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On the Linear Partial Differential Equation of the First Order

By Takashi KASUGA

Introduction

In this paper, we shall treat the following partial differential equation

$$\frac{\partial z}{\partial x} + f(x, y, z) \frac{\partial z}{\partial y} = g(x, y, z)$$

without the usual condition of the total differentiability on the solution $z(x, y)$. (For the simpler equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} f(x, y) = 0$$

see our previous paper, Kasuga [2]).

In the following, we shall denote by D a fixed open set in R^3 (Euclidean space defined by the three coordinates x, y, z), by G its projection on the (x, y) -plane, by $f(x, y, z)$, $g(x, y, z)$ two fixed continuous functions defined on D , which have continuous f_y, f_z, g_y, g_z . Evidently G is open in the (x, y) -plane.

We shall consider the partial differential equation

$$\frac{\partial z}{\partial x} + f(x, y, z) \frac{\partial z}{\partial y} = g(x, y, z). \quad (1)$$

With (1) we shall associate the simultaneous ordinary differential equations

$$\left\{ \begin{array}{l} \frac{dy}{dx} = f(x, y, z) \\ \frac{dz}{dx} = g(x, y, z) \end{array} \right. \quad (2)$$

$$(3)$$

The curves representing the solutions of (2), (3) which are prolonged as far as possible on both sides in an open subset D_1 of D , will be called characteristic curves (of (1)) in D_1 . Through any point (x_0, y_0, z_0)

in D_1 , there passes one and only one characteristic curve in D_1 .¹⁾ We represent it by

$$y = \varphi(x, x_0, y_0, z_0, D_1) \quad z = \psi(x, x_0, y_0, z_0, D_1) \\ \alpha(x_0, y_0, z_0, D_1) < x < \beta(x_0, y_0, z_0, D_1)$$

($\alpha(x_0, y_0, z_0, D_1), \beta(x_0, y_0, z_0, D_1)$ may be $-\infty, +\infty$ respectively). Sometimes we abbreviate it as $C(x_0, y_0, z_0, D_1)$. Evidently $C(x_0, y_0, z_0, D_1) \supset C(x_0, y_0, z_0, D_2)$, if D_1, D_2 are two open subsets of D such that $D_1 \supset D_2$ and $(x_0, y_0, z_0) \in D_2$. In the following, if an interval (open, closed, or half-open) is contained in the interval $\alpha(x_0, y_0, z_0, D_1) < x < \beta(x_0, y_0, z_0, D_1)$, then we say that the characteristic curve $C(x_0, y_0, z_0, D_1)$ and its projection on the (x, y) -plane, $y = \varphi(x, x_0, y_0, z_0, D_1)$, are defined for that interval.

Let us consider a continuous function $z(x, y)$ defined on G , which has $\partial z / \partial x$ and $\partial z / \partial y$ (not necessarily continuous), except at most at the points of an enumerable set, in G (in the following we suppose always that the above conditions are satisfied by $z(x, y)$). We denote by S the surface represented by $z = z(x, y)$. Then we obtain the following two theorems.

Theorem 1. *If $S \subset D$, and $z(x, y)$ satisfies (1) almost everywhere in G , then any characteristic curve in D which has a point in common with the surface S , is totally contained in S .*

Theorem 2. *If $S \subset D$, and $z(x, y)$ satisfies (1) almost everywhere in G and moreover if $z(x, y)$ has $\partial z / \partial y$ (not necessarily continuous) everywhere in G , then $z(x, y)$ is totally differentiable and satisfies (1) everywhere in G .*

Remark 1. If the domain G , where $z(x, y)$ is defined, is not the projection of D on the (x, y) -plane but is only a part of the projection, theorem 1, 2 hold, if we substitute D by the set of the points of D whose projections on the (x, y) -plane are contained in G .

Remark 2. In the premises of Theorem 2, the condition that $z(x, y)$ has $\partial z / \partial y$ everywhere in G , can not be omitted, as the following example shows it.

Example. $D = R^3$, $G =$ the whole (x, y) -plane, the differential equation is

$$\frac{\partial z}{\partial x} + zx \frac{\partial z}{\partial y} = 0$$

1) Cf. Kamke [1], § 16, Nr. 79, Satz 4.

and a solution $z(x, y)$ is implicitly defined by

$$y = \frac{1}{2}zx^2 + z^3.$$

$z(x, y)$ is one-valued, continuous and has $\partial z/\partial x$ in the whole (x, y) -plane, but has $\partial z/\partial y$ except at $(0, 0)$ and also satisfies (1) except at $(0, 0)$. For this example, the premises of Theorem 1 are fulfilled, but those of Theorem 2 are not fulfilled.

§ 1. Some lemmas

In this chapter, the notations are the same as in the introduction and we assume that $z(x, y)$ satisfies the premises of Theorem 1.

1. Let us denote by K the set of the points (x_0, y_0) of G such that the characteristic curve $C\{x_0, y_0, z(x_0, y_0), D\}$ is contained in S , in a neighbourhood of $\{x_0, y_0, z(x_0, y_0)\}$. We denote by F the set $\overline{G-K} \cdot G$ (by $\overline{G-K}$ we denote the closure of $G-K$ in the (x, y) -plane). Evidently F is closed in G .

We shall often use the following lemmas.

Lemma 1. *If the characteristic curve $C\{x_0, y_0, z(x_0, y_0), D\}$ is defined for $\alpha_0 \leq x \leq x_0$ (or $x_0 \leq x \leq \alpha_0$) ($\alpha_0 \neq x_0$) and its projection on the (x, y) -plane, $y = \varphi\{x, x_0, y_0, z(x_0, y_0), D\}$ is contained in K for $\alpha_0 < x \leq x_0$ (or $x_0 \leq x < \alpha_0$), then $C\{x_0, y_0, z(x_0, y_0), D\}$ is contained in S for $\alpha_0 \leq x \leq x_0$ (or $x_0 \leq x \leq \alpha_0$).*

Proof. We denote by ξ_0 the nearest point to α_0 among the points ξ in the interval $\alpha_0 \leq x \leq x_0$ (or $x_0 \leq x \leq \alpha_0$) such that $C\{x_0, y_0, z(x_0, y_0), D\}$ is contained in S for $\xi \leq x \leq x_0$ (or $x_0 \leq x \leq \xi$). Such ξ_0 exists by the continuity of $\varphi\{x, x_0, y_0, z(x_0, y_0), D\}$ and $\psi\{x, x_0, y_0, z(x_0, y_0), D\}$ as functions of x and by the continuity of $z(x, y)$. If the lemma were false, then $\xi_0 \neq \alpha_0$. We put $\eta_0 = \varphi\{\xi_0, x_0, y_0, z(x_0, y_0), D\}$. Then $C\{x_0, y_0, z(x_0, y_0), D\}$ passes through $\{\xi_0, \eta_0, z(\xi_0, \eta_0)\}$ and is contained in S in some neighbourhood of the point $\{\xi_0, \eta_0, z(\xi_0, \eta_0)\}$, since $(\xi_0, \eta_0) \in K$. This is inconsistent with the definition of ξ_0 and the lemma is proved.

Lemma 2. *Let us denote by D_+ an open subset of D . We denote by F' the set of the points $(x_0, y_0, z_0) (\in D)$ such that $C(x_0, y_0, z_0, D_+)$ is totally contained in S . Then F' is closed in D_+ .*

Proof. We take a point $(\xi_0, \eta_0, \zeta_0) \in \bar{F}' \cdot D_+$ (here we denote by \bar{F}'

the closure of F' in R^3). Then if $C(\xi_0, \eta_0, \varsigma_0, D_1)$ is defined for $\alpha_0 \leq x \leq \beta_0$, $C(x_0, y_0, z_0, D_1)$ is defined for $\alpha_0 \leq x \leq \beta_0$, for any point $(x_0, y_0, z_0) (\in F')$ in a neighbourhood of the point $(\xi_0, \eta_0, \varsigma_0)$ and

$$\begin{aligned}\varphi(x, x_0, y_0, z_0, D_1) &\longrightarrow \varphi(x, \xi_0, \eta_0, \varsigma_0, D_1) \\ \psi(x, x_0, y_0, z_0, D_1) &\longrightarrow \psi(x, \xi_0, \eta_0, \varsigma_0, D_1)\end{aligned}$$

uniformly in the interval $\alpha_0 \leq x \leq \beta_0$, as $(x_0, y_0, z_0) \rightarrow (\xi_0, \eta_0, \varsigma_0)$.²⁾ From this and by the continuity of $z(x, y)$, $C(\xi_0, \eta_0, \varsigma_0, D_1)$ is totally contained in S , that is, $(\xi_0, \eta_0, \varsigma_0) \in F'$, q. e. d.

§ 2. Proof of Theorem 1.

In this chapter, the notations are the same as in the introduction and § 1, and we assume that $z(x, y)$ satisfies the premises of Theorem 1.

2. Domain Q . If F is empty, that is, if $G = K$, we can conclude by Lemma 1 that the characteristic curve in D passing through any point of S is totally contained in S and the theorem is established. Suppose therefore, if possible, that $F \neq \emptyset$. We denote by H the enumerable set consisting of the points of G at which $z(x, y)$ is not derivable with respect to x and with respect to y simultaneously. If we denote by F_n , for each positive integer n , the set of the points (x, y) of G such that

$$\begin{aligned}|z(x+h, y) - z(x, y)| &\leq |h|n \\ |z(x, y+k) - z(x, y)| &\leq |k|n\end{aligned}$$

whenever $|h|, |k| \leq 1/n$, $(x+h, y), (x, y+k) \in G$, then the sets F_n cover $G-H$ and each of the sets F_n is closed in G by dint of the continuity of $z(x, y)$.

If a point (x_0, y_0) of G has an open neighbourhood V in G such that every point of V belongs to K except (x_0, y_0) , then by Lemma 1, $C\{x_0, y_0', z(x_0, y_0'), (V \times R) \cdot D\}$ (we denote by $V \times R$ the set of the points of R^3 whose projections on the (x, y) -plane belong to V) is totally contained in S , when $(x_0, y_0') \in V$ and $y_0 \neq y_0'$. Hence by Lemma 2, $C\{x_0, y_0, z(x_0, y_0), (V \times R) \cdot D\}$ is totally contained in S , since $\{x_0, y_0', z(x_0, y_0')\} \rightarrow \{x_0, y_0, z(x_0, y_0)\}$ (as $y_0' \rightarrow y_0$) by the continuity of $z(x, y)$. Therefore also (x_0, y_0) belongs to K , that is, F can contain no isolated point, F is perfect in G .

Thus $F-H$ is not empty and of the second category in itself as it

2) Cf. Kamke, § 17, Nr. 84, Satz 3.

is a G_δ set in R^2 . Therefore there must exist a positive integer N and an open square $Q: |x-a| < L, |y-b| < L$ such that $0 < L < 1/(2N)$, $\bar{Q} \subset G$, $(a, b) \in (F-H) \cdot Q \subset F_N$ (\bar{Q} is the closure of Q in the (x, y) -plane). Then $(a, b) \in F \cdot Q \subset F_N$, since the closure of $F-H$ in G is F by the perfectness of F in G and the enumerability of H . Hence if $(x, y) \in F \cdot Q$ and $(x+h, y), (x, y+k) \in Q$,

$$\begin{cases} |z(x+h, y) - z(x, y)| \leq |h|N \\ |z(x, y+k) - z(x, y)| \leq |k|N, \end{cases} \quad (4)$$

by the definition of F_N and Q .

3. **Domains Q_1, Q_2, Q_3 .** We take an open cube $Q_1: |x-a| < L_1, |y-b| < L_1, |z-z(a, b)| < L_1$ such that $0 < L_1 \leq L$ and $\bar{Q}_1 \subset D$ (by \bar{Q}_1 we denote the closure of Q_1 in R^3). Then by the continuity of f_y, f_z, g_y, g_z , there is a positive number M_1 such that

$$|f_y|, |f_z|, |g_y|, |g_z| < M_1 \quad \text{in } Q_1.$$

Again we take a parallelepiped $Q_2: |x-a| < L_2, |y-b| < L_2, |z-z(a, b)| < L_3$, which satisfies the following conditions:

- i) $0 < L_2, L_3 \leq L_1$, that is, $Q_2 \subset Q_1$,
- ii) $|z(x, y) - z(a, b)| < L_3$ for $|x-a| < L_2, |y-b| < L_2$, that is, S is contained in Q_2 for $|x-a| < L_2, |y-b| < L_2$,
- iii) any characteristic curve $C(x_0, y_0, z_0, Q_1)$ where $(x_0, y_0, z_0) \in Q_2$ is defined for $|x-a| < L_2$,

$$\text{iv)} \quad \begin{cases} \frac{\exp(4M_1L_2) + 2N\{\exp(4M_1L_2) - 1\}}{2N\{2 - \exp(4M_1L_2)\} - \{\exp(4M_1L_2) - 1\}} \leq \frac{2}{3N} \\ 2N\{2 - \exp(4M_1L_2)\} - \{\exp(4M_1L_2) - 1\} > 0. \end{cases} \quad (5)$$

The conditions i), iii), iv) can be realized, if we take L_2, L_3 sufficiently small (iii) by the boundedness of f, g in \bar{Q}) and the condition ii) can be realized if we take L_2 still smaller (by the continuity of $z(x, y)$).

We denote by Q_3 the open square: $|x-a| < L_2, |y-b| < L_2$. Evidently $Q_3 \subset Q$. If we take any point (x_0, y_0) belonging to Q_3 , $\{x_0, y_0, z(x_0, y_0)\}$ belongs to Q_2 and $C\{x_0, y_0, z(x_0, y_0), Q_1\}$ is defined for $|x-a| < L_2$, by the conditions ii), iii) on Q_2 . We denote by K_1 the set of the points (x_0, y_0) of Q_3 such that the curve $y = \varphi\{x, x_0, y_0, z(x_0, x_0), Q_2\}$ has no point in common with $F \cdot Q_3$. We denote by E the set $\bar{K}_1 \cdot Q_3$ (\bar{K}_1 is the closure of K_1 in the (x, y) -plane). If $(x_0, y_0) \in K_1$, then by Lemma 1, $C\{x_0, y_0, z(x_0, y_0), Q_2\}$ is totally contained in S . Therefore, by Lemma 2 and by the continuity of $z(x, y)$, $C\{x_0, y_0, z(x_0, y_0), Q_2\}$ is totally contained

in S , if $(x_0, y_0) \in E$. Hence, $(a, b) \notin F$, if $E = Q_3$. But since $(a, b) \in F$, so $Q_3 - E$ is not empty. Evidently $Q_3 - E$ is open.

We shall prove that $(Q_3 - E) \cdot F$ is also not empty. We take a point $(c, d) \in Q_3 - E$. If $(c, d) \in F$, the proposition is already proved. Therefore we assume that $(c, d) \notin F$. The curve $y = \varphi\{x, c, d, z(c, d), Q_2\}$ has at least a point in common with $F \cdot Q_3$, by the definition of E , and obviously

$$L_2 + a \geq \beta\{c, d, z(c, d), Q_2\} > c > \alpha\{c, d, z(c, d), Q_2\} \geq L_2 - a.$$

Hence, as $F \cdot Q$ is closed in Q , there is the nearest point of F to (c, d) on the portion of the curve $y = \varphi\{x, c, d, z(c, d), Q_2\}$ for $\alpha\{c, d, z(c, d), Q_2\} < x \leq c$ or for $c \leq x < \beta\{c, d, z(c, d), Q_2\}$. We denote it by (a_1, b_1) . If $(a_1, b_1) \notin E$, then $(a_1, b_1) \in (Q_3 - E) \cdot F$ and the above proposition is established. Suppose therefore that $(a_1, b_1) \in E$.

Again by Lemma 1, $C\{c, d, z(c, d), Q_2\}$ is contained in S for the interval $a_1 \leq x \leq c$ or $c \leq x \leq a_1$ and so $C\{c, d, z(c, d), Q_2\} = C\{a_1, b_1, z(a_1, b_1), Q_2\}$. On the other hand, we assume that $(a_1, b_1) \in E$, so in any neighbourhood of the point (a_1, b_1) , there is a point (x_0, y_0) which belongs to K_1 . As $C\{a_1, b_1, z(a_1, b_1), Q_2\} (= C\{c, d, z(c, d), Q_2\})$ is defined for $a_1 \leq x \leq c$ or $c \leq x \leq a_1$, $C\{x_0, y_0, z(x_0, y_0), Q_2\}$ is also defined for $a_1 \leq x \leq c$ or $c \leq x \leq a_1$ if $(x_0, y_0) (\in K_1)$ belongs to a neighbourhood of (a_1, b_1) , and $\varphi\{x, x_0, y_0, z(x_0, y_0), Q_2\} \rightarrow \varphi\{x, c, d, z(c, d), Q_2\} (= \varphi\{x, a_1, b_1, z(a_1, b_1), Q_2\})$ uniformly in the interval $a_1 \leq x \leq c$ or $c \leq x \leq a_1$, as $(x_0, y_0) (\in K_1) \rightarrow (a_1, b_1)^{3)}$ (since $\{x_0, y_0, z(x_0, y_0)\} \rightarrow \{a_1, b_1, z(a_1, b_1)\}$ as $(x_0, y_0) \rightarrow (a_1, b_1)$ by the continuity of $z(x, y)$). On the other hand, as it is proved before, $C\{x_0, y_0, z(x_0, y_0), Q_2\}$ is totally contained in S , if $(x_0, y_0) \in K_1$. From this and by the definition of K_1 , the curve $y = \varphi\{x, x_0, y_0, z(x_0, y_0), Q_2\}$ is totally contained in K_1 , if $(x_0, y_0) \in K_1$. Therefore there is a point which belongs to K_1 in any neighbourhood of (c, d) , that is, $(c, d) \in E$. But this is a contradiction, since $(c, d) \in Q_3 - E$. Thus in any case, $F \cdot (Q_3 - E)$ is not empty and there is at least one point $(a_1, b_1) \in F \cdot (Q_3 - E)$.

4. Domain Q_4 . As $Q_3 - E$ is open and $F \cdot (Q_3 - E)$ is not empty, we can take an open square $Q_4: |x - a_1| < L_4, |y - b_1| < L_4$ such that $Q_4 \subset Q_3 - E$ and $(a_1, b_1) \in F \cdot Q_4$. Obviously $Q_4 \subset Q_3 \subset Q$.

We take any pair of points $(x_1, \bar{y}_1), (x_1, y_1)$ with the same x coordinate, in Q_4 . We shall prove

$$|z(x_1, \bar{y}_1) - z(x_1, y_1)| \leq 2N |\bar{y}_1 - y_1|.$$

3) Cf. Kamke, §17, Nr. 84, Satz 3.

Suppose, if possible, that

$$|z(x_1, \bar{y}_1) - z(x_1, y_1)| > 2N|\bar{y}_1 - y_1| \quad (6)$$

If $(x_1, \bar{y}_1) \in F$ or $(x_1, y_1) \in F$, then $|z(x_1, \bar{y}_1) - z(x_1, y_1)| \leq N|\bar{y}_1 - y_1|$, by (4) and as $Q_4 \subset Q$. So we may assume that $(x_1, \bar{y}_1) \notin F$, $(x_1, y_1) \notin F$ and $y_1 < \bar{y}_1$.

By the way of the construction of Q_1, Q_2, Q_3, Q_4 , the characteristic curves $C\{x_1, y_1, z(x_1, y_1), Q_1\}$ and $C\{x_1, \bar{y}_1, z(x_1, \bar{y}_1), Q_1\}$ are defined for $|x-a| < L_2$. So their projection on the (x, y) -plane $y = \varphi\{x, x_1, y_1, z(x_1, y_1), Q_1\}$ and $y = \varphi\{x, x_1, \bar{y}_1, z(x_1, \bar{y}_1), Q_1\}$ are defined for $|x-a| < L_2$ and contained in Q .

As $(x_1, y_1), (x_1, \bar{y}_1) \in Q_4 \subset Q_3 - E$, and $F \cdot Q$ is closed in Q , on either side of x_1 there is the nearest x to x_1 in the interval $|x-a| < L_2$ such that either $(x, \varphi\{x, x_1, y_1, z(x_1, y_1), Q_1\}) \in F \cdot Q$ or $(x, \varphi\{x, x_1, \bar{y}_1, z(x_1, \bar{y}_1), Q_1\}) \in F \cdot Q$ is satisfied. We denote it by x_2 and $\varphi\{x_2, x_1, y_1, z(x_1, y_1), Q_1\}$, $\varphi\{x_2, x_1, \bar{y}_1, z(x_1, \bar{y}_1), Q_1\}$, $\psi\{x_2, x_1, y_1, z(x_1, y_1), Q_1\}$, $\psi\{x_2, x_1, \bar{y}_1, z(x_1, \bar{y}_1), Q_1\}$ respectively by y_2, \bar{y}_2, z_2 and \bar{z}_2 . Then by Lemma 1, $C\{x_1, y_1, z(x_1, y_1), Q_1\}$, $C\{x_1, \bar{y}_1, z(x_1, \bar{y}_1), Q_1\}$ are contained in S for the interval $x_1 \leq x \leq x_2$ or $x_2 \leq x \leq x_1$. Hence $z_2 = z(x_2, y_2)$, $\bar{z}_2 = z(x_2, \bar{y}_2)$. Moreover $|x_2 - a| < L_2$, $(x_2, y_2), (x_2, \bar{y}_2) \in Q$ and either $(x_2, y_2) \in F \cdot Q$ or $(x_2, \bar{y}_2) \in F \cdot Q$.

In the following we denote by P_x , the plane parallel to the (y, z) -plane which cuts the x -axis at the point whose x coordinate is x .

By the way of the construction of Q_1, Q_2 and by the continuity of f_y, f_z, g_y , and g_z , $y = \varphi(x, x_1, \eta, \varsigma, Q_1)$ and $z = \psi(x, x_1, \eta, \varsigma, Q_1)$ define a bicontinuous one to one mapping $A_x: (\eta, \varsigma) \rightarrow (y, z)$ of the domain $|\eta - b| < L_2$, $|\varsigma - z(a, b)| < L_3$ on the plane P_{x_1} (its y, z coordinates we denote by η, ς respectively), onto some domain on the plane P_x , for any fixed x in the interval $|x-a| < L_2$ ⁴⁾ (in the following, (η, ς) will always belong to the domain: $|\eta - b| < L_2$, $|\varsigma - z(a, b)| < L_3$). Moreover this domain on P_x is contained in Q_1 and continuous $\partial y / \partial \eta, \partial y / \partial \varsigma, \partial z / \partial \eta, \partial z / \partial \varsigma$ exist.⁵⁾ From (2), (3) we have⁶⁾

$$\begin{aligned} \frac{d}{dx} \left(\frac{\partial y}{\partial \eta} \right) &= f_y \frac{\partial y}{\partial \eta} + f_z \frac{\partial z}{\partial \eta} & \frac{d}{dx} \left(\frac{\partial z}{\partial \eta} \right) &= g_y \frac{\partial y}{\partial \eta} + g_z \frac{\partial z}{\partial \eta} \\ \frac{d}{dx} \left(\frac{\partial y}{\partial \varsigma} \right) &= f_y \frac{\partial y}{\partial \varsigma} + f_z \frac{\partial z}{\partial \varsigma} & \frac{d}{dx} \left(\frac{\partial z}{\partial \varsigma} \right) &= g_y \frac{\partial y}{\partial \varsigma} + g_z \frac{\partial z}{\partial \varsigma} \end{aligned}$$

for $|x-a| < L_2$. In Q_1 , $|f_y|, |f_z|, |g_y|, |g_z| < M$, so

4) Cf. Kamke, §17, Nr. 84, Satz 3.

5), 6) Cf. Kamke, §18, Nr. 87, Satz 1 and its "zusatz".

$$\left| \frac{d}{dx} \left(\frac{\partial y}{\partial \eta} - 1 \right) \right| + \left| \frac{d}{dx} \left(\frac{\partial z}{\partial \eta} \right) \right| \leq 2M_1 \left(\left| \frac{\partial y}{\partial \eta} - 1 \right| + \left| \frac{\partial z}{\partial \eta} \right| + 1 \right)$$

$$\left| \frac{d}{dx} \left(\frac{\partial y}{\partial \varsigma} \right) \right| + \left| \frac{d}{dx} \left(\frac{\partial z}{\partial \varsigma} - 1 \right) \right| \leq 2M_1 \left(\left| \frac{\partial y}{\partial \varsigma} \right| + \left| \frac{\partial z}{\partial \varsigma} - 1 \right| + 1 \right)$$

for $|x-a| < L_2$. Hence⁷⁾

$$\left| \frac{\partial y}{\partial \eta} - 1 \right| + \left| \frac{\partial z}{\partial \eta} \right| \leq \exp(2M_1|x-x_1|) - 1$$

$$\left| \frac{\partial y}{\partial \varsigma} \right| + \left| \frac{\partial z}{\partial \varsigma} - 1 \right| \leq \exp(2M_1|x-x_1|) - 1$$

for $|x-a| < L_2$, as $\partial y/\partial \eta - 1$, $\partial z/\partial \eta$, $\partial y/\partial \varsigma$, $\partial z/\partial \varsigma - 1$ vanish at $x = x_1$. As $|x_2-a| < L_2$, the above inequalities subsist for $x = x_2$. Hence as $|x_1-a| < L_2$, $|x_2-a| < L_2$ and by (5) $2 - \exp(4M_1L_2) > 0$,

$$\left\{ \begin{array}{l} \left| \frac{\partial z}{\partial \eta} \right|, \left| \frac{\partial y}{\partial \varsigma} \right| \leq \exp(4M_1L_2) - 1 \\ 0 < 2 - \exp(4M_1L_2) \leq \frac{\partial z}{\partial \varsigma}, \frac{\partial y}{\partial \eta} \leq \exp(4M_1L_2) \end{array} \right. \quad (7)$$

for $x = x_2$.

By the way of the construction of Q_2 , the segment T of straight line on the plane P_{x_1} :

$$\varsigma - z(x_1, y_1) = t(\eta - y_1) \quad y_1 \leq \eta \leq \bar{y}_1$$

where

$$t = \frac{z(x_1, \bar{y}_1) - z(x_1, y_1)}{\bar{y}_1 - y_1}$$

which joins the points $\{y_1, z(x_1, y_1)\}$ and $\{\bar{y}_1, z(x_1, \bar{y}_1)\}$ is totally contained in the domain $|\eta - b| < L_2$, $|\varsigma - z(a, b)| < L_3$ on the plane P_{x_1} . By (6)

$$|t| > 2N. \quad (8)$$

We denote by T' the image of T on the plane P_{x_2} by the mapping A_{x_2} . T' is represented by

$$y = \varphi\{x_2, x_1, \eta, z(x_1, y_1) + t(\eta - y_1), Q_1\} = \lambda(\eta)$$

$$z = \psi\{x_2, x_1, \eta, z(x_1, y_1) + t(\eta - y_1), Q_1\} = \mu(\eta)$$

$$\bar{y}_1 \geq \eta \geq y_1 \quad (\eta \text{ is taken as parameter})$$

and $y_2 = \lambda(y_1)$, $z_2 = \mu(y_1)$, $\bar{y}_2 = \lambda(\bar{y}_1)$, $\bar{z}_2 = \mu(\bar{y}_1)$. As it can be shown

7) Cf. Kamke, § 17, Nr. 85, Hilfssatz 3.

easily, $d\lambda/d\eta$, $d\mu/d\eta$ exist and are continuous, and by (7), (8), (5)

$$\begin{aligned} t \neq 0, \left| \frac{1}{t} \frac{d\mu}{d\eta} \right| &= \left| \frac{\partial z}{\partial \eta} \frac{1}{t} + \frac{\partial z}{\partial \varsigma} \right| \geq \left| \frac{\partial z}{\partial \varsigma} \right| - \left| \frac{1}{t} \frac{\partial z}{\partial \eta} \right| \geq 2 - \exp(4M_1 L_2) \\ &- \frac{\exp(4M_1 L_2) - 1}{2N} = \frac{2N \{2 - \exp(4M_1 L_2)\} - \{\exp(4M_1 L_2) - 1\}}{2N} > 0, \\ \left| \frac{1}{t} \frac{d\lambda}{d\eta} \right| &= \left| \frac{\partial y}{\partial \eta} \frac{1}{t} + \frac{\partial y}{\partial \varsigma} \right| \leq \frac{\exp(4M_1 L_2) + \exp(4M_1 L_2) - 1}{2N} \\ &= \frac{\exp(4M_1 L_2) + 2N \{\exp(4M_1 L_2) - 1\}}{2N}. \end{aligned}$$

Hence by (5)

$$\begin{aligned} \frac{d\mu}{d\eta} \neq 0, \left| \frac{d\lambda}{d\eta} \middle/ \frac{d\mu}{d\eta} \right| \\ \leq \frac{\exp(4M_1 L_2) + 2N \{\exp(4M_1 L_2) - 1\}}{2N \{2 - \exp(4M_1 L_2)\} - \{\exp(4M_1 L_2) - 1\}} \leq \frac{2}{3N} \end{aligned}$$

Therefore we can represent T' as

$$y = \gamma(z) \quad z_2 \geq z \geq \bar{z}_2 \quad \text{or} \quad \bar{z}_2 \geq z \geq z_2$$

($z_2 \neq \bar{z}_2$ as $d\mu/d\eta \neq 0$) and $\gamma(z)$ satisfies following conditions :

$y_2 = \gamma(z_2)$, $\bar{y}_2 = \gamma(\bar{z}_2)$, continuous $d\gamma/dz$ exist and

$$|d\gamma/dz| \leq 2/(3N) \quad \text{for} \quad z_2 \geq z \geq \bar{z}_2 \quad \text{or} \quad \bar{z}_2 \geq z \geq z_2.$$

Hence

$$\left| \frac{\bar{y}_2 - y_2}{\bar{z}_2 - z_2} \right| \leq \frac{2}{3N} \quad \bar{z}_2 \neq z_2.$$

As it is proved before, $\bar{z}_2 = z(x_2, \bar{y}_2)$ and $z_2 = z(x_2, y_2)$. So

$$\left| \frac{z(x_2, \bar{y}_2) - z(x_2, y_2)}{\bar{y}_2 - y_2} \right| \geq \frac{3N}{2}, \quad \bar{y}_2 \neq y_2.$$

But $(x_2, \bar{y}_2), (x_2, y_2) \in Q$ and either $(x_2, \bar{y}_2) \in F \cdot Q$ or $(x_2, y_2) \in F \cdot Q$. This contradicts (4).

Thus we have proved

$$|z(x_1, \bar{y}_1) - z(x_1, y_1)| \leq 2N |\bar{y}_1 - y_1| \quad (9)$$

for any pair of points $(x_1, \bar{y}_1), (x_1, y_1)$ in Q_4 with the same x coordinate.

5. Domains Q_5, Q_6 . We now consider the following ordinary differential equation whose right side is defined and continuous on G ,

$$\frac{dy}{dx} = f\{x, y, z(x, y)\}. \quad (10)$$

$f\{x, y, z(x, y)\}$ is defined and continuous on $\bar{Q}_4 \subset \bar{Q} \subset G$, so there is a positive M such that

$$|f\{x, y, z(x, y)\}| < M \quad \text{in } Q_4. \quad (11)$$

In Q_1 , $|f_y|, |f_z| < M_1$, so $|f(x, \bar{y}, \bar{z}) - f(x, y, z)| \leq M_1(|\bar{y} - y| + |\bar{z} - z|)$ if $(x, \bar{y}, \bar{z}), (x, y, z) \in Q_1$. On the other hand $\{x, y, z(x, y)\} \in Q_2 \subset Q_1$. If $(x, y) \in Q_4 \subset Q_3$. Therefore by (9) if $(x_1, y_1), (x_1, \bar{y}_1) \in Q_4$,

$$\begin{aligned} & |f\{x_1, \bar{y}_1, z(x_1, \bar{y}_1)\} - f\{x_1, y_1, z(x_1, y_1)\}| \leq M_1(|\bar{y}_1 - y_1| + |z(x_1, \bar{y}_1) \\ & - z(x_1, y_1)|) \leq M_1(1 + 2N)|\bar{y}_1 - y_1|. \end{aligned} \quad (12)$$

Hence the right side of (10) satisfies Lipschitz condition on Q_4 . Let us write $l = L_4/(M+1)$. We denote by η_1 any number such that $|\eta_1 - b_1| \leq l$. Then $\eta_1 + lM \leq b_1 + L_4$, $\eta_1 - lM \geq b_1 - L_4$. Thus for any η_1 there exists a unique solution of (10) defined for $|x - a_1| < l$ which passes through (a_1, η_1) and lies in Q_4 .⁸⁾ We denote it by $y = \chi(x, \eta_1)$. Hence if we denote by Q_5 the domain defined by:

$$\chi(x, b_1 - l) < y < \chi(x, b_1 + l) \quad |x - a_1| < l,$$

the curves $y = \chi(x, \eta_1)$ fill up Q_5 simple-fold, when η_1 takes all values in the open interval $|\eta_1 - b_1| < l$,⁹⁾ and $(a_1, b_1) \in Q_5 \subset Q_4$.

By (11), (12), for any two $\eta_1, \bar{\eta}_1$ in the interval $|\eta_1 - b_1| < l$ and any x in the interval $|x - a_1| < l$,¹⁰⁾

$$\begin{aligned} & |\bar{\eta}_1 - \eta_1| \leq |\chi(x, \bar{\eta}_1) - \chi(x, \eta_1)| \exp \{M_1(1 + 2N)|x - a_1|\} \\ & \leq |\chi(x, \bar{\eta}_1) - \chi(x, \eta_1)| \exp \{M_1(1 + 2N)l\} \end{aligned} \quad (13)$$

We denote by Q_6 the open square: $|\xi_1 - a_1| < l$, $|\eta_1 - b_1| < l$ in the (ξ_1, η_1) -plane. We denote by A the one to one mapping of Q_6 onto Q_5 defined by

$$x = \xi_1 \quad y = \chi(\xi_1, \eta_1).$$

Then A is bicontinuous¹¹⁾ and by (13) we can easily conclude that A_1^{-1} maps any null set in Q_5 to a null set in Q_6 .

6. $z(x, y)$ in the domain Q_5 . We take any pair of points $(x_3, y_3), (x_4, y_4)$ belonging to Q_5 . Then $\chi(x_4, \eta_4) = y_4$ for an η_4 in the open interval $|\eta_1 - b_1| < l$. Now we denote by (x_5, y_5) :

8), 9) Cf. Kamke, § 6, Nr. 30, Satz 1, § 10, Nr. 47, Satz 4, and § 12, Nr. 54, Satz 3.

10) Cf. Kamke, § 11, Nr. 51, Satz 1.

11) Cf. Kamke, § 10, Nr. 47, Satz 4.

Case I. The nearest point of F to (x_4, y_4) on the portion of the continuous curve $y = \chi(x, \eta_4)$ for $x_3 \leq x \leq x_4$ or $x_4 \leq x \leq x_3$, if it contains some points of F (such (x_5, y_5) exists in this case, as $F \cdot Q$ is closed in Q),

Case II. The point $x_5 = x_3$, $y_5 = \chi(x_3, \eta_4)$, if that portion contains no point of F .

The characteristic curve $C\{x_4, y_4, z(x_4, y_4), Q_1\}$ is defined for $|x - a_1| < l$, as $(x_4, y_4) \in Q_3$ and the interval $|x - a_1| < l$ is contained in the interval $|x - a| < L_2$. We shall prove that in both Cases the portion of the curve $y = \chi(x, \eta_4)$ for the interval $x_4 \leq x \leq x_5$ or $x_5 \leq x \leq x_4$ is contained in the curve $y = \varphi\{x, x_4, y_4, z(x_4, y_4), Q_1\}$ and the portion of $C\{x_4, y_4, z(x_4, y_4), Q_1\}$ for the interval $x_4 \leq x \leq x_5$ or $x_5 \leq x \leq x_4$ is contained in S . If $(x_4, y_4) \in F \cdot Q_5$, then $x_5 = x_4$, so the proposition is obvious. Hence we assume that $(x_4, y_4) \notin F \cdot Q_5$.

Suppose, if possible, that the proposition were false. We denote by x_6 the nearest point to x_5 among the points ξ in the interval $x_5 \leq x \leq x_4$ or $x_4 \leq x \leq x_5$ such that: the portion of the curve $y = \chi(x, \eta_4)$ for the interval $x_4 \leq x \leq \xi$ or $\xi \leq x \leq x_4$ is contained in the curve $y = \varphi\{x, x_4, y_4, z(x_4, y_4), Q_1\}$ and the portion of $C\{x_4, y_4, z(x_4, y_4), Q_1\}$ for the same interval $x_4 \leq x \leq \xi$ or $\xi \leq x \leq x_4$ is contained in S . Such x_6 exists by the continuity of $\chi(x, \eta_4)$, $\varphi\{x, x_4, y_4, z(x_4, y_4), Q_1\}$, $\psi\{x, x_4, y_4, z(x_4, y_4), Q_1\}$ and $z(x, y)$. We denote $\chi(x_6, \eta_4)$ by y_6 . Evidently $x_5 \neq x_6$, as the above proposition is supposed false. By the definition of (x_5, y_5) , (x_6, y_6) , $C\{x_4, y_4, z(x_4, y_4)\}$ passes through the point $\{x_6, y_6, z(x_6, y_6)\}$ and $(x_6, y_6) \notin F$. Hence $C\{x_4, y_4, z(x_4, y_4)\}$ is contained in S in some neighbourhood of the point $\{x_6, y_6, z(x_6, y_6)\}$. Also from this, the curve $y = \varphi\{x, x_4, y_4, z(x_4, y_4), Q_1\}$ passes through (x_6, y_6) and satisfies (10) in some neighbourhood of $x = x_6$. Hence, the curve $y = \chi(x, \eta_4)$ is contained in the curve $y = \varphi\{x, x_4, y_4, z(x_4, y_4), Q_1\}$ in some neighbourhood of (x_6, y_6) , by the uniqueness of the solution in Q_4 of (10) passing through (x_6, y_6) . These are inconsistent with the definition of (x_6, y_6) . The above proposition is thus established.

By the above proposition, the curve $y = \chi(x, \eta_4)$ $z = z\{x, \chi(x, \eta_4)\}$ satisfies (2), (3) and is contained in Q_1 for the interval $x_4 \leq x \leq x_5$ or $x_5 \leq x \leq x_4$ (if $x_4 \neq x_5$). On \bar{Q}_1 , $g(x, y, z)$ is defined and continuous. Hence there is a positive M_2 such that $|g(x, y, z)| \leq M_2$ on Q_1 . Therefore

$$\frac{dz\{x, \chi(x, \eta_4)\}}{dx} = |g[x, \chi(x, \eta_4), z\{x, \chi(x, \eta_4)\}]| \leq M_2$$

for $x_5 \leq x \leq x_4$ or $x_4 \leq x \leq x_5$.

$$\text{Thus} \quad |z(x_5, y_5) - z(x_4, y_4)| \leq M_2 |x_5 - x_4| \leq M_2 |x_3 - x_4| \quad (14)$$

(if $x_4 = x_5$, this is obvious).

Now $y = \chi(x, \eta_4)$ is a solution of (10) contained in Q_4 and $|f\{x, y, z(x, y)\}| < M$ on Q_4 .

Thus $|y_4 - y_5| = |\chi(x_4, \eta_4) - \chi(x_5, \eta_4)| \leq M|x_4 - x_5| \leq M|x_3 - x_4|$.

Hence $|y_3 - y_5| \leq |y_3 - y_4| + |y_4 - y_5| \leq |y_3 - y_4| + M|x_3 - x_4|$. (15)

We have

$$|z(x_3, y_5) - z(x_5, y_5)| \leq N|x_3 - x_5| \leq N|x_3 - x_4| \quad (16)$$

in Case I, by (4) and as $(x_5, y_5) \in F \cdot Q$, $(x_3, y_5) \in Q$, and in Case II, simply as $x_3 = x_5$. Also we have by (9)

$$|z(x_3, y_3) - z(x_3, y_5)| \leq 2N|y_3 - y_5| \quad (17)$$

as $(x_3, y_3), (x_3, y_5) \in Q_4$.

By (14), (15), (16), (17),

$$\begin{aligned} |z(x_3, y_3) - z(x_4, y_4)| &\leq |z(x_3, y_3) - z(x_3, y_5)| + |z(x_3, y_5) - z(x_5, y_5)| \\ &\quad + |z(x_5, y_5) - z(x_4, y_4)| \leq 2N|y_3 - y_5| + N|x_3 - x_4| + M_2|x_3 - x_4| \\ &\leq 2N|y_3 - y_4| + (2NM + N + M_2)|x_3 - x_4| \\ &\leq (2N + 2NM + M_2)(|y_3 - y_4| + |x_3 - x_4|). \end{aligned}$$

Hence if we denote $2NM + 2N + M_2$ by M_3 ,

$$\limsup_{(x, y) \rightarrow (x_3, y_3)} \frac{|z(x, y) - z(x_3, y_3)|}{|x - x_3| + |y - y_3|} \leq M_3 \quad (18)$$

whenever $(x_3, y_3) \in Q_5$.

7. *Completion of the proof.* From (18), $z(x, y)$ is totally differentiable almost everywhere in Q_5 , by Stepanoff's theorem on almost everywhere total differentiability.¹²⁾ Moreover $z(x, y)$ fulfills (1) almost everywhere in G and, as we have seen in section 5, A_1^{-1} maps any null set in Q_5 to a null set in Q_6 . Hence if we write $\zeta_1(\xi_1, \eta_1) = z\{\xi_1, \chi(\xi_1, \eta_1)\}$ for $(\xi_1, \eta_1) \in Q_6$,

$$\left\{ \begin{aligned} \frac{\partial}{\partial \xi_1} \zeta_1(\xi_1, \eta_1) &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \frac{\partial \chi}{\partial \xi_1} \\ &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} f\{\xi_1, \chi(\xi_1, \eta_1), \zeta_1(\xi_1, \eta_1)\} = g\{(\xi_1, \chi(\xi_1, \eta_1), \zeta_1(\xi_1, \eta_1))\} \end{aligned} \right. \quad (19)$$

almost everywhere in Q_6 .

12) Cf. Saks [3], pp. 238-239.

parallelepiped $R_2: |x-x_1| < r_2, |y-y_1| < r_2, |z-z_1| < r_3$ such that

- i) $r_2, r_3 \leq r_1$, that is, $R_2 \subset R_1$,
- ii) $|z(x, y) - z(x_1, y_1)| < r_3$ for $|x-x_1| < r_2, |y-y_1| < r_2$,
- iii) $C(x_0, y_0, z_0, R_1)$ is defined for the interval $|x-x_1| < r_2$ whenever $(x_0, y_0, z_0) \in R_2$.

i), iii) can be realized if we take r_2, r_3 sufficiently small (iii) by the boundedness of f, g in R_1) and ii) can be realized if we take r_2 still smaller. We denote by R_3 the open square: $|x-x_1| < r_2, |y-y_1| < r_2$.

If we take any point (x_2, y_2) which belongs to R_3 and denote $z(x_2, y_2)$ by z_2 , then by ii) and iii), $(x_2, y_2, z_2) \in R_2$ and $C(x_2, y_2, z_2, R_1)$ is defined for $|x-x_1| < r_2$. We denote $\varphi(x_1, x_2, y_2, z_2, R_1)$, $\psi(x_1, x_2, y_2, z_2, R_1)$ by y_3, z_3 respectively. Then $z_3 = z(x_1, y_3)$, since $C(x_2, y_2, z_2, R_1)$ is totally contained in S by Theorem 1. By the continuity of $z(x, y)$ and of $\varphi(x, x_0, y_0, z_0, R_1)$, $\psi(x, x_0, y_0, z_0, R_1)$ with respect to all the arguments x, x_0, y_0, z_0 ,¹³⁾

$$\begin{cases} \varphi(x, x_2, y_2, z_2, R_1) \longrightarrow \varphi(x_1, x_1, y_1, z_1, R_1) = y_1 \\ \psi(x, x_2, y_2, z_2, R_1) \longrightarrow \psi(x_1, x_1, y_1, z_1, R_1) = z_1 \\ \text{as } x \rightarrow x_1, x_2 \rightarrow x_1, y_2 \rightarrow y_1. \end{cases} \quad (23)$$

Hence by the continuity of $f(x, y, z)$, $g(x, y, z)$,

$$\begin{aligned} f\{x, \varphi(x, x_2, y_2, z_2, R_1), \psi(x, x_2, y_2, z_2, R_1)\} &\longrightarrow f(x_1, y_1, z_1), \\ g\{x, \varphi(x, x_2, y_2, z_2, R_1), \psi(x, x_2, y_2, z_2, R_1)\} &\longrightarrow g(x_1, y_1, z_1) \\ \text{as } x \rightarrow x_1, x_2 \rightarrow x_1, y_2 \rightarrow y_1. \end{aligned}$$

On the other hand, by (2), (3)

$$\begin{aligned} y_2 - y_3 &= \int_{x_1}^{x_2} f\{x, \varphi(x, x_2, y_2, z_2, R_1), \psi(x, x_2, y_2, z_2, R_1)\} dx \\ z_2 - z_3 &= \int_{x_1}^{x_2} g\{x, \varphi(x, x_2, y_2, z_2, R_1), \psi(x, x_2, y_2, z_2, R_1)\} dx. \end{aligned}$$

Therefore we have

$$\begin{cases} y_2 - y_3 = (x_2 - x_1)\{f(x_1, y_1, z_1) + \rho_1(x_2, y_2)\} \\ z_2 - z_3 = (x_2 - x_1)\{g(x_1, y_1, z_1) + \rho_2(x_2, y_2)\} \\ \rho_1(x_2, y_2), \rho_2(x_2, y_2) \rightarrow 0 \text{ as } (x_2, y_2) \rightarrow (x_1, y_1). \end{cases} \quad (24)$$

By the assumption, $z(x, y)$ has $\partial z / \partial y$ at (x_1, y_1) and by (23) $y_3 = \varphi(x_1, x_2, y_2, z_2, R_1) \rightarrow y_1$ as $(x_2, y_2) \rightarrow (x_1, y_1)$. Hence we have

13) Cf. Kamke [1], § 17, Nr. 84, Satz 3.

Also by (18) (if we write $x_3 = \xi_3$, $y_3 = \chi(\xi_3, \eta_3)$)

$$\left\{ \begin{aligned} \limsup_{\xi_1 \rightarrow \xi_3} \frac{|\varsigma_1(\xi_1, \eta_3) - \varsigma_1(\xi_3, \eta_3)|}{|\xi_1 - \xi_3|} &\leq \left(\limsup_{(x, y) \rightarrow (x_3, y_3)} \frac{|z(x, y) - z(x_3, y_3)|}{|x - x_3| + |y - y_3|} \right) \\ &\times \left(\limsup_{\xi_1 \rightarrow \xi_3} \frac{|\xi_1 - \xi_3| + |\chi(\xi_1, \eta_3) - \chi(\xi_3, \eta_3)|}{|\xi_1 - \xi_3|} \right) \leq M_3 \left(1 + \left| \frac{\partial \chi}{\partial \xi_1}(\xi_3, \eta_3) \right| \right) \\ &= M_3 [1 + |f\{x_3, y_3, z(x_3, y_3)\}|] \leq M_3 (1 + M) \end{aligned} \right. \quad (20)$$

for any $(\xi_3, \eta_3) \in Q_6$. Therefore by Fubini's theorem $\varsigma_1(\xi_1, \eta_1)$ satisfies (19) almost everywhere in the interval $|\xi_1 - a_1| < l$, as a function of ξ_1 , for almost all η_1 in the interval $|\eta_1 - b_1| < l$ and by (20) $\varsigma_1(\xi_1, \eta_1)$ is absolutely continuous as a function of ξ_1 in the interval $|\xi_1 - a_1| < l$ for all η_1 in the interval $|\eta_1 - b_1| < l$. Hence for any ξ_1 in the interval $|\xi_1 - a_1| < l$,

$$\varsigma_1(\xi_1, \eta_1) - \varsigma_1(a_1, \eta_1) = \int_{a_1}^{\xi_1} g\{\xi_1, \chi(\xi_1, \eta_1), \varsigma_1(\xi_1, \eta_1)\} d\xi_1 \quad (21)$$

for almost all η_1 in the interval $|\eta_1 - b_1| < l$. By the continuity of $z(x, y)$, $g(x, y, z)$, $\chi(\xi_1, \eta_1)$, accordingly of $\varsigma_1(\xi_1, \eta_1)$, $g\{\xi_1, \chi(\xi_1, \eta_1), \varsigma_1(\xi_1, \eta_1)\}$, (21) is established for any $(\xi_1, \eta_1) \in Q_6$. Hence by the continuity of $g\{\xi_1, \chi(\xi_1, \eta_1), \varsigma_1(\xi_1, \eta_1)\}$,

$$\frac{\partial \varsigma_1(\xi_1, \eta_1)}{\partial \xi_1} = g\{\xi_1, \chi(\xi_1, \eta_1), \varsigma_1(\xi_1, \eta_1)\} \quad (22)$$

for any $(\xi_1, \eta_1) \in Q_6$.

By the definition of $\chi(\xi_1, \eta_1)$, $\varsigma_1(\xi_1, \eta_1)$ and by (22), for any η_1 in the interval $|\eta_1 - b_1| < l$, the curve $y = \chi(x, \eta_1)$, $z = \varsigma_1(x, \eta_1)$ satisfies (2), (3) in the interval $|x - a_1| < l$, that is, is a characteristic curve in $D \cdot (Q_5 \times R)$, and is contained totally in S . On the other hand, the curves $y = \chi(x, \eta_1)$ fill up Q_5 , when η_1 takes all values in the open interval $|\eta_1 - b_1| < l$. This is however excluded, since $(a_1, b_1) \in F \cdot Q_5 \neq 0$. We thus arrive at a contradiction and this completes the proof of Theorem 1.

§ 3. Proof of Theorem 2.

Now we shall prove Theorem 2 by the use of Theorem 1.

In this chapter the notations are the same as in the introduction and we assume that $z(x, y)$ satisfies the premises of Theorem 2.

8. We take an arbitrary but fixed point (x_1, y_1) which belongs to G . We denote $z(x_1, y_1)$ by z_1 . We take an open cube $R_1: |x - x_1| < r_1$, $|y - y_1| < r_1$, $|z - z_1| < r_1$ such that $\bar{R}_1 \subset D$. Again we take an open

$$\left\{ \begin{array}{l} z_3 - z_1 = z(x_1, y_3) - z(x_1, y_1) = (y_3 - y_1) \left\{ \frac{\partial z(x_1, y_1)}{\partial y} + \rho_3(x_2, y_2) \right\} \\ \rho_3(x_2, y_2) \rightarrow 0 \text{ as } (x_2, y_2) \rightarrow (x_1, y_1). \end{array} \right. \quad (25)$$

By (24), (25) we have

$$\begin{aligned} z(x_2, y_2) - z(x_1, y_1) &= z_2 - z_1 = (z_2 - z_3) + (z_3 - z_1) \\ &= (x_2 - x_1) \{g(x_1, y_1, z_1) + \rho_2(x_2, y_2)\} + (y_3 - y_1) \left\{ \frac{\partial z(x_1, y_1)}{\partial y} + \rho_3(x_2, y_2) \right\} \\ &= (x_2 - x_1) \left\{ g(x_1, y_1, z_1) - f(x_1, y_1, z_1) \frac{\partial z(x_1, y_1)}{\partial y} + \rho_4(x_2, y_2) \right\} \\ &\quad + (y_2 - y_1) \left\{ \frac{\partial z(x_1, y_1)}{\partial y} + \rho_3(x_2, y_2) \right\} \\ \rho_3(x_2, y_2), \rho_4(x_2, y_2) &\rightarrow 0 \text{ as } (x_2, y_2) \rightarrow (x_1, y_1). \end{aligned}$$

Thus the total differentiability of $z(x, y)$ at any point (x_1, y_1) of G is proved. At the same time, we obtain, as the value of $\partial z / \partial x$ at (x_1, y_1) ,

$$g\{x_1, y_1, z(x_1, y_1)\} - f\{x_1, y_1, z(x_1, y_1)\} \frac{\partial z(x_1, y_1)}{\partial y}.$$

Hence

$$\frac{\partial z(x_1, y_1)}{\partial x} + f\{x_1, y_1, z(x_1, y_1)\} \frac{\partial z(x_1, y_1)}{\partial y} = g\{x_1, y_1, z(x_1, y_1)\}$$

at any point (x_1, y_1) of G .

This completes the proof of Theorem 2.

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