



Title	A non-stable secondary operation and homotopy classification of maps.
Author(s)	Nomura, Yasutoshi
Citation	Osaka Journal of Mathematics. 1969, 6(1), p. 117-134
Version Type	VoR
URL	https://doi.org/10.18910/8899
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

A NON-STABLE SECONDARY OPERATION AND HOMOTOPY CLASSIFICATION OF MAPS

Dedicated to Professor A. Komatu on his 60th birthday

YASUTOSHI NOMURA

(Received October 26, 1968)

1. Introduction

Let $p: E \rightarrow A$ be the principal fibration with classifying map $\theta: A \rightarrow B$ and let $q: T \rightarrow E$ be the principal fibration induced by a map $\rho: E \rightarrow C$. We assume that A , B and C are H -spaces. Given a map $v: X \rightarrow E$, I. M. James and E. Thomas [5, 7] have defined the homomorphisms

$$\begin{aligned}\Delta(\theta, u): [X, \Omega A] &\rightarrow [X, \Omega B] \\ \Delta_p(\rho, v): [X, \Omega^2 B] &\rightarrow [X, \Omega C],\end{aligned}$$

where u is the composite $p \circ v$, Ω is the loop functor and $[Y, Z]$ denotes the set of based homotopy classes of based maps $Y \rightarrow Z$.

The action $\Omega C \times T \rightarrow T$ of the principal fibration q induces the function

$$[X, \Omega C] \times [X, T] \rightarrow [X, T],$$

the image of $(\tau, w) \in [X, \Omega C] \times [X, T]$ under which is denoted by $\tau \cdot w$. The subgroup

$$I(w) = \{\tau \in [X, \Omega C]; \tau \cdot w = w\}$$

of $[X, \Omega C]$ is called the *isotropy group* of w under the action of $[X, \Omega C]$ on $[X, T]$. Our first main result is the following:

Theorem A. *Suppose $w: X \rightarrow T$ is a lifting of v . If $\Delta(\theta, u)$ is injective, then $I(w)$ coincides with the image of $\Delta_p(\rho, v)$.*

This is obtained as a direct consequence of a property (Theorem 4.2) of a non-stable secondary operation $\Phi_\theta(\rho, v)$ which is inspired by an operation due to N. Shimada [14, p. 141].

The prime concern in this paper is to examine a few situations to which Theorem A is applicable.

Consider first the real projective space $P_n(R)$, where the dimension n is odd > 1 . Let X be a *path-connected* $(n+1)$ -dimensional complex and let δ^*

denote the Bockstein homomorphism associated with $0 \rightarrow Z \rightarrow Z \rightarrow Z_2 \rightarrow 0$. We define

$$\varphi: [X, P_n(R)] \rightarrow H^1(X; Z_2)$$

by $\varphi(g) = g^* \iota$, where $g: X \rightarrow P_n(R)$ and ι denotes the generator of $H^1(P_n(R); Z_2)$. The following extends a result due to P. Olum [10].

Theorem B. *Let u be an element of $H^1(X; Z_2)$ such that $(\delta^* u)^{(n+1)/2} = 0$. Then $\varphi^{-1}(u)$ is equivalent to:*

$$\begin{aligned} H^n(X; Z) \times H^{n+1}(X; Z_2) / (Sq^2 + u^2 \cup) H^{n-1}(X; Z) & \quad \text{for } n \equiv 1 \pmod{4} \\ H^n(X; Z) \times H^{n+1}(X; Z_2) / Sq^2 H^{n-1}(X; Z) & \quad \text{for } n \equiv 3 \pmod{4} \end{aligned}$$

Next, let η be the reduced stable class of the canonical line bundle over the real projective space and let $N_n(\xi; X)$ denote the number of classes of n -plane bundles over X which are stably equivalent to a stable reduced bundle ξ over X . The following is a partial extension of a result due to I. M. James and E. Thomas [5].

Theorem C. *Let k be an integer and let $\binom{k}{n+1}$ be even. Then*

- (1) *for $n \equiv 1 \pmod{4}$ and $\binom{k}{2}$ odd, $N_n(k\eta; P_{n+1}(R)) = 1$ or 2 according as $\binom{k-1}{n-1}$ is odd or even;*
- (2) *$N_n(k\eta; P_{n+1}(R)) = 2$ if $n \equiv 1 \pmod{4}$, $\binom{k}{2}$ even and $\binom{k-1}{n-1}$ odd;*
- (3) *$N_n(k\eta; P_{n+2}(R)) = 2$ if $n \equiv 3 \pmod{4}$ and $\binom{k-1}{n-1}$ odd.*

Finally, let $\mathcal{E}(X)$ denote the group of homotopy classes of homotopy equivalences of X whose group structure is induced by map-composition. Consider the tower

$$\begin{array}{ccccc} \Omega C & \longrightarrow & T & & \\ & & \downarrow q & & \\ \Omega B & \longrightarrow & E & \xrightarrow{\rho} & C \\ & & \downarrow p & & \\ & & K(\pi, n) & \xrightarrow{\theta} & B, \end{array}$$

where B and C are H -spaces such that $\pi_r(B) \neq 0$ only for $n+2 \leq r \leq m$ ($n > 1$) and $\pi_s(C) \neq 0$ only for $m+1 \leq s \leq m+n-1$. The following theorem can be obtained by applying the results of [9] to the fibration q and observing the fact $H^{n-1}(T; \pi) = 0$.

Theorem D. *The following sequence of groups and homomorphisms is exact:*

$$[T, \Omega^2 B] \xrightarrow{\Delta_p(\rho, q)} q^*[E, \Omega C] \rightarrow \mathcal{E}(T) \rightarrow \mathcal{E}(E) \times \mathcal{E}(\Omega C)$$

in which the image of the last homomorphism consists of $(g, \Omega h) \in \mathcal{E}(E) \times \mathcal{E}(\Omega C)$ such that $\rho g \simeq h \rho$.

The results stated in sections 2, 3 and 4 can be dualized in a cofibre space and will be considered elsewhere.

2. Preliminaries

We shall here fix the notations and recall some definitions given in [5, 6, 7]. We work in the category of spaces with basepoints (usually denoted by $*$) and basepoint preserving maps. The spaces considered are assumed to have the homotopy type of a CW complex. We blur the distinction between maps and their homotopy classes. We use the additive notation for path-composition and path-inversion. The suspension functor is denoted by S .

For a space X , let $\Omega^* X$ denote the space of free loops in X . One has a fibration

$$\Omega X \xrightarrow{i_X} \Omega^* X \xrightarrow{r_X} X$$

with section $s_X: X \rightarrow \Omega^* X$ given by $s_X(x) = \text{the constant loop at } x \in X$. A map $f: X \rightarrow Y$ induces the map $\Omega^* f: \Omega^* X \rightarrow \Omega^* Y$ in an obvious way.

Lemma 2.1. $(i_X)_* \xi = (i_X)_* \xi'$ for $\xi, \xi' \in [V, \Omega X]$ if and only if ξ and ξ' are conjugate to each other.

This can be proved directly or by replacing i_X by a principal fibration.

Corollary 2.2. (Theorem 2.6 of [5]) *The group $[V, \Omega X]$ is abelian if and only if $(i_X)_*: [V, \Omega X] \rightarrow [V, \Omega^* X]$ is injective.*

Given a fibration $p: E \rightarrow A$ and a map $u: X \rightarrow A$, we denote by $[X, E; u]$ the set of u -homotopy classes of u -maps $X \rightarrow E$ (see [6])

Let $p: E \rightarrow A$ be a fibration with fibre inclusion $j: F \rightarrow E$ and let $q: T \rightarrow E$ be the principal fibration with classifying map $\rho: E \rightarrow C$. Given a map $w: X \rightarrow T$, let $v = q \circ w$ and $u = p \circ v$. We define the u -isotropy group $I_u(w)$ of w by setting

$$I_u(w) = \{\tau \in [X, \Omega C]; \quad \tau \cdot w = w \text{ in } [X, T; u]\}.$$

q induces the function

$$q_*: [X, T; u] \rightarrow [X, E; u]$$

and there is a bijection between $q_*^{-1}(v)$ and the totality of left cosets $[X, \Omega C]/I_u(w)$. The following is obvious:

Proposition 2.3. (1) *For the trivial map $*$: $X \rightarrow T$, $I_*(*)$ is the image of*

- $(\Omega\rho)_*(\Omega j)_*: [X, \Omega F] \rightarrow [X, \Omega C];$
 (2) $I_u(\tau \cdot w) = \tau + I_u(w) - \tau$ for $\tau \in [X, \Omega C];$
 (3) $g^* I_u(w) \subset I_{u \circ g}(g^* w)$ for $g: Y \rightarrow X;$
 (4) Let $i: \Omega C \rightarrow T$ denote the inclusion. Then $I_*(i) = 1 + (\Omega\rho)_*(\Omega j)_*[\Omega C, \Omega F] - 1$, where 1 is the identity map of ΩC .

Let $\Omega_p^* E$ denote the subspace of $\Omega^* E$ consisting of free loops λ such that $(\Omega^* p)\lambda \in s_A(A)$. Then one obtains a fibration $r: \Omega_p^* E \rightarrow E$ with fibre ΩF .

Theorem 2.4. $\tau \in [X, \Omega C]$ is ρ -correlated (see §3 of [7]) to $v \in [X, E; u]$, if and only if there exists an $\eta \in [X, T; u]$ such that $\tau \in I_u(\eta)$ and $q_* \eta = v$.

Proof. The “if” part is proved in Lemma (3.3) of [7]. We shall prove the “only if” part. Assume that there exists a $\psi \in [X, \Omega_p^* E; u]$ such that $r_* \psi = v$ and $\rho'_* \psi = (i_C)_* \tau$ in the following commutative diagram

$$\begin{array}{ccc} [X, \Omega_p^* E; u] & \xrightarrow{r_*} & [X, E; u] \\ \downarrow \rho'_* & & \downarrow \rho_* \\ [X, \Omega C] & \xrightarrow{(i_C)_*} [X, \Omega^* C] & \xrightarrow{(r_C)_*} [X, C], \end{array}$$

where $\rho': \Omega_p^* E \rightarrow \Omega^* C$ denotes the restriction of $\Omega^* \rho$ to $\Omega_p^* E$.

Since r is a fibration, we may assume $\psi: X \rightarrow \Omega_p^* E$ is a lift of v . Let $F_t: X \rightarrow \Omega^* C$ be a homotopy with $F_0 = i_C \tau$, $F_1 = \rho'_* \psi$. Consider $\gamma: X \rightarrow C^I$ defined by $\gamma(x)(t) = r_C \circ F_t(x)$, $x \in X$, $0 \leq t \leq 1$. It is easy to see that $\tau = \gamma + \rho'_* \psi - \gamma$ in $[X, \Omega C]$. Take $\eta: X \rightarrow T$ given by $\eta(x) = (v, \gamma)$; then it follows that $(\gamma + \rho'_* \psi - \gamma) \cdot \eta = \eta$ in $[X, T; u]$. This proves the only if part.

Corollary 2.5. Suppose $p: E \rightarrow A$ is a principal fibration with fibre F in the sense of [12]. Then $[X, \Omega F] = 0$ implies $I_u(w) = 0$.

Proof. Let $\mu: F \times E \rightarrow E$ denote the action map and let $\mu': \Omega F \times E \rightarrow \Omega_p^* E$ be the induced map defined as in §4 of [7]. Then, by Theorem (4.1) of [7],

$$\mu'_*: [X, \Omega F] \times [X, E; u] \rightarrow [X, \Omega_p^* E; u]$$

is bijective. Note that $\mu'\{*, 1_E\}: E \rightarrow \Omega_p^* E$ is p -homotopic to the canonical section $s: E \rightarrow \Omega_p^* E$ of r , so that both s and r induce the bijections between $[X, E; u]$ and $[X, \Omega_p^* E; u]$ because of $[X, \Omega F] = 0$.

Now let $\tau \in [X, \Omega C]$ be ρ -correlated to v , i.e., there is an element $\psi \in [X, \Omega_p^* E; u]$ such that $r_* \psi = v$, $\rho'_* \psi = (i_C)_* \tau$. Then, since $q_* w = v$,

$$(i_C)_* \tau = \rho'_* s_* v = (s_C)_* \rho_* v = 0,$$

which implies $\tau = 0$ by virtue of $\ker (i_C)_* = 0$.

Taking $A = *$ in the above situation, we obtain

Corollary 2.6. $\tau \in [X, \Omega C]$ is ρ -correlated to $v \in [X, E]$ (see §2 of [5]), if and only if there is an element $\eta \in [X, T]$ such that $\tau \in I(\eta)$ and $q_*\eta = v$. If E is an H -space with $[X, \Omega E] = 0$, then $I(\eta) = 0$ for any lifting η of $v: X \rightarrow E$.

3. The homomorphisms $\Delta(\theta, u)$ and $\Delta_p(\rho, v)$

Consider the situation

$$(3.1) \quad \begin{array}{ccccc} \Omega C & \xrightarrow{i} & T & & \\ & & \downarrow q & & \\ F = \Omega B & \xrightarrow{j} & E & \xrightarrow{\rho} & C \\ & & \downarrow p & \searrow \theta & \\ & & A & \longrightarrow & B, \end{array}$$

in which p and q are the principal fibrations with classifying maps θ and ρ respectively, and B and C are H -spaces with multiplications $t: B \times B \rightarrow B$ and $n: C \times C \rightarrow C$. Let $\mu: F \times E \rightarrow E$ denote the action of F on E .

In case A is an H -space with multiplication $m: A \times A \rightarrow A$ and there is given a map $v: X \rightarrow E$ with $u = p \circ v$, I. M. James and E. Thomas have defined in [5, 7] the homomorphisms

$$\begin{aligned} \Delta(\theta, u): [X, \Omega A] &\rightarrow [X, \Omega B], \\ \Delta_p(\rho, v): [X, \Omega F] &\rightarrow [X, \Omega C] \end{aligned}$$

as follows. Let

$$\begin{aligned} \mu': (\Omega F \times E, \Omega F \times F) &\rightarrow (\Omega_p^* E, \Omega^* F) \\ m': \Omega A \times A &\rightarrow \Omega^* A, \quad t': \Omega B \times B \rightarrow \Omega^* B, \quad n': \Omega C \times C \rightarrow \Omega^* C \end{aligned}$$

denote the "right translations" determined by μ, m, t and n ; then the equations

$$\begin{aligned} (\Omega^* \theta)_* m'_* \{\alpha, u\} &= t'_* \{\Delta(\theta, u) \alpha, \theta_* u\} & \text{for } \alpha \in [X, \Omega A] \\ \rho'_* \mu'_* \{\beta, v\} &= n'_* \{\Delta_p(\rho, v) \beta, \rho_* v\} & \text{for } \beta \in [X, \Omega F] \end{aligned}$$

determine $\Delta(\theta, u)\alpha$ and $\Delta_p(\rho, v)\beta$ uniquely by virtue of Theorem 2.7 of [5]. The following is proven in [7, Theorem (4.2)]:

Theorem 3.2. $\tau \in [X, \Omega C]$ is ρ -correlated to $v \in [X, E; u]$ if and only if, first, $\rho_* v = 0$ and, secondly, τ lies in the image of $\Delta_p(\rho, v)$. Thus, $I_u(w) = \Delta_p(\rho, v)[X, \Omega F]$ for any lift w of v .

The homomorphism $\Delta(\theta, u)$ has also been introduced by J. W. Rutter [13], who has examined various properties of it. In the similar way we can obtain analogous theorems for $\Delta_p(\rho, v)$, so that the proofs are mostly omitted.

Triviality Theorem 3.3. $\Delta_p(\rho, *) = (\Omega\rho)_*(\Omega j)_*$. More generally, if $j \circ \vartheta = v$ for a map $\vartheta: X \rightarrow F$, then $\Delta_p(\rho, v) = \Delta(\rho j, \vartheta)$.

In order to state the next theorem we define the *dual Hopf invariant* $v(\rho) \in [F \times E, C]$ of ρ to be

$$v(\rho) = -(j \circ p_1)^* \rho + \mu^* \rho - p_2^* \rho.$$

where $p_1: F \times E \rightarrow F$ and $p_2: F \times E \rightarrow E$ are the projections. If $v(\rho) = 0$, we say that ρ is *primitive with respect to μ* .

Primitivity Theorem 3.4. If ρ is primitive with respect to μ , then $\Delta_p(\rho, v) = (\Omega\rho)_*(\Omega j)_*$.

Theorem 3.5. If X is an H cogroup (see [15]), then $\Delta_p(\rho, v) = (\Omega\rho)_*(\Omega j)_*$.

Composition Theorem 3.6. Let $\alpha: C \rightarrow D$ be a map to an H space. Then $\Delta_p(\alpha\rho, v) = \Delta(\alpha, \rho v) \Delta_p(\rho, v)$.

Cartesian Product Theorem 3.7. Let $\rho_1: E \rightarrow C_1$, $\rho_2: E \rightarrow C_2$ be maps to H -spaces and let $\{\rho_1, \rho_2\} = (\rho_1 \times \rho_2) \circ d: E \rightarrow C_1 \times C_2$ be the composite with diagonal map d . Then

$$\Delta_p(\{\rho_1, \rho_2\}, v)\beta = (\Delta_p(\rho_1, v)\beta, \Delta_p(\rho_2, v)\beta).$$

Additivity Theorem 3.8. Let $\rho_1, \rho_2: E \rightarrow C$ be maps. Then

$$\Delta_p(\rho_1 + \rho_2, v)\beta = \Delta_p(\rho_1, v)\beta + (\rho_1 v) \square \Delta_p(\rho_2, v)\beta,$$

$$\Delta_p(-\rho, v)\beta = -(-\rho v) \square \Delta_p(\rho, v)\beta,$$

where $(\rho_1 v) \square$ and $(-\rho v) \square$ are the endomorphisms as defined in [13, p. 383].

Cup Product Theorem 3.9. Given $\rho \in H^a(E; \pi)$ and $\rho' \in H^b(E; \pi')$, let $\rho \cup \rho'$ denote the cup product with respect to a pairing $\pi \otimes \pi' \rightarrow G$. Then

$$\Delta_p(\rho \cup \rho', v)\beta = \Delta_p(\rho, v)\beta \cup v^* \rho' + (-1)^a v^* \rho \cup \Delta_p(\rho', v)\beta.$$

The following theorem will be useful in computing $\Delta_p(\rho, v)$ in terms of deviation of ρ from primitivity and corresponds to Corollary 1.4 of [5] or Theorem 2.4.1 of [13].

Theorem 3.10. Suppose C is homotopy abelian or $\rho_* v = 0$. Then

$$\Delta_p(\rho, v)\beta = (\Omega\rho)_*(\Omega j)_* \beta + \Delta_{p p_2}(v(\rho), \{*, v\})\{\beta, *\}.$$

Assume further that C is an Eilenberg-MacLane space and that

$$v(\rho) = \sum u_i \times v_i + \sum \delta^*(u'_j \times v'_j),$$

where δ^* is the Bockstein. Then

$$\Delta_p(\rho, v)\beta = (\Omega\rho)_*(\Omega j)_*\beta + \sum (\Omega u_i)\beta \cup v^*v_i - \sum \delta^*(\Omega u'_j)\beta \cup v^*v'_j.$$

Proof. We imitate the proof of Theorem 2.4.1 of Rutter [13]. Consider the diagram

$$\begin{array}{ccccc} & & F \times E & \xrightarrow{\rho\mu} & C \\ & \nearrow \{\ast, v\} & \downarrow p p_2 & \searrow \rho p_2 & \\ X & \xrightarrow{u} & A & \xrightarrow{\{\ast, \theta\}} & B \times B \end{array}$$

Note that the action $\bar{\mu}: (F \times F) \times (F \times E) \rightarrow F \times E$ of the principal fibration $p \circ p_2$ is given by $\bar{\mu}(x, x'; x'', y) = (x + x'', \mu(x', y))$. Thus, the right translation $\bar{\mu}': \Omega(F \times F) \times (F \times E) \rightarrow \Omega_{pp_2}^*(F \times E)$ satisfies the following:

$$\begin{aligned} (\rho\mu)'_*\bar{\mu}'_*(\{\beta, \ast\}, \{\ast, v\}) &= \rho'_*\mu'_*(\{\beta, v\}), \\ (\rho p_2)'_*\bar{\mu}'_*(\{\beta, \ast\}, \{\ast, v\}) &= (s_C)_*\rho_*v, \\ p'_1*\mu'_*(\{\beta, \ast\}, \{\ast, v\}) &= (i_F)_*\beta. \end{aligned}$$

Using these and by 3.8 and 3.6 we have that

$$\begin{aligned} \Delta_p(\rho, v)\beta &= \Delta_{pp_2}(\rho\mu, \{\ast, v\})\{\beta, \ast\} \\ &= \Delta_{pp_2}(\rho j p_1, \{\ast, v\})\{\beta, \ast\} + \Delta_{pp_2}(\nu(\rho), \{\ast, v\})\{\beta, \ast\} \\ &\quad + \Delta_{pp_2}(\rho p_2, \{\ast, v\})\{\beta, \ast\} \\ &= \Delta(\rho j, \ast)\Delta_{pp_2}(p_1, \{\ast, v\})\{\beta, \ast\} + \Delta_{pp_2}(\nu(\rho), \{\ast, v\})\{\beta, \ast\} \\ &= (\Omega\rho)_*(\Omega j)_*\beta + \Delta_{pp_2}(\nu(\rho), \{\ast, v\})\{\beta, \ast\}. \end{aligned}$$

Now it follows from 3.9, 3.6 and 1.4.1 of [13] that

$$\begin{aligned} \Delta_{pp_2}(u_i \times v_i, \{\ast, v\})\{\beta, \ast\} &= \Delta_{pp_2}(p_1^*u_i, \{\ast, v\})\{\beta, \ast\} \cup v_i p_2\{\ast, v\} \\ &\quad + (-1)^{\dim u_i} u_i p_1\{\ast, v\} \cup \Delta_{pp_2}(p_2^*v_i, \{\ast, v\})\{\beta, \ast\} \\ &= \Delta(u_i, \ast)\Delta_{pp_2}(p_1, \{\ast, v\})\{\beta, \ast\} \cup v^*v_i \\ &= (\Omega u_i)\beta \cup v^*v_i. \end{aligned}$$

Similarly,

$$\Delta_{pp_2}(\delta^*(u'_j \times v'_j), \{\ast, v\})\{\beta, \ast\} = (\Omega\delta^*)(\Omega u'_j)_*\beta \cup v^*v'_j.$$

This completes the proof of 3.10.

In [7], James and Thomas have called $\pi = p \circ q$ of (3.1) a *stable decomposition* of π if there exists a map $c: F \times A \rightarrow C$ such that the composite

$$A \xrightarrow{i_2} F \times A \xrightarrow{c} C$$

is null-homotopic and $\rho\mu \simeq c(1 \times p) + \rho p_*$, where i_2 denotes the injection. Note that, if ρ is primitive with respect to μ , then $\pi = p \circ q$ is stable with $c = \rho j p_*$, where p_* denotes the projection $F \times A \rightarrow F$. As stated in p. 104 of [7], for $v: X \rightarrow E$ liftable to T , $\tau \cdot v$ is liftable to T if and only if $c_*\{\tau, p_*v\} = 0$. The following theorem can also be proved in a way similar to 3.10.

Theorem 3.11. (James and Thomas [7]) *If (3.1) is a stable decomposition of $\pi = p \circ q$, then*

$$\Delta_p(\rho, v)\beta = c'_*\{\beta, p_*v\},$$

where $c': \Omega F \times A \rightarrow \Omega C$ is determined by the composite

$$\Omega F \times A \xrightarrow{\text{injection}} \Omega(F \times A) \times F \times A \xrightarrow{\approx} \Omega^*(F \times A) \xrightarrow{\Omega^*c} \Omega^*C$$

4. Secondary operation $\Phi_\theta(\rho, v)$ and proof of Theorem A

Consider the situation (3.1) in which A, B and C are H -spaces with multiplications m, t and n respectively. Given a map $v: X \rightarrow E$, we set $u = p \circ v$. We shall now define a sort of secondary operation

$$\Phi_\theta(\rho, v): \ker \Delta(\theta, u) \rightarrow \text{coker } \Delta_p(\rho, v)$$

as follows.

Take an element $\alpha \in [X, \Omega A]$ such that $(\Omega^*\theta)_*m'_*\{\alpha, u\} = 0$; then it follows from the next Sublemma (i) that there exists an element $\psi \in [X, \Omega^*E; u]$ such that

$$(4.1) \quad (\Omega^*p)_*\psi = hm'_*\{\alpha, u\} \quad \text{and} \quad (r_E)_*\psi = v \in [X, E; u],$$

where $h: [X, \Omega^*A; m\{*, u\}] \rightarrow [X, \Omega^*A; u]$ is the canonical bijection. The coset of $\gamma \in [X, \Omega C]$ determined by $(\Omega^*\rho)_*\psi = n'_*\{\gamma, \rho_*v\}$ is, by definition, $\Phi_\theta(\rho, v)\alpha$.

Observe that, if there exists another ψ' such that $(\Omega^*p)_*\psi' = (\Omega^*p)_*\psi$ in $[X, \Omega^*A; u]$ and $(r_E)_*\psi' = v$ in $[X, E; u]$, then we may assume $(r_E)_*\psi' = v = (r_E)_*\psi$ as maps and, applying Sublemma (ii) to $\varphi = \psi + (-\psi')$, we conclude that $(\Omega^*\rho)_*\psi - (\Omega^*\rho)_*\psi'$ lies in the image of $\Delta_p(\rho, v)$. This ensures that $\Phi_\theta(\rho, v)$ is well defined.

Sublemma. (i) *Given an element $\beta \in [X, \Omega^*A; u]$ lying in the image of $(\Omega^*p)_*; [X, \Omega^*E] \rightarrow [X, \Omega^*A]$, there exists a $\psi \in [X, \Omega^*E; u]$ such that $(\Omega^*p)_*\psi = \beta$ and $(r_E)_*\psi = v$.*

(ii) *If $\varphi \in [X, \Omega^*E; u]$ satisfies $(\Omega^*p)_*\varphi = (s_A)_*u$, then φ is contained in the image of the natural map $[X, \Omega_p^*E; u] \rightarrow [X, \Omega^*E; u]$.*

Proof. (i) Let $\hat{\mu}: F \times \Omega^*E \rightarrow \Omega^*E$ be the map induced by the action $\mu: F \times E \rightarrow E$. It is easily verified that

$$(\Omega^*p)\hat{\mu}(x, \gamma) = (\Omega^*p)\gamma, \quad \mu(1 \times r_E) = r_E\hat{\mu} \quad (x \in F, \gamma \in \Omega^*E)$$

By assumption we can take $\tilde{\psi}: X \rightarrow \Omega^*E$ with $(\Omega^*p)\tilde{\psi} \simeq \beta$, so that there is a u -map $\psi_0: X \rightarrow \Omega^*E$ with $(\Omega^*p)\psi_0 = \beta$, since Ω^*p is a fibration. Choose $\omega: X \rightarrow F$ such that $\mu\{\omega, r_E\psi_0\}$ is u -homotopic to v . Then $\psi = \hat{\mu}\{\omega, \psi_0\}$ has the desired property.

(ii) This is a simple application of homotopy lifting property.

Theorem *A* in the introduction follows immediately from 3.2 and the following theorem which states a main property of $\Phi_\theta(\rho, v)$.

Theorem 4.2. *If $w: X \rightarrow T$ is a lifting of v , then the image of $\Phi_\theta(\rho, v)$ coincides with the factor group $I(w)/I_u(w)$.*

Proof. Let $\gamma \in [X, \Omega C]$ lie in the coset $\Phi_\theta(\rho, v)\alpha$. Since $\rho_*v = 0$, we have that

$$(\Omega^*\rho)_*\psi = (i_C)_*\gamma \quad \text{and} \quad (r_E)_*\psi = v,$$

which shows that γ is ρ -correlated to v . Hence it follows from Theorem 3.3 of [5] that γ lies in $I(w)$.

Conversely, suppose $\gamma \in I(w)$. Then, by Theorem 3.2 of [5], there exists $\psi' \in [X, \Omega^*E]$ such that $(\Omega^*\rho)_*\psi' = (i_C)_*\gamma$ and $(r_E)_*\psi' = v$ in $[X, E]$. By the homotopy lifting property of r_E we see that $\psi \simeq \psi'$ for a map $\psi: X \rightarrow \Omega^*E$ with $r_E\psi = v$. Then there is an $\alpha \in [X, \Omega A]$ with $h^{-1}(\Omega^*p)_*\psi = m'_*\{\alpha, u\}$ and, moreover, $(\Omega^*\theta)_*m'_*\{\alpha, u\} = 0$. This means that γ lies in the coset $\Phi_\theta(\rho, v)\alpha$.

As a special case of Theorem *A* we obtain

Corollary 4.3. *If $[X, \Omega A] = 0$, then $I(w) = I_u(w)$ for any lifting w of v .*

REMARK. The conclusion of 4.3 remains valid without assuming that B is an H -space, as shown in what follows. Since $I_u(w) \subset I(w)$, it suffices to prove $I(w) \subset I_u(w)$. Let $\tau \in I(w)$, then, by Theorem 3.2 of [5], there is a $\psi \in [X, \Omega^*E]$ such that $(\Omega^*\rho)_*\psi = (i_C)_*\tau$ and $(r_E)_*\psi = v$. As above we may represent ψ by a v -map $\tilde{\psi}: X \rightarrow \Omega^*E$. The assumption implies that $(r_A)_*: [X, \Omega^*A] \rightarrow [X, A]$ is a bijection with inverse $(s_A)_*$ and hence there is a homotopy deforming $(\Omega^*p)\tilde{\psi}$ into a map $X \rightarrow s_A(A)$. Since Ω^*p is a fibration, it follows that there is a map $\psi_0: X \rightarrow \Omega_p^*E$ homotopic to $\tilde{\psi}$ in Ω^*E . Now the composite $\Omega_p^*E \xrightarrow{r} E \xrightarrow{p} A$ is a fibration, so that we can find a map $\psi_1: X \rightarrow \Omega_p^*E$ such that $pr\psi_1 = u$ and $\psi_0 \simeq \psi_1$ in Ω_p^*E . Consequently, if we can show that $r\psi_1: X \rightarrow E$ is u -homotopic to v , then we infer from 2.4 that $\tau \in I_u(w)$. Now, since $pr\psi_1 = p v$, there is an $\omega: X \rightarrow F$ such that $\omega \cdot v$ and $r\psi_1$ are u -homotopic, whence $r\psi_1 \simeq r_E\tilde{\psi} = v$ implies $\omega \in I(v)$. Thus, by 2.6, $\omega = 0$, which shows that $r\psi_1$ is u -homotopic to v .

Now let

$$p_*: [X, E] \rightarrow [X, A], \quad q_*: [X, T] \rightarrow [X, E]$$

be the induced functions and let $v: X \rightarrow E$ be liftable to T . We set $u = p \circ v$. We see that $(p \circ q)_*^{-1}(u)$ coincides with the union

$$\cup q_*^{-1}(\omega \cdot v),$$

where ω runs over the cosets in $[X, F]/\Delta(\theta, u)[X, \Omega A]$ such that $\nu(\rho)_*\{\omega, v\} + (\rho j)_*\omega = 0$. We conclude from Theorem A that

A Classification Theorem 4.4. *Let $\pi = p \circ q$ in (3.1) be a stable decomposition with $c: F \times A \rightarrow C$ such that $\Delta(\theta, u)$ is injective. Then $(p \circ q)_*^{-1}(u)$ is equivalent to the product*

$$\begin{aligned} & \{\omega \in [X, F]; c_*\{\omega, u\} = 0\} / \Delta(\theta, u)[X, \Omega A] \\ & \times [X, \Omega C] / \{c'_*\{\beta, u\}; \beta \in [X, \Omega F]\}. \end{aligned}$$

Finally we list some properties of $\Phi_\theta(\rho, v)$.

Triviality Theorem 4.5. $\Phi_\theta(\rho, v)$ is the usual (stable) secondary operation Φ determined by $\Omega\rho: \Omega E \rightarrow \Omega C$ and the image of $\Phi_\theta(\rho, v)$ is $(\Omega\rho)_*[X, \Omega E] / \Delta_\theta(\rho, v)[X, \Omega F]$ in each of the following cases:

- (i) v is the constant map;
- (ii) X is an H cogroup;
- (iii) $p_*v = 0$, ρ is primitive with respect to μ and C is homotopy abelian;
- (iv) θ is primitive and ρ is primitive with respect to H structure of E .

Proof. (i) follows from the fact that $[X, \Omega^*A; *]$ can be identified with $[X, \Omega A]$.

In order to prove (ii), note that the following diagram is homotopy-commutative:

$$\begin{array}{ccccccc} X & \xrightarrow{d} & X \times X & \xrightarrow{(\Omega\rho)\omega \times \rho v} & \Omega C \times C & \xrightarrow{n'} & \Omega^*C \\ & \searrow g & \uparrow & & \uparrow & & \uparrow d' \\ & & X \vee X & \xrightarrow{(\Omega\rho)\omega \vee \rho v} & \Omega C \vee C & \xrightarrow{i_C \vee s_C} & \Omega^*C \vee \Omega^*C \end{array}$$

where g is H' structure map, d the diagonal map, d' the folding map and $\omega: X \rightarrow \Omega E$. Let ψ be an element of $[X, \Omega^*E; u]$ which corresponds to $(i_E)_*\omega + (s_E)_*v \in [X, \Omega^*E; *+u]$ under the canonical isomorphism. Then,

$$(\Omega^*\rho)_*\psi = (i_C)_*(\Omega\rho)_*\omega + (s_C)_*\rho_*v = n'_*\{(\Omega\rho)_*\omega, \rho_*v\}$$

and we see that $(\Omega^*p)_*\psi$ corresponds to $(i_A)_*(\Omega p)_*\omega + (s_A)_*u$, i.e., $(\Omega^*p)_*\psi = hm'_*\{\alpha, u\}$ with $\alpha = (\Omega p)_*\omega$. These show that $(\Omega\rho)_*\omega$ represents $\Phi_\theta(\rho, v)\alpha$.

We prove (iii). Given $\alpha \in [X, \Omega A]$ with $(\Omega\theta)_*\alpha = 0$ (which is equivalent

to $(\Omega^*\theta)_*m'_*\{\alpha, *\}=0$), take $\gamma \in [X, \Omega E]$ such that $(\Omega p)_*\gamma = \alpha$. Since $p_*v=0$, there is a $\tau \in [X, F]$ with $j_*\tau=v$. Using the map $\hat{\mu}: F \times \Omega^*E \rightarrow \Omega^*E$ in the proof of Sublemma, we set $\psi = \hat{\mu}_*\{\tau, (i_E)_*\gamma\} \in [X, \Omega^*E; *]$; then, $(r_E)_*\psi = v$. Thus,

$$\begin{aligned} (\Omega^*\rho)_*\psi &= (\Omega^*\rho)_*\hat{\mu}_*\{\tau, (i_E)_*\gamma\} = \bar{n}_*\{\rho_*j_*\tau, (\Omega\rho)_*\gamma\} \\ &= n'_*\{(\Omega\rho)_*\gamma, \rho_*v\}, \end{aligned}$$

where $\bar{n}: C \times \Omega C \rightarrow \Omega^*C$ is the "left translation". This proves (iii). The proof of (iv) is left to the reader.

Composition Theorem 4.6. *Let $\sigma: C \rightarrow D$ be a map of C to an H space D . Let*

$$\hat{\Delta}(\sigma, \rho v): \text{coker } \Delta_p(\rho, v) \rightarrow \text{coker } \Delta_p(\sigma\rho, v)$$

denote the homomorphism induced by $\Delta(\sigma, \rho v)$. Then

$$\Phi_\theta(\sigma\rho, v) = \hat{\Delta}(\sigma, \rho v)\Phi_\theta(\rho, v).$$

Cartesian Product Theorem 4.7. *Suppose ρ_1 and ρ_2 are as in 3.7. Then*

$$\Phi_\theta(\{\rho_1, \rho_2\}, v)\alpha = (\Phi_\theta(\rho_1, v)\alpha, \Phi_\theta(\rho_2, v)\alpha).$$

Cup Product Theorem 4.8. *Let ρ, ρ' and $\rho \cup \rho'$ be as in 3.9. Then*

$$\Phi_\theta(\rho \cup \rho', v)\alpha = \Phi_\theta(\rho, v)\alpha \cup \rho'v + (-1)^a \rho v \cup \Phi_\theta(\rho', v)\alpha.$$

Naturality Theorem 4.9. *Let α lie in $\ker \Delta(\theta, u)$ and let $f: Y \rightarrow X$ be a map. Then*

$$\Phi_\theta(\rho, f^*v)(f^*\alpha) = f^*\Phi_\theta(\rho, v)\alpha \bmod \Delta_p(\rho, f^*v)[Y, \Omega F].$$

5. Proof of Theorem B

Consider a Postnikov tower for $P_n(R)$, n odd > 1 . In (3.1) we take

$$A = K(Z_2, 1), \quad B = K(Z, n+1), \quad C = K(Z_2, n+2), \quad \theta = (\delta^*\iota_1)^{(n+1)/2},$$

where $\iota_1 \in H^1(Z_2, 1; Z_2)$ is the fundamental class; then we have the first two stages and $H^{n+2}(E; Z_2) = Z_2$ whose generator is the second invariant ρ with $j^*\rho = Sq^2\iota_n$, ι_n being the non-zero element of $H^n(Z, n; Z_2)$. We claim that

$$\mu^*(\rho) = \begin{cases} Sq^2\iota_n \times 1 + 1 \times \rho + \iota_n \times p^*\iota_1^2 & \text{for } n \equiv 1 \pmod{4} \\ Sq^2\iota_n \times 1 + 1 \times \rho & \text{for } n \equiv 3 \pmod{4} \end{cases}$$

and hence (3.1) in this case is a stable decomposition with $c = Sq^2\iota_n \times 1 + \iota_n \times \iota_1^2$ or $Sq^2\iota_n \times 1$.

Now it follows from Cartan's formula that

$$Sq^2(\delta^*\iota_1)^{\langle n+1 \rangle/2} = Sq^2(\iota_1^{n+1}) = \frac{(n+1)n}{2} \iota_1^{n+3} = \begin{cases} \iota_1^{n+3} & \text{if } n \equiv 1 \pmod{4} \\ 0 & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

We shall use an exact sequence due to E. Thomas [16, p. 187]. We see from the above equality that, for the morphism $\tau: H^{n+2}(F \times E, E) \rightarrow H^{n+3}(A)$,

$$\tau(Sq^2\iota_n \times 1 + \iota_n \times p^*\iota_1^2) = 0 \quad \text{or} \quad \tau(Sq^2\iota_n \times 1) = 0$$

according as $n \equiv 1$ or $3 \pmod{4}$, and hence there is a $\bar{\rho} \in H^{n+2}(E; Z_2)$ such that

$$\bar{\mu}(\bar{\rho}) = Sq^2\iota_n \times 1 + \iota_n \times p^*\iota_1^2 \quad \text{or} \quad Sq^2\iota_n \times 1$$

for the operator $\bar{\mu}: H^{n+2}(E) \rightarrow H^{n+2}(F \times E, E)$ (which is essentially equal to $\mu^* - p_2^*$). Since $\ker p^* = \ker l^*$ in $\dim n+2$ for a map $l: P_n(R) \rightarrow A$ representing ι , we infer that $\rho = \bar{\rho}$, which proves our assertion. Theorem B now follows from 4.4 and the fact $[X, P_n(R)] \approx [X, T]$.

6. Proof of Theorem C

Consider the first two stages of a Moore-Postnikov tower for the inclusion $BO(n) \subset BO$ between the classifying spaces for n -plane bundles and stable ones:

$$\begin{array}{ccccc} & & T & & \\ & & \downarrow q & & \\ K(Z_2, n) & \xrightarrow{j} & E & \longrightarrow & K(Z_2, m) \\ & & \downarrow p & & \\ & & BO & \xrightarrow{w_{n+1}} & K(Z_2, n+1), \end{array}$$

where $m = n+2$ or $n+3$ according as $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$, $n > 2$, and w_i denotes the universal Stiefel-Whitney class of dimension i . As shown in [7, p. 110], $p \circ q$ forms a stable decomposition with

$$c = \begin{cases} Sq^2\iota_n \times 1 + \iota_n \times w_2 & \text{if } n \equiv 1 \pmod{4} \\ Sq^2Sq^1\iota_n \times 1 + Sq^1\iota_n \times (w_1^2 + w_2) & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

where $\iota_n \in H^n(Z_2, n; Z_2)$ is the fundamental class (This can also be shown using Thomas' exact sequence). Thus, we conclude from 4.4 that, for a stable bundle $\xi \in \widetilde{KO}(X)$ with $w_{n+1}(\xi) = 0$ such that $\Delta(w_{n+1}, \xi): \widetilde{KO}^{-1}(X) \rightarrow H^n(X; Z_2)$ is injective,

(a) if $n \equiv 1 \pmod{4}$ and $\dim X \leq n+1$, then $N_n(\xi; X)$ is equal to the cardinal of the direct product:

$$\text{Coker } \Delta(w_{n+1}, \xi) \times H^{n+1}(X; Z_2) / (Sq^2 + w_2(\xi) \cup) H^{n-1}(X; Z_2).$$

(b) if $n \equiv 3 \pmod{4}$ and $\dim X \leq n+2$, then $N_n(\xi; X)$ is equal to the cardinal

of the direct product:

$$\text{Coker } \Delta(w_{n+1}, \xi) \times H^{n+2}(X; Z_2) / (Sq^2 + (w_1(\xi)^2 + w_2(\xi))) \cup Sq^1 H^{n-1}(X; Z_2).$$

Now take $X = P_{n+1}(R)$ or $P_{n+2}(R)$ according as $n \equiv 1$ or $3 \pmod{4}$. Then $\widetilde{KO}^{-1}(X) = Z_2$ by [3] and it follows from a formula of [5, p. 489] that $\Delta(w_{n+1}, k\eta)$ is injective for $\binom{k-1}{n-1}$ odd and $w_{n+1}(k\eta) = 0$ for $\binom{k}{n+1}$ even. Theorem C will be obtained by observing that

$$w_2(k\eta) = \binom{k}{2} x^2, \quad Sq^2 x^{n-1} = 0 \text{ for } n \equiv 1 \pmod{4}, \quad Sq^1 x^{n-1} = 0 \text{ for odd } n,$$

where x denotes the non-zero element of $H^1(X; Z_2)$.

Note that, for $n \equiv 1 \pmod{4}$ and $\binom{k}{2}$ odd, $\Delta_p(\rho, v): H^{n-1}(P_{n+1}(R); Z_2) \rightarrow H^{n+1}(P_{n+1}(R); Z_2)$ is surjective for any lift v of $k\eta$ and hence $q_*: [P_{n+1}(R), T] \rightarrow [P_{n+1}(R), E]$ is bijective.

As another illustration, let $n \equiv 1 \pmod{4}$ and let X be $P_{(n+1)/2}(C)$, the complex projective space of complex dimension $\frac{n+1}{2}$. Then $\widetilde{KO}^{-1}(X) = 0$ by [3]. Since $H^n(X; Z_2) = 0$ and $Sq^2(y^{(n-1)/2}) = 0$ for the non-zero element y of $H^2(X; Z_2)$, we see that $N_n(\xi; X) = 1$ or 2 according as $w_2(\xi) \neq 0$ or $w_2(\xi) = 0$.

7. Further examples

7.1. Suppose that, in (3.1), A is an $(n-1)$ -connected space such that $\pi_k(A) = 0$ for $k \geq n+n'-2$ ($n' > n \geq 2$), $B = K(\pi', n'+1)$ and $C = K(G, n+n')$. Assume $\rho \in H^{n+n'}(E; G)$ represents $(\bar{p} + \psi \cup)_p(\theta)$ for $\bar{p} \in H^{n+n'+1}(\pi', n'+1; G)$, $\psi \in H^n(A; \pi)$, where the cup product is taken with respect to the Whitehead product pairing $\pi \otimes \pi' \rightarrow G$ in T , $\pi = \pi_n(T)$. Then it is proven by F. P. Peterson [11] that

$$\mu^*(\rho) = j^* \rho \times 1 + 1 \times \rho + x' \times p^*(\psi), \quad x' \in H^{n'}(\pi', n; \pi'),$$

where $j^* \rho$ is the suspension of $\bar{p}$ and $j^* \rho(x') = j^* \rho(\iota')$, ι' being the fundamental class of $H^{n'}(\pi', n'; \pi')$. Thus, the tower $p \circ q$ in this case is a stable decomposition with $c = j^* \rho \times 1 + x' \times \psi$ (cf. Theorem 3.1 of [4]). Hence, $\Delta_p(\rho, v)\beta = \Omega(\rho j)_* \beta + (\Omega x') \beta \cup u^* \psi$, $\beta \in H^{n'-1}(X; \pi')$.

In case $A = K(\pi, n)$, we can take the basic classes ι' and $\iota \in H^n(\pi, n; \pi)$ for x' and ψ respectively.

7.2. Consider a Postnikov tower for the usual lens space $L = S^{2n+1}/Z_p$, where p is an odd prime. We take

$$A = K(Z_p, 1), \quad B = K(Z, 2n+2), \quad C = K(Z_2, 2n+3),$$

and let $\theta = (\delta^* \iota)^{n+1}$ where δ^* is the Bockstein associated with $0 \rightarrow Z \rightarrow Z \rightarrow Z_p \rightarrow 0$ and ι denotes the basic class of $H^1(Z_p, 1; Z_p)$, and let ρ denote the generator of $H^{2n+3}(E; Z_2) = Z_2$.

Given a *path-connected* $(2n+2)$ -dimensional complex X , we have $\Delta(\theta, u) = 0$ by virtue of $H^1(SX; Z_p) = 0$ and $j^* \rho = Sq^2 \bar{\iota}$, $\bar{\iota}$ being the basic class of $H^{2n+1}(Z, 2n+1; Z)$. Since $H^{2n+3}(F \times E, F \vee E; Z_2) = 0$, ρ is primitive with respect to the action μ , so that $\Delta_p(\rho, v) = Sq^2: H^{2n}(X; Z) \rightarrow H^{2n+2}(X; Z_2)$. Thus, we see from 4.4 that $[X, L]$ is equivalent to the product

$$\{u \in H^1(X; Z_p); (\delta^* u)^{n+1} = 0\} \times H^{2n+1}(X; Z) \times H^{2n+2}(X; Z_2) / Sq^2 H^{2n}(X; Z).$$

This extends a result of P. Olum [10].

7.3. Consider a Postnikov tower for the n -sphere S^n , $n \geq 4$. We take

$$A = K(Z, n), \quad B = K(Z_2, n+2), \quad C = K(Z_2, n+3)$$

and $Sq^2 \iota$ and the unique non-zero element of $H^{n+3}(E; Z_2) = Z_2$ for θ and ρ , where ι is the basic class of $H^n(Z, n; Z)$. Then,

$$\begin{aligned} \Delta(\theta, u) &= Sq^2: H^{n-1}(X; Z) \rightarrow H^{n+1}(X; Z_2) \\ \Delta_p(\rho, v) &= Sq^2: H^n(X; Z_2) \rightarrow H^{n+2}(X; Z_2). \end{aligned}$$

Let X be a complex with $\dim X \leq n+2$; then $v: X \rightarrow E$ is always liftable to T . We conclude

(1) (Nakaoka [8, p. 94, Theorem 4]) Assume $Sq^2: H^n(X; Z_2) \rightarrow H^{n+2}(X; Z_2)$ is surjective; then $I(w) = I_u(w) = H^{n+2}(X; Z_2)$ and hence it follows that $[X, S^n]$ is equivalent to

$$\{u \in H^n(X; Z); Sq^2 u = 0\} \times H^{n+1}(X; Z_2) / Sq^2 H^{n-1}(X; Z).$$

(2) Assume $Sq^2: H^{n-1}(X; Z) \rightarrow H^{n+1}(X; Z_2)$ is injective; then it follows from 4.4 that $[X, S^n]$ is equivalent to the product

$$\begin{aligned} \{u \in H^n(X; Z); Sq^2 u = 0\} \times H^{n+1}(X; Z_2) / Sq^2 H^{n-1}(X; Z) \\ \times H^{n+2}(X; Z_2) / Sq^2 H^n(X; Z_2). \end{aligned}$$

7.4. Consider a Postnikov tower for the complex projective space $P_m(C)$. Let

$$A = K(Z, 2), \quad B = K(Z, 2m+2), \quad C = K(Z_2, 2m+3),$$

and let $\theta = \iota^{m+1}$, where $\iota \in H^2(Z, 2; Z)$ is the basic class, and ρ be the unique non-zero element $\psi_m \in H^{2m+3}(E; Z_2)$ (cf. [12]). Then $j^* \rho = Sq^2 \iota_{2m+1}$, where ι_{2m+1} is the generator of $H^{2m+1}(Z, 2m+1; Z_2)$. The dual Hopf invariant $\nu(\theta)$ with respect to H -structure of A is $(\iota \times 1 + 1 \times \iota)^{m+1}$ and hence

$$\Delta(\theta, u)\alpha = (m+1)\alpha \cup u^m, \quad \alpha \in H^1(X; Z) \quad (\text{cf. [15, p. 452]})$$

We see from 7.1 and [11] that

$$\mu^*(\rho) = \begin{cases} \rho j \times 1 + 1 \times \rho & \text{if } m \text{ is odd} \\ \rho j \times 1 + 1 \times \rho + \bar{\iota}_{2m+1} \times p^* \iota & \text{if } m \text{ is even,} \end{cases}$$

where the cross product is taken with respect to the nontrivial pairing $Z \otimes Z \rightarrow Z_2$ and $\bar{\iota}_{2m+1}$ denotes the basic class of $H^{2m+1}(Z, 2m+1; Z)$.

Given a $(2m+2)$ -dimensional complex X , we assume that $(m+1)\alpha \cup u^m = 0$ implies $\alpha = 0$ for $\alpha \in H^1(X; Z)$ and a fixed $u \in H^2(X; Z)$. Then it follows from 4.4 that, for the function $\varphi: [X, P_m(C)] \rightarrow H^2(X; Z)$ assigning f^*z to $f: X \rightarrow P_m(C)$, z being a generator of $H^2(P_m(C); Z)$, $\varphi^{-1}(u)$ is equivalent to

$$\begin{aligned} & H^{2m+1}(X; Z)/(m+1)u^m \cup H^1(X; Z) \times H^{2m+2}(X; Z_2)/Sq^2 H^{2m}(X; Z) \quad \text{for } m \text{ odd} \\ & H^{2m+1}(X; Z)/(m+1)u^m \cup H^1(X; Z) \times H^{2m+2}(X; Z_2)/(Sq^2 + u \cup) H^{2m}(X; Z) \\ & \hspace{15em} \text{for } m \text{ even.} \end{aligned}$$

It seems likely that, for $m=1$ and $\dim X=4$, our $\Phi_\theta(\rho, v)$ coincides with Φ_{C_2} introduced by N. Shimada [14, p. 141].

7.5. Let n be an even integer and let

$$\begin{array}{ccccc} K(Z_2, 2n+2) & \longrightarrow & T & & \\ & & \downarrow q & & \\ K(Z, 2n+1) & \xrightarrow{j} & E & \xrightarrow{\rho} & K(Z_2, 2n+3) \\ & & \downarrow p & & \\ & & BU & \xrightarrow{c_{n+1}} & K(Z, 2n+2) \end{array}$$

be part of a Moore-Postnikov tower for $BU(n) \subset BU$ between the classifying spaces for the unitary groups $U(n)$ and U , where c_{n+1} denotes the universal $(n+1)$ th Chern class. It is readily shown that $H^{2n+3}(E; Z_2) = Z_2$ is generated by ρ with $j^*\rho = Sq^2 \iota_{2n+1}$, where ι_{2n+1} is the generator of $H^{2n+1}(Z, 2n+1; Z_2)$.

Since, for the realification $\hat{\gamma}$ of the canonical bundle γ over BU ,

$$Sq^2 c_{n+1} = Sq^2 w_{2n+2}(\hat{\gamma}) = w_2(\hat{\gamma}) \cup c_{n+1}$$

where the cup product is with respect to the non-trivial pairing $Z_2 \otimes Z \rightarrow Z_2$, Thomas' exact sequence reveals that

$$\mu^* \rho = Sq^2 \iota_{2n+1} \times 1 + 1 \times \rho + \iota_{2n+1} \times p^* w_2(\hat{\gamma}).$$

Hence it follows from 4.4 that, for a complex X such that $\dim X \leq 2n+2$ and $\Delta(c_{n+1}, u): \tilde{K}^{-1}(X) \rightarrow H^{2n+1}(X; Z)$ is injective for $u \in \tilde{K}(X)$ with $c_{n+1}(u) = 0$, the number of n -dimensional complex vector bundles over X which are stably

equivalent to u , is equal to the cardinal of the direct product

$$\text{coker } \Delta(c_{n+1}, u) \times H^{2n+2}(X; Z_2) / (Sq^2 + w_2(u) \cup) H^{2n}(X; Z),$$

where $\hat{u}$ is the realification of u .

For example, let $X = P_{2n+2}(R)$. Since $\tilde{K}^{-1}(X) = 0$ by Theorem 3.3 of [2] and since $K(X)$ consists of elements $k\nu$ ($k=0, 1, \dots, 2^{n-1}-1$), ν denoting the complexification of the canonical line bundle λ (see [1]), the number of classes of n -dimensional complex plane bundles which are stably equivalent to $k\nu$, is equal to 2 or 4 according as k is odd or even. This follows by observing that $Sq^2 x^{2n} = 0$ for the generator $x \in H^1(X; Z_2)$ and $w_2(k\hat{\nu}) = kw_2(2\lambda) = kx^2$.

8. Appendix: the group of fibre homotopy equivalences

Given a fibration $f: Y \rightarrow Z$, we denote by $\mathcal{E}(Y; f)$ the group of fibre homotopy classes of fibre homotopy equivalences of Y .

In the situation (3.1) we shall assume that $\pi_k(A) \neq 0$ only for $n \leq k \leq n'-1$, $\pi_k(F) \neq 0$ only for $n \leq k \leq n'-1$ and $\pi_r(\Omega C) \neq 0$ only for $n' \leq r \leq n'+n-1$ ($n > 1$). It is easily shown that there is an exact sequence

$$1 \rightarrow q^*[E, \Omega C] \rightarrow \mathcal{E}(T; q) \rightarrow \mathcal{E}(\Omega C).$$

We shall study $\mathcal{E}(T; p \circ q)$. First we need

Lemma 8.1. *The functions*

$$q^*: [E, E; p] \rightarrow [T, E; p \circ q], \quad i_*: [\Omega C, \Omega C] \rightarrow [\Omega C, T]$$

are bijective.

Proof. Introduce the commutative diagram

$$\begin{array}{ccc} [E, F] & \xrightarrow{q^*} & [T, F] \\ T_1 \downarrow & & \downarrow T_2 \\ [E, E; p] & \xrightarrow{q^*} & [T, E; p \circ q], \end{array}$$

where the vertical bijections T_1 and T_2 are given by

$$T_1(\tau) = \mu_*\{\tau, 1_E\}, \quad T_2(\omega) = \mu_*\{\omega, q\},$$

1_X being the identity map of X . Since the upper q^* is bijective, so is the bottom q^* . The second assertion can be proved by a classical obstruction argument or by using a Moore-Postnikov tower for i .

In the light of Lemma 8.1 we can now define homomorphisms

$$J: \mathcal{E}(T; p \circ q) \rightarrow \mathcal{E}(E; p), \quad J_0: \mathcal{E}(T; p \circ q) \rightarrow \mathcal{E}(\Omega C)$$

by requiring, for $g \in \mathcal{E}(T; p \circ q)$,

$$q_*g = q^*J(g) \quad \text{in} \quad [T, E; p \circ q], \quad i_*J_0(g) = i^*g.$$

Let

$$\Delta: \ker i^* \rightarrow \mathcal{E}(T; p \circ q)$$

denote the homomorphism defined by $\Delta(\tau) = \tau \cdot 1_T$, where i^* is the homomorphism in the exact sequence

$$[\Omega C, \Omega C] \xleftarrow{i^*} [T, \Omega C] \xleftarrow{q^*} [E, \Omega C].$$

Theorem 8.2. *The following sequence of groups and homomorphisms is exact:*

$$[T, \Omega F] \xrightarrow{\Delta_p(\rho, q)} q^*[E, \Omega C] \xrightarrow{\Delta} \mathcal{E}(T; p \circ q) \xrightarrow{\{J, J_0\}} \mathcal{E}(E; p) \times \mathcal{E}(\Omega C),$$

in which the image of $\{J, J_0\}$ consists of $(g, \Omega h) \in \mathcal{E}(E; p) \times \mathcal{E}(\Omega C)$ such that $\rho g \simeq h\rho$.

Proof. The exactness at the second term follows from the fact that the image of $\Delta_p(\rho, q)$ coincides with $I_{p \circ q}(1_T)$ by 3.2. We shall prove the exactness at the third term.

Let $g: T \rightarrow T$ be a homotopy equivalence such that $gi \simeq i$, $pqq = pq$ and $qg \simeq q$ by a pq -homotopy $H_t: T \rightarrow E$, $0 \leq t \leq 1$, with $H_0 = qg$, $H_1 = q$. By the homotopy lifting property there exists a homotopy $\tilde{H}_t: T \rightarrow T$ with $\tilde{H}_0 = g$, $q\tilde{H}_t = H_t$. Since $pq\tilde{H}_t = pH_t = pq$, $\tilde{H}_t$ is a pq -homotopy. Put $g' = \tilde{H}_1$, then $qg' = q$ and so g' is q -homotopic to $\tau \cdot 1_T$ for some $\tau: T \rightarrow \Omega C$. Since

$$(\tau i) \cdot i \simeq g' i \simeq gi \simeq i$$

and $[\Omega C, \Omega E] = 0$, it follows from 2.3 that $I(i) = 0$ and hence $i^*\tau = 0$.

The assertion about the image of $\{J, J_0\}$ can be proved by an argument similar to Theorem 2.9 of [9], noting that, if $qg \simeq gq$ by a pq -homotopy, we can replace g by $\hat{g}$ which is pq -homotopic to g and which is such that $q\hat{g} = gq$.

Consider the situation in which $A = K(\pi, n)$, $B = K(\pi', n' + 1)$ and $C = K(G, n + n')$, $1 < n < n'$ in (3.1). Theorem 8.2, together with 7.1, gives rise to an exact sequence

$$1 \rightarrow R \rightarrow \mathcal{E}(T; p \circ q) \rightarrow \mathcal{E}(E; p) \times \text{Aut } G,$$

where R denotes the factor group

$$H^{n+n'-1}(E; G)/p^*(\Omega(\rho j) + \iota_n \cup) H^{n'-1}(\pi, n; \pi'),$$

ι_n being the basic class in $H^n(\pi, n; \pi)$ and the cup product being taken with respect to the Whitehead product pairing of T .

OSAKA UNIVERSITY

References

- [1] J.F. Adams: *Vector fields on spheres*, Ann. of Math. **75** (1962), 603–632.
- [2] M. Fujii: *Ku-groups of Dold manifolds*, Osaka J. Math. **3** (1966), 49–64.
- [3] M. Fujii: *K_0 -groups of projective spaces*, Osaka J. Math. **4** (1967), 141–149.
- [4] I.M. James: *A relation between Postnikov classes*, Quart. J. Math. Oxford (2), **17** (1966), 269–280.
- [5] I.M. James and E. Thomas: *An approach to the enumeration problem for non-stable vector bundles*, J. Math. Mech. **14** (1965), 485–506.
- [6] I.M. James and E. Thomas: *Note on the classification of cross-sections*, Topology **4** (1966), 351–359.
- [7] I.M. James and E. Thomas: *On the enumeration of cross-sections*, Topology **5** (1966), 95–114.
- [8] M. Nakaoka: *Exact sequences $\sum_p(K, L)$ and their application* J. Inst. Polytech. Osaka City Univ. **3** (1952), 83–100.
- [9] Y. Nomura: *Homotopy equivalences in a principal fibre space*, Math. Z. **92** (1966), 380–388.
- [10] P. Olum: *Cocycle formulas for homotopy classification; maps into projective and lens spaces*, Trans. Amer. Math. Soc. **103** (1962), 30–44.
- [11] F.P. Peterson: *Whitehead products and the cohomology structure of principal fibre spaces*, Amer. J. Math. **82** (1960), 649–652.
- [12] F.P. Peterson and N. Stein: *Secondary cohomology operations; two formulas*, Amer. J. Math. **81** (1959), 281–305.
- [13] J.W. Rutter: *A homotopy classification of maps into an induced fibre space*, Topology **6** (1967), 379–403.
- [14] N. Shimada: *Homotopy classification of mappings of a 4-dimensional complex into a 2-dimensional sphere*, Nagoya Math. J. **5** (1953), 127–144.
- [15] E.H. Spanier: *Algebraic Topology*, McGraw-Hill, New York, 1966.
- [16] E. Thomas: *Postnikov invariants and higher order cohomology operations*, Ann. of Math. **85** (1967), 184–217.