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## ON THE LENGTHS OF THE SECOND FUNDAMENTAL FORMS OF $R$ -SPACES

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### Introduction

The aim of this paper is to study the lengths of the second fundamental forms of a certain class of homogeneous submanifolds, called  $R$ -spaces, minimally imbedded into a unit sphere  $S$ . Among these submanifolds, we find Veronese surfaces and generalized Clifford surfaces. These have been characterized as minimal submanifolds with second fundamental form of minimal positive constant length by Chern-Do Carmo-Kobayashi [2]. Also Simons [9] discusses the lengths of the second fundamental forms of submanifolds in  $S$ .

Our main results are as follows. Let  $\|A\|^2$  be the square of the length of the second fundamental form of an  $R$ -space  $N$  minimally imbedded into  $S$ . Then if  $N$  is regular (See section 2),  $\|A\|^2$  is a certain multiple of  $\dim N$ . If  $N$  is symmetric (See section 4), then  $\|A\|^2$  is a rational number. These results are independent of the choice of an invariant Riemannian metric on  $N$ .

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### 1. Preliminaries

1.1. Let  $(\mathfrak{g}, \sigma)$  be an orthogonal symmetric Lie algebra of compact type. Put  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ , where  $\mathfrak{k}$  (resp.  $\mathfrak{p}$ ) is the eigenspace of  $\sigma$  corresponding to the eigenvalue 1 (resp.  $-1$ ). Let  $\text{Aut}(\mathfrak{g})$  be the group of automorphisms of  $\mathfrak{g}$ . Identifying the Lie algebra of  $\text{Aut}(\mathfrak{g})$  with  $\mathfrak{g}$ , let  $K$  be the connected Lie subgroup of  $\text{Aut}(\mathfrak{g})$  corresponding to the Lie subalgebra  $\mathfrak{k}$  of  $\mathfrak{g}$ . Then  $K$  leaves the subspace  $\mathfrak{p}$  invariant. Let  $(\ , \ )$  be an inner product on  $\mathfrak{g}$  invariant under  $\text{Aut}(\mathfrak{g})$ . Then  $K$  acts as an isometry group on the Euclidean space  $\mathfrak{p}$  with the inner product  $(\ , \ )$ , the restriction of the inner product  $(\ , \ )$  on  $\mathfrak{g}$  to  $\mathfrak{p}$ . Let  $S$  be the unit sphere of  $\mathfrak{p}$ , and  $H$  an element of  $S$ . Let  $N$  be the orbit of  $K$  through  $H$ . Denoting by  $L$  the stabilizer of  $H$  in  $K$ , the space  $N$  may be identified with the quotient space  $K/L$ , which is called an  $R$ -space.

1.2. Let  $\mathfrak{a}$  be a maximal abelian subspace in  $\mathfrak{p}$ . We shall identify  $\mathfrak{a}$  with

the dual space  $\alpha^*$  of  $\alpha$  by the map  $\iota: \alpha \rightarrow \alpha^*$ ,  $\iota(X)(Y) = (Y, X)$  for  $X, Y \in \alpha$ . For  $\lambda \in \alpha$ , we define the subspace  $\mathfrak{k}_\lambda$  and  $\mathfrak{p}_\lambda$  of  $\mathfrak{g}$  as follows:

$$\begin{aligned}\mathfrak{k}_\lambda &= \{X \in \mathfrak{k}; \operatorname{ad}(H)^2 X = -(\lambda, H)^2 X, \text{ for any } H \in \alpha\}, \\ \mathfrak{p}_\lambda &= \{X \in \mathfrak{p}; \operatorname{ad}(H)^2 X = -(\lambda, H)^2 X, \text{ for any } H \in \alpha\}.\end{aligned}$$

Then  $\mathfrak{k}_{-\lambda} = \mathfrak{k}_\lambda$ ,  $\mathfrak{p}_{-\lambda} = \mathfrak{p}_\lambda$  and  $\mathfrak{p}_0 = \alpha$ . If we put

$$\mathfrak{r} = \{\lambda \in \alpha; \lambda \neq 0, \mathfrak{p}_\lambda \neq \{0\}\},$$

$\mathfrak{r}$  is a root system in  $\alpha$  (Satake [7]). The root system  $\mathfrak{r}$  is called the *restricted root system* of  $(\mathfrak{g}, \sigma)$ . We choose a linear order in  $\alpha$  and fix it once for all. We denote by  $\mathfrak{r}^+$  the set of positive roots in  $\mathfrak{r}$  with respect to this linear order in  $\alpha$ . Then we have the following orthogonal decomposition of  $\mathfrak{k}$  and  $\mathfrak{p}$  with respect to the inner product  $(\ , \ )$  (cf. Helgason [3]):

$$(1.1) \quad \mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in \mathfrak{r}^+} \mathfrak{k}_\lambda, \quad \mathfrak{p} = \alpha + \sum_{\lambda \in \mathfrak{r}^+} \mathfrak{p}_\lambda.$$

1.3. Let  $(M, h)$  and  $(M', g)$  be Riemannian manifolds, and  $f: M \rightarrow M'$  an isometric immersion. Let  $T_x(M)$  be the tangent space of  $M$  at a point  $x \in M$ , and  $T_x^\perp(M)$  the orthogonal complement of  $T_x(M)$  in  $T_{f(x)}(M')$ . Let  $A: T_x^\perp(M) \times T_x(M) \rightarrow T_x(M)$  be the Weingarten form at  $x \in M$ . Let  $\{e_1, \dots, e_n\}$  (resp.  $\{f_1, \dots, f_m\}$ ) be an orthonormal basis of  $T_x(M)$  (resp.  $T_x^\perp(M)$ ). Then the square of the length of the second fundamental form  $\|A\|^2(x)$  at  $x$  is given by

$$\|A\|^2(x) = \sum_{p=1}^n \sum_{q=1}^m |A_{f_q} e_p|^2,$$

where  $|X|^2 = g(X, X)$  for  $X \in T_{f(x)}(M')$ . Let  $\rho(x)$  be the scalar curvature of  $M$  at  $x$ .

**Lemma 1.** *If the immersion  $f: M \rightarrow M'$  is minimal and  $M'$  is a space form with the sectional curvature  $c$ , then we have*

$$(1.2) \quad \rho(x) = n(n-1)c - \|A\|^2(x),$$

where  $n = \dim M$ .

Proof. If  $c > 0$ , Simons [9] proves the formula. In the general case, we can prove the formula in the same way as in Simons [9].

## 2. Second fundamental forms of $R$ -spaces

2.1. As in section 1, we assume that the point  $H$  is contained in the unit sphere  $S$ . Moreover we may assume that  $H \in S \cap \alpha$  and  $(\lambda, H) \geq 0$  for any  $\lambda \in \mathfrak{r}^+$ , by virtue of the following lemma.

**Lemma 2** (Helgason [3]). *For any  $X \in \mathfrak{p}$ , there exists an element  $k \in K$  such*

that  $kX \in \alpha$  and  $(\lambda, kX) \geq 0$  for any  $\lambda \in \mathfrak{r}^+$ .

We identify the tangent space  $T_H(N)$  of  $N$  at  $H$  with a subspace of  $\mathfrak{p}$  in a canonical manner. Then we have  $T_H(N) = [\mathfrak{k}, H]$ . Put

$$\mathfrak{r}_1^+ = \{\lambda \in \mathfrak{r}^+; (\lambda, H) = 0\}, \mathfrak{r}_2^+ = \{\lambda \in \mathfrak{r}^+; (\lambda, H) > 0\}.$$

The tangent space  $T_H(N)$  and the orthogonal complement  $T_H^\perp(N)$  in  $T_H(S)$  are given by

$$(2.1) \quad T_H(N) = \sum_{\lambda \in \mathfrak{r}_2^+} \mathfrak{p}_\lambda,$$

$$(2.2) \quad T_H^\perp(N) = \alpha_H + \sum_{\lambda \in \mathfrak{r}_1^+} \mathfrak{p}_\lambda,$$

where  $\alpha_H = \{X \in \alpha; (X, H) = 0\}$ .

We shall call the submanifold  $N$  *regular*, if  $\mathfrak{r}_2^+ = \mathfrak{r}^+$ .

2.2. Let  $\Delta$  be the fundamental root system of  $\mathfrak{r}$  with respect to the order in  $\alpha$ . Put

$$\Delta_1 = \{\lambda \in \Delta; \lambda \in \mathfrak{r}_1^+\}.$$

Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{g}$  containing  $\alpha$ . Let  $\tilde{\mathfrak{g}}$  be the complexification of  $\mathfrak{g}$ , and  $\tilde{\mathfrak{h}}$  the subspace of  $\tilde{\mathfrak{g}}$  spanned by  $\mathfrak{h}$ . The inner product  $(\cdot, \cdot)$  on  $\mathfrak{g}$  can be extended uniquely to a complex symmetric bilinear form, denoted also by  $(\cdot, \cdot)$  on  $\tilde{\mathfrak{g}}$ . Let  $\tilde{\mathfrak{r}}$  be the root system of  $\tilde{\mathfrak{g}}$  relative to  $\tilde{\mathfrak{h}}$ . An element  $\alpha \in \tilde{\mathfrak{h}}$  belongs to  $\tilde{\mathfrak{r}}$ , if  $\alpha \neq 0$  and there exists a non-zero vector  $X \in \tilde{\mathfrak{g}}$  such that  $[H, X] = (\alpha, H)X$  for any  $H \in \tilde{\mathfrak{h}}$ . Let  $\mathfrak{h}_0$  be the real part of  $\tilde{\mathfrak{h}}$ , i.e. the real subspace of  $\tilde{\mathfrak{h}}$  spanned by  $\mathfrak{r}$ . Note that then  $\mathfrak{h}_0 = \sqrt{-1} \mathfrak{h}$ . We denote by the same letter  $\sigma$  the conjugation of  $\tilde{\mathfrak{g}}$  with respect to  $\mathfrak{k} + \sqrt{-1} \mathfrak{p}$ . We choose a  $\sigma$ -order in  $\mathfrak{h}_0$  in the sense of Satake [7] which has the following property. Let  $\tilde{\Delta}$  be the fundamental system with respect to this order in  $\mathfrak{h}_0$ , and denote by  $p$  the projection of  $\mathfrak{h}_0$  onto  $\sqrt{-1} \alpha$ . Then  $\sqrt{-1} \Delta = p(\tilde{\Delta}) - \{0\}$ . We denote the Satake diagram of  $\tilde{\Delta}$  also by  $\tilde{\Delta}$ . Put  $\tilde{\Delta}_1 = p^{-1}(\sqrt{-1} \Delta_1)$ . It is known (Takeuchi [11]) that isomorphic pairs  $(\tilde{\Delta}, \tilde{\Delta}_1)$  of Satake diagrams gives rise to isomorphic pairs  $(K, L)$ : We say that the pair  $(\tilde{\Delta}, \tilde{\Delta}_1)$  is isomorphic to the pair  $(\tilde{\Delta}', \tilde{\Delta}_1')$  if there exists an isomorphism  $\varphi$  of  $\tilde{\Delta}$  onto  $\tilde{\Delta}'$  such that  $\varphi$  maps  $\tilde{\Delta}_1$  onto  $\tilde{\Delta}_1'$ , and that the pair  $(K, L)$  is isomorphic to the pair  $(K', L')$  if there exists an isomorphism  $f$  of  $K$  onto  $K'$  such that  $f$  maps  $L$  onto  $L'$ .

2.3. Let  $\Delta_1$  be a subsystem of  $\Delta$ . Put

$$A(\Delta_1) = \left\{ H \in \alpha \cap S; \begin{array}{l} (\lambda, H) \geq 0, \text{ for any } \lambda \in \mathfrak{r}^+, \\ \{\lambda \in \Delta; (\lambda, H) = 0\} = \Delta_1 \end{array} \right\}.$$

Then there exists an element  $H \in A(\Delta_1)$  such that the orbit of  $K$  through  $H$  is minimal in  $S$ . This follows easily from Hsiang-Lawson [4] (Corollary 1.8). If  $(g, \sigma)$  is irreducible and the pair  $(K, L)$  is symmetric, then for the subsystem  $\Delta_1$  of  $\Delta$  obtained from  $N=K/L$  as in 2.2 the set  $A(\Delta_1)$  consists of only one element (cf. Takeuchi [11]). Therefore in this case the submanifold  $N$  is minimal.

2.4. Let  $A: T_H^\perp(N) \times T_H(N) \rightarrow T_H(N)$  be the Weingarten form of the submanifold  $N$  of  $S$  at  $H$ . The following proposition is due to Takagi-Takahashi [10].

**Proposition 3.** For  $X_\lambda \in \mathfrak{p}_\lambda$ ,  $\lambda \in \mathfrak{r}_2^+$ , the Weingarten form  $A$  is given by

$$\begin{aligned} A_{Z_0} X_\lambda &= -\frac{(\lambda, Z_0)}{(\lambda, H)} X_\lambda, \quad \text{if } Z_0 \in \mathfrak{a}_H, \\ A_{Z_\mu} X_\lambda &= -\frac{1}{(\lambda, H)^2} [[H, X_\lambda], Z_\mu], \quad \text{if } Z_\mu \in \mathfrak{p}_\mu, \mu \in \mathfrak{r}_1^+. \end{aligned}$$

There exists an orthonormal basis  $\{X_{\lambda \cdot 1}, \dots, X_{\lambda \cdot m_\lambda}\}$  (resp.  $\{Y_{\lambda \cdot 1}, \dots, Y_{\lambda \cdot m_\lambda}\}$ ) of  $\mathfrak{p}_\lambda$  (resp.  $\mathfrak{k}_\lambda$ ) such that

$$(2.3) \quad \begin{cases} [H, X_{\lambda \cdot p}] = -(\lambda, H) Y_{\lambda \cdot p}, \\ [H, Y_{\lambda \cdot p}] = (\lambda, H) X_{\lambda \cdot p} \end{cases} \quad \text{for any } H \in \mathfrak{a},$$

where  $m_\lambda$  is the multiplicity of  $\lambda \in \mathfrak{r}^+$ , i.e.  $m_\lambda = \dim \mathfrak{p}_\lambda$ .

**Proposition 4.** The square of the length of the second fundamental form  $\|A\|^2$  at  $H$  is given by

$$(2.4) \quad \|A\|^2 = -n + \sum_{\lambda \in \mathfrak{r}_2^+} \frac{1}{(\lambda, H)^2} (m_\lambda |\lambda|^2 + \sum_{p=1}^{m_\lambda} \sum_{\mu \in \mathfrak{r}_1^+} \sum_{q=1}^{m_\mu} |X_{(\lambda \cdot p; \mu \cdot q)}|^2).$$

Here  $n = \dim N$ ,  $X_{(\lambda \cdot p; \mu \cdot q)} = [Y_{\lambda \cdot p}, X_{\mu \cdot q}]$  and  $|X|^2 = (X, X)$  for  $X \in \mathfrak{g}$ . In particular when  $N$  is regular, we have

$$(2.5) \quad \|A\|^2 = -n + \sum_{\lambda \in \mathfrak{r}^+} m_\lambda \frac{|\lambda|^2}{(\lambda, H)^2}.$$

**Proof.** Let  $\{H, H_1, \dots, H_l\}$  be an orthonormal basis of  $\mathfrak{a}$ . Applying Proposition 3 and (2.3), we have

$$\begin{aligned} \|A\|^2 &= \sum_{\lambda \in \mathfrak{r}_2^+} \sum_{p=1}^{m_\lambda} \left( \sum_{k=1}^l |A_{H_k} X_{\lambda \cdot p}|^2 + \sum_{\mu \in \mathfrak{r}_1^+} \sum_{q=1}^{m_\mu} |A_{X_{\mu \cdot q}} X_{\lambda \cdot p}|^2 \right) \\ &= \sum_{\lambda \in \mathfrak{r}_2^+} \frac{1}{(\lambda, H)^2} (m_\lambda \sum_{k=1}^l (\lambda, H_k)^2) \end{aligned}$$

$$\begin{aligned}
& + \sum_{p=1}^{m_\lambda} \sum_{\mu \in \mathbf{r}_1^+} \sum_{q=1}^{m_\mu} |[Y_{\lambda \cdot p}, X_{\mu \cdot q}]|^2) \\
& = \sum_{\lambda \in \mathbf{r}_2^+} \frac{1}{(\lambda, H)^2} (m_\lambda (|\lambda|^2 - (\lambda, H)^2) \\
& \quad + \sum_{p=1}^{m_\lambda} \sum_{\mu \in \mathbf{r}_1^+} \sum_{q=1}^{m_\mu} |X_{(\lambda \cdot p; \mu \cdot q)}|^2) \\
& = -n + \sum_{\lambda \in \mathbf{r}_2^+} \frac{1}{(\lambda, H)^2} (m_\lambda |\lambda|^2 \\
& \quad + \sum_{p=1}^{m_\lambda} \sum_{\mu \in \mathbf{r}_1^+} \sum_{q=1}^{m_\mu} |X_{(\lambda \cdot p; \mu \cdot q)}|^2),
\end{aligned}$$

which proves the first formula of the proposition. The second formula (2.5) is the immediate consequence of (2.4).

2.5. Let  $\alpha: T_H(N) \times T_H(N) \rightarrow T_H^\perp(N)$  be the second fundamental form at  $H$ . Then we have (cf. Kobayashi-Nomizu [5])

$$(2.6) \quad (\alpha(X, Y), Z) = (A_Z X, Y) \quad \text{for } X, Y \in T_H(N) \text{ and } Z \in T_H^\perp(N).$$

**Proposition 5.** *The submanifold  $N$  of  $S$  is minimal if and only if the following condition is satisfied:*

$$(2.7) \quad \sum_{\lambda \in \mathbf{r}_2^+} \frac{m_\lambda}{(\lambda, H)} \lambda = nH.$$

Proof. By definition,  $N$  is minimal if and only if

$$\sum_{\lambda \in \mathbf{r}_2^+} \sum_{p=1}^{m_\lambda} \alpha(X_{\lambda \cdot p}, X_{\lambda \cdot p}) = 0.$$

By (2.6) and Proposition 3, we have

$$\begin{aligned}
\alpha(X_{\lambda \cdot p}, X_{\lambda \cdot p}) & = \sum_{k=1}^l (A_{H_k} X_{\lambda \cdot p}, X_{\lambda \cdot p}) H_k \\
& \quad + \sum_{\mu \in \mathbf{r}_1^+} \sum_{q=1}^{m_\mu} (A_{X_{\mu \cdot q}} X_{\lambda \cdot p}, X_{\lambda \cdot p}) X_{\mu \cdot q} \\
& = -\frac{1}{(\lambda, H)} \sum_{k=1}^l (\lambda, H_k) H_k \\
& = H - \frac{1}{(\lambda, H)} \lambda.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
0 & = \sum_{\lambda \in \mathbf{r}_2^+} \sum_{p=1}^{m_\lambda} \alpha(X_{\lambda \cdot p}, X_{\lambda \cdot p}) = \sum_{\lambda \in \mathbf{r}_2^+} m_\lambda \left( H - \frac{1}{(\lambda, H)} \lambda \right) \\
& = nH - \sum_{\lambda \in \mathbf{r}_2^+} \frac{m_\lambda}{(\lambda, H)} \lambda,
\end{aligned}$$

which proves the proposition.

2.6. Assume that the algebra  $\mathfrak{g}$  decomposes into the direct sum  $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$  of two ideals  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  invariant under  $\sigma$ . For  $i=1, 2$ , let  $\mathfrak{g}_i = \mathfrak{k}_i + \mathfrak{p}_i$ , where  $\mathfrak{k}_i = \mathfrak{g}_i \cap \mathfrak{k}$  and  $\mathfrak{p}_i = \mathfrak{g}_i \cap \mathfrak{p}$ , and put  $S_i = S \cap \mathfrak{p}_i$ ,  $\alpha_i = \alpha \cap \mathfrak{p}_i$ . Assume that an element  $H_i \in \alpha_i \cap S_i$  satisfies  $(\lambda, H_i) \geq 0$  for any  $\lambda \in \mathfrak{r}^+$ . Let  $N_i$  be the orbit of  $K$  through  $H_i$ , and suppose that the submanifold  $N_i$  of  $S_i$  is minimal. Let  $\|A_i\|^2$  be the square of the second fundamental form of the submanifold  $N_i$  of  $S_i$ . Then we have

**Proposition 6.** *Assume that the submanifold  $N$  is the orbit of  $K$  through  $H = \sqrt{\frac{n_1}{n}} H_1 + \sqrt{\frac{n_2}{n}} H_2$ , where  $n_i = \dim N_i$ . Then  $N$  is a minimal submanifold of the unit sphere  $S$  and we have*

$$(2.8) \quad \|A\|^2 = n \left( 1 + \frac{1}{n_1} \|A_1\|^2 + \frac{1}{n_2} \|A_2\|^2 \right).$$

Proof. Put  $(\mathfrak{r}_i)_s^+ = \mathfrak{r}_s^+ \cap \mathfrak{p}_i$ ,  $i, s=1, 2$ . By (2.7) we have

$$\sum_{\lambda \in (\mathfrak{r}_i)_2^+} \frac{m_\lambda}{(\lambda, H_i)} \lambda = n_i H_i.$$

Hence

$$\begin{aligned} \sum_{\lambda \in \mathfrak{r}_2^+} \frac{m_\lambda}{(\lambda, H)} \lambda &= \sum_{\lambda \in (\mathfrak{r}_1)_2^+} \frac{m_\lambda}{(\lambda, H)} \lambda + \sum_{\lambda \in (\mathfrak{r}_2)_2^+} \frac{m_\lambda}{(\lambda, H)} \lambda \\ &= \sqrt{nm_1} H_1 + \sqrt{nm_2} H_2 \\ &= nH, \end{aligned}$$

which proves the minimality of  $N$ . By (2.4) we have

$$\begin{aligned} \|A\|^2 &= -n + \sum_{\lambda \in (\mathfrak{r}_1)_2^+} \frac{1}{(\lambda, H)^2} (m_\lambda |\lambda|^2 \\ &\quad + \sum_{p=1}^{m_\lambda} \sum_{\mu \in (\mathfrak{r}_1)_1^+} \sum_{q=1}^{m_\mu} |X_{(\lambda \cdot p \cdot \mu \cdot q)}|^2) \\ &\quad + \sum_{\lambda \in (\mathfrak{r}_2)_2^+} \frac{1}{(\lambda, H)^2} (m_\lambda |\lambda|^2 \\ &\quad + \sum_{p=1}^{m_\lambda} \sum_{\mu \in (\mathfrak{r}_2)_1^+} \sum_{q=1}^{m_\mu} |X_{(\lambda \cdot p \cdot \mu \cdot q)}|^2) \\ &= -n + \frac{n}{n_1} (\|A_1\|^2 + n_1) + \frac{n}{n_2} (\|A_2\|^2 + n_2) \\ &= n \left( 1 + \frac{1}{n_1} \|A_1\|^2 + \frac{1}{n_2} \|A_2\|^2 \right), \end{aligned}$$

which proves (2.8).

2.7. **EXAMPLE.** Let  $(\mathfrak{g}, \sigma)$  be the orthogonal symmetric Lie algebra corresponding to a symmetric pair  $(SU(3), SO(3))$ . Then if  $N$  is not regular, the pair  $(K, L)$  is either  $(SO(3), S(O(1) \times O(2)))$  or  $(SO(3), S(O(2) \times O(1)))$ . In these cases the submanifolds  $N$  are minimal, and they are isometric. They are the so-called Veronese surfaces. Applying (2.4) and (2.7), we get

$$\|A\|^2 = \begin{cases} 6, & \text{if } N \text{ is regular and minimal,} \\ \frac{4}{3}, & \text{if } N \text{ is the Veronese surface.} \end{cases}$$

### 3. The case where the submanifold $N$ is regular

3.1. In this section we assume that the submanifold  $N$  is regular. Put

$$\mathfrak{s} = \{\lambda \in \mathfrak{r}; 2\lambda \notin \mathfrak{r}\} \quad \text{and} \quad \mathfrak{s}^+ = \{\lambda \in \mathfrak{s}; \lambda \in \mathfrak{r}^+\}.$$

Then  $\mathfrak{s}$  is a reduced root system. For  $\lambda \in \mathfrak{s}^+$ , put  $k_\lambda = m_\lambda + m_{\lambda/2}$ , where  $m_{\lambda/2} = 0$  unless  $\frac{\lambda}{2} \in \mathfrak{r}$ . Then by Proposition 4, we get

$$(3.1) \quad \|A\|^2 = -n + \sum_{\lambda \in \mathfrak{s}^+} k_\lambda \frac{|\lambda|^2}{(\lambda, H)^2},$$

and the submanifold  $N$  is minimal if and only if

$$(3.2) \quad \sum_{\lambda \in \mathfrak{s}^+} \frac{k_\lambda}{(\lambda, H)} \lambda = nH,$$

by Proposition 5.

**Theorem 1.** *If the submanifold  $N$  is regular and minimal, then*

$$(3.3) \quad \|A\|^2 = n(|\mathfrak{s}^+| - 1).$$

*Proof.* By (3.1) it is sufficient to show that

$$\sum_{\lambda \in \mathfrak{s}^+} k_\lambda \frac{|\lambda|^2}{(\lambda, H)^2} = n \cdot |\mathfrak{s}^+|.$$

On the other hand, we have

$$\left( \sum_{\lambda \in \mathfrak{s}^+} \frac{1}{(\lambda, H)} \lambda, nH \right) = n \cdot |\mathfrak{s}^+|.$$

Therefore by (3.2) it is sufficient to prove

$$(3.4) \quad \sum_{\lambda \in \mathfrak{s}^+} k_\lambda \frac{|\lambda|^2}{(\lambda, H)^2} = \left( \sum_{\lambda \in \mathfrak{s}^+} \frac{1}{(\lambda, H)} \lambda, \sum_{\mu \in \mathfrak{s}^+} \frac{k_\mu}{(\mu, H)} \mu \right).$$

To prove the formula, we prepare two lemmas. Let  $V$  be an  $h$ -dimensional real



vector space. Let  $\Phi$  be a reduced root system in  $V$ , and  $W$  the Weyl group of  $\Phi$ . Let  $(\cdot, \cdot)$  be an inner product on  $V$  invariant under  $W$ . We choose a linear order in  $V$ . Let  $\Phi^+$  be the set of positive roots with respect to this order. For  $\lambda \in \Phi^+$ , put

$$\Phi_\lambda^+ = \{\xi \in \Phi^+; \xi = s\lambda \text{ for some } s \in W\}.$$

We can take a subset  $\Lambda$  of  $\Phi^+$  such that the union  $\Phi^+ = \bigcup_{\lambda \in \Lambda} \Phi_\lambda^+$  is disjoint. For  $\lambda \in \Lambda$  and  $H \in V$  such that  $(\eta, H) \neq 0$  for any  $\eta \in \Phi$ , put

$$K(\lambda, H) = \sum_{\xi \in \Phi_\lambda^+} \frac{1}{(\xi, H)} \xi.$$

**Lemma 7.**  $|K(\lambda, H)|^2 = \sum_{\xi \in \Phi_\lambda^+} \frac{|\xi|^2}{(\xi, H)^2}.$

*Proof.* Since

$$|K(\lambda, H)|^2 = \sum_{\xi \in \Phi_\lambda^+} \frac{|\xi|^2}{(\xi, H)^2} + 2 \sum_{\substack{\xi, \eta \in \Phi_\lambda^+ \\ \xi < \eta}} \frac{(\xi, \eta)}{(\xi, H)(\eta, H)},$$

it suffices to prove

$$(3.5) \quad \sum_{\substack{\xi, \eta \in \Phi_\lambda^+ \\ \xi < \eta}} \frac{(\xi, \eta)}{(\xi, H)(\eta, H)} = 0.$$

Assume that  $\xi, \eta \in \Phi_\lambda^+$  and  $\xi < \eta$ . Then  $|\xi| = |\eta| = |\lambda|$ . If  $(\xi, \eta) > 0$  (resp.  $< 0$ ), we have  $(\xi, \eta) = \frac{|\lambda|^2}{2}$  (resp.  $-\frac{|\lambda|^2}{2}$ ) (cf. Serre [8]). Suppose  $(\xi, \eta) < 0$ . Then  $(\xi, \xi + \eta) = |\xi|^2 + (\xi, \eta) = \frac{|\lambda|^2}{2}$ , and similarly  $(\eta, \xi + \eta) = \frac{|\lambda|^2}{2}$ . It follows easily

$$(3.6) \quad \frac{(\xi, \eta)}{(\xi, H)(\eta, H)} + \frac{(\xi, \xi + \eta)}{(\xi, H)(\xi + \eta, H)} + \frac{(\eta, \xi + \eta)}{(\eta, H)(\xi + \eta, H)} = 0.$$

Put

$$\begin{aligned} A^+ &= \{(\xi; \eta) \in \Phi_\lambda^+ \times \Phi_\lambda^+; (\xi, \eta) > 0, \xi < \eta\}, \\ A^- &= \{(\xi; \eta) \in \Phi_\lambda^+ \times \Phi_\lambda^+; (\xi, \eta) < 0, \xi < \eta\}. \end{aligned}$$

We define a mapping  $f$  of  $A^+$  to  $A^-$  by

$$f(\xi; \eta) = \begin{cases} (\xi; \eta - \xi), & \text{if } \xi < \eta - \xi, \\ (\eta - \xi; \xi), & \text{if } \eta - \xi < \xi. \end{cases}$$

Let  $S_\xi$  be the symmetry with respect to  $\xi$ . Then, if  $(\xi, \eta) > 0$ ,  $(\xi, \eta) = \frac{|\lambda|^2}{2}$  and so  $S_\xi(\eta) = \eta - \xi$ . Therefore the above mapping is well-defined. If  $(\xi, \eta) < 0$ ,

then  $(\xi, \eta) = -\frac{|\lambda|^2}{2}$  and so  $S_\xi(\eta) = \xi + \eta$ . Therefore we have easily

$$(3.7) \quad f^{-1}(\xi; \eta) = \{(\xi; \xi + \eta), (\eta; \xi + \eta)\}.$$

This, together with (3.6), implies (3.5). The proof of Lemma 7 is completed.

**Lemma 8.**  $(K(\lambda, H), K(\mu, H)) = 0$  for  $\lambda, \mu \in \Lambda, \lambda \neq \mu$ .

Proof. We have

$$(K(\lambda, H), K(\mu, H)) = \sum_{\xi \in \Phi_\lambda^+} \sum_{\eta \in \Phi_\mu^+} \frac{(\xi, \eta)}{(\xi, H)(\eta, H)}.$$

If  $\lambda$  and  $\mu$  are contained in the different irreducible components of  $\Phi$ , the formula is trivially true, and so we may assume that the root system  $\Phi$  is irreducible. Then if  $\alpha, \beta \in \Phi$  are such that  $|\alpha| = |\beta|$ , there exists an element  $s \in W$  such that  $\beta = s\alpha$ . Therefore we have  $|\lambda| \neq |\mu|$ . We may assume  $|\lambda| < |\mu|$ . Since the root system  $\Phi$  is reduced, we have  $|\mu|^2 = 2|\lambda|^2$  or  $3|\lambda|^2$  (cf. Serre [8]).

In the case of  $|\mu|^2 = 3|\lambda|^2$ ,  $\Phi$  is of type  $G_2$  and we may assume that  $\Lambda$  is a fundamental root system of  $\Phi$ . Then we have  $(\lambda, \mu) = -\frac{3}{2}|\lambda|^2$ ,  $\Phi_\lambda^+ = \{\lambda, \lambda + \mu, 2\lambda + \mu\}$  and  $\Phi_\mu^+ = \{\mu, 3\lambda + \mu, 3\lambda + 2\mu\}$ . In this case the proof is straightforward.

In the case of  $|\mu|^2 = 2|\lambda|^2$ , assume that  $\xi \in \Phi_\lambda^+$  and  $\eta \in \Phi_\mu^+$ . If  $(\xi, \eta) > 0$  (resp.  $< 0$ ), then we have  $(\xi, \eta) = |\lambda|^2$  (resp.  $-|\lambda|^2$ ) (cf. Serre [8]). If  $(\xi, \eta) < 0$ , it follows easily

$$(3.8) \quad \frac{(\xi, \eta)}{(\xi, H)(\eta, H)} + \frac{(\xi + \eta, \eta)}{(\xi + \eta, H)(\eta, H)} + \frac{(\xi + \eta, 2\xi + \eta)}{(\xi + \eta, H)(2\xi + \eta, H)} + \frac{(\xi, 2\xi + \eta)}{(\xi, H)(2\xi + \eta, H)} = 0.$$

Put

$$\begin{aligned} A^+ &= \{(\xi; \eta) \in \Phi_\lambda^+ \times \Phi_\mu^+; (\xi, \eta) > 0\}, \\ A^- &= \{(\xi; \eta) \in \Phi_\lambda^+ \times \Phi_\mu^+; (\xi, \eta) < 0\}. \end{aligned}$$

We define a mapping  $f$  of  $A^+$  to  $A^-$  by

$$f(\xi; \eta) = \begin{cases} (\xi; \eta - 2\xi), & \text{if } \eta - 2\xi \in \Phi^+, \\ (\xi - \eta; \eta), & \text{if } 2\xi - \eta \in \Phi^+ \text{ and } \xi - \eta \in \Phi^+, \\ (\eta - \xi; 2\xi - \eta), & \text{if } 2\xi - \eta \in \Phi^+ \text{ and } \eta - \xi \in \Phi^+. \end{cases}$$

If  $(\xi, \eta) > 0$ , then  $(\xi, \eta) = |\lambda|^2$  and so  $S_\xi(\eta) = \eta - 2\xi$ ,  $S_\eta(\xi) = \xi - \eta$ . Therefore the above mapping  $f$  is well-defined. If  $(\xi, \eta) < 0$ , then  $(\xi, \eta) = -|\lambda|^2$  and so

$S_\xi(\eta) = \eta + 2\xi$ ,  $S_\eta(\xi) = \xi + \eta$ . Therefore we have easily

$$(3.9) \quad f^{-1}(\xi; \eta) = \{(\xi; 2\xi + \eta), (\xi + \eta; \eta), (\xi + \eta; 2\xi + \eta)\}.$$

This, together with (3.8), implies the assertion, thus completing the proof of the lemma.

We return to the proof of Theorem 1. Taking  $\mathfrak{s}, \mathfrak{s}^+$  for  $\Phi, \Phi^+$ , let  $\Lambda \subset \mathfrak{s}^+$  be as above. Since  $k_\xi = k_\lambda$  for  $\lambda \in \Lambda$ ,  $\xi \in \mathfrak{s}_\lambda^+$ , we have

$$\begin{aligned} & \left( \sum_{\lambda \in \mathfrak{s}^+} \frac{1}{(\lambda, H)} \lambda, \sum_{\mu \in \mathfrak{s}^+} \frac{k_\mu}{(\mu, H)} \mu \right) = \left( \sum_{\lambda \in \Lambda} K(\lambda, H), \sum_{\mu \in \Lambda} k_\mu K(\mu, H) \right) \\ & = \sum_{\lambda \in \Lambda} k_\lambda |K(\lambda, H)|^2 + \sum_{\substack{\lambda, \mu \in \Lambda \\ \lambda \neq \mu}} k_\mu (K(\lambda, H), K(\mu, H)). \end{aligned}$$

Applying Lemma 7 and Lemma 8, we get (3.4), and this proves Theorem 1.

#### 4. The case where the pair $(K, L)$ is symmetric

4.1. Let  $\tilde{\mathfrak{g}}$  be the complexification of  $\mathfrak{g}$ . For a subspace  $\mathfrak{v}$  of  $\mathfrak{g}$ , we denote by  $\tilde{\mathfrak{v}}$  the subspace of  $\tilde{\mathfrak{g}}$  spanned by  $\mathfrak{v}$ . Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{g}$  invariant under  $\sigma$ . Put  $\tilde{\mathfrak{h}} = \mathfrak{h}^+ + \mathfrak{h}^-$ , where  $\mathfrak{h}^+ = \mathfrak{k} \cap \mathfrak{h}$  and  $\mathfrak{h}^- = \mathfrak{p} \cap \mathfrak{h}$ . We denote also by  $(\ , \ )$  the symmetric  $\mathbb{C}$ -bilinear form on  $\tilde{\mathfrak{g}}$  which is the extension of the inner product  $(\ , \ )$  on  $\mathfrak{g}$ . Let  $\tilde{\mathfrak{r}}$  be the root system of  $\tilde{\mathfrak{g}}$  relative to  $\tilde{\mathfrak{h}}$ . An element  $\alpha \in \tilde{\mathfrak{h}}$  belongs to  $\tilde{\mathfrak{r}}$ , if  $\alpha \neq 0$  and there exists a non-zero vector  $X \in \tilde{\mathfrak{g}}$  such that  $[H, X] = (\alpha, H)X$  for any  $H \in \tilde{\mathfrak{h}}$ . We have the root space decomposition

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} + \sum_{\alpha \in \tilde{\mathfrak{r}}} \tilde{\mathfrak{g}}_\alpha,$$

where  $\tilde{\mathfrak{g}}_\alpha$  is the eigenspace belonging to  $\alpha \in \tilde{\mathfrak{r}}$ . Let  $\tau$  be the conjugation of  $\tilde{\mathfrak{g}}$  with respect to  $\mathfrak{g}$ . We can choose a Weyl canonical basis  $\{E_\alpha; \alpha \in \tilde{\mathfrak{r}}\}$  such that  $\tau E_\alpha = E_{-\alpha}$  for each  $\alpha \in \tilde{\mathfrak{r}}$  (cf. Serre [8]). We denote also by the same letter  $\sigma$  the conjugation of  $\tilde{\mathfrak{g}}$  with respect to  $\mathfrak{k} + \sqrt{-1}\mathfrak{p}$ . Then we have  $\sigma(\tilde{\mathfrak{r}}) = \tilde{\mathfrak{r}}$  and  $\sigma(\tilde{\mathfrak{g}}_\alpha) = \tilde{\mathfrak{g}}_{\sigma\alpha}$ . Put  $\sigma E_\alpha = \rho_\alpha E_{\sigma\alpha}$  for each  $\alpha \in \tilde{\mathfrak{r}}$ , and define  $\tilde{\mathfrak{r}}_0 = \{\alpha \in \tilde{\mathfrak{r}}; \sigma\alpha = -\alpha\}$ . Then we have easily  $|\rho_\alpha| = 1$  for any  $\alpha \in \tilde{\mathfrak{r}}$  and  $\rho_\alpha = \rho_{-\alpha} = \pm 1$  for  $\alpha \in \tilde{\mathfrak{r}}_0$ . Put

$$\tilde{\mathfrak{r}}_0^+ = \{\alpha \in \tilde{\mathfrak{r}}_0; \rho_\alpha = 1\}, \quad \tilde{\mathfrak{r}}_0^- = \{\alpha \in \tilde{\mathfrak{r}}_0; \rho_\alpha = -1\}.$$

Then we have the following decompositions

$$(4.1) \quad \tilde{\mathfrak{k}} = \tilde{\mathfrak{h}}^+ + \sum_{\alpha \in \tilde{\mathfrak{r}}_0^+} \tilde{\mathfrak{g}}_\alpha + \sum_{\alpha \in \tilde{\mathfrak{r}} - \tilde{\mathfrak{r}}_0} \mathbb{C}(E_\alpha + \sigma E_{-\alpha}),$$

$$(4.2) \quad \tilde{\mathfrak{p}} = \tilde{\mathfrak{h}}^- + \sum_{\alpha \in \tilde{\mathfrak{r}}_0^-} \tilde{\mathfrak{g}}_\alpha + \sum_{\alpha \in \tilde{\mathfrak{r}} - \tilde{\mathfrak{r}}_0} \mathbb{C}(E_\alpha - \sigma E_{-\alpha}),$$

where the last summations in (4.1) and (4.2) run over all unordered pairs  $(\alpha, \sigma\alpha)$

such that  $\alpha \in \tilde{\tau} - \tilde{\tau}_0$ . Put

$$\tilde{\tau}_1 = \{\alpha \in \tilde{\tau}; \sigma\alpha = \alpha\}.$$

The following lemma is an easy consequence of (4.1).

**Lemma 9.**  $\mathfrak{h}^+$  is maximal abelian subspace of  $\mathfrak{k}$ , if and only if the set  $\tilde{\tau}_1$  is empty.

In the following, let  $\mathfrak{h}^+$  be a maximal abelian subspace of  $\mathfrak{k}$ . By Lemma 9 we obtain the following lemma.

**Lemma 10** (Murakami [6]).

$$\alpha + \sigma\alpha \notin \tilde{\tau} \quad \text{for any } \alpha \in \tilde{\tau}.$$

Since the group  $K$  is compact, we can consider the root system of  $\tilde{\mathfrak{k}}$  relative to  $\tilde{\mathfrak{h}}^+$ , say  $\tilde{\Sigma}$ . Put  $\bar{\alpha} = \frac{1}{2}(\alpha - \sigma\alpha)$  for each  $\alpha \in \tilde{\tau}$ . By (4.1) and Lemma 9 we have

**Lemma 11** (Murakami [6]).

$$(4.3) \quad \tilde{\Sigma} = \{\bar{\alpha}; \alpha \in \tilde{\tau}\}.$$

**Lemma 12.** For  $\alpha \in \tilde{\tau}$ , we have

$$(4.4) \quad \frac{(\alpha, \alpha)}{(\bar{\alpha}, \bar{\alpha})} = \begin{cases} 1, & \text{if } \sigma\alpha = -\alpha, \\ 2, & \text{if } \sigma\alpha \neq -\alpha, (\sigma\alpha, \alpha) = 0, \\ 4, & \text{if } \sigma\alpha \neq -\alpha, (\sigma\alpha, \alpha) \neq 0. \end{cases}$$

Proof. Since  $(\sigma\alpha, \sigma\alpha) = (\alpha, \alpha)$  and  $\bar{\alpha} \neq 0$ , we have

$$(4.5) \quad \frac{(\alpha, \alpha)}{(\bar{\alpha}, \bar{\alpha})} = \frac{4(\alpha, \alpha)}{(\alpha - \sigma\alpha, \alpha - \sigma\alpha)} = \frac{4}{2 - \frac{2(\sigma\alpha, \alpha)}{(\alpha, \alpha)}}.$$

Since  $(\sigma\alpha, \sigma\alpha) = (\alpha, \alpha)$  and  $\sigma\alpha \neq \alpha$ , we have  $\frac{2(\alpha, \sigma\alpha)}{(\alpha, \alpha)} = -2, \pm 1$  or 0, and

$\frac{2(\alpha, \sigma\alpha)}{(\alpha, \alpha)} = -2$  if and only if  $\sigma\alpha = -\alpha$  (cf. Serre [8]). Suppose  $\sigma\alpha \neq -\alpha$ .

Since  $\alpha + \sigma\alpha \notin \tilde{\tau}$  by Lemma 10, we must have  $\frac{2(\sigma\alpha, \alpha)}{(\alpha, \alpha)} \geq 0$  (cf. Serre [8]).

Therefore for each  $\alpha \in \tilde{\tau}$  we have

$$(4.6) \quad \frac{2(\sigma\alpha, \alpha)}{(\alpha, \alpha)} = -2, 1 \text{ or } 0.$$

This, together with (4.5), completes the proof.

4.2. We define two  $K$ -invariant Riemann metrics  $g$  and  $g'$  on the quotient space  $K/L$  as follows: The metric  $g$  is induced from the imbedding  $\varphi: K/L \rightarrow S$ ,  $\varphi(kL) = kH$  for  $k \in K$ . The other metric  $g'$  is induced from the  $K$ -invariant inner product  $(\ , \ )$  on  $\mathfrak{k}$ , the restriction of the inner product  $(\ , \ )$  on  $\mathfrak{g}$  to  $\mathfrak{k}$ .

**Lemma 13** (Takeuchi-Kobayashi [12]). *If the orthogonal symmetric Lie algebra  $(\mathfrak{g}, \sigma)$  is irreducible and the pair  $(K, L)$  is symmetric, then we have*

$$(4.7) \quad g = (\lambda, H)^2 g',$$

where  $\Delta - \Delta_1 = \{\lambda\}$ .

REMARK. Under the assumptions of Lemma 13, we have  $(\xi, H)^2 = (\eta, H)^2$  for any  $\xi, \eta \in \mathfrak{r}_2^+$ .

Let  $\rho$  (resp.  $\rho'$ ) be the scalar curvature with respect to the metric  $g$  (resp.  $g'$ ). Under the assumptions of Lemma 13, (4.7) implies

$$(4.8) \quad \rho = \frac{1}{(\lambda, H)^2} \rho'.$$

Suppose that  $(\mathfrak{g}, \sigma)$  is irreducible and the pair  $(K, L)$  is symmetric. Let  $\theta$  be the involutive automorphism of  $K$  defining the symmetric pair  $(K, L)$ . Then  $\mathfrak{k} = \mathfrak{l} + \mathfrak{m}$ , where  $\mathfrak{l}$  (resp.  $\mathfrak{m}$ ) is the eigenspace of  $\theta$  corresponding to the eigenvalue 1 (resp.  $-1$ ), and  $\mathfrak{l}$  is the Lie algebra of  $L$ . We have the following decomposition (cf. Helgason [3]):

$$\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{k}_1 + \cdots + \mathfrak{k}_r,$$

where each  $\mathfrak{k}_j$  is an ideal of  $\mathfrak{k}$  invariant under  $\theta$ ,  $(\mathfrak{k}_0, \theta)$  is of Euclidean type, and  $(\mathfrak{k}_i, \theta)$ ,  $i=1, \dots, r$ , is irreducible of compact type. Put  $\mathfrak{l}_j = \mathfrak{k}_j \cap \mathfrak{l}$  and  $\mathfrak{m}_j = \mathfrak{k}_j \cap \mathfrak{m}$ ,  $j=0, 1, \dots, r$ . Then  $\mathfrak{k}_j = \mathfrak{l}_j + \mathfrak{m}_j$ . Let  $\mathfrak{b}_j$  be a maximal abelian subspace of  $\mathfrak{m}_j$ , and  $\Sigma_j$  the restricted root system of  $(\mathfrak{k}_j, \theta)$  ( $j=0, 1, \dots, r$ ). For each  $\mathfrak{b}_j$ , we choose a linear order in  $\mathfrak{b}_j$ . Let  $\Sigma_j^+$  be the set of positive roots in  $\Sigma_j$  with respect to this order.

**Lemma 14.** *We have*

$$(4.9) \quad \rho' = \sum_{i=1}^r \frac{h_i}{b_i} \sum_{\omega \in \Sigma_i^+} m_\omega |\omega|^2,$$

where  $h_j = \dim \mathfrak{m}_j$ ,  $b_j = \dim \mathfrak{b}_j$  ( $j=0, 1, \dots, r$ ), and  $m_\omega$  is the multiplicity of  $\omega \in \Sigma_i^+$ .

Proof. Put  $\mathfrak{b} = \mathfrak{b}_0 + \mathfrak{b}_1 + \cdots + \mathfrak{b}_r$ . For  $\omega \in \Sigma_i^+$ ,  $i=1, \dots, r$ , we define the subspace  $\mathfrak{m}_\omega$  as follows:

$$\mathfrak{m}_\omega = \{X \in \mathfrak{m}; ad(H)^2 X = -(\omega, H)^2 X \text{ for any } H \in \mathfrak{b}\}.$$

Then we have the decomposition

$$\mathfrak{m} = \sum_{j=0}^r \mathfrak{b}_j + \sum_{i=1}^r \sum_{\omega \in \Sigma_i^+} \mathfrak{m}_\omega.$$

Let  $S(, )$  be the Ricci tensor of  $(K/L, g')$ . Since  $(\mathfrak{k}_0, \theta)$  is of Euclidean type and  $(\mathfrak{k}_i, \theta)$ ,  $i=1, \dots, r$ , is irreducible, there exist constants  $c_j$ ,  $j=0, 1, \dots, r$ , such that

$$(4.10) \quad S(X, Y) = c_j(X, Y) \quad \text{for any } X, Y \in \mathfrak{m}_j,$$

where we identify the tangent space  $T_0(K/L)$  at the origin with  $\mathfrak{m}$ . Let  $\{H_{j \cdot 1}, \dots, H_{j \cdot b_j}\}$  (resp.  $\{X_{\omega \cdot 1}, \dots, X_{\omega \cdot m_\omega}\}$ ) be an orthonormal basis of  $\mathfrak{b}_j$  (resp.  $\mathfrak{m}_\omega$ ) with respect to  $(, )$ . By (4.10) we have

$$(4.11) \quad \begin{aligned} \rho' &= \sum_{j=0}^r \left( \sum_{p=1}^{b_j} S(H_{j \cdot p}, H_{j \cdot p}) \right) + \sum_{\omega \in \Sigma_i^+} \sum_{q=1}^{m_\omega} S(X_{\omega \cdot q}, X_{\omega \cdot q}) \\ &= \sum_{i=1}^r c_i h_i \end{aligned}$$

because  $c_0=0$ . Let  $R$  be the curvature tensor of  $(K/L, g')$ . Then we have, (cf. Helgason [3])

$$R(X, Y)Z = -[[X, Y], Z] \text{ for any } X, Y, Z \in \mathfrak{m}.$$

Therefore for  $1 \leq i \leq r$ , we have

$$\begin{aligned} c_i &= S(H_{i \cdot p}, H_{i \cdot p}) \\ &= \sum_{j=0}^r \left( \sum_{q=1}^{b_j} (R(H_{j \cdot q}, H_{i \cdot p})H_{i \cdot p}, H_{j \cdot q}) \right. \\ &\quad \left. + \sum_{\omega \in \Sigma_i^+} \sum_{q=1}^{m_\omega} (R(X_{\omega \cdot q}, H_{i \cdot p})H_{i \cdot p}, X_{\omega \cdot q}) \right) \\ &= \sum_{\omega \in \Sigma_i^+} m_\omega (\omega, H_{i \cdot p})^2. \end{aligned}$$

So we get

$$(4.12) \quad \begin{aligned} b_i c_i &= \sum_{p=1}^{b_i} S(H_{i \cdot p}, H_{i \cdot p}) \\ &= \sum_{\omega \in \Sigma_i^+} m_\omega |\omega|^2. \end{aligned}$$

The formulas (4.11) and (4.12) imply (4.9) in the lemma.

**Theorem 2.** *If the orthogonal symmetric Lie algebra  $(\mathfrak{g}, \sigma)$  is irreducible and the pair  $(K, L)$  is symmetric, then the square of the length of the second fundamental form  $\|A\|^2$  is a rational number.*

Proof. By (1.2) it is sufficient to show that  $\rho$  is rational. By (4.8) and (4.9) we have

$$(4.13) \quad \rho = \frac{1}{(\lambda, H)^2} \sum_{i=1}^r \frac{h_i}{b_i} \sum_{\omega \in \Sigma_i^+} m_\omega |\omega|^2.$$

Let  $\mathfrak{h}_j$  be a Cartan subalgebra of  $\mathfrak{k}_j$  containing  $\mathfrak{b}_j$ , and  $\tilde{\Sigma}_j$  the root system of  $\tilde{\mathfrak{k}}_j$  relative to  $\tilde{\mathfrak{h}}_j$  ( $j=0, 1, \dots, r$ ). Put  $\mathfrak{h}^+ = \mathfrak{h}_0 + \mathfrak{h}_1 + \dots + \mathfrak{h}_r$ . Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{g}$  containing  $\mathfrak{h}^+$  and  $\tilde{\mathfrak{r}}$  the root system of  $\tilde{\mathfrak{g}}$  relative to  $\tilde{\mathfrak{h}}$ . For  $i=1, \dots, r$ , let  $B_i$  be the Killing form of  $\mathfrak{k}_i$ . Note that the restriction of the inner product  $(\ , \ )$  to  $\mathfrak{k}_i$  is a positive multiple of  $-B_i$ , because  $\Sigma_i$  is irreducible and  $(\ , \ )$  is invariant under  $\text{Aut}(\mathfrak{k}_i)$ . By the relation between  $\tilde{\Sigma}_i$  and  $\Sigma_i$  given by Araki [1] (the proof of Proposition 2.1), for  $\omega \in \Sigma_i$  there exists a root  $\beta \in \tilde{\Sigma}_i$  such that

$$\frac{-(\beta, \beta)}{(\omega, \omega)} = 1, 2 \text{ or } 4.$$

By (4.3) there exists a root  $\alpha \in \tilde{\mathfrak{r}}$  such that  $\beta = \alpha$ , and we have by (4.4)

$$\frac{(\alpha, \alpha)}{(\beta, \beta)} = 1, 2 \text{ or } 4.$$

Since  $\mathfrak{r}$  is irreducible and the inner product  $(\ , \ )$  on  $\mathfrak{g}$  is invariant under  $\text{Aut}(\mathfrak{g})$ ,  $\frac{-(\alpha, \alpha)}{(\lambda, \lambda)}$  is rational by the same reason as above. Therefore  $\frac{|\omega|^2}{|\lambda|^2}$  is rational.

By (4.13) it is now sufficient to show that  $\frac{|\lambda|^2}{(\lambda, H)^2}$  is rational. Let  $\Delta = \{\lambda_1, \lambda_2, \dots, \lambda_{l+1} = \lambda\}$  and put

$$\begin{aligned} a_{ij} &= (\lambda_i, \lambda_j), \quad i, j = 1, \dots, l+1, \\ A_0 &= 1, \quad A_s = |a_{ij}|_{i,j=1,\dots,s}, \quad s = 1, \dots, l+1. \end{aligned}$$

Then by induction on  $j$ , we have easily  $A_j > 0$ ,  $j=0, 1, \dots, l+1$ . Put

$$\xi = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1l+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{l1} & a_{l2} & \cdots & a_{ll+1} \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{l+1} \end{vmatrix}.$$

Then we have easily  $(\lambda_i, \xi) = 0$  ( $i=1, \dots, l$ ),  $(\lambda_{l+1}, \xi) = A_{l+1}$  and  $(\xi, \xi) = (\lambda_{l+1}, \xi) A_l$ .

Since  $H$  is a multiple of  $\xi$ , we have  $(\lambda_{l+1}, H)^2 = \frac{A_{l+1}}{A_l}$ . Since  $\mathfrak{r}$  is irreducible, we

have  $\frac{a_{ij}}{|\lambda|^2} = \frac{(\lambda_i, \lambda_j)}{(\lambda_{l+1}, \lambda_{l+1})}$  and these are rational numbers. Hence we have

$$\frac{|\lambda|^2}{(\lambda, H)^2} = \frac{\frac{1}{|\lambda|^{2l}} A_l}{\frac{1}{|\lambda|^{2(l+1)}} A_{l+1}} = \frac{\begin{vmatrix} \frac{a_{11}}{|\lambda|^2} & \frac{a_{12}}{|\lambda|^2} & \dots & \frac{a_{1l}}{|\lambda|^2} \\ \dots & \dots & \dots & \dots \\ \frac{a_{l1}}{|\lambda|^2} & \frac{a_{l2}}{|\lambda|^2} & \dots & \frac{a_{ll}}{|\lambda|^2} \end{vmatrix}}{\begin{vmatrix} \frac{a_{11}}{|\lambda|^2} & \frac{a_{12}}{|\lambda|^2} & \dots & \frac{a_{1l+1}}{|\lambda|^2} \\ \dots & \dots & \dots & \dots \\ \frac{a_{l1}}{|\lambda|^2} & \frac{a_{l2}}{|\lambda|^2} & \dots & \frac{a_{ll+1}}{|\lambda|^2} \\ \frac{a_{l+1,1}}{|\lambda|^2} & \frac{a_{l+1,2}}{|\lambda|^2} & \dots & \frac{a_{l+1,l+1}}{|\lambda|^2} \end{vmatrix}}$$

and this is a rational number. This completes the proof of Theorem 2.

**Corollary.** *If the submanifold  $N$  is minimal and the pair  $(K, L)$  is symmetric, then  $\|A\|^2$  is a rational number.*

**Proof.** Suppose that  $\mathfrak{g}$  decomposes into the direct sum  $\mathfrak{g} = \mathfrak{g}_1 + \dots + \mathfrak{g}_r$  of ideals  $\mathfrak{g}_i$  invariant under  $\sigma$  and  $(\mathfrak{g}_i, \sigma)$  is irreducible. Put  $\mathfrak{g}_i = \mathfrak{k}_i + \mathfrak{p}_i$ ,  $S_i = S \cap \mathfrak{p}_i$ ,  $\mathfrak{a}_i = \mathfrak{a} \cap \mathfrak{p}_i$  and  $(\mathfrak{r}_i)_s^+ = \mathfrak{r}_s^+ \cap \mathfrak{p}_i$  ( $i=1, \dots, r, s=1, 2$ ), where  $\mathfrak{k}_i = \mathfrak{k} \cap \mathfrak{g}_i$  and  $\mathfrak{p}_i = \mathfrak{p} \cap \mathfrak{g}_i$ . Let  $H = a_1 H_1 + \dots + a_r H_r$ , where  $H_i \in S_i \cap \mathfrak{a}_i$  and  $(\lambda, H_i) \geq 0$  for any  $\lambda \in \mathfrak{r}^+$ . Let  $N_i$  be the orbit of  $K$  through  $H_i$ . Then by Takeuchi [11] and the remark in 2.3, the submanifold  $N_i$  of  $S_i$  is a symmetric space and minimal is  $S_i$ . Put  $n_i = \dim N_i$ . By (2.7) we have

$$\begin{aligned} nH &= \sum_{\lambda \in \mathfrak{r}_2^+} \frac{m_\lambda}{(\lambda, H)} \lambda \\ &= \sum_{i=1}^r \sum_{\lambda \in (\mathfrak{r}_i)_2^+} \frac{m_\lambda}{(\lambda, a_i H_i)} \lambda \\ &= \sum_{i=1}^r \frac{n_i}{a_i} H_i. \end{aligned}$$

Therefore we have  $a_i = \sqrt{\frac{n_i}{n}}$ . Applying Theorem 2 and (2.8), the corollary follows by induction on  $r$ .

4.3. We give the table of  $\|A\|^2$  in the following cases:

- (1) The orthogonal symmetric Lie algebra  $(\mathfrak{g}, \sigma)$  is irreducible.
- (2) The pair  $(K, L)$  is symmetric.

Here  $S'(O(p-1) \times O(q-1))$  is the subgroup of  $SO(p) \times SO(q)$  consisting of matrices of the form



$$\begin{pmatrix} \varepsilon & O \\ O & A \\ & \varepsilon & O \\ & O & B \end{pmatrix}, \quad \varepsilon = \pm 1, \quad A \in O(p-1), \quad B \in O(q-1).$$

$(g, \sigma)$	$N$	$\dim N$	$\ A\ ^2$
A	$SU(p+q)/S(U(p) \times U(q))$	$2pq$	$2pq(pq-1)$
B	$SO(2n+1)/SO(2) \times SO(2n-1)$	$2(2n-1)$	$4(n-1)(2n-1)$
C	$Sp(n)/U(n)$	$n(n+1)$	$\frac{1}{2}n(n+1)(n-1)(n+2)$
D	(1) $SO(2n)/SO(2) \times SO(2n-2)$	$4(n-1)$	$4(n-1)(2n-3)$
	(2) $SO(2n)/U(n)$	$n(n-1)$	$\frac{1}{2}n(n-1)(n+1)(n-2)$
$E_6$	symmetric space of type EIII	32	$32 \times 15$
$E_7$	symmetric space of type EVII	54	$54 \times 26$
AI	$SO(p+q)/S(O(p) \times O(q))$	$pq$	$\frac{1}{2}pq\left(\frac{pq(p+q+2)}{p+q}-2\right)$
AII	$Sp(p+q)/Sp(p) \times Sp(q)$	$4pq$	$4pq\left(\frac{2pq(p+q-1)}{p+q}-1\right)$
AIII	$U(n)$	$n^2$	$\frac{1}{2}n^2(n-1)(n+1)$
BDI	(1) $SO(p) \times SO(q)/S'(O(p-1) \times O(q-1))$	$p+q-2$	$2(p-1)(q-1)$
	(2) $SO(p)$	$\frac{1}{2}p(p-1)$	$\frac{1}{2}p(p-1)(p-2)(p+2)$
DIII	$U(2n)/Sp(n)$	$n(2n-1)$	$n(n-1)^2(2n+1)$
CI	$U(n)/O(n)$	$\frac{1}{2}n(n+1)$	$\frac{1}{8}n(n-1)(n+2)^2$
CII	$Sp(n)$	$n(2n+1)$	$n(n-1)(n+1)(2n+1)$
EI	$\mathfrak{f}$ is of type $C_4$ $\mathfrak{l}$ is of type $C_2 \times C_2$	16	$\frac{16 \times 25}{3}$
EIV	$F_4/\text{Spin}(9)$	16	$16 \times 3$
EV	$\mathfrak{f}$ is of type $A_7$ $\mathfrak{l}$ is of type $C_4$	27	$27 \times 14$
EVII	$\mathfrak{f}$ is of type $\mathbf{R} \times E_6$ $\mathfrak{l}$ is of type $F_4$	27	$26 \times 9$

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