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A NON-STABLE SECONDARY OPERATION AND HOMOTOPY CLASSIFICATION OF MAPS

Dedicated to Professor A. Komatu on his 60th birthday

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(Received October 26, 1968)

1. Introduction

Let $p: E \to A$ be the principal fibration with classifying map $\theta: A \to B$ and let $q: T \to E$ be the principal fibration induced by a map $\rho: E \to C$. We assume that A, B and C are H-spaces. Given a map $v: X \to E$, I. M. James and E. Thomas [5, 7] have defined the homomorphisms

$$\Delta(\theta, u) \colon [X, \Omega A] \to [X, \Omega B]$$
$$\Delta_{b}(\rho, v) \colon [X, \Omega^{2}B] \to [X, \Omega C],$$

where u is the composite $p \circ v$, Ω is the loop functor and [Y, Z] denotes the set of based homotopy classes of based maps $Y \rightarrow Z$.

The action $\Omega C \times T \rightarrow T$ of the principal fibration q induces the function

$$[X, \Omega C] \times [X, T] \rightarrow [X, T],$$

the image of $(\tau, w) \in [X, \Omega C] \times [X, T]$ under which is denoted by $\tau \cdot w$. The subgroup

$$\mathbf{I}(w) = \{ \tau \in [X, \Omega C]; \tau \cdot w = w \}$$

of $[X, \Omega C]$ is called the *isotropy group* of w under the action of $[X, \Omega C]$ on [X, T]. Our first main result is the following:

Theorem A. Suppose $w: X \to T$ is a lifting of v. If $\Delta(\theta, u)$ is injective, then I(w) coincides with the image of $\Delta_{t}(\rho, v)$.

This is obtained as a direct consequence of a property (Theorem 4.2) of a non-stable secondary operation $\Phi_{\theta}(\rho, v)$ which is inspired by an operation due to N. Shimada [14, p. 141].

The prime concern in this paper is to examine a few situations to which Theorem A is applicable.

Consider first the real projective space $P_n(R)$, where the dimension *n* is odd>1. Let X be a *path-connected* (n+1)-dimensional complex and let δ^*

denote the Bockstein homomorphism associated with $0 \rightarrow Z \rightarrow Z \rightarrow Z_2 \rightarrow 0$. We define

$$\varphi \colon [X, P_n(R)] \to H^1(X; Z_2)$$

by $\varphi(g) = g^*\iota$, where $g: X \to P_n(R)$ and ι denotes the generator of $H^1(P_n(R); Z_2)$. The following extends a result due to P. Olum [10].

Theorem B. Let u be an element of $H^1(X; Z_2)$ such that $(\delta^* u)^{(n+1)/2} = 0$. Then $\varphi^{-1}(u)$ is equivalent to:

$$H^{n}(X; Z) \times H^{n+1}(X; Z_{2})/(Sq^{2}+u^{2} \cup)H^{n-1}(X; Z) \quad for \quad n \equiv 1 \pmod{4}$$

$$H^{n}(X; Z) \times H^{n+1}(X; Z_{2})/Sq^{2}H^{n-1}(X; Z) \quad for \quad n \equiv 3 \pmod{4}$$

Next, let η be the reduced stable class of the canonical line bundle over the real projective space and let $N_n(\xi; X)$ denote the number of classes of *n*-plane bundles over X which are stably equivalent to a stable reduced bundle ξ over X. The following is a partial extension of a result due to I. M. James and E. Thomas [5].

Theorem C. Let k be an integer and let
$$\binom{k}{n+1}$$
 be even. Then
(1) for $n \equiv 1 \pmod{4}$ and $\binom{k}{2}$ odd, $N_n(k\eta; P_{n+1}(R)) = 1$ or 2 according as
 $\binom{k-1}{n-1}$ is odd or even;
(2) $N_n(k\eta; P_{n+1}(R)) = 2$ if $n \equiv 1 \pmod{4}$, $\binom{k}{2}$ even and $\binom{k-1}{n-1}$ odd;
(3) $N_n(k\eta; P_{n+2}(R)) = 2$ if $n \equiv 3 \pmod{4}$ and $\binom{k-1}{n-1}$ odd.

Finally, let $\mathscr{E}(X)$ denote the group of homotopy classes of homotopy equivalences of X whose group structure is induced by map-composition. Consider the tower

$$\Omega C \longrightarrow T$$

$$\downarrow q$$

$$\Omega B \longrightarrow E \xrightarrow{\rho} C$$

$$\downarrow p$$

$$K(\pi, n) \xrightarrow{\theta} B$$

where B and C are H-spaces such that $\pi_r(B) \neq 0$ only for $n+2 \leq r \leq m$ (n>1)and $\pi_s(C) \neq 0$ only for $m+1 \leq s \leq m+n-1$. The following theorem can be obtained by applying the results of [9] to the fibration q and observing the fact $H^{n-1}(T; \pi) = 0$.

Theorem D. The following sequence of groups and homomorphisms is exact:

$$[T, \Omega^{2}B] \xrightarrow{\Delta_{\mathfrak{p}}(\rho, q)} q^{*}[E, \Omega C] \to \mathscr{E}(T) \to \mathscr{E}(E) \times \mathscr{E}(\Omega C)$$

in which the image of the last homomorphism consists of $(\bar{g}, \Omega h) \in \delta(E) \times \delta(\Omega C)$ such that $\rho \bar{g} \simeq h \rho$.

The results stated in sections 2, 3 and 4 can be dualized in a cofibre space and will be considered elsewhere.

2. Preliminaries

We shall here fix the notations and recall some definitions given in [5, 6, 7]. We work in the category of spaces with basepoints (usually denoted by *) and basepoint preserving maps. The spaces considered are assumed to have the homotopy type of a CW complex. We blur the distinction between maps and their homotopy classes. We use the additive notation for path-composition and path-inversion. The suspension functor is denoted by S.

For a space X, let Ω^*X denote the space of free loops in X. One has a fibration

$$\Omega X \xrightarrow{i_X} \Omega^* X \xrightarrow{r_X} X$$

with section $s_X: X \to \Omega^* X$ given by $s_X(x)$ =the constant loop at $x \in X$. A map $f: X \to Y$ induces the map $\Omega^* f: \Omega^* X \to \Omega^* Y$ in an obvious way.

Lemma 2.1. $(i_X)_*\xi = (i_X)_*\xi'$ for $\xi, \xi' \in [V, \Omega X]$ if and only if ξ and ξ' are conjugate to each other.

This can be proved directly or by replacing i_x by a principal fibration.

Corollary 2.2. (Theorem 2.6 of [5]) The group $[V, \Omega X]$ is abelian if and only if $(i_X)_*: [V, \Omega X] \rightarrow [V, \Omega^* X]$ is injective.

Given a fibration $p: E \rightarrow A$ and a map $u: X \rightarrow A$, we denote by [X, E; u] the set of *u*-homotopy classes of *u*-maps $X \rightarrow E$ (see [6])

Let $p: E \to A$ be a fibration with fibre inclusion $j: F \to E$ and let $q: T \to E$ be the principal fibration with classifying map $\rho: E \to C$. Given a map $w: X \to T$, let $v=q \circ w$ and $u=p \circ v$. We define the *u*-isotropy group $I_u(w)$ of w by setting

$$\mathbf{I}_{u}(w) = \{ \tau \in [X, \Omega C]; \quad \tau \cdot w = w \text{ in } [X, T; u] \}.$$

q induces the function

$$q_*: [X, T; u] \rightarrow [X, E; u]$$

and there is a bijection between $q_*^{-1}(v)$ and the totality of left cosets $[X, \Omega C]/I_*(w)$. The following is obvious:

Proposition 2.3. (1) For the trivial map $*: X \rightarrow T$, $I_*(*)$ is the image of

 $(\Omega \rho)_*(\Omega j)_*: [X, \Omega F] \rightarrow [X, \Omega C];$

- (2) $I_u(\tau \cdot w) = \tau + I_u(w) \tau$ for $\tau \in [X, \Omega C];$
- (3) $g^*I_u(w) \subset I_{u \circ g}(g^*w)$ for $g: Y \to X$;

(4) Let i: $\Omega C \rightarrow T$ denote the inclusion. Then $I_*(i)=1+(\Omega \rho)_*(\Omega j)_*[\Omega C, \Omega F]-1$, where 1 is the identity map of ΩC .

Let Ω_p^*E denote the subspace of Ω^*E consisting of free loops λ such that $(\Omega^*p)\lambda \in s_A(A)$. Then one obtains a fibration $r: \Omega_p^*E \to E$ with fibre ΩF .

Theorem 2.4. $\tau \in [X, \Omega C]$ is ρ -correlated (see §3 of [7]) to $v \in [X, E; u]$, if and only if there exists an $\eta \in [X, T; u]$ such that $\tau \in I_u(\eta)$ and $q_*\eta = v$.

Proof. The "if" part is proved in Lemma (3.3) of [7]. We shall prove the "only if" part. Assume that there exists a $\psi \in [X, \Omega_p^* E; u]$ such that $r_*\psi=v$ and $\rho'_*\psi=(i_c)_*\tau$ in the following commutative diagram

where $\rho': \Omega_p^* E \to \Omega^* C$ denotes the restriction of $\Omega^* \rho$ to $\Omega_p^* E$.

Since r is a fibration, we may assume $\psi: X \to \Omega_p^* E$ is a lift of v. Let $F_t: X \to \Omega^* C$ be a homotopy with $F_0 = i_C \tau$, $F_1 = \rho' \psi$. Consider $\gamma: X \to C^T$ defined by $\gamma(x)(t) = r_C \circ F_t(x), x \in X, 0 \le t \le 1$. It is easy to see that $\tau = \gamma + \rho' \psi - \gamma$ in $[X, \Omega C]$. Take $\eta: X \to T$ given by $\eta(x) = (v, \gamma)$; then it follows that $(\gamma + \rho' \psi - \gamma) \cdot \eta = \eta$ in [X, T; u]. This proves the only if part.

Corollary 2.5. Suppose $p: E \rightarrow A$ is a principal fibration with fibre F in the sense of [12]. Then $[X, \Omega F]=0$ implies $I_u(w)=0$.

Proof. Let $\mu: F \times E \to E$ denote the action map and let $\mu': \Omega F \times E \to \Omega_p^* E$ be the induced map defined as in §4 of [7]. Then, by Theorem (4.1) of [7],

 $\mu'_*: [X, \Omega F] \times [X, E; u] \rightarrow [X, \Omega_p^*E; u]$

is bijective. Note that $\mu'\{*, 1_E\}: E \to \Omega_p^* E$ is *p*-homotopic to the canonical section $s: E \to \Omega_p^* E$ of *r*, so that both *s* and *r* induce the bijections between [X, E; u] and $[X, \Omega_p^* E; u]$ because of $[X, \Omega F] = 0$.

Now let $\tau \in [X, \Omega C]$ be ρ -correlated to v, i.e., there is an element $\psi \in [X, \Omega_p^* E; u]$ such that $r_* \psi = v$, $\rho'_* \psi = (i_c)_* \tau$. Then, since $q_* w = v$,

$$(i_c)_* \tau =
ho'_* s_* v = (s_c)_*
ho_* v = 0$$
,

which implies $\tau = 0$ by virtue of ker $(i_c)_* = 0$.

Taking A = * in the above situation, we obtain

Corollary 2.6. $\tau \in [X, \Omega C]$ is ρ -correlated to $v \in [X, E]$ (see §2 of [5]), if and only if there is an element $\eta \in [X, T]$ such that $\tau \in I(\eta)$ and $q_*\eta = v$. If E is an H-space with $[X, \Omega E] = 0$, then $I(\eta) = 0$ for any lifting η of $v: X \rightarrow E$.

3. The homomorphisms $\Delta(\theta, u)$ and $\Delta_{p}(\rho, v)$

Consider the situation

in which p and q are the principal fibrations with classifying maps θ and ρ respectively, and B and C are H-spaces with multiplications $t: B \times B \rightarrow B$ and $n: C \times C \rightarrow C$. Let $\mu: F \times E \rightarrow E$ denote the action of F on E.

In case A is an H-space with multiplication $m: A \times A \rightarrow A$ and there is given a map $v: X \rightarrow E$ with $u = p \circ v$, I. M. James and E. Thomas have defined in [5, 7] the homomorphisms

$$\Delta(\theta, u) \colon [X, \Omega A] \to [X, \Omega B],$$

$$\Delta_{\theta}(\rho, v) \colon [X, \Omega F] \to [X, \Omega C]$$

as follows. Let

$$\mu': (\Omega F \times E, \ \Omega F \times F) \to (\Omega_p^* E, \ \Omega^* F)$$
$$m': \Omega A \times A \to \Omega^* A, \ t': \ \Omega B \times B \to \Omega^* B, \ n': \ \Omega C \times C \to \Omega^* C$$

denote the "right translations" determined by μ , m, t and n; then the equations

$$(\Omega^*\theta)_*m'_*\{\alpha, u\} = t'_*\{\Delta(\theta, u)\alpha, \theta_*u\} \quad \text{for} \quad \alpha \in [X, \Omega A]$$
$$\rho'_*\mu'_*\{\beta, v\} = n'_*\{\Delta_*(\rho, v)\beta, \rho_*v\} \quad \text{for} \quad \beta \in [X, \Omega F]$$

determine $\Delta(\theta, u)\alpha$ and $\Delta_{\rho}(\rho, v)\beta$ uniquely by virtue of Theorem 2.7 of [5]. The following is proven in [7, Theorem (4.2)]:

Theorem 3.2. $\tau \in [X, \Omega C]$ is ρ -correlated to $v \in [X, E; u]$ if and only if, first, $\rho_* v = 0$ and, secondly, τ lies in the image of $\Delta_p(\rho, v)$. Thus, $I_u(w) = \Delta_p(\rho, v)$ $[X, \Omega F]$ for any lift w of v.

The homomorphism $\Delta(\theta, u)$ has also been introduced by J. W. Rutter [13], who has examined various properties of it. In the similar way we can obtain analogous theorems for $\Delta_p(\rho, v)$, so that the proofs are mostly omitted.

Triviality Theorem 3.3. $\Delta_{p}(\rho, *) = (\Omega \rho)_{*}(\Omega j)_{*}$. More generally, if $j \circ \hat{v} = v$ for a map $\hat{v}: X \to F$, then $\Delta_{p}(\rho, v) = \Delta(\rho j, \hat{v})$.

In order to state the next theorem we define the dual Hopf invariant $\nu(\rho) \in [F \times E, C]$ of ρ to be

$$u(
ho) = -(j \circ p_1)^*
ho + \mu^*
ho - p_2^*
ho$$

where $p_1: F \times E \to F$ and $p_2: F \times E \to E$ are the projections. If $\nu(\rho) = 0$, we say that ρ is *primitive with respect to* μ .

Primitivity Theorem 3.4. If ρ is primitive with respect to μ , then $\Delta_p(\rho, v) = (\Omega \rho)_*(\Omega j)_*$.

Theorem 3.5. If X is an H cogroup (see [15]), then $\Delta_{p}(\rho, v) = (\Omega \rho)_{*}(\Omega j)_{*}$.

Composition Theorem 3.6. Let $\alpha : C \to D$ be a map to an H space. Then $\Delta_p(\alpha \rho, v) = \Delta(\alpha, \rho v) \Delta_p(\rho, v)$.

Cartesian Product Theorem 3.7. Let $\rho_1: E \to C_1$, $\rho_2: E \to C_2$ be maps to H-spaces and let $\{\rho_1, \rho_2\} = (\rho_1 \times \rho_2) \circ d: E \to C_1 \times C_2$ be the composite with diagonal map d. Then

$$\Delta_{p}(\{
ho_{1},\,
ho_{2}\},\,v)eta=(\Delta_{p}(
ho_{1},\,v)eta,\,\Delta_{p}(
ho_{2},\,v)eta)\,.$$

Additivity Theorem 3.8. Let $\rho_1, \rho_2: E \rightarrow C$ be maps. Then

$$egin{aligned} &\Delta_{p}(
ho_{1}+
ho_{2},\,v)eta=\Delta_{p}(
ho_{1},\,v)eta+(
ho_{1}v)_{\Box}\Delta_{p}(
ho_{2},\,v)eta\,,\ &\Delta_{p}(-
ho,\,v)eta=-(-
ho v)_{\Box}\Delta_{p}(
ho,\,v)eta\,, \end{aligned}$$

where $(\rho_1 v)_{\Box}$ and $(-\rho v)_{\Box}$ are the endomorphisms as defined in [13, p. 383].

Cup Product Theorem 3.9. Given $\rho \in H^a(E; \pi)$ and $\rho' \in H^b(E; \pi')$, let $\rho \cup \rho'$ denote the cup product with respect to a pairing $\pi \otimes \pi' \to G$. Then

$$\Delta_p(
ho\cup
ho',\,v)eta=\Delta_p(
ho,\,v)eta\cup v^*
ho'+(-1)^av^*
ho\cup\Delta_p(
ho',\,v)eta\,.$$

The following theorem will be useful in computing $\Delta_{\rho}(\rho, v)$ in terms of deviation of ρ from primitivity and corresponds to Corollary 1.4 of [5] or Theorem 2.4.1 of [13].

Theorem 3.10. Suppose C is homotopy abelian or $\rho_*v=0$. Then

$$\Delta_{p}(
ho, v)eta = (\Omega
ho)_{*}(\Omega j)_{*}eta + \Delta_{pp_2}(
u(
ho), \{*, v\})\{eta, *\}$$

Assume further that C is an Eilenberg-MacLane space and that

$$u(
ho) = \sum u_i \times v_i + \sum \delta^*(u'_j \times v'_j),$$

where δ^* is the Bockstein. Then

$$\Delta_{p}(\rho, v)\beta = (\Omega\rho)_{*}(\Omega j)_{*}\beta + \sum (\Omega u_{i})\beta \cup v^{*}v_{i} - \sum \delta^{*}(\Omega u_{j}')\beta \cup v^{*}v_{j}'.$$

Proof. We imitate the proof of Theorem 2.4.1 of Rutter [13]. Consider the diagram



Note that the action $\overline{\mu}$: $(F \times F) \times (F \times E) \to F \times E$ of the principal fibration $p \circ p_2$ is given by $\overline{\mu}(x, x'; x'', y) = (x + x'', \mu(x', y))$. Thus, the right translation $\overline{\mu}'$: $\Omega(F \times F) \times (F \times E) \to \Omega^*_{pp_2}(F \times E)$ satisfies the following:

$$\begin{aligned} (\rho\mu)'_{*}\overline{\mu}'_{*}(\{\beta, *\}, \{*, v\}) &= \rho'_{*}\mu'_{*}\{\beta, v\}, \\ (\rhop_{2})'_{*}\overline{\mu}'_{*}(\{\beta, *\}, \{*, v\}) &= (s_{C})_{*}\rho_{*}v, \\ p'_{1*}\mu'_{*}(\{\beta, *\}, \{*, v\}) &= (i_{F})_{*}\beta. \end{aligned}$$

Using these and by 3.8 and 3.6 we have that

$$\begin{split} \Delta_p(\rho, v)\beta &= \Delta_{pp_2}(\rho\mu, \{*, v\})\{\beta, *\} \\ &= \Delta_{pp_2}(\rho jp_1, \{*, v\})\{\beta, *\} + \Delta_{pp_2}(\nu(\rho), \{*, v\})\{\beta, *\} \\ &+ \Delta_{pp_2}(\rho p_2, \{*, v\})\{\beta, *\} \\ &= \Delta(\rho j, *)\Delta_{pp_2}(p_1, \{*, v\})\{\beta, *\} + \Delta_{pp_2}(\nu(\rho), \{*, v\})\{\beta, *\} \\ &= (\Omega\rho)_*(\Omega j)_*\beta + \Delta_{pp_2}(\nu(\rho), \{*, v\})\{\beta, *\} \,. \end{split}$$

Now it follows from 3.9, 3.6 and 1.4.1 of [13] that

$$\begin{aligned} \Delta_{pp_2}(u_i \times v_i, \{*, v\})\{\beta, *\} &= \Delta_{pp_2}(p_1^*u_i, \{*, v\})\{\beta, *\} \cup v_i p_2\{*, v\} \\ &+ (-1)^{\dim^u i} u_i p_1\{*, v\} \cup \Delta_{pp_2}(p_2^*v_i, \{*, v\})\{\beta, *\} \\ &= \Delta(u_i, *) \Delta_{pp_2}(p_1, \{*, v\})\{\beta, *\} \cup v^*v_i \\ &= (\Omega u_i)\beta \cup v^*v_i . \end{aligned}$$

Similarly,

$$\Delta_{pp_2}(\delta^{\sharp}(u'_{j} \times v'_{j}), \{*, v\})\{\beta, *\} = (\Omega \delta^{\sharp})_{*}(\Omega u'_{j})_{*}\beta \cup v^{*}v'_{j}$$

This completes the proof of 3.10.

In [7], James and Thomas have called $\pi = p \circ q$ of (3.1) a stable decomposition of π if there exists a map $c: F \times A \rightarrow C$ such that the composite

$$A \xrightarrow{i_2} F \times A \xrightarrow{c} C$$

is null-homotopic and $\rho\mu \simeq c(1 \times p) + \rho p_2$, where i_2 denotes the injection. Note that, if ρ is primitive with respect to μ , then $\pi = p \circ q$ is stable with $c = \rho j p_1$, where p_1 denotes the projection $F \times A \rightarrow F$. As stated in p. 104 of [7], for $v: X \rightarrow E$ liftable to $T, \tau \cdot v$ is liftable to T if and only if $c_*\{\tau, p_*v\}=0$. The following theorem can also be proved in a way similar to 3.10.

Theorem 3.11. (James and Thomas [7]) If (3.1) is a stable decomposition of $\pi = p \circ q$, then

$$\Delta_{p}(
ho, v)eta=c_{st}'\{eta, p_{st}v\}$$
 ,

where $c': \Omega F \times A \rightarrow \Omega C$ is determined by the composite

$$\Omega F \times A \xrightarrow{injection} \Omega(F \times A) \times F \times A \xrightarrow{\approx} \Omega^*(F \times A) \xrightarrow{\Omega^*c} \Omega^*C$$

4. Secondary operation $\Phi_{\theta}(\rho, v)$ and proof of Theorem A

Consider the situation (3.1) in which A, B and C are H-spaces with multiplications m, t and n respectively. Given a map $v: X \rightarrow E$, we set $u = p \circ v$. We shall now define a sort of secondary operation

$$\Phi_{\theta}(\rho, v)$$
: ker $\Delta(\theta, u) \to \operatorname{coker} \Delta_{\rho}(\rho, v)$

as follows.

Take an element $\alpha \in [X, \Omega A]$ such that $(\Omega^* \theta)_* m'_* \{\alpha, u\} = 0$; then it follows from the next Sublemma (i) that there exists an element $\psi \in [X, \Omega^* E; u]$ such that

(4.1)
$$(\Omega^* p)_* \psi = hm'_* \{\alpha, u\}$$
 and $(r_E)_* \psi = v \in [X, E; u],$

where $h: [X, \Omega^*A; m\{*, u\}] \rightarrow [X, \Omega^*A; u]$ is the canonical bijection. The coset of $\gamma \in [X, \Omega C]$ determined by $(\Omega^*\rho)_* \psi = n'_* \{\gamma, \rho_* v\}$ is, by definition, $\Phi_{\theta}(\rho, v) \alpha$.

Observe that, if there exists another ψ' such that $(\Omega^* p)\psi' = (\Omega^* p)_*\psi$ in $[X, \Omega^* A; u]$ and $(r_E)_*\psi' = v$ in [X, E; u], then we may assume $(r_E)\psi' = v = (r_E)\psi$ as maps and, applying Sublemma (ii) to $\varphi = \psi + (-\psi')$, we conclude that $(\Omega^* \rho)_* \psi - (\Omega^* \rho)_* \psi'$ lies in the image of $\Delta_p(\rho, v)$. This ensures that $\Phi_{\theta}(\rho, v)$ is well defined.

Sublemma. (i) Given an element $\beta \in [X, \Omega^*A; u]$ lying in the image of $(\Omega^*p)_*; [X, \Omega^*E] \rightarrow [X, \Omega^*A]$, there exists a $\psi \in [X, \Omega^*E; u]$ such that $(\Omega^*p)_*\psi = \beta$ and $(r_E)_*\psi = v$.

(ii) If $\varphi \in [X, \Omega^*E; u]$ satisfies $(\Omega^*p)_*\varphi = (s_A)_*u$, then φ is contained in the image of the natural map $[X, \Omega_p^*E; u] \rightarrow [X, \Omega^*E; u]$.

Proof. (i) Let $\hat{\mu}: F \times \Omega^* E \to \Omega^* E$ be the map induced by the action $\mu: F \times E \to E$. It is easily verified that

 $(\Omega^*p) \mathcal{A}(x, \gamma) = (\Omega^*p)\gamma, \ \mu(1 \times r_E) = r_E \mathcal{A} \ (x \in F, \gamma \in \Omega^*E)$

By assumption we can take $\tilde{\psi}: X \to \Omega^* E$ with $(\Omega^* p) \tilde{\psi} \simeq \beta$, so that there is a *u*-map $\psi_0: X \to \Omega^* E$ with $(\Omega^* p) \psi_0 = \beta$, since $\Omega^* p$ is a fibration. Choose $\omega: X \to F$ such that $\mu\{\omega, r_E\psi_0\}$ is *u*-homotopic to *v*. Then $\psi = \beta\{\omega, \psi_0\}$ has the desired property.

(ii) This is a simple application of homotopy lifting property.

Theorem A in the introduction follows immediately from 3.2 and the following theorem which states a main property of $\Phi_{\theta}(\rho, v)$.

Theorem 4.2. If $w: X \to T$ is a lifting of v, then the image of $\Phi_{\theta}(\rho, v)$ coincides with the factor group $I(w)/I_{u}(w)$.

Proof. Let $\gamma \in [X, \Omega C]$ lie in the coset $\Phi_{\theta}(\rho, v)\alpha$. Since $\rho_* v = 0$, we have that

$$(\Omega^* \rho)_* \psi = (i_c)_* \gamma \quad \text{and} \quad (r_E)_* \psi = v ,$$

which shows that γ is ρ -correlated to v. Hence it follows from Theorem 3.3 of [5] that γ lies in I(w).

Conversely, suppose $\gamma \in I(w)$. Then, by Theorem 3.2 of [5], there exists $\psi' \in [X, \Omega^*E]$ such that $(\Omega^*\rho)_*\psi' = (i_c)_*\gamma$ and $(r_E)_*\psi' = v$ in [X, E]. By the homotopy lifting property of r_E we see that $\psi = \psi'$ for a map $\psi: X \to \Omega^*E$ with $r_E\psi = v$. Then there is an $\alpha \in [X, \Omega A]$ with $h^{-1}(\Omega^*p)_*\psi = m'_*(\alpha, u)$ and, more-over, $(\Omega^*\theta)_*m'_*(\alpha, u) = 0$. This means that γ lies in in the coset $\Phi_{\theta}(\rho, v)\alpha$.

As a special case of Theorem A we obtain

Corollary 4.3. If $[X, \Omega A] = 0$, then $I(w) = I_u(w)$ for any lifting w of v.

REMARK. The conclusion of 4.3 remains valid without assuming that B is an H-space, as shown in what follows. Since $I_u(w) \subset I(w)$, it suffices to prove $I(w) \subset I_u(w)$. Let $\tau \in I(w)$, then, by Theorem 3.2 of [5], there is a $\psi \in [X, \Omega^*E]$ such that $(\Omega^*\rho)_*\psi = (i_C)_*\tau$ and $(r_E)_*\psi = v$. As above we may represent ψ by a v-map $\tilde{\psi}: X \to \Omega^*E$. The assumption implies that $(r_A)_*: [X, \Omega^*A] \to [X, A]$ is a bijection with inverse $(s_A)_*$ and hence there is a homotopy deforming $(\Omega^*p)\tilde{\psi}$ into a map $X \to s_A(A)$. Since Ω^*p is a fibration, it follows that there is a map $\psi_0: X \to \Omega_p^*E$ homotopic to $\tilde{\psi}$ in Ω^*E . Now the composite $\Omega_p^*E \xrightarrow{r} E \xrightarrow{p} A$ is a fibration, so that we can find a map $\psi_1: X \to \Omega_p^*E$ such that $pr\psi_1=u$ and $\psi_0 \simeq \psi_1$ in Ω_p^*E . Consequently, if we can show that $r\psi_1: X \to E$ is u-homotopic to v, then we infer from 2.4 that $\tau \in I_u(w)$. Now, since $pr\psi_1 = pv$, there is an $\omega: X \to F$ such that $\omega \cdot v$ and $r\psi_1$ are u-homotopic, whence $r\psi_1 \simeq r_E \tilde{\psi} = v$ implies $\omega \in I(v)$. Thus, by 2.6, $\omega = 0$, which shows that $r\psi_1$ is u-homotopic to v. Now let

 $p_*: [X, E] \rightarrow [X, A], \quad q_*: [X, T] \rightarrow [X, E]$

be the induced functions and let $v: X \rightarrow E$ be liftable to T. We set $u=p \circ v$. We see that $(p \circ q)_*^{-1}(u)$ coincides with the union

$$\cup q_*^{-1}(\omega \cdot v)$$
,

where ω runs over the cosets in $[X, F]/\Delta(\theta, u)[X, \Omega A]$ such that $\nu(\rho)_*\{\omega, v\} + (\rho j)_*\omega = 0$. We conclude from Theorem A that

A Classification Theorem 4.4. Let $\pi = p \circ q$ in (3.1) be a stable decomposition with $c: F \times A \rightarrow C$ such that $\Delta(\theta, u)$ is injective. Then $(p \circ q)_*^{-1}(u)$ is equivalent to the product

$$\{\omega \in [X, F]; c_*\{\omega, u\} = 0\} / \Delta(\theta, u)[X, \Omega A] \\ \times [X, \Omega C] / \{c'_*\{\beta, u\}; \beta \in [X, \Omega F]\}.$$

Finally we list some properties of $\Phi_{\theta}(\rho, v)$.

Triviality Theorem 4.5. $\Phi_{\theta}(\rho, v)$ is the usual (stable) secondary operation Φ determined by $\Omega\rho: \Omega E \rightarrow \Omega C$ and the image of $\Phi_{\theta}(\rho, v)$ is $(\Omega\rho)_*[X, \Omega E] / \Delta_{\rho}(\rho, v)[X, \Omega F]$ in each of the following cases:

- (i) v is the constant map;
- (ii) X is an H cogroup;
- (iii) $p_*v=0$, ρ is primitive with respect to μ and C is homotopy abelian;
- (iv) θ is primitive and ρ is primitive with respect to H structure of E.

Proof. (i) follows from the fact that $[X, \Omega^*A; *]$ can be identified with $[X, \Omega A]$.

In order to prove (ii), note that the following diagram is homotopycommutative:



where g is H' structure map, d the diagonal map, d' the folding map and $\omega: X \to \Omega E$. Let ψ be an element of $[X, \Omega^* E; u]$ which corresponds to $(i_E)_* \omega + (s_E)_* v \in [X, \Omega^* E; *+u]$ under the canonical isomorphism. Then,

$$(\Omega^*
ho)_*\psi = (i_c)_*(\Omega
ho)_*\omega + (s_c)_*
ho_*v = n'_*\{(\Omega
ho)_*\omega, \,
ho_*v\}$$

and we see that $(\Omega^* p)_* \psi$ corresponds to $(i_A)_* (\Omega p)_* \omega + (s_A)_* u$, i.e., $(\Omega^* p)_* \psi = hm'_* \{\alpha, u\}$ with $\alpha = (\Omega p)_* \omega$. These show that $(\Omega \rho)_* \omega$ represents $\Phi_{\theta}(\rho, v) \alpha$.

We prove (iii). Given $\alpha \in [X, \Omega A]$ with $(\Omega \theta)_* \alpha = 0$ (which is equivalent

to $(\Omega^*\theta)_*m'_*\{\alpha, *\}=0$, take $\gamma \in [X, \Omega E]$ such that $(\Omega p)_*\gamma = \alpha$. Since $p_*v=0$, there is a $\tau \in [X, F]$ with $j_*\tau = v$. Using the map $\beta: F \times \Omega^* E \to \Omega^* E$ in the proof of Sublemma, we set $\psi = \beta_*\{\tau, (i_E)_*\gamma\} \in [X, \Omega^* E; *]$; then, $(r_E)_*\psi = v$. Thus,

$$egin{aligned} &(\Omega^*
ho)_*\psi = (\Omega^*
ho)_*\mu_*\{ au,\,(i_E)_*\gamma\} = ar n_*\{
ho_*j_* au,\,(\Omega
ho)_*\gamma\} \ &= n'_*\{(\Omega
ho)_*\gamma,\,
ho_*v\}\,, \end{aligned}$$

where $\bar{n}: C \times \Omega C \rightarrow \Omega^* C$ is the "left translation". This proves (iii). The proof of (iv) is left to the reader.

Composition Theorem 4.6. Let $\sigma: C \rightarrow D$ be a map of C to an H space D. Let

$$\hat{\Delta}(\sigma, \rho v)$$
: coker $\Delta_{\flat}(\rho, v) \rightarrow \text{coker } \Delta_{\flat}(\sigma \rho, v)$

denote the homomorphism induced by $\Delta(\sigma, \rho v)$. Then

$$\Phi_{ heta}(\sigma
ho,\,v) = \hat{\Delta}(\sigma,\,
ho v) \Phi_{ heta}(
ho,\,v)\,.$$

Cartesian Product Theorem 4.7. Suppose ρ_1 and ρ_2 are as in 3.7. Then

 $\Phi_{ heta}(\{
ho_1,\
ho_2\},\ v)lpha=(\Phi_{ heta}(
ho_1,\ v)lpha,\ \Phi_{ heta}(
ho_2,\ v)lpha)\,.$

Cup Product Theorem 4.8. Let ρ , ρ' and $\rho \cup \rho'$ be as in 3.9. Then

$$\Phi_{ heta}(
ho\cup
ho',\,v)lpha=\Phi_{ heta}(
ho,\,v)lpha\cup
ho'v\!+\!(-1)^{a}
ho v\cup\Phi_{ heta}(
ho',\,v)lpha\,.$$

Naturality Theorem 4.9. Let α lie in ker $\Delta(\theta, u)$ and let $f: Y \rightarrow X$ be a map. Then

$$\Phi_{\theta}(\rho, f^*v)(f^*\alpha) = f^*\Phi_{\theta}(\rho, v)\alpha \mod \Delta_{p}(\rho, f^*v)[Y, \Omega F].$$

5. Proof of Theorem B

Consider a Postnikov tower for $P_n(R)$, n odd > 1. In (3.1) we take

$$A = K(Z_2, 1), \quad B = K(Z, n+1), \quad C = K(Z_2, n+2), \quad \theta = (\delta^* \iota_1)^{(n+1)/2}$$

where $\iota_1 \in H^1(Z_2, 1; Z_2)$ is the fundamental class; then we have the first two stages and $H^{n+2}(E; Z_2) = Z_2$ whose generator is the second invariant ρ with $j^*\rho = Sq^2\iota_n$, ι_n being the non-zero element of $H^n(Z, n; Z_2)$. We claim that

$$\mu^{*}(\rho) = \begin{cases} Sq^{2}\iota_{n} \times 1 + 1 \times \rho + \iota_{n} \times p^{*}\iota_{1}^{2} & \text{for} \quad n \equiv 1 \text{ (4)} \\ Sq^{2}\iota_{n} \times 1 + 1 \times \rho & \text{for} \quad n \equiv 3 \text{ (4)} \end{cases}$$

and hence (3.1) in this case is a stable decomposition with $c=Sq^2\iota_n\times 1+\iota_n\times \iota_1^2$ or $Sq^2\iota_n\times 1$.

Now it follows from Cartan's formula that

$$Sq^{2}(\delta^{*}\iota_{1})^{(n+1)/2} = Sq^{2}(\iota_{1}^{n+1}) = \frac{(n+1)n}{2}\iota_{1}^{n+3} = \begin{cases} \iota_{1}^{n+3} & \text{if } n \equiv 1 \ (4) \\ 0 & \text{if } n \equiv 3 \ (4) \end{cases}$$

We shall use an exact sequence due to E. Thomas [16, p. 187]. We see from the above equality that, for the morphism $\tau: H^{n+2}(F \times E, E) \rightarrow H^{n+3}(A)$,

$$\tau(Sq^2\iota_n \times 1 + \iota_n \times p^*\iota_1^2) = 0 \quad \text{or} \quad \tau(Sq^2\iota_n \times 1) = 0$$

according as $n \equiv 1$ or 3 (4), and hence there is a $\tilde{\rho} \in H^{n+2}(E; \mathbb{Z}_2)$ such that

$$\overline{\mu}(\tilde{
ho}) = Sq^2\iota_n imes 1 + \iota_n imes p^*\iota_1^2 \quad ext{or} \quad Sq^2\iota_n imes 1$$

for the operator $\overline{\mu}: H^{n+2}(E) \to H^{n+2}(F \times E, E)$ (which is essentially equal to $\mu^* - p_2^*$). Since ker $p^* = \ker l^*$ in dim n+2 for a map $l: P_n(R) \to A$ representing ι , we infer that $\rho = \tilde{\rho}$, which proves our assertion. Theorem B now follows from 4.4 and the fact $[X, P_n(R)] \approx [X, T]$.

6. Proof of Theorem C

Consider the first two stages of a Moore-Postnikov tower for the inclusion $BO(n) \subset BO$ between the classifying spaces for *n*-plane bundles and stable ones:

$$K(Z_2, n) \xrightarrow{j} E \longrightarrow K(Z_2, m)$$

$$\downarrow p$$

$$BO \xrightarrow{w_{n+1}} K(Z_2, n+1),$$

where m=n+2 or n+3 according as $n\equiv 1$ (4) or $n\equiv 3$ (4), n>2, and w_i denotes the universal Stiefel-Whitney class of dimension *i*. As shown in [7, p. 110], $p \circ q$ forms a stable decomposition with

$$c = \begin{cases} Sq^{2}\iota_{n} \times 1 + \iota_{n} \times w_{2} & \text{if } n \equiv 1 \ (4) \\ Sq^{2}Sq^{1}\iota_{n} \times 1 + Sq^{1}\iota_{n} \times (w_{1}^{2} + w_{2}) & \text{if } n \equiv 3 \ (4) \,, \end{cases}$$

where $\iota_n \in H^n(Z_2, n; Z_2)$ is the fundamental class (This can also be shown using Thomas' exact sequence). Thus, we conclude from 4.4 that, for a stable bundle $\xi \in \widetilde{KO}(X)$ with $w_{n+1}(\xi) = 0$ such that $\Delta(w_{n+1}, \xi) \colon \widetilde{KO}^{-1}(X) \to H^n(X; Z_2)$ is injective,

(a) if $n \equiv 1$ (4) and dim $X \leq n+1$, then $N_n(\xi; X)$ is equal to the cardinal of the direct product:

Coker $\Delta(w_{n+1}, \xi) \times H^{n+1}(X; Z_2) / (Sq^2 + w_2(\xi) \cup) H^{n-1}(X; Z_2)$.

(b) if $n \equiv 3$ (4) and dim $X \leq n+2$, then $N_n(\xi; X)$ is equal to the cardinal

of the direct product:

Coker $\Delta(w_{n+1}, \xi) \times H^{n+2}(X; Z_2) / (Sq^2 + (w_1(\xi)^2 + w_2(\xi)) \cup)Sq^1 H^{n-1}(X; Z_2)$.

Now take $X=P_{n+1}(R)$ or $P_{n+2}(R)$ according as $n \equiv 1$ or 3 (4). Then $\widetilde{KO}^{-1}(X)=Z_2$ by [3] and it follows from a formula of [5, p. 489] that $\Delta(w_{n+1}, k\eta)$ is injective for $\binom{k-1}{n-1}$ odd and $w_{n+1}(k\eta)=0$ for $\binom{k}{n+1}$ even. Theorem C will be obtained by observing that

$$w_2(k\eta) = {k \choose 2} x^2$$
, $Sq^2 x^{n-1} = 0$ for $n \equiv 1$ (4), $Sq^1 x^{n-1} = 0$ for odd n ,

where x denotes the non-zero element of $H^1(X; \mathbb{Z}_2)$.

Note that, for $n \equiv 1$ (4) and $\binom{k}{2}$ odd, $\Delta_p(\rho, v)$: $H^{n-1}(P_{n+1}(R); Z_2) \rightarrow H^{n+1}(P_{n+1}(R); Z_2)$ is surjective for any lift v of $k\eta$ and hence $q_*: [P_{n+1}(R), T] \rightarrow [P_{n+1}(R), E]$ is bijective.

As another illustration, let $n \equiv 1$ (4) and let X be $P_{(n+1)/2}(C)$, the complex projective space of complex dimension $\frac{n+1}{2}$. Then $\widetilde{KO}^{-1}(X) = 0$ by [3]. Since $H^n(X; Z_2) = 0$ and $Sq^2(y^{(n-1)/2}) = 0$ for the non-zero element y of $H^2(X; Z_2)$, we see that $N_n(\xi; X) = 1$ or 2 according as $w_2(\xi) \neq 0$ or $w_2(\xi) = 0$.

7. Further examples

7.1. Suppose that, in (3.1), A is an (n-1)-connected space such that $\pi_k(A)=0$ for $k \ge n+n'-2$ $(n'>n\ge 2)$, $B=K(\pi', n'+1)$ and C=K(G, n+n'). Assume $\rho \in H^{n+n'}(E; G)$ represents $(\bar{\rho}+\psi \cup)_{\rho}(\theta)$ for $\bar{\rho} \in H^{n+n'+1}(\pi', n'+1; G)$, $\psi \in H^n(A; \pi)$, where the cup product is taken with respect to the Whitehead product pairing $\pi \otimes \pi' \to G$ in T, $\pi = \pi_n(T)$. Then it is proven by F. P. Peterson [11] that

$$\mu^*(\rho) = j^* \rho \times 1 + 1 \times \rho + x' \times p^*(\psi) , \qquad x' \in H^{n'}(\pi', n; \pi') ,$$

where $j^*\rho$ is the suspension of $\bar{\rho}$ and $j^*\rho(x')=j^*\rho(\iota')$, ι' being the fundamental class of $H^{n'}(\pi', n'; \pi')$. Thus, the tower $p \circ q$ in this case is a stable decomposition with $c=j^*\rho \times 1+x' \times \psi$ (cf. Theorem 3.1 of [4]). Hence, $\Delta_p(\rho, v)\beta =$ $\Omega(\rho j)_*\beta + (\Omega x')\beta \cup u^*\psi$, $\beta \in H^{n'-1}(X; \pi')$.

In case $A = K(\pi, n)$, we can take the basic classes ι' and $\iota \in H^n(\pi, n; \pi)$ for x' and ψ respectively.

7.2. Consider a Postnikov tower for the usual lens space $L=S^{2n+1}/Z_p$, where p is an odd prime. We take

$$A = K(Z_p, 1), \quad B = K(Z, 2n+2), \quad C = K(Z_2, 2n+3),$$

and let $\theta = (\delta^* \iota)^{n+1}$ where δ^* is the Bockstein associated with $0 \to Z \to Z \to Z_p \to 0$ and ι denotes the basic class of $H^1(Z_p, 1; Z_p)$, and let ρ denote the generator of $H^{2n+3}(E; Z_2) = Z_2$.

Given a path-connected (2n+2)-dimensional complex X, we have $\Delta(\theta, u) = 0$ by virtue of $H^1(SX; Z_p) = 0$ and $j^*\rho = Sq^2\overline{\iota}$, $\overline{\iota}$ being the basic class of $H^{2n+1}(Z, 2n+1; Z)$. Since $H^{2n+3}(F \times E, F \vee E; Z_2) = 0$, ρ is primitive with respect to the action μ , so that $\Delta_p(\rho, v) = Sq^2 : H^{2n}(X; Z) \to H^{2n+2}(X; Z_2)$. Thus, we see from 4.4 that [X, L] is equivalent to the product

$$\{u \in H^{1}(X; Z_{p}); (\delta^{*}u)^{n+1} = 0\} \times H^{2n+1}(X; Z) \times H^{2n+2}(X; Z_{2}) / Sq^{2}H^{2n}(X; Z) .$$

This extends a result of P. Olum [10].

7.3. Consider a Postnikov tower for the *n*-sphere S^n , $n \ge 4$. We take

$$A = K(Z, n), \quad B = K(Z_2, n+2), \quad C = K(Z_2, n+3)$$

and $Sq^2\iota$ and the unique non-zero element of $H^{n+3}(E; Z_2) = Z_2$ for θ and ρ , where ι is the basic class of $H^n(Z, n; Z)$. Then,

$$\Delta(\theta, u) = Sq^2 \colon H^{n-1}(X; Z) \to H^{n+1}(X; Z_2)$$
$$\Delta_p(\rho, v) = Sq^2 \colon H^n(X; Z_2) \to H^{n+2}(X; Z_2) .$$

Let X be a complex with dim $X \leq n+2$; then $v: X \rightarrow E$ is always liftable to T. We conclude

(1) (Nakaoka [8, p. 94, Theorem 4]) Assume $Sq^2: H^n(X; Z_2) \rightarrow H^{n+2}(X; Z_2)$ is surjective; then $I(w) = I_u(w) = H^{n+2}(X; Z_2)$ and hence it follows that $[X, S^n]$ is equivalent to

$$\{u \in H^n(X; Z); Sq^2u = 0\} \times H^{n+1}(X; Z_2)/Sq^2H^{n-1}(X; Z)$$

(2) Assume $Sq^2: H^{n-1}(X; Z) \rightarrow H^{n+1}(X; Z_2)$ is injective; then it follows from 4.4 that $[X, S^n]$ is equivalent to the product

$$\{u \in H^{n}(X; Z); Sq^{2}u = 0\} \times H^{n+1}(X; Z_{2})/Sq^{2}H^{n-1}(X; Z) \times H^{n+2}(X; Z_{2})/Sq^{2}H^{n}(X; Z_{2}).$$

7.4. Consider a Postnikov tower for the complex projective space $P_m(C)$. Let

$$A = K(Z, 2), \quad B = K(Z, 2m+2), \quad C = K(Z_2, 2m+3),$$

and let $\theta = \iota^{m+1}$, where $\iota \in H^2(Z, 2; Z)$ is the basic class, and ρ be the unique nonzero element $\psi_m \in H^{2m+3}(E; Z_2)$ (cf. [12]). Then $j^*\rho = Sq^2\iota_{2m+1}$, where ι_{2m+1} is the generator of $H^{2m+1}(Z, 2m+1; Z_2)$. The dual Hopf invariant $\nu(\theta)$ with respect to *H*-structure of *A* is $(\iota \times 1 + 1 \times \iota)^{m+1}$ and hence

$$\Delta(\theta, u)\alpha = (m+1)\alpha \cup u^m, \ \alpha \in H^1(X; Z) \qquad \text{(cf. [15, p. 452])}$$

We see from 7.1 and [11] that

$$\mu^*(\rho) = \begin{cases} \rho j \times 1 + 1 \times \rho & \text{if } m \text{ is odd} \\ \rho j \times 1 + 1 \times \rho + \overline{\iota}_{2m+1} \times p^* \iota & \text{if } m \text{ is even}, \end{cases}$$

where the cross product is taken with respect to the nontrivial pairing $Z \otimes Z \rightarrow Z_2$ and $\bar{\iota}_{2m+1}$ denotes the basic class of $H^{2m+1}(Z, 2m+1; Z)$.

Given a (2m+2)-dimensional complex X, we assume that $(m+1)\alpha \cup u^m = 0$ implies $\alpha = 0$ for $\alpha \in H^1(X; Z)$ and a fixed $u \in H^2(X; Z)$. Then it follows from 4.4 that, for the function $\varphi \colon [X, P_m(C)] \to H^2(X; Z)$ assigning f^*z to $f \colon X \to P_m(C)$, z being a generator of $H^2(P_m(C); Z)$, $\varphi^{-1}(u)$ is equivalent to

$$\begin{array}{ll} H^{2m+1}(X;\,Z)/(m+1)u^m \cup H^1(X;\,Z) \times H^{2m+2}(X;\,Z_2)/Sq^2H^{2m}(X;\,Z) & \text{for } m \text{ odd} \\ H^{2m+1}(X;\,Z)/(m+1)u^m \cup H^1(X;\,Z) \times H^{2m+2}(X;\,Z_2)/(Sq^2+u \cup)H^{2m}(X;\,Z) \\ & \text{for } m \text{ even} \,. \end{array}$$

It seems likely that, for m=1 and dim X=4, our $\Phi_{\theta}(\rho, v)$ coincides with Φ_{c_2} introduced by N. Shimada [14, p. 141].

7.5. Let *n* be an even integer and let

$$K(Z_{2}, 2n+2) \longrightarrow T$$

$$\downarrow q$$

$$K(Z, 2n+1) \longrightarrow E \xrightarrow{\rho} K(Z_{2}, 2n+3)$$

$$\downarrow p$$

$$BU \xrightarrow{c_{n+1}} K(Z, 2n+2)$$

be part of a Moore-Postnikov tower for $BU(n) \subset BU$ between the classifying spaces for the unitary groups U(n) and U, where c_{n+1} denotes the universal (n+1) th Chern class. It is readily shown that $H^{2n+3}(E; Z_2) = Z_2$ is generated by ρ with $j^*\rho = Sq^2\iota_{2n+1}$, where ι_{2n+1} is the generator of $H^{2n+1}(Z, 2n+1; Z_2)$.

Since, for the realification $\hat{\gamma}$ of the canonical bundle γ over BU,

$$Sq^{2}c_{n+1} = Sq^{2}w_{2n+2}(\hat{\gamma}) = w_{2}(\hat{\gamma}) \cup c_{n+1}$$

where the cup product is with respect to the non-trivial pairing $Z_2 \otimes Z \rightarrow Z_2$, Thomas' exact sequence reveals that

$$\mu^*
ho = Sq^2\iota_{2n+1} imes 1 + 1 imes
ho + \iota_{2n+1} imes p^*w_2(\hat{\gamma})$$

Hence it follows from 4.4 that, for a complex X such that dim $X \leq 2n+2$ and $\Delta(c_{n+1}, u): \tilde{K}^{-1}(X) \to H^{2n+1}(X; Z)$ is injective for $u \in \tilde{K}(X)$ with $c_{n+1}(u)=0$, the number of *n*-dimensional complex vector bundles over X which are stably

equivalent to u, is equal to the cardinal of the direct product

coker
$$\Delta(c_{n+1}, u) \times H^{2n+2}(X; Z_2)/(Sq^2 + w_2(\hat{u}) \cup)H^{2n}(X; Z)$$
,

where \hat{u} is the realification of u.

For example, let $X=P_{2n+2}(R)$. Since $\tilde{K}^{-1}(X)=0$ by Theorem 3.3 of [2] and since K(X) consists of elements $k\nu$ $(k=0, 1, \dots, 2^{n-1}-1)$, ν denoting the complexification of the canonical line bundle λ (see [1]), the number of classes of *n*-dimensional complex plane bundles which are stably equivalent to $k\nu$, is equal to 2 or 4 according as k is odd or even. This follows by observing that $Sq^2x^{2n}=0$ for the generator $x \in H^1(X; Z_2)$ and $w_2(k\hat{\nu})=kw_2(2\lambda)=kx^2$.

8. Appendix: the group of fibre homotopy equivalences

Given a fibration $f: Y \rightarrow Z$, we denote by $\mathcal{E}(Y; f)$ the group of fibre homotopy classes of fibre homotopy equivalences of Y.

In the situation (3.1) we shall assume that $\pi_k(A) \neq 0$ only for $n \leq k \leq n'-1$, $\pi_k(F) \neq 0$ only for $n \leq k \leq n'-1$ and $\pi_r(\Omega C) \neq 0$ only for $n' \leq r \leq n'+n-1$ (n>1). It is easily shown that there is an exact sequence

$$1 \to q^*[E, \Omega C] \to \mathcal{E}(T; q) \to \mathcal{E}(\Omega C) .$$

We shall study $\mathcal{E}(T; p \circ q)$. First we need

Lemma 8.1. The functions

$$q^* \colon [E, E; p] \to [T, E; p \circ q], \quad i_* \colon [\Omega C, \Omega C] \to [\Omega C, T]$$

are bijective.

Proof. Introduce the commutative diagram

$$\begin{matrix} [E, F] & \xrightarrow{q^*} [T, F] \\ T_1 & \downarrow & \downarrow T_2 \\ [E, E; p] & \xrightarrow{q^*} [T, E; p \circ q], \end{matrix}$$

where the vertical bijections T_1 and T_2 are given by

$$T_1(\tau) = \mu_* \{ \tau, 1_E \}, \quad T_2(\omega) = \mu_* \{ \omega, q \},$$

 1_x being the identity map of X. Since the upper q^* is bijective, so is the bottom q^* . The second assertion can be proved by a classical obstruction argument or by using a Moore-Postnikov tower for *i*.

In the light of Lemma 8.1 we can now define homomorphisms

$$J: \mathcal{E}(T; p \circ q) \to \mathcal{E}(E; p), \quad J_0: \mathcal{E}(T; p \circ q) \to \mathcal{E}(\Omega C)$$

by requiring, for $g \in \mathcal{E}(T; p \circ q)$,

 $q_*g = q^*J(g)$ in $[T, E; p \circ q], \quad i_*J_0(g) = i^*g.$

Let

$$\Delta \colon \ker i^* \to \mathcal{E} (T; p \circ q)$$

denote the homomorphism defined by $\Delta(\tau) = \tau \cdot 1_T$, where i^* is the homomorphism in the exact sequence

$$[\Omega C, \Omega C] \xleftarrow{i^*} [T, \Omega C] \xleftarrow{q^*} [E, \Omega C].$$

Theorem 8.2. The following sequence of groups and homomorphisms is exact:

$$[T, \Omega F] \xrightarrow{\Delta_p(\rho, q)} q^*[E, \Omega C] \xrightarrow{\Delta} \mathcal{E}(T; p \circ q) \xrightarrow{\{J, J_0\}} \mathcal{E}(E; p) \times \mathcal{E}(\Omega C),$$

in which the image of $\{J, J_0\}$ consists of $(\bar{g}, \Omega h) \in \mathcal{E}(E; p) \times \mathcal{E}(\Omega C)$ such that $\rho \bar{g} \simeq h \rho$.

Proof. The exactness at the second term follows from the fact that the image of $\Delta_p(\rho, q)$ coincides with $I_{p \circ q}(1_T)$ by 3.2. We shall prove the exactness at the third term.

Let $g: T \to T$ be a homotopy equivalence such that $gi \simeq i$, pqg = pq and $qg \simeq q$ by a pq-homotopy $H_t: T \to E$, $0 \le t \le 1$, with $H_0 = qg$, $H_1 = q$. By the homotopy lifting property there exists a homotopy $\hat{H}_t: T \to T$ with $\hat{H}_0 = g$, $q\hat{H}_t = H_t$. Since $pq\hat{H}_t = pH_t = pq$, \hat{H}_t is a pq-homotopy. Put $g' = \hat{H}_1$, then qg' = q and so g' is q-homotopic to $\tau \cdot 1_T$ for some $\tau: T \to \Omega C$. Since

$$(\tau i) \cdot i \simeq g' i \simeq g i \simeq i$$

and $[\Omega C, \Omega E] = 0$, it follows from 2.3 that I(i)=0 and hence $i^*\tau=0$.

The assertion about the image of $\{J, J_0\}$ can be proved by an argument similar to Theorem 2.9 of [9], noting that, if $qg \simeq \bar{g}q$ by a pq-homotopy, we can replace g by \hat{g} which is pq-homotopic to g and which is such that $q\hat{g} = \bar{g}q$.

Consider the situation in which $A = K(\pi, n)$, $B = K(\pi', n'+1)$ and C = K(G, n+n'), 1 < n < n' in (3.1). Theorem 8.2, together with 7.1, gives rise to an exact sequence

$$1 \to R \to \mathcal{E}(T; p \circ q) \to \mathcal{E}(E; p) \times \operatorname{Aut} G,$$

where R denotes the factor group

$$H^{n+n'-1}(E; G)/p^*(\Omega(\rho j)+\iota_n \cup)H^{n'-1}(\pi, n; \pi'),$$

 ι_n being the basic class in $H^n(\pi, n; \pi)$ and the cup product being taken with respect to the Whitehead product pairing of T.

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