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ON SOME ASYMPTOTIC PROPERTIES OF THE SOLUTION FOR A STOCHASTIC DIFFERENTIAL EQUATION ON HILBERT SPACES

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1. Introduction

Let \((H, \rho)\) and \((K, q)\) be separable Hilbert spaces with inner products \(\rho\) and \(q\) respectively. \(\sigma_2(H,K)\) denotes the Hilbert space of totality of Hilbert-Schmidt operators from \(H\) into \(K\).

For a cylindrical Brownian motion \(B\) on \(H\), we consider the following stochastic differential equation on \(K\):

\[
\begin{cases}
    dX(t) = G(X(t))dB(t) + b(X(t))dt \\
    X(0) = x,
\end{cases}
\]

where \(G: K \to \sigma_2(H,K)\) is Borel measurable and so is \(b: K \to K\). Moreover \(G\) and \(b\) satisfy the following Lipschitz condition \((A_1)\):

\[
(A_1) \quad \text{there exists a positive constant } \alpha \text{ such that}
\]

\[
\begin{cases}
    q(b(x) - b(y)) \leq \alpha q(x - y) \\
    ||G(x) - G(y)|| \leq \alpha q(x - y),
\end{cases}
\]

where \(||\cdot||\) mean the Hilbert-Schmidt norm in \(\sigma_2(H,K)\) and \(q^2(x) = q(x, x)\).

Therefore according to M. Yor [7] and Y. Miyahara [6], we have

**Proposition.** There exists a unique solution \(X\) of (1.1), which is a diffusion with generator \(L\):

\[
Lf(x) = q(f'(x), b(x)) + (1/2)trace(G^*(x)f''(x)G(x)) .
\]

Moreover \(X\) has continuous paths, i.e. with probability 1,

\[
q(X(t) - X(s)) \to 0 \text{ as } t-s \to 0.
\]

In the finite dimensional case, A. Friedman [1] investigated various asymptotic properties of \(X(t)\), and especially he gave recurrence and transience criteria in terms of \(G\) and \(b\). The purpose of this paper is to extend Friedman’s results
on recurrence to the infinite dimensional case.

The following auxiliary theorem will be proved in §2.

**Theorem 1.** Suppose the condition \((A_2)\).

\((A_2)\) For any \(R > 0\), there exist \(z_R \in K\), and two positive constants \(\gamma_R\) and \(\gamma_R\) such that

\[
p^2(G^*(x)z_R)\gamma_R + q(b(x), z_R) \geq \gamma_R \quad \text{whenever } q(x) \leq R,
\]

where \(G^*\) is the conjugate operator of \(G\) and \(p^2(h) = p(h, h)\) for \(h \in H\).

Then for any \(x \in K\)

\[
P_x\{\limsup_{t \to \infty} q(X(t)) = \infty\} = 1
\]

holds.

According to A. Friedman [1], we define

\[
A(x, y) = p^2(G^*(x)y)/q^2(y), \quad B(x) = ||G^*(x)||^2,
\]

\[
C(x, y) = 2q(b(x), y),
\]

\[
S(x, y) = \frac{B(x) + C(x, y)}{A(x, y)} - 1 \quad \text{and} \quad S(x) = S(x, x).
\]

Let us introduce the non-degeneracy condition \((A_3)\);

\((A_3)\) \(p(G^*(x)y) > 0\) for any non-zero \(y \in K\) and \(x \in K\).

So the condition \((A_3)\) implies that

\[
A(x, y) > 0 \quad \text{for } y \neq 0.
\]

In §3 we will prove the following Theorems 2 and 3, using the similar method as A. Friedman [1].

**Theorem 2.** Besides \((A_2)\) and \((A_3)\), we assume \((A_4)\).

\((A_4)\) There exist a positive constant \(R_0\) and a continuous function \(\varepsilon_1\) on \([0, \infty)\), such that

\[
S(x) \geq 1 + \varepsilon_1(q(x)) \quad \text{whenever } q(x) \geq R_0,
\]

and

\[
\int_{R_0}^\infty (1/t) \exp \left[ - \int_{R_0}^t \varepsilon_1(s)/s \, ds \right] dt < +\infty.
\]

Then the solution \(X\) is transient, i.e.

\[
P_x(\lim_{t \to \infty} q(X(t)) = \infty) = 1 \quad \text{for any } x \in K.
\]
Theorem 3. Besides (A_2) and (A_3), we assume (A_5).

(A_5) For any \( z \in K \), there exist a positive constant \( R_z \) and a continuous function \( \varepsilon_z \) on \([0, \infty)\) such that

\[ S(x, x-z) \leq 1 + \varepsilon_z(q(x-z)) \quad \text{whenever} \quad q(x-z) > R_z, \]

and for some \( R^* > 0 \),

\[ \int_{R^*}^{\infty} \left( \frac{1}{t} \right) \exp \left[ -\int_{R^*}^{t} \varepsilon_z(s)/s \, ds \right] dt = \infty. \]

Then \( X \) is recurrent, i.e., for any ball \( B_d(z) = \{y: q(y-z) \leq \alpha\} \), \( \alpha > 0 \)

\[ P_x(X(t_1) \in B_d(z) \text{ for some } t_1 < t_2 < \cdots \uparrow \infty) = 1 \text{ for any } x \in K. \]

See Funaki [2] for a result related to Theorem 3. Moreover, we have from the separability of \( K \).

Corollary. Under the conditions (A_2), (A_3) and (A_5),

\[ P_x(\text{closure of } \{X(t); t \in [0, \infty)\} = K) = 1 \text{ for any } x \in K. \]

Finally, as a simple example, we treat an Ornstein-Uhlenbeck type process

\[ dX(t) = GdB(t) - cX(t)dt, \]

where \( G \in \sigma_{\mathbb{R}}(H, K) \), and \( c \) is a real constant, and will discuss how the asymptotic behavior of \( X(t) \) depends on the constant \( c \).

2. Proof of Theorem 1

First of all, we will recall the definition of the solution of (1.1), according to M. Yor [7]. Let \( (\Omega, \mathcal{F}, P) \) be a complete probability space. A cylindrical Brownian motion \( B \) on \( H \) means a Wiener process \( (0, \mathcal{F}) \) on \( H \), namely \( B: [0, \infty) \times H \rightarrow \mathcal{R}^1 \) satisfies the following conditions

1. \( B(0, \cdot, \cdot) = 0 \),
2. \( B(\cdot, h, \cdot)/\|h\| \) is a one-dimensional Brownian motion for \( h \neq 0 \),
3. for any \( t \in [0, \infty) \) and \( \lambda, \mu \in \mathcal{R}^1 \),

\[ B(t, \lambda h + \mu k, \cdot) = \lambda B(t, h, \cdot) + \mu B(t, k, \cdot) \quad \text{a.s..} \]

Put \( \mathcal{F}_t := \text{the } \sigma\text{-field generated by } \{B(s, h, \cdot); s \leq t, h \in H\} \). A \( K \)-valued process \( X \) is called a solution of (1.1) if

1. \( X \) is \( \mathcal{F}_t \)-progressively measurable,
2. \( E\left[ \int_0^T q^2(X(s)) \, ds \right] < \infty \) for any \( T > 0 \),

and
for any $t$

$$X(t) = x + \int_0^t G(X(s))dB(s) + \int_0^t b(X(s))ds \quad \text{a.s.}$$

holds, where the second term is the stochastic integral and the third term is the Bochner integral.

Proof of Theorem 1. Fix $R > 0$ arbitrarily. By (A2) we can take $z_R, \gamma_R$ and $\Gamma_R$ of (1.4). So $z_R$ is not zero. Put $\xi_1 = z_R/q(z_R)$ and define $\phi$ by

$$\phi(x) = A[\delta^{(R+1)} - \delta^{(x, t_1)}],$$

where $A$ and $\alpha$ are some positive constants, which will be determined later. Then $\phi$ is Fréchet differentiable at any order and its first and second derivatives are given by

$$\phi'(x)[h] = -A\alpha e^{\delta^{(x, t_1)}} q(h, \xi_1)$$

and

$$\phi''(x)[h_1, h_2] = -A\alpha^2 e^{\delta^{(x, t_1)}} q(h_1, \xi_1)q(h_2, \xi_1),$$

namely $\phi'(x) \in K^*$ can be regarded as

$$\phi'(x) = -A\alpha e^{\delta^{(x, t_1)}} \xi_1 \in K$$

and $\phi''(x) \in \mathcal{L}(K \rightarrow K^*)$ (linear map from $K$ into $K^*$) can be regarded as

$$\phi''(x) = -A\alpha^2 e^{\delta^{(x, t_1)}} \xi_1 \otimes \xi_1.$$ 

Therefore $G^*(x)\phi''(x)G(x) \in \mathcal{L}(H \rightarrow H)$, and

$$\text{trace}(G^*(x)\phi''(x)G(x))$$

$$= \sum_{i=1}^{m} p(e_i, G^*(x)\phi''(x)G(x)e_i)$$

$$= -A\alpha^2 e^{\delta^{(x, t_1)}} \sum_{i=1}^{m} p^2(G^*(x)\xi_1, e_i)$$

$$= -A\alpha^2 e^{\delta^{(x, t_1)}} p^2(G^*(x)\xi_1),$$

where $\{e_i; i=1, 2, \ldots\}$ is an ONB in $H$. Hence we see that from (2.4) and (2.6)

$$L\phi(x) = q(\phi'(x), b(x)) + (1/2)\text{trace}(G^*(x)\phi''(x)G(x))$$

$$= -A\alpha e^{\delta^{(x, t_1)}}[q(b(x), \xi_1) + (1/2)p^2(G^*(x)\xi_1)] \alpha.$$ 

Put

$$\alpha = 2\Gamma_R q(z_R) \quad \text{and} \quad A = \exp\{2\Gamma_R q(z_R)R\} / 2\Gamma_R \gamma_R.$$ 

Then recalling the definition of $\xi_1$ and (1.4), we have

$$L\phi(x) \leq -1 \quad \text{whenever} \quad q(x) \leq R.$$
Since \( \phi \) is smooth, Itô’s formula derives

\[
\phi(X(t)) = \phi(x) + \int_0^t L\phi(X(s))ds + \int_0^t \langle G^*(X(s))\phi'(X(s)), dB(s) \rangle
\]

where \( \langle G^*(X(s))\phi'(X(s)), dB(s) \rangle = \sum_{i=1}^{\infty} \phi(G^*(X(s))\phi'(X(s)), e_i)dB(s, e_i) \).

Let \( \tau = \tau_R \) be the first exit time from the ball \( B_R := \{ y \in K; q(y) \leq R \} \), i.e.

\[
\tau = \begin{cases} 
\inf \{ t > 0; X(t) \notin B_R \}, & \text{if the above set is empty.} \\
\infty & \text{otherwise.} 
\end{cases}
\]

Then (2.10) yields

\[
E_x(\phi(X(t \wedge \tau))) = \phi(x) + E_x \left[ \int_0^{t \wedge \tau} L\phi(X(s))ds \right] \leq \phi(x) - E_x(t \wedge \tau).
\]

On the other hand, if \( q(y) \leq R \), then we have

\[
0 \leq \phi(y) \leq Ae^{-(R+1)},
\]

the right hand side of which we denote by \( M \). Therefore we can get from (2.11) and (2.12),

\[
E_x(t \wedge \tau) \leq \phi(x) \leq M \quad \text{for } x \in B_R.
\]

Since \( t \wedge \tau \) is increasing to \( \tau \) as \( t \to \infty \), the monotone convergence theorem implies

\[
E_x(\tau) = \lim_{t \to \infty} E_x(t \wedge \tau) \leq M.
\]

Hence we have

\[
P_x(\tau_R < \infty) = 1,
\]

that is

\[
P_x(\sup_{r \geq \delta} q(X(t)) \geq R) = 1 \quad \text{for } x \in B_R.
\]

Since \( R \) is arbitrary, we complete the proof.

### 3. Proof of Theorems 2 and 3

To prove Theorem 2, define functions \( \theta, I, F \) and \( f \) by

\[
\theta(r) := 1 + e_1(r), \quad r \geq R_0, \quad I(s) := \int_{R_0}^s \theta(t)/t \, dt, \\
F(r) := \int_r^\infty e^{-t(\theta)} \, dt, \quad r \geq R_0, \quad f(x) := F(q(x)) \quad q(x) \geq R_0.
\]

Then the condition (A4) means “\( F(r) < \infty \)” and we can easily calculate Freéchet derivatives \( f' \) and \( f'' \):
(3.1) \[ f'(x)[h] = F'(q(x))q(x, h)/q(x) \]
and
(3.2) \[ f''(x)[h_1, h_2] = F''(q(x))q(x, h_1)q(x, h_2)/q^2(x) \]
\[ + F'(q(x))q(h_1, h_2)/q(x) \]
\[ - F'(q(x))q(x, h_1)q(x, h_2)/q^2(x). \]

Hence we get

(3.3) \[ \text{trace}(G^*(x)f''(x)G(x)) \]
\[ = \sum_{j=1}^{\infty} p(e_j, G^*(x)f''(x)G(x)e_j) \]
\[ = \sum_{j=1}^{\infty} q(G(x)e_j, f''(x)G(x)e_j) \]
\[ = \left[ F''(q(x))/q^2(x) \right] \sum_{j=1}^{\infty} q^2(G(x)e_j, x) + \left[ F'(q(x))/q(x) \right] \sum_{j=1}^{\infty} q^2(G(x)e_j) \]
\[ - \left[ F'(q(x))/q^2(x) \right] \sum_{j=1}^{\infty} q^2(G(x)e_j, x) \]

and

(3.4) \[ Lf(x) = (1/2)F''(q(x))A(x, x) + (1/2)F'(q(x)) \{ B(x) - A(x, x) + C(x, x) \} /q(x) \]
\[ = (1/2)A(x, x)F''(q(x)) + (1/2)A(x, x)F'(q(x)) \{ S(x) - \theta(q(x)) \} /q(x) \]
\[ = (1/2)A(x, x)F''(q(x)) \{ S(x) - \theta(q(x)) \} /q(x) \leq 0. \]

For any fixed \( \alpha < \beta < R < \beta \), we put

\[ \tau_\alpha^\beta : = \left\{ \begin{array}{cl}
\inf \{ t > 0; q(X(t)) \in (\alpha, \beta) \} \\
\infty & \text{if the above set is empty},
\end{array} \right. \]

\[ \Omega^*(\alpha) : = \{ \omega \in \Omega; X(\tau_R + t_n, \omega) \in B_\alpha \text{ for some } t_1 < t_2 < \cdots < \infty \}, \]

\[ \Omega(\alpha) : = \{ \omega \in \Omega; X(t, \omega) \in B_\alpha \text{ for some } t \}, \]

where \( \tau_R \) is the exit time from the ball \( B_R \).

Recalling Theorem 1, we have

(3.6) \[ P_y(\tau_\alpha^\beta < \infty) = 1 \text{ for any } y \in K. \]

Hence, using Itô's formula and (3.4), we get for \( \alpha < q(y) < \beta \)

(3.7) \[ E_y \{ f(X(\tau_\alpha^\beta)) \} - f(y) = E_y \left\{ \int_0^{\tau_\alpha^\beta} Lf(X(s)) \, ds \right\} \leq 0. \]

Appealing to the definition of \( f \), we have for \( \alpha < q(y) < \beta \)

(3.8) \[ F(\alpha)P_y(q(X(\tau_\alpha^\beta)) = \alpha) + F(\beta)P_y(q(X(\tau_\alpha^\beta)) = \beta) \leq F(q(y)). \]

Therefore we have
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\[ P_\gamma(q(X(\tau_\alpha))) = \alpha \leq F(q(y))/F(\alpha) \quad \text{for} \quad \alpha < q(y) < \beta. \]

Since \( P_\gamma(q(X(\tau_\alpha)) = \alpha) \) is increasing to \( P_\gamma(\Omega(\alpha)) \) as \( \beta \uparrow \infty \), we see that
\[(3.9) \quad P_\gamma(\Omega(\alpha)) \leq F(q(y))/F(\alpha) \quad \text{for} \quad \alpha < q(y). \]

Using the strong Markov property of \( X \), we get
\[(3.10) \quad P_\gamma(\Omega^*(\alpha)) \leq E_x[P_{X(\tau_\beta)}(\Omega(\alpha))] \leq E_x[F(q(X(\tau_\beta)))/F(\alpha)]
= E_x[F(R)/F(\alpha)] = F(R)/F(\alpha) \]
for \( q(x) < R \). Tending \( R \) to \( \infty \), we conclude
\[(3.11) \quad P_\gamma(\Omega^*(\alpha)) = 0 \quad \text{for any} \quad x \in K, \]

since \( \lim_{R \to \infty} F(R) = 0 \). Therefore we have
\[(3.12) \quad P_\gamma(\liminf_{t \to \infty} q(X(t)) \geq \alpha) = 1 \quad \text{for any} \quad \alpha. \]

This completes the proof of Theorem 2.

For the proof of Theorem 3, we will show for simplicity
\[(3.13) \quad P_\gamma(X(t_0) \in B_\alpha \quad \text{for some} \quad t_1 < t_2 < \cdots \uparrow \infty) = 1. \]

Because we can apply the similar argument for \( B_\alpha(z) \), replacing \( X(t) \) by \( X(t) - z \).
Define \( \theta, I, F \) and \( f \) by
\[
\begin{align*}
\theta(r) &= 1 + \epsilon_2(r) \quad r \geq R_0, \quad I(s) = \int_{R_0}^{s} \theta(t)/t \, dt \\
F(r) &= -\int_{R_0}^{r} e^{-I(t)} \, ds, \quad f(x) = F(q(x)).
\end{align*}
\]

Then the condition \( (A_5) \) implies that
\[(3.14) \quad \lim_{r \to \infty} F(r) = -\infty. \]

Since \( f \) is twice Fréchet differentiable, in the same way as (3.4) we have
\[(3.15) \quad LF(x) \geq (1/2)A(x, x)F'(q(x)) \{S(x) - \theta(q(x))\} / q(x) \geq 0. \]

From this, as we derived (3.8) from (3.6) and (3.7), we get
\[(3.16) \quad F(\alpha)P_\gamma(q(X(\tau_\alpha))) = \alpha + F(\beta)P_\gamma(q(X(\tau_\beta))) = \beta \geq F(q(y)). \]

Tending \( \beta \) to \( \infty \), we get from (3.14)
\[(3.17) \quad \lim_{\beta \to \infty} P_\gamma(q(X(\tau_\beta)) = \beta) = 0. \]

Therefore (3.6) derives
\[(3.18) \quad P_\gamma(\Omega^*(\alpha)) = \lim_{\beta \to \infty} P_\gamma(q(X(\tau_\beta)) = \alpha) = 1 \quad \text{for any} \quad \alpha > 0. \]
For any fixed \( \alpha < R_1 < R_2 < \cdots \uparrow \infty \), we define stopping times \( t_m \) and \( \sigma_m \) by
\[
\begin{align*}
t_1 &= \inf \{ t; X(t) \in B_{a} \}, \quad \sigma_1 = \inf \{ t; t > t_1 \text{ and } X(t) \in \partial B_{a}\}, \\
t_m &= \inf \{ t; t > \sigma_{m-1} \text{ and } X(t) \in B_{a} \}, \quad \sigma_m = \inf \{ t; t > t_m \text{ and } X(t) \in \partial B_{a} \}.
\end{align*}
\]
Then we can easily see that
\[
(3.19) \quad P_x(t_1 < \infty) = P_x(\Omega(\alpha)) = 1 \text{ for any } x \in K
\]
and that from Theorem 1, for \( m = 1, 2, \ldots \),
\[
(3.20) \quad P_x(\sigma_m < \infty) = 1 \text{ for any } x \in K.
\]
Again using the strong Markov property, we get by (3.19)
\[
(3.21) \quad P_x(t_2 < \infty) = E_x[P_x(\sigma_1)(t_1 < \infty)] = 1 \text{ for any } x \in K.
\]
Assume that \( P_x(t_m < \infty) = 1 \) for any \( x \in K \). Then (3.20) and the strong Markov property derive
\[
(3.22) \quad P_x(t_{m+1} < \infty) = E_x[P_x(\sigma_m)(t_m < \infty)] = 1 \text{ for any } x \in K.
\]
Therefore we see that for \( m = 1, 2, \ldots \),
\[
P_x(t_m < \infty) = 1 \text{ for any } x \in K.
\]
Since \( t_1 < \sigma_1 < t_2 < \sigma_2 < \cdots \), we can show that
\[
(3.23) \quad \lim_{m \to \infty} t_m = \infty \text{ a.s.}
\]
by virtue of continuity (1.3) of \( X \). Now we complete the proof.

**Example.** Consider an Ornstein-Uhlenbeck type stochastic differential equation
\[
(3.24) \quad dX(t) = GdB(t) - cX(t)dt.
\]
We assume the condition \( (A_3) \), i.e.
\[
(3.25) \quad p^2(G^*z) > 0 \quad \text{if } z \neq 0.
\]
For any fixed normalized \( z \in K \), we put \( \Gamma_R := (1 + |c| R)/p^2(G^*z) \). Then
\[
p^2(G^*z)\Gamma_R - cq(x, z) \geq 1 + |c| R - |c| q(x)q(y) \geq 1,
\]
namely \( (A_3) \) holds.

Now, first assume that \( c \) is negative. Then we can easily see that
\[
S(x) = (||G||^2 - 2cq^2(x))q^2(x)/p^2(G^*x) - 1
\]
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\[ \geq (||G||^2 - 2cq(x))/||G||^2 - 1 \]
\[ = -2cq(x)/||G||^2. \]

Putting \( R_0 = ||G||/\sqrt{|c|} \), we have
\[ S(x) \geq 2 \quad \text{whenever} \quad q(x) \geq R_0, \]
namely the constant function \( c_1 \equiv 1 \) satisfies (1.7) and (1.8). So \((A_4)\) holds in this case.

Next, consider the case when \( c \) is positive. Fix \( z \in K \) arbitrarily. Then we have
\[ A(x, x-z) = p^2(G^*(x-z))/(2q(x-z)), \quad C(x, x-z) = -2cq(x-z) - 2cq(x, x-z) \]
and
\[ S(x, x-z) = [||G||^2 - 2cq(x-z) - 2cq(x, x-z)]/p^2(G^*(x-z)) - 1. \]

Hence
\[ S(x, x-z) - 1 \leq [||G||^2 - 2cq(x-z) - 2cq(x, x-z)]q(x-z)/p^2(G^*(x-z)) \]
\[ \leq [||G||^2 - 2cq(x-z) + 2cq(x)q(x-z)]q(x-z)/p^2(G^*(x-z)). \]

Consider the quadratic form \( Q(\xi) = ||G||^2 - 2c\xi^2 + 2cq(\xi)\). Since \( c \) is positive, there exists \( R_0 > 0 \) such that
\[ Q(\xi) < 0 \quad \text{for} \quad \xi > R_0. \]
Therefore \( S(x, x-z) \leq 1 \) whenever \( q(x-z) > R_0 \). Setting \( c_2 = 0 \) and \( R^* = R_0 \), we have \((A_5)\).

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References


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