_a-1$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$E(a-1,1)$</td>
<td>$E(a,1)$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$E(a-3,m_a-3)$</td>
<td>$E(a-2,m_a-3)$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$E(3,1)$</td>
<td>$E(3,1)$</td>
</tr>
<tr>
<td>$-(m_a+2)$</td>
<td>$-(m_a+2)$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$E(1,m_a)$</td>
<td>$E(1,m_a)$</td>
</tr>
<tr>
<td>$m_1-1$</td>
<td>$m_1-1$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$E(1,m_a-1)$</td>
<td>$E(1,m_a-1)$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$E(1,1)$</td>
<td>$E(1,1)$</td>
</tr>
</tbody>
</table>
\( \mu(i, j) \) defined inductively by:

\[
\begin{align*}
\mu(0, m_0) & := 1, \quad \mu(1, j) = j \quad \text{for } 1 \leq j \leq m_1, \\
\mu(i, 1) & = \mu(i-1, m_{i-1}) + \mu(i-2, m_{i-2}) \quad \text{for } 1 < i \leq a, \\
\mu(i, j) & = \mu(i, j-1) + \mu(i-1, m_{i-1}) \quad \text{for } 1 < j \leq m_i.
\end{align*}
\]

On the other hand, the integer \( \alpha \) is regained by the method as indicated in the appendix of [10; p. 76] from the above development of \( \alpha/(\alpha-\beta) \) into a continued fraction.

2.11. Note that \( V-S^* \) consists of nonsingular components crossing normally. It is also easy to see that there exists a unique contraction \( \sigma: V \to V_0 \), where

(i) \( \varphi_0: V_0 \to C \) is a relatively minimal ruled surface;

(ii) Let \( M_0 := \sigma \ast M^0 \) and \( M_\infty := \sigma \ast M^\prime \). Then

\( (M_0^\ast) = -(b+N) \) and

\( (M_\infty^\ast) = b+N. \)

The canonical divisor \( K_{V_0} \) is given by

\[
K_{V_0} \sim -M_0 - M_\infty + \varphi_0^\ast(K_C) \quad \text{and} \quad M_\infty \sim M_0 + \varphi_0^\ast(\delta),
\]

where \( K_C \) is the canonical divisor of \( C \) and \( \delta \) is a divisor on \( C \) with \( \deg(\delta) = b+N \). In effect, \( V_0 = \text{Proj}(O_C \oplus O_C(\delta)) \), and \( M_0 \) and \( M_\infty \) correspond to the zero section and the infinite section of \( V_0 \), respectively.

Each irreducible component \( E(i, j) \) of the singular fiber has the contribution \( k(i, j) \) in the canonical divisor \( K_V \) determined inductively as follows:

\[
\begin{align*}
k(0, m_0) & := 0, \quad k(1, j) = j \quad \text{for } 1 \leq j \leq m_1, \\
k(i, 1) & = k(i-1, m_{i-1}) + k(i-2, m_{i-2}) + 1 \quad \text{for } 1 < i \leq a, \\
k(i, j) & = k(i, j-1) + k(i-1, m_{i-1}) + 1 \quad \text{for } 1 < j \leq m_i.
\end{align*}
\]

On the other hand, \( E(i, j) \) has multiplicity \( n(i, j) \) in \( \sigma^\ast(M_\infty) \), which is determined by

\[
\begin{align*}
n(0, m_0) & := 0, \quad n(1, j) = 1 \quad \text{for } 1 \leq j \leq m_1, \\
n(i, 1) & = n(i-1, m_{i-1}) + n(i-2, m_{i-2}) \quad \text{for } 1 < i \leq a, \\
n(i, j) & = n(i, j-1) + n(i-1, m_{i-1}) \quad \text{for } 1 < j \leq m_i.
\end{align*}
\]

Let \( D \) be the reduced effective divisor such that \( \text{Supp}(D) = V-S^* \). Then it is straightforward to show that the coefficient \( \nu(i, j) \) of \( E(i, j) \) in \( D + K_V - \Phi \) is given by,

\[
\nu(i, j) = \begin{cases} 
0 & \text{if } (i, j) \neq (a, m_a) \\
-1 & \text{if } (i, j) = (a, m_a).
\end{cases}
\]

Therefore we have:
\[
D + K_Y \sim \sum_{i=1}^{N} \Phi_i - \sum_{i=1}^{N} F_i + \varphi^*(K_c)
\]
\[
\geq \sum_{i=1}^{N} \left(1 - \frac{1}{\alpha_i}\right) \Phi_i + \varphi^*(K_c),
\]
where \(\alpha_i\) is the multiplicity of \(F_i\) in \(\Phi_i\). Let
\[
A := \left(\sum_{P \in e} \left(1 - \frac{1}{\alpha_P}\right) \pi^*(P) + \varphi^*(K_c) \cdot M_0\right).
\]
Note that \(\alpha_i\) has one of the values \((q_1, q_2), (q_2, q_3)\) and \((q_3, q_1)\) (cf. 2.9) and that \(A\) is, in effect, equal to
\[
\left(\sum_{P \in e} \left(1 - \frac{1}{\alpha_P}\right) \pi^*(P) + \varphi^*(K_c) \cdot M_0\right),
\]
where \(\pi^*(P) = \alpha_P F_P\) with \(F_P \approx A_k^\perp\). Then we can calculate \(A\) as follows:
\[
A = \frac{d(q_1, q_3)}{q_1 q_3} + \frac{d(q_2, q_3)}{q_2 q_3} + \frac{d(q_3, q_1)}{q_3 q_1} - \frac{d(q_1, q_3)}{q_1 q_2} \frac{1}{(q_1, q_3)}
\]
\[
- \frac{d(q_2, q_3)}{q_2 q_3} \frac{1}{(q_2, q_3)} - \frac{d(q_3, q_1)}{q_3 q_1} \frac{1}{(q_3, q_1)} + 2g - 2
\]
\[
= \frac{d^2}{q_1 q_2 q_3} \left(1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3}\right).
\]
We have clearly \(\kappa(S^*) = 1\) if \(A > 0\), because \(D + K_Y\) is linearly equivalent to a divisor supported by fibers and the components contained in fibers of \(\varphi\). If \(A = 0\) we have \(\kappa(S^*) = 0\) (cf. 2.7). If \(A < 0\), i.e., \(1 < \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}\), we have the following under an additional assumption \(2 \leq p_1 \leq p_2 \leq p_3\): \(\{p_1, p_2, p_3\} = \{2, 2, n\}\) \((n \geq 2)\), \(\{2, 3, 3\}\), \(\{2, 3, 4\}\) or \(\{2, 3, 5\}\). In each of the above four cases for \(A < 0\), the foregoing arguments of evaluating \(D + K_Y\) shows that \(\kappa(S^*) = -\infty\); note that if \(A < 0\) then \(g = 0\). This completes the proof of Theorem 2.8.

2.12. If \(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1\), the surface \(S_{p_1, p_2, p_3}\) is the quotient variety of \(A_k^\perp\) with respect to a linear action of a Kleinian subgroup \(G\) of \(GL(2, k)\) (cf. Brieskorn [1]). In effect, \(G\) acts freely on \((0)\). Hence there exists an étale finite morphism \(\rho: A_k^\perp \rightarrow S^*\), and \(A_k^\perp \rightarrow (0)\) is algebraically simply connected.

Suppose that the ground field \(k\) is the field \(C\) of complex numbers. Let \(U\) be the universal covering space of \(S_{p_1, p_2, p_3}^*\). Then it is known(*) that:

(*) This was communicated by Dr. A. Fujiki.
where \( D \) is a unit disc.

2.13. For later use, we shall prove:

**Lemma.** Suppose that \( \tilde{r}(S^*_{p_1,p_2,p_3}) > 0 \) and \( C \cong P^1_\mathbb{C} \). Then \( \pi : S^* \to C \) has three or more multiple fibers.

**Proof.** We have the inequalities,

\[
\frac{(q_2, q_3) + (q_3, q_1) + (q_1, q_3)}{p_1} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.
\]

(c.f. 2.6 and 2.8). Hence it is impossible that \((q_2, q_3) = (q_3, q_1) = (q_1, q_3) = 1\). If \((q_2, q_3) > 1, (q_3, q_1) > 1\) and \((q_1, q_3) > 1\), \( \pi \) has three or more multiple fibers. We shall consider the cases where one or two of \((q_2, q_3)\), \((q_3, q_1)\) and \((q_1, q_3)\) equal 1.

Assume first that \((q_2, q_3) = 1, (q_3, q_1) > 1\) and \((q_1, q_3) > 1\). Suppose that \(d(q_3, q_1)/q_3q_1 = d(q_1, q_3)/q_2q_3 = 1\). Then \(q_3 = p_1(q_1, q_3)\) and \(q_2 = p_1(q_1, q_3)\). Hence \((q_2, q_3)\) is divisible by \(p_1\). Since \(p_1 > 1\), this contradicts the assumption that \((q_2, q_3) = 1\).

Hence \(d(q_3, q_1) > 1\) or \(d(q_1, q_3) > 1\). Thus \( \pi \) has three or more multiple fibers.

Consider next the case where \((q_2, q_3) = (q_3, q_1) = 1\) and \((q_1, q_3) > 1\). Then the above inequalities imply that \((q_1, q_3) > p_3\). Hence \(q_3(q_1, q_3) > d\), and

\[
1 \geq (q_1, q_3) > \frac{d}{q_3q_3}.
\]

However, since \((q_2, q_3) = 1\), \( d \) is divisible by \(q_3q_3\). This is a contradiction. Thus this case does not occur. The other cases can be treated in a similar fashion.

Q.E.D.

2.14. We shall prove the following:

**Theorem.** (1) If \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \leq 1 \), then there are no non-constant morphisms from \( A^1_\mathbb{C} \) to \( S^*_{p_1,p_2,p_3} \).

(2) If \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1 \), then there are dominant morphisms from \( A^1_\mathbb{C} \) to \( S^*_{p_1,p_2,p_3} \).

**Proof.** (1) If \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \), \( S^* \) is an \( A^1_\mathbb{C} \)-bundle over a nonsingular
elliptic curve C. Thus, if \( f: \mathbb{A}^1 \rightarrow \mathbb{S}^* \) is a non-constant morphism, \( f(\mathbb{A}^1) \) is contained in a fiber of \( \pi \), which is isomorphic to \( \mathbb{A}^1 \). This is impossible. So, we may assume that \( \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} < 1 \), i.e., \( \kappa(\mathbb{S}^*) > 0 \). Let \( f: \mathbb{A}^1 \rightarrow \mathbb{S}^* \) be a non-constant morphism if such a morphism exists at all. If \( f \) is dominant, we may assume without loss of generality that \( r = 2 \). Then we have

\[-\infty = \kappa(\mathbb{A}^1) \geq \kappa(\mathbb{S}^*) = 1,
\]

which is impossible. Hence \( f(\mathbb{A}^1) \) is a rational curve with at most one place at infinity, and \( f(\mathbb{A}^1) \) is not contained in any fiber of \( \pi \). Thus we have a dominant morphism

\[ \psi: = \pi \cdot f: \mathbb{A}^1 \rightarrow \mathbb{S}^* \rightarrow C. \]

Hence \( C \) is isomorphic to \( \mathbb{P}^1 \), and \( \psi(\mathbb{A}^1) \) is isomorphic to \( \mathbb{A}^1 \) or \( \mathbb{P}^1 \). Consider first the case where \( \psi(\mathbb{A}^1) = \mathbb{A}^1 \). By 2.13, there exist points \( P, Q \) of \( C \) such that \( P, Q \in \psi(\mathbb{A}^1) \) and that \( \pi*P \) and \( \pi*Q \) are multiple fibers of multiplicity \( \mu \) and \( \nu \), respectively. Choose an inhomogeneous coordinate \( t \) of \( \mathbb{A}^1 \) such that \( P \) and \( Q \) are defined by \( t = 0 \) and \( t = 1 \), respectively. Then there exist non-constant polynomials \( g \) and \( h \) in \( R := \mathbb{K}[u_1, \ldots, u_r] \) such that \( \psi*(t) = g^\mu \) and \( \psi*(t-1) = h^\nu \). This implies that \( \{ x = g, y = h \} \) is a solution of the Diophantine equation

\[ x^\mu - y^\nu = 1. \]

This contradicts Theorem 1.2. Consider next the case where \( \psi(\mathbb{A}^1) = \mathbb{P}^1 \). In order to prove, by reductio ad absurdum, the non-existence of such a non-constant morphism as \( \psi \), we may assume, by embedding the ground field \( k \) into the field \( \mathbb{C} \) of complex numbers in a suitable way, that \( k = \mathbb{C} \). Restricting \( \psi \) onto a suitable line \( \mathbb{A}^1 \subset \mathbb{A}^n \), we may assume that \( r = 1 \). Then the Nevanlinna theory (cf. Hayman [3]) implies that

\[ \sum_{i=1}^{N} \left( 1 - \frac{1}{\alpha_i} \right) - 2 \leq 0, \]

where \( N \) is the number of multiple fibers of \( \pi \) and \( \alpha_i \)'s are multiplicities. The left-hand side of the above inequality is, in effect, equal to \( A \) in 2.11. Hence we have \( \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} \geq 1 \). This is a contradiction. Thus there are no non-constant morphisms \( f: \mathbb{A}^1 \rightarrow \mathbb{S}^* \) provided \( \kappa(\mathbb{S}^*) \geq 0 \).

(2) We may assume that \( p_1 \leq p_2 \leq p_3 \). Then \( \{ p_1, p_2, p_3 \} \) is one of the following triplets: \( \{ 2, 2, n \} \) \( (n \geq 2) \), \( \{ 2, 3, 3 \} \), \{ 2, 3, 4 \}, \{ 2, 3, 5 \}. Except in the case where \( \{ p_1, p_2, p_3 \} = \{ 2, 3, 5 \} \), one can easily find a solution \( \{ x_1 = f_1, x_2 = f_2, x_3 = f_3 \} \) of the equation

\[ x_1^2 + x_2^2 + x_3^2 = 0 \]
in a polynomial ring \( R := k[u_1, \ldots, u_r] \) such that the subvarieties \( \{ f_i = 0 \} \) (\( 1 \leq i \leq 3 \)) have no common points in \( A_3^r \) and that trans. \( \text{deg}_k(f_1, f_2, f_3) = 2 \). Then the assignment \( x_i \mapsto f_i \) (\( 1 \leq i \leq 3 \)) gives rise to a dominant morphism \( f: A_3^r \to S^* \). For example, if \( \{ p_1, p_2, p_3 \} = \{ 2, 2, 2 \} \), such a solution is given by
\[
\begin{align*}
x_1 &= \frac{\xi^2 + \eta^2}{2}, \\
x_2 &= \frac{\xi^2 - \eta^2}{2\sqrt{-1}}, \\
x_3 &= \sqrt{-1} \cdot \xi \eta,
\end{align*}
\]
where \( \xi, \eta \) are polynomials in \( R \) such that \( \{ \xi = 0 \} \) and \( \{ \eta = 0 \} \) have no common points in \( A_3^r \) and that \( \text{trans.} \text{deg}_k(\xi, \eta) = 2 \). The case where \( \{ p_1, p_2, p_3 \} = \{ 2, 3, 5 \} \) seems more subtle.(\(^*\)) We look for a dominant morphism \( f: A_3^r \to S^* \). Since \( A_3^r \) is algebraically simply connected, such a morphism \( f \) (if it exists at all) is factored by a dominant morphism \( \bar{f}: A_3^2 \to A_3^2 \) (0) such that \( f = \rho \cdot \bar{f} \) (cf. 2.12). Conversely, if a dominant morphism \( \bar{f} \) is given, \( f := \rho \cdot \bar{f} \) is a required dominant morphism. Hence we have only to find a dominant morphism \( \bar{f}: A_3^2 \to A_3^2 \). Such a morphism \( \bar{f} \) exists because a dominant morphism \( f: A_3^2 \to S_{7,2,2}^* \) provides one. Note that this argument works also for the other cases. Q.E.D.

2.15. We shall prove:

Theorem. Let \( \Sigma_{p_1, p_2, p_3} \) be the nonsingular surface defined in 2.4. Assume that \( \{ p_1, p_2, p_3, p_4 \} \) is one of the following quadruplets: \( \{ 2, 2, 2, 2s + 1 \} \) (\( s \geq 1 \)), \( \{ 2, 2, 3, 4 \} \), \( \{ 2, 2, 3, 5 \} \), i.e., those in the examples in 2.5 with \( g(\Sigma_{p_1, p_2, p_3}) = 0 \). Then there are no non-constant morphisms from \( A_3^r \) to \( \Sigma_{p_1, p_2, p_3} \).

Proof. We only consider the case where \( \{ p_1, p_2, p_3, p_4 \} = \{ 2, 2, 2, 3 \} \). The other cases can be treated in a similar fashion. Suppose that \( f: A_3^r \to S^* \) is a non-constant morphism. With the notations of 2.4, \( C \) is then isomorphic to \( P^1 \). Let \( \psi := \pi \cdot f \). Then \( \psi(A_3) \) is isomorphic to \( A_3 \) or \( P^1 \). The case where \( \psi(A_3) = A_3 \) is impossible because \( \pi \) has four multiple fibers of multiplicity 3 (cf. 2.5 and the proof of Theorem 2.14). Hence \( \psi(A_3) = P^1 \). Let \( 3F_i \) (\( 1 \leq i \leq 4 \)) be the multiple fibers of \( \pi \). Then \( f^*(F_i) \) is defined by \( f_i = 0 \) with \( f_i \in R := k[u_1, \ldots, u_r] \). Since \( 3F_1 \sim 3F_2 \sim 3F_3 \), for example, we have a relation
\[
f_3^3 = f_1^3 + bf_1^3,
\]
where \( b \in k^* \). Since \( f^*(F_1) \cap f^*(F_2) \cap f^*(F_3) = \phi \), we can define a non-constant morphism
\[
g: A_3^r \to S_{3,3,3}^* \subset \text{Spec}(k[x_1, x_2, x_3]/(x_1^3 + x_2^3 + x_3^3))
\]
by \( g^*(x_1) = b/3f_1, g^*(x_2) = f_2 \) and \( g^*(x_3) = -f_3 \). This is impossible because \( S_{3,3,3}^*/G_m \) is an elliptic curve. Q.E.D.

(\(^*\)) For the following argument, the author owes Dr. A. Fujiki.
3. Regular subrings in a polynomial ring

3.1. Let \( A \) be a finitely generated, two-dimensional, regular \( k \)-algebra contained in a polynomial ring \( R := k[u_1, \cdots, u_r] \) of dimension \( r \). Let \( X := \text{Spec}(A) \) and let \( A^\ast := \text{Spec}(R) \). Then the inclusion \( A \hookrightarrow R \) gives rise to a dominant morphism \( f: A^\ast \rightarrow X \). By restricting \( f \) onto a linear plane \( L \) in \( A^\ast \) which meets general fibers of \( f \) in finitely many points, we have a dominant morphism \( f_L: L \approx A^\ast \rightarrow X \). This implies that \( A \) is a \( k \)-subalgebra of the two-dimensional polynomial ring. Thus we may assume without loss of generality that \( r = 2 \).

Since \( f: A^\ast \rightarrow X \) is generically finite, we have \( \kappa(X) = -\infty \), which follows from the inequality of logarithmic Kodaira dimensions,

\[
\kappa(X) \leq \kappa(A^\ast) = -\infty.
\]

This implies that \( X \) contains a cylinderlike open set \( U \approx U_0 \times A^\ast \), where \( U_0 \) is an affine curve (cf. Miyanishi-Sugie [8]; Fujita [2]). The projection \( p: U \rightarrow U_0 \) is induced from a dominant morphism \( \rho: X \rightarrow P_1^1 \), where \( U_0 \) is an open set of \( P_1^1 \). Then \( \rho(X) \approx A^\ast_1 \) or \( \rho(X) = P_1^1 \). Indeed, if \( P_1^1 \rightarrow \rho(X) \) consists of more than one point, we may write \( \rho(X) = \text{Spec}(k[t, h(t)^{-1}]) \), where \( t \) is an inhomogeneous coordinate of \( P_1^1 \) and \( h(t) \in k[t] - k \); then \( k[t, h(t)^{-1}] \) is a \( k \)-subalgebra of \( A \) (and, hence, of \( k[u_1, u_2] \)); this contradicts the fact that \( A^\ast = k^\ast \).

Summing up, we have the following:

**Lemma.** Let \( X := \text{Spec}(A) \) be a nonsingular affine surface. Then \( A \) is contained in a polynomial ring as a \( k \)-subalgebra if and only if there exists a dominant morphism \( f: A^\ast \rightarrow X \). In this case, we have:

1. \( A^\ast = k^\ast \);
2. There exists an \( A^1 \)-fibration \( \rho: X \rightarrow Y \), where \( Y \approx A^\ast_1 \) or \( P_1^1 \);
3. Every fiber of \( \rho \) is supported by a disjoint union of irreducible curves, each of which is isomorphic to \( A^1 \).

For the last assertion, see Miyanishi [7].

3.2. A fiber \( \rho^\ast(P) \) of \( \rho \) is a singular fiber if either \( \rho^{-1}(P) \) is reducible or \( \rho^\ast(P) \) is irreducible and non-reduced. Write \( \rho^\ast(P) = \sum_{i=1}^{s} n_i C_i \), where \( C_i \approx A^1 \) and \( n_i > 0 \). \( \rho^\ast(P) \) is called a singular fiber of the first kind if \( s \geq 2 \) and \( n_i = 1 \) for some \( i \); \( \rho^\ast(P) \) is called a singular fiber of the second kind if \( n_i \geq 2 \) for every \( i \). Let \( \mu := \text{G.C.D.} \ (n_1, \cdots, n_s) \). If \( \mu > 1 \), the fiber \( \rho^\ast(P) \) is called a multiple fiber and \( \mu \) is called the multiplicity.

3.3. We shall prove:

**Theorem.** Let \( X := \text{Spec}(A) \) be a nonsingular surface with an \( A^1 \)-fibration \( \rho: X \rightarrow Y \), where \( Y \approx A^\ast_1 \). Then \( A \) is contained in a polynomial ring as a \( k \)-sub-
algebra if and only if \( \rho \) has at most one singular fiber of the second kind.

Proof. (I) Let \( f: A \rightarrow X \) be a dominant morphism. Then note that 
\( \rho \circ f(\mathbf{A}^1_i) = Y \). Suppose that \( \rho \) has two singular fibers of the second kind \( \rho^*(P) \) and \( \rho^*(Q) \). Then \( f^* \rho^*(P) \) and \( f^* \rho^*(Q) \) are defined by the equations
\[
g_{11}^i \cdots g_{m^*}^i = 0 \quad \text{and} \quad h_{11}^i \cdots h_{n^*}^i = 0
\]
respectively, where \( g_1, \ldots, g_m \) and \( h_1, \ldots, h_n \) are non-constant polynomials in \( k[u_1, u_2] \) and where \( a_i \geq 2 \) (\( 1 \leq i \leq m \)) and \( b_j \geq 2 \) (\( 1 \leq j \leq n \)). We may choose an inhomogeneous coordinate \( t \) of \( Y := \text{Spec}(k[t]) \) in such a way that the points \( P \) and \( Q \) are defined by \( t = 0 \) and \( t = -1 \), respectively. Then we have a relation
\[
g_{11}^i \cdots g_{m^*}^i - h_{11}^i \cdots h_{n^*}^i = 1.
\]
This is impossible by virtue of Theorem 1.2. Therefore \( \rho \) has at most one singular fiber of the second kind provided \( A \) is contained in a polynomial ring as a \( k \)-subalgebra.

(II) We shall prove the "if" part of the theorem. Let \( \rho^*(P) = \sum_{i=1}^n n_i C_i \) be a singular fiber of the first kind. We shall show that after replacing \( X \) by a suitable affine open set with an \( \mathbf{A}^1 \)-fibration similar to that for \( X, \rho^*(P) \) can be assumed to be an irreducible and reduced fiber. For this purpose, embed \( X \) into a nonsingular projective surface \( V \) as a dense open set. Then \( V - X \) consists only of components of codimension 1. Since \( X \) is affine, there exists an effective ample divisor \( D \) on \( V \) such that \( \text{Supp}(D) = V - X \). For \( \rho^*(P) = \sum_{i=1}^n n_i C_i \), suppose that \( n_i = 1 \). Then there exists an ample divisor \( D' \) on \( V \) such that \( \text{Supp}(D') = (V - X) \cup \bigcup_{i=1}^n \overline{C}_i \), where \( \overline{C}_i \) is the closure of \( C_i \) in \( V \). Replace \( X \) by \( X' := X - \text{Supp}(D') \). Then \( X' \) is an affine open set of \( X \) and \( \rho' := \rho|_{x'}: X' \rightarrow Y \) is an \( \mathbf{A}^1 \)-fibration over \( Y \) for which the fiber \( \rho'^*(P) \) is irreducible and reduced.

Performing this operation to all singular fibers of the first kind of \( \rho \), we may assume that \( \rho \) has no singular fibers of the first kind. Let \( \rho^*(P) \) denote anew a singular fiber of the second kind if such a fiber exists at all. If \( \rho^*(P) \) is reducible, we may delete all irreducible components but one by replacing \( X \) by a smaller affine open set with an \( \mathbf{A}^1 \)-fibration over \( Y \) similar to that for \( X \). Hence we may assume that \( \rho^*(P) \) is an irreducible multiple fiber, i.e., \( \rho^*(P) = nC \) with \( C \cong \mathbf{A}^1 \) and \( n \geq 2 \).

Write \( Y := \text{Spec}(k[t]) \), and assume that the point \( P \) is defined by \( t = 0 \). Let \( Z := \text{Spec}(k[\tau]) \rightarrow Y \) be the morphism defined by \( t = \tau^r \), which is a finite covering ramifying totally over \( P \). Let \( W \) be the normalization of \( X \times_{Y} \). Then \( W \) is a nonsingular affine surface, and the canonical surjective morphism...
\[ \sigma : W \to Z \] is an \( A^1 \)-fibration over \( Z \). This can be seen as follows. Let \( x \) be a point of \( X \) lying over the point \( P \), and find a system of local coordinates \(( \xi, \eta )\) around \( x \) such that the curve \( C \) is defined by \( \xi = 0 \). Then we have a relation \( \xi^a = a \tau \), where \( a \) is a unit in \( O_{x,t} \). Then \( \xi/\tau \) is regular at every point \( x \) of \( W \) lying over \( x \). Analytically, \( W \) around \( x \) is defined as a hypersurface \( (\xi/\tau)^a = a \) in the \((\xi/\tau, \tau, \eta )\)-space. By the Jacobian criterion of smoothness, \( W \) is nonsingular at every point \( x \) lying over \( x \). It is easy to see that \( W \) is nonsingular at every point of \( W \) lying over \( X - P \). Hence \( W \) is nonsingular. By construction, general fibers of \( \sigma \) are isomorphic to \( A^1 \). Let \( \tilde{P} \) be the point of \( Z \) lying over \( P \). Every fiber of \( \sigma \) except the fiber \( \sigma^* \tilde{P} \) is irreducible and reduced, while \( \sigma^* \tilde{P} \) is reduced and reducible with \( n \) irreducible components. Let \( W' \) be an affine open set of \( W \) obtained by deleting all components of \( \sigma^* \tilde{P} \) except one. Then \( \sigma' := \sigma|_{W'} : W' \to Z \) is an \( A^1 \)-bundle over \( Z \approx A^1 \), whence \( W' \) is isomorphic to \( A^1 \) (cf. Kambayashi-Miyanishi [5]). Let \( f \) be the composite of the natural morphisms

\[ f : A^1 \to W' \to W \to X \times Z \to X. \]

Since \( f \) is apparently a dominant morphism, \( A \) is contained in a polynomial ring as a \( k \)-subalgebra. Q.E.D.

3.4. Corollary. Let \( X \) be a nonsingular affine surface which satisfies the condition in Theorem 3.3. Then the torsion part \( \text{Pic}(X)_{\text{tor}} \) of the Picard group of \( X \) is a cyclic group.

Proof. Let \( \rho : X \to Y \) be the \( A^1 \)-fibration as in Theorem 3.3. Let \( \rho^* P_i \) \((0 \leq i \leq m)\) exhaust all singular fibers of \( \rho \); if there exists a singular fiber of the second kind, we let \( \rho^* P_0 \) denote it. Write \( \rho^* P_i = \sum_{1 \leq j \leq s_i} n_{ij} C_{ij} \), where \( C_{ij} \approx A^1 \) and \( n_{ij} > 0 \). Then, since \( Y \approx A^1 \), the Picard group \( \text{Pic}(X) \) of \( X \) is an abelian group with the following generators and relations:

\[ \{\xi_{ij} | 0 \leq i \leq m, 1 \leq j \leq s_i\} \text{ and } \sum_{1 \leq j \leq s_i} n_{ij} \xi_{ij} = 0 \text{ for } 0 \leq i \leq m. \]

It is then clear that \( \text{Pic}(X) \approx \prod_{i=0}^{m} G_i \), where \( G_i \) is an abelian group with generators and relations given as above with \( i \) fixed and with \( 1 \leq j \leq s_i \). Since \((n_{01}, \ldots, n_{0r_0}) = 1 \) for \( i \geq 1 \) by assumption, we have \( G_i \approx Z^{\oplus (s_i - 1)} \). Let \( \mu = (n_{01}, \ldots, n_{0r_0}) \). Then \( G_0 \approx Z^{\mu} \oplus Z^{\oplus (r_0 - 1)} \). Hence we have \( \text{Pic}(X)_{\text{tor}} \approx Z^{\mu} \). Q.E.D.

3.5. We shall prove:

**Theorem.** Let \( X := \text{Spec}(A) \) be a nonsingular affine surface with an \( A^1 \)-fibration \( \rho : X \to Y \), where \( Y \approx P^1 \). Assume that \( A \) is contained in a polynomial ring as a \( k \)-subalgebra. Then the fibration \( \rho \) has at most three multiple fibers. If
$\rho$ has three multiple fibers, their multiplicities $\{\mu_1, \mu_2, \mu_3\}$ are given, up to permutation, by one of the following triplets: $\{2, 2, n\}$ $(n \geq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ and $\{2, 3, 5\}$.

Proof. Suppose that $\rho$ has three or more multiple fibers. Let $\rho^* P_i = \mu_i F_i$ $(1 \leq i \leq 3)$ be a multiple fiber of multiplicity $\mu_i > 1$. Let $f: A^*_i := \text{Spec}(k[u, u_2]) \to X$ be a dominant morphism as in 3.1. Then $\rho \cdot f(A^*_i) \cong A^*_i$ or $\rho \cdot f(A^*_i) = Y$. If $\rho \cdot f(A^*_i) \cong A^*_i$, we may assume that $P_1, P_2 \in \rho \cdot f(A^*_i)$. However, this assumption leads to a contradiction by the argument in the step (I) of the proof of Theorem 3.3. Hence $\rho \cdot f(A^*_i) = Y$. Then $f^* F_i$ $(1 \leq i \leq 3)$ is defined by an equation $f_i = 0$, where $f_i$ is a non-constant polynomial in $k[u, u_2]$. Since $\mu_1 f^* F_i \sim \mu_2 f^* F_i \sim \mu_3 f^* F_i$ (linear equivalence), we have

$$\frac{f_{i, 1}}{f_{i, 1}^{a_i}} = a\frac{f_{i, 2}}{f_{i, 2}^{a_i}} + b,$$

where $a, b \in k^*$. Without loss of generality, we may assume that $a = b = -1$. Namely, we have a relation

$$f_{i, 1}^{a_i} + f_{i, 2}^{a_i} + f_{i, 3}^{a_i} = 0.$$

Note that $f^*(F_i) \cap f^*(F_j) = \emptyset$ whenever $i \neq j$. The assignment $x_i \mapsto f_i$ defines a non-constant morphism

$$\psi: A^*_i \to S^*_{\mu_1, \mu_2, \mu_3} \subset \text{Spec}(k[x_1, x_2, x_3]/(x_1^{a_i} + x_2^{a_i} + x_3^{a_i})).$$

Hence $\{\mu_1, \mu_2, \mu_3\}$ is, up to permutation, one of the following triplets: $\{2, 2, n\}$ $(n \geq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ and $\{2, 3, 5\}$ (cf. 2.14).

Suppose that $\rho$ has four multiple fibers $\rho^* P_i = \mu_i F_i$ with multiplicity $\mu_i$ $(1 \leq i \leq 4)$. Let $f^* F_i$ be defined by $f_i = 0$, where $f_i$ is a non-constant polynomial in $k[u, u_2]$. Then we obtain relations of the following form:

$$f_{i, 1}^{a_i} + f_{i, 2}^{a_i} + f_{i, 3}^{a_i} = 0,$$

$$af_{i, 1}^{a_i} + f_{i, 2}^{a_i} + f_{i, 3}^{a_i} = 0,$$

where $a \in k - \{0, 1\}$. In view of the above observations on possible multiplicities of three multiple fibers of $\rho$, we know that $\{\mu_1, \mu_2, \mu_3, \mu_4\}$ is, up to permutation, one of the following quadruplets: $\{2, 2, 2, n\}$ $(n \geq 2)$, $\{2, 2, 3, 3\}$, $\{2, 2, 3, 4\}$ and $\{2, 2, 3, 5\}$. The induced relations provide a non-constant morphism

$$\psi: A^*_i \to S^*_{\mu_1, \mu_2, \mu_3, \mu_4}.$$

This is impossible by 2.5 and 2.15.

Q.E.D.

3.6 Corollary. Let $X$ be the same surface as in 3.5. Then $\text{Pic}(X)_{\text{tor}}$ has at most two cyclic components. If $\text{Pic}(X)_{\text{tor}}$ has two cyclic components, it is of the form:
Proof. An argument similar to that in Corollary 3.4.

3.7. We shall prove:

**Theorem.** Let $X := \text{Spec}(A)$ be a nonsingular affine surface with an $A^1$-fibration $\rho : X \to Y$, where $Y \cong \mathbb{P}^1$. Assume that $\rho$ satisfies the following conditions:

1. $\rho$ has no singular fibers of the second kind but at most three multiple fibers with a single irreducible component;

2. If $\rho$ has three multiple fibers, the set of multiplicities $\{\mu_1, \mu_2, \mu_3\}$ is one of the following triplets: $\{2, 2, n\}$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ and $\{2, 3, 5\}$.

Then $A$ is contained in a polynomial ring as a $k$-subalgebra.

Proof. (I) By performing the same operation as we did in the second step of the proof of Theorem 3.3, we may assume that $\rho$ has no singular fibers of the first kind. Suppose that $\rho$ has at most two multiple fibers. Let $P$ be a point of $Y$ such that $\rho^*P$ is a multiple fiber (if such a fiber exists at all), and let $X' := X - \rho^{-1}(P)$. Then the nonsingular affine surface $X'$ with an $A^1$-fibration $\rho' := \rho|_{X'}$ over $Y' := Y - \{P\}$ has at most one singular fiber of the second kind. By Theorem 3.3, there exist a dominant morphism $A^1 \to X'$, and hence a dominant morphism $A^1 \to X$. Therefore $A$ is contained in a polynomial ring as a $k$-subalgebra.

(II) Suppose that $\rho$ has three multiple fibers $\rho^*P_i = \mu_i F_i$ $(1 \leq i \leq 3)$ with multiplicity $\mu_i$. We consider first the case where $\{\mu_1, \mu_2, \mu_3\} = \{2, 2, n\}$ $(n \geq 2)$. Let $Y'' \to Y$ be a double covering of $Y$ which ramifies over the points $P_1$ and $P_2$; then $Y'' \cong \mathbb{P}^1$. Let $X'$ be the normalization of $X \times_Y Y'$ and let $\rho' : X' \to Y'$ be the natural projection. Then $X'$ is a nonsingular affine surface and $\rho'$ is an $A^1$-fibration over $Y'$ (cf. the proof of Theorem 3.3). Moreover, $\rho'^*P'_i$ $(i = 1, 2)$ is a reduced singular fiber with two irreducible components, $P'_i$ being the unique point of $Y'$ lying over $P_i$, and $\rho'^*Q_i$ $(i = 1, 2)$ is a multiple fiber of multiplicity $n$ with single irreducible component, $Q_1$ and $Q_2$ being two points of $Y'$ lying over $P_3$. Replacing $X'$ by an affine open set, we may assume that $\rho'$ has no singular fibers of the first kind. Let $Y'' \to Y'$ be an $n$-ple covering which ramifies totally over $Q_1$ and $Q_2$, let $X''$ be the normalization of $X' \times_Y Y''$, and let $\rho'' : X'' \to Y''$ be the natural projection. Then $X''$ is a nonsingular affine surface and $\rho''$ is an $A^1$-fibration over $Y'' \cong \mathbb{P}^1$. The fibration $\rho''$ has two reduced singular fibers $\rho''*Q'_i$ $(i = 1, 2)$ with $n$ irreducible components, where $Q'_i$ $(i = 1, 2)$ is the unique point of $Y''$ lying over $Q_i$. Then, by virtue of the step (I), there exist a dominant morphism $A^1 \to X''$, and hence a dominant morphism $A^1 \to X$. Therefore, $A$ is contained in a polynomial ring as a $k$-subalgebra.
(III) The other cases except the last one can be treated in a similar fashion, that is, by choosing suitable multiple coverings $P_i \to P$ and then taking the normalizations of the fiber products with respect to such multiple coverings. The following diagram will indicate roughly the necessary steps:

\[
\begin{array}{c}
\{2, 3, 3\} \quad \text{triple covering} \\
\downarrow \\
\{2, 2, 2\} \quad \text{the former case,}
\end{array}
\]

\[
\begin{array}{c}
\{2, 3, 4\} \quad \text{double covering} \\
\downarrow \\
\{2, 3, 3\} \quad \text{the former case.}
\end{array}
\]

(IV) In the case where $\{\mu_1, \mu_2, \mu_3\} = \{2, 3, 5\}$, we know by the theory of Kleinian singularities that there exists a ramified covering $\tau: Y' \to Y$ of degree 60 with 30 points over $P_1$ with ramification index 2, 20 points over $P_2$ with ramification index 3 and 12 points over $P_3$ with ramification index 5, where $Y' \cong P^4$. Let $X'$ be the normalization of $X \times Y'$ and $\rho': X' \to Y'$ be the natural $A^1$-fibration. Then $\rho'$ has no multiple fibers of the second kind. So, we are done. Q.E.D.

References


