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On the Linear Partial Differential Equation of the First Order

By Takashi KASUGA

Introduction

In this paper, we shall treat the following partial differential equation

$$\frac{\partial z}{\partial x}+f(x,y,z)\frac{\partial z}{\partial y}=g(x,y,z)$$

without the usual condition of the total differentiability on the solution $z(x,y)$. (For the simpler equation

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}f(x,y)=0$$

see our previous paper, Kasuga [2]).

In the following, we shall denote by D a fixed open set in R^3 (Euclidean space defined by the three coordinates x,y,z), by G its projection on the (x,y)-plane, by $f(x,y,z)$, $g(x,y,z)$ two fixed continuous functions defined on D, which have continuous f_y, f_z, g_y, g_z. Evidently G is open in the (x,y)-plane.

We shall consider the partial differential equation

$$\frac{\partial z}{\partial x}+f(x,y,z)\frac{\partial z}{\partial y}=g(x,y,z). \tag{1}$$

With (1) we shall associate the simultaneous ordinary differential equations

$$\begin{cases} \dfrac{dy}{dx}=f(x,y,z) & (2) \\[2ex] \dfrac{dz}{dx}=g(x,y,z) & (3) \end{cases}$$

The curves representing the solutions of (2), (3) which are prolonged as far as possible on both sides in an open subset D_1 of D, will be called characteristic curves (of (1)) in D_1. Through any point (x_0, y_0, z_0)

in D_1, there passes one and only one characteristic curve in D_1.[1] We represent it by

$$y = \varphi(x, x_0, y_0, z_0, D_1) \qquad z = \psi(x, x_0, y_0, z_0, D_1)$$
$$\alpha(x_0, y_0, z_0, D_1) < x < \beta(x_0, y_0, z_0, D_1)$$

($\alpha(x_0, y_0, z_0, D_1)$, $\beta(x_0, y_0, z_0, D_1)$ may be $-\infty$, $+\infty$ respectively). Sometimes we abbreviate it as $C(x_0, y_0, z_0, D_1)$. Evidently $C(x_0, y_0, z_0, D_1) \supset C(x_0, y_0, z_0, D_2)$, if D_1, D_2 are two open subsets of D such that $D_1 \supset D_2$ and $(x_0, y_0, z_0) \in D_2$. In the following, if an interval (open, closed, or half-open) is contained in the interval $\alpha(x_0, y_0, z_0, D_1) < x < \beta(x_0, y_0, z_0, D_1)$, then we say that the characteristic curve $C(x_0, y_0, z_0, D_1)$ and its projection on the (x, y)-plane, $y = \varphi(x, x_0, y_0, z_0, D_1)$, are defined for that interval.

Let us consider a continuous function $z(x, y)$ defined on G, which has $\partial z/\partial x$ and $\partial z/\partial y$ (not necessarily continuous), except at most at the points of an enumerable set, in G (in the following we suppose always that the above conditions are satisfied by $z(x, y)$). We denote by S the surface represented by $z = z(x, y)$. Then we obtain the following two theorems.

Theorem 1. *If $S \subset D$, and $z(x, y)$ satisfies* (1) *almost everywhere in G, then any characteristic curve in D which has a point in common with the surface S, is totally contained in S.*

Theorem 2. *If $S \subset D$, and $z(x, y)$ satisfies* (1) *almost everywhere in G and moreover if $z(x, y)$ has $\partial z/\partial y$ (not necessarily continuous) everywhere in G, then $z(x, y)$ is totally differentiable and satisfies* (1) *everywhere in G.*

Remark 1. If the domain G, where $z(x, y)$ is defined, is not the projection of D on the (x, y)-plane but is only a part of the projection, theorem 1, 2 hold, if we substitute D by the set of the points of D whose projections on the (x, y)-plane are contained in G.

Remark 2. In the premises of Theorem 2, the condition that $z(x, y)$ has $\partial z/\partial y$ everywhere in G, can not be omitted, as the following example shows it.

Example. $D = R^3$, $G =$ the whole (x, y)-plane, the differential equation is

$$\frac{\partial z}{\partial x} + zx\frac{\partial z}{\partial y} = 0$$

1) Cf. Kamke [1], § 16, Nr. 79, Satz 4.

and a solution $z(x, y)$ is implicitly defined by

$$y = \frac{1}{2} zx^2 + z^3.$$

$z(x, y)$ is one-valued, continuous and has $\partial z/\partial x$ in the whole (x, y)-plane, but has $\partial z/\partial y$ except at $(0, 0)$ and also satisfies (1) except at $(0, 0)$. For this example, the premises of Theorem 1 are fulfilled, but those of Theorem 2 are not fulfilled.

§ 1. Some lemmas

In this chapter, the notations are the same as in the introduction and we assume that $z(x, y)$ satisfies the premises of Theorem 1.

1. Let us denote by K the set of the points (x_0, y_0) of G such that the characteristic curve $C\{x_0, y_0, z(x_0, y_0), D\}$ is contained in S, in a neighbourhood of $\{x_0, y_0, z(x_0, y_0)\}$. We denote by F the set $\overline{G-K}\cdot G$ (by $\overline{G-K}$ we denote the closure of $G-K$ in the (x, y)-plane). Evidently F is closed in G.

We shall often use the following lemmas.

Lemma 1. *If the characteristic curve $C\{x_0, y_0, z(x_0, y_0), D\}$ is defined for $\alpha_0 \leqq x \leqq x_0$ (or $x_0 \leqq x \leqq \alpha_0$) ($\alpha_0 \neq x_0$) and its projection on the (x, y)-plane, $y = \varphi\{x, x_0, y_0, z(x_0, y_0), D\}$ is contained in K for $\alpha_0 < x \leqq x_0$ (or $x_0 \leqq x < \alpha_0$), then $C\{x_0, y_0, z(x_0, y_0), D\}$ is contained in S for $\alpha_0 \leqq x \leqq x_0$ (or $x_0 \leqq x \leqq \alpha_0$).*

Proof. We denote by ξ_0 the nearest point to α_0 among the points ξ in the interval $\alpha_0 \leqq x \leqq x_0$ (or $x_0 \leqq x \leqq \alpha_0$) such that $C\{x_0, y_0, z(x_0, y_0), D\}$ is contained in S for $\xi \leqq x \leqq x_0$ (or $x_0 \leqq x \leqq \xi$). Such ξ_0 exists by the continuity of $\varphi\{x, x_0, y_0, z(x_0, y_0), D\}$ and $\psi\{x, x_0, y_0, z(x_0, y_0), D\}$ as functions of x and by the continuity of $z(x, y)$. If the lemma were false, then $\xi_0 \neq \alpha_0$. We put $\eta_0 = \varphi\{\xi_0, x_0, y_0, z(x_0, y_0), D\}$. Then $C\{x_0, y_0, z(x_0, y_0), D\}$ passes through $\{\xi_0, \eta_0, z(\xi_0, \eta_0)\}$ and is contained in S in some neighbourhood of the point $\{\xi_0, \eta_0, z(\xi_0, \eta_0)\}$, since $(\xi_0, \eta_0) \in K$. This is inconsistent with the definition of ξ_0 and the lemma is proved.

Lemma 2. *Let us denote by D_1 an open subset of D. We denote by F' the set of the points (x_0, y_0, z_0) ($\in D$) such that $C(x_0, y_0, z_0, D_1)$ is totally contained in S. Then F' is closed in D_1.*

Proof. We take a point $(\xi_0, \eta_0, \zeta_0) \in \bar{F}'\cdot D_1$ (here we denote by $\bar{F}'$

the closure of F' in R^3). Then if $C(\xi_0, \eta_0, \zeta_0, D_1)$ is defined for $\alpha_0 \leqq x \leqq \beta_0$, $C(x_0, y_0, z_0, D_1)$ is defined for $\alpha_0 \leqq x \leqq \beta_0$, for any point (x_0, y_0, z_0) $(\in F')$ in a neighbourhood of the point (ξ_0, η_0, ζ_0) and

$$\varphi(x, x_0, y_0, z_0, D_1) \longrightarrow \varphi(x, \xi_0, \eta_0, \zeta_0, D_1)$$
$$\psi(x, x_0, y_0, z_0, D_1) \longrightarrow \psi(x, \xi_0, \eta_0, \zeta_0, D_1)$$

uniformly in the interval $\alpha_0 \leqq x \leqq \beta_0$, as $(x_0, y_0, z_0) \rightarrow (\xi_0, \eta_0, \zeta_0)$.[2] From this and by the continuity of $z(x, y)$, $C(\xi_0, \eta_0, \zeta_0, D_1)$ is totally contained in S, that is, $(\xi_0, \eta_0, \zeta_0) \in F'$, q. e. d.

§ 2. Proof of Theorem 1.

In this chapter, the notations are the same as in the introduction and § 1, and we assume that $z(x, y)$ satisfies the premises of Theorem 1.

2. **Domain** Q. If F is empty, that is, if $G = K$, we can conclude by Lemma 1 that the characteristic curve in D passing through any point of S is totally contained in S and the theorem is established. Suppose therefore, if possible, that $F \neq 0$. We denote by H the enumerable set consisting of the points of G at which $z(x, y)$ is not derivable with respect to x and with respect to y simultaneously. If we denote by F_n, for each positive integer n, the set of the points (x, y) of G such that

$$|z(x+h, y)-z(x, y)| \leqq |h| n$$
$$|z(x, y+k)-z(x, y)| \leqq |k| n$$

whenever $|h|, |k| \leqq 1/n, (x+h, y), (x, y+k) \in G$, then the sets F_n cover $G-H$ and each of the sets F_n is closed in G by dint of the continuity of $z(x, y)$.

If a point (x_0, y_0) of G has an open neighbourhood V in G such that every point of V belongs to K except (x_0, y_0), then by Lemma 1, $C\{x_0, y_0', z(x_0, y_0'), (V \times R) \cdot D\}$ (we denote by $V \times R$ the set of the points of R^3 whose projections on the (x, y)-plane belong to V) is totally contained in S, when $(x_0, y_0') \in V$ and $y_0 \neq y_0'$. Hence by Lemma 2, $C\{x_0, y_0, z(x_0, y_0), (V \times R) \cdot D\}$ is totally contained in S, since $\{x_0, y_0', z(x_0, y_0')\} \rightarrow \{x_0, y_0, z(x_0, y_0)\}$ (as $y_0' \rightarrow y_0$) by the continuity of $z(x, y)$. Therefore also (x_0, y_0) belongs to K, that is, F can contain no isolated point, F is perfect in G.

Thus $F-H$ is not empty and of the second category in itself as it

2) Cf. Kamke, § 17, Nr. 84, Satz 3.

is a G_δ set in R^2. Therefore there must exist a positive integer N and an open square $Q: |x-a|<L, |y-b|<L$ such that $0<L<1/(2N)$, $\bar{Q}\subset G$, $(a,b)\in(F-H)\cdot Q\subset F_N$ ($\bar{Q}$ is the closure of Q in the (x,y)-plane). Then $(a,b)\in F\cdot Q\subset F_N$, since the closure of $F-H$ in G is F by the perfectness of F in G and the enumerability of H. Hence if $(x,y)\in F\cdot Q$ and $(x+h,y)$, $(x,y+k)\in Q$,

$$\begin{cases} |z(x+h,y)-z(x,y)|\leqq|h|N \\ |z(x,y+k)-z(x,y)|\leqq|k|N, \end{cases} \tag{4}$$

by the definition of F_N and Q.

3. **Domains** Q_1, Q_2, Q_3. We take an open cube $Q_1: |x-a|<L_1$, $|y-b|<L_1$, $|z-z(a,b)|<L_1$ such that $0<L_1\leqq L$ and $\bar{Q}_1\subset D$ (by $\bar{Q}_1$ we denote the closure of Q_1 in R^3). Then by the continuity of f_y, f_z, g_y, g_z, there is a positive number M_1 such that

$$|f_y|,\ |f_z|,\ |g_y|,\ |g_z|<M_1 \qquad \text{in } Q_1.$$

Again we take a parallelepiped $Q_2: |x-a|<L_2$, $|y-b|<L_2$, $|z-z(a,b)|<L_3$, which satisfies the following conditions:

i) $0<L_2, L_3\leqq L_1$, that is, $Q_2\subset Q_1$,

ii) $|z(x,y)-z(a,b)|<L_3$ for $|x-a|<L_2$, $|y-b|<L_2$, that is, S is contained in Q_2 for $|x-a|<L_2$, $|y-b|<L_2$,

iii) any characteristic curve $C(x_0, y_0, z_0, Q_1)$ where $(x_0, y_0, z_0)\in Q_2$ is defined for $|x-a|<L_2$,

iv)
$$\begin{cases} \dfrac{\exp(4M_1L_2)+2N\{\exp(4M_1L_2)-1\}}{2N\{2-\exp(4M_1L_2)\}-\{\exp(4M_1L_2)-1\}}\leqq\dfrac{2}{3N} \\ 2N\{2-\exp(4M_1L_2)\}-\{\exp(4M_1L_2)-1\}>0. \end{cases} \tag{5}$$

The conditions i), iii), iv) can be realized, if we take L_2, L_3 sufficiently small (iii) by the boundedness of f, g in $\bar{Q}$) and the condition ii) can be realized if we take L_2 still smaller (by the continuity of $z(x,y)$).

We denote by Q_3 the open square: $|x-a|<L_2$, $|y-b|<L_2$. Evidently $Q_3\subset Q$. If we take any point (x_0, y_0) belonging to Q_3, $\{x_0, y_0, z(x_0, y_0)\}$ belongs to Q_2 and $C\{x_0, y_0, z(x_0, y_0), Q_1\}$ is defined for $|x-a|<L_2$, by the conditions ii), iii) on Q_2. We denote by K_1 the set of the points (x_0, y_0) of Q_3 such that the curve $y=\varphi\{x, x_0, y_0, z(x_0, x_0), Q_2\}$ has no point in common with $F\cdot Q_3$. We denote by E the set $\bar{K}_1\cdot Q_3$ ($\bar{K}_1$ is the closure of K_1 in the (x,y)-plane). If $(x_0, y_0)\in K_1$, then by Lemma 1, $C\{x_0, y_0, z(x_0, y_0), Q_2\}$ is totally contained in S. Therefore, by Lemma 2 and by the continuity of $z(x,y)$, $C\{x_0, y_0, z(x_0, y_0), Q_2\}$ is totally contained

in S, if $(x_0, y_0) \in E$. Hence, $(a, b) \notin F$, if $E = Q_3$. But since $(a, b) \in F$, so $Q_3 - E$ is not empty. Evidently $Q_3 - E$ is open.

We shall prove that $(Q_3 - E) \cdot F$ is also not empty. We take a point $(c, d) \in Q_3 - E$, If $(c, d) \in F$, the proposition is already proved. Therefore we assume that $(c, d) \notin F$. The curve $y = \varphi\{x, c, d, z(c, d), Q_2\}$ has at least a point in common with $F \cdot Q_3$, by the definition of E, and obviously

$$L_2 + a \geqq \beta\{c, d, z(c, d), Q_2\} > c > \alpha\{c, d, z(c, d), Q_2\} \geqq L_2 - a .$$

Hence, as $F \cdot Q$ is closed in Q, there is the nearest point of F to (c, d) on the portion of the curve $y = \varphi\{x, c, d, z(c, d), Q_2\}$ for $\alpha\{c, d, z(c, d), Q_2\} < x \leqq c$ or for $c \leqq x < \beta\{c, d, z(c, d), Q_2\}$. We denote it by (a_1, b_1). If $(a_1, b_1) \notin E$, then $(a_1, b_1) \in (Q_3 - E) \cdot F$ and the above proposition is established. Suppose therefore that $(a_1, b_1) \in E$.

Again by Lemma 1, $C\{c, d, z(c, d), Q_2\}$ is contained in S for the interval $a_1 \leqq x \leqq c$ or $c \leqq x \leqq a_1$ and so $C\{c, d, z(c, d), Q_2\} = C\{a_1, b_1, z(a_1, b_1), Q_2\}$. On the other hand, we assume that $(a_1, b_1) \in E$, so in any neighbourhood of the point (a_1, b_1), there is a point (x_0, y_0) which belongs to K_1. As $C\{a_1, b_1, z(a_1, b_1), Q_2\}$ $(= C\{c, d, z(c, d), Q_2\})$ is defined for $a_1 \leqq x \leqq c$ or $c \leqq x \leqq a_1$, $C\{x_0, y_0, z(x_0, y_0), Q_2\}$ is also defined for $a_1 \leqq x \leqq c$ or $c \leqq x \leqq a_1$ if $(x_0, y_0)\,(\in K_1)$ belongs to a neighbourhood of (a_1, b_1), and $\varphi\{x, x_0, y_0, z(x_0, y_0), Q_2\} \to \varphi\{x, c, d, z(c, d), Q_2\}$ $(= \varphi\{x, a_1, b_1, z(a_1, b_1), Q_2\})$ uniformly in the interval $a_1 \leqq x \leqq c$ or $c \leqq x \leqq a_1$, as $(x_0, y_0)\,(\in K_1) \to (a_1, b_1)$[3] (since $\{x_0, y_0, z(x_0, y_0)\} \to \{a_1, b_1, z(a_1, b_1)\}$ as $(x_0, y_0) \to (a_1, b_1)$ by the continuity of $z(x, y)$). On the other hand, as it is proved before, $C\{x_0, y_0, z(x_0, y_0), Q_2\}$ is totally contained in S, if $(x_0, y_0) \in K_1$. From this and by the definition of K_1, the curve $y = \varphi\{x, x_0, y_0, z(x_0, y_0), Q_2\}$ is totally contained in K_1, if $(x_0, y_0) \in K_1$. Therefore there is a point which belongs to K_1 in any neighbourhood of (c, d), that is, $(c, d) \in E$. But this is a contradiction, since $(c, d) \in Q_3 - E$. Thus in any case, $F \cdot (Q_3 - E)$ is not empty and there is at least one point $(a_1, b_1) \in F \cdot (Q_3 - E)$.

4. **Domain** Q_4. As $Q_3 - E$ is open and $F \cdot (Q_3 - E)$ is not empty, we can take an open square $Q_4 : |x - a_1| < L_4$, $|y - b_1| < L_4$ such that $Q_4 \subset Q_3 - E$ and $(a_1, b_1) \in F \cdot Q_4$. Obviously $Q_4 \subset Q_3 \subset Q$.

We take any pair of points $(x_1, \bar{y}_1)$, (x_1, y_1) with the same x coordinate, in Q_4. We shall prove

$$|z(x_1, \bar{y}_1) - z(x_1, y_1)| \leqq 2N |\bar{y}_1 - y_1| .$$

3) Cf. Kamke, §17, Nr. 84, Satz 3.

Suppose, if possible, that

$$|z(x_1, \bar{y}_1)-z(x_1, y_1)|>2N|\bar{y}_1-y_1| \tag{6}$$

If $(x_1, \bar{y}_1)\in F$ or $(x_1, y_1)\in F$, then $|z(x_1, \bar{y}_1)-z(x_1, y_1)|\leqq N|\bar{y}_1-y_1|$, by (4) and as $Q_4 \subset Q$. So we may assume that $(x_1, \bar{y}_1)\notin F$, $(x_1, y_1)\notin F$ and $y_1<\bar{y}_1$.

By the way of the construction of Q_1, Q_2, Q_3, Q_4, the characteristic curves $C\{x_1, y_1, z(x_1, y_1), Q_1\}$ and $C\{x_1, \bar{y}_1, z(x_1, \bar{y}_1), Q_1\}$ are defined for $|x-a|<L_2$. So their projection on the (x, y)-plane $y=\varphi\{x, x_1, y_1, z(x_1, y_1), Q_1\}$ and $y=\varphi\{x, x_1, \bar{y}_1, z(x_1, \bar{y}_1), Q_1\}$ are defined for $|x-a|<L_2$ and contained in Q.

As $(x_1, y_1), (x_1, \bar{y}_1)\in Q_4 \subset Q_3-E$, and $F\cdot Q$ is closed in Q, on either side of x_1 there is the nearest x to x_1 in the interval $|x-a|<L_2$ such that either $(x, \varphi\{x, x_1, y_1, z(x_1, y_1), Q_1\})\in F\cdot Q$ or $(x, \varphi\{x, x_1, \bar{y}_1, z(x_1, \bar{y}_1), Q_1\})\in F\cdot Q$ is satisfied. We denote it by x_2 and $\varphi\{x_2, x_1, y_1, z(x_1, y_1), Q_1\}$, $\varphi\{x_2, x_1, \bar{y}_1, z(x_1, \bar{y}_1), Q_1\}$, $\psi\{x_2, x_1, y_1, z(x_1, y_1), Q_1\}$, $\psi\{x_2, x_1, \bar{y}_1, z(x_1, \bar{y}_1), Q_1\}$ respectively by $y_2, \bar{y}_2, z_2$ and $\bar{z}_2$. Then by Lemma 1, $C\{x_1, y_1, z(x_1, y_1), Q_1\}$, $C\{x_1, \bar{y}_1, z(x_1, \bar{y}_1), Q_1\}$ are contained in S for the interval $x_1\leqq x\leqq x_2$ or $x_2\leqq x\leqq x_1$. Hence $z_2=z(x_2, y_2)$, $\bar{z}_2=z(x_2, \bar{y}_2)$. Moreover $|x_2-a|<L_2$, $(x_2, y_2), (x_2, \bar{y}_2)\in Q$ and either $(x_2, y_2)\in F\cdot Q$ or $(x_2, \bar{y}_2)\in F\cdot Q$.

In the following we denote by P_x, the plane parallel to the (y, z)-plane which cuts the x-axis at the point whose x coordinate is x.

By the way of the construction of Q_1, Q_2 and by the continuity of f_y, f_z, g_y, and g_z, $y=\varphi(x, x_1, \eta, \varsigma, Q_1)$ and $z=\psi(x, x_1, \eta, \varsigma, Q_1)$ define a bicontinuous one to one mapping $A_x: (\eta, \zeta)\rightarrow(y, z)$ of the domain $|\eta-b|<L_2$, $|\varsigma-z(a, b)|<L_3$ on the plane P_{x_1} (its y, z coordinates we denote by η, ς respectively), onto some domain on the plane P_x, for any fixed x in the interval $|x-a|<L_2$[4] (in the following, (η, ς) will always belong to the domain: $|\eta-b|<L_2$, $|\varsigma-z(a, b)|<L_3$). Moreover this domain on P_x is contained in Q_1 and continuous $\partial y/\partial\eta$, $\partial y/\partial\varsigma$, $\partial z/\partial\eta$, $\partial z/\partial\varsigma$ exist.[5] From (2), (3) we have[6]

$$\frac{d}{dx}\left(\frac{\partial y}{\partial\eta}\right)=f_y\frac{\partial y}{\partial\eta}+f_z\frac{\partial z}{\partial\eta} \qquad \frac{d}{dx}\left(\frac{\partial z}{\partial\eta}\right)=g_y\frac{\partial y}{\partial\eta}+g_z\frac{\partial z}{\partial\eta}$$

$$\frac{d}{dx}\left(\frac{\partial y}{\partial\varsigma}\right)=f_y\frac{\partial y}{\partial\varsigma}+f_z\frac{\partial z}{\partial\varsigma} \qquad \frac{d}{dx}\left(\frac{\partial z}{\partial\varsigma}\right)=g_y\frac{\partial y}{\partial\varsigma}+g_z\frac{\partial z}{\partial\varsigma}$$

for $|x-a|<L_2$. In Q_1, $|f_y|, |f_z|, |g_y|, |g_z|<M$, so

4) Cf. Kamke, §17, Nr. 84, Satz 3.

5), 6) Cf. Kamke, § 18, Nr. 87, Satz 1 and its "zusatz".

$$\left|\frac{d}{dx}\left(\frac{\partial y}{\partial \eta}-1\right)\right|+\left|\frac{d}{dx}\left(\frac{\partial z}{\partial \eta}\right)\right|\leqq 2M_1\left(\left|\frac{\partial y}{\partial \eta}-1\right|+\left|\frac{\partial z}{\partial \eta}\right|+1\right)$$
$$\left|\frac{d}{dx}\left(\frac{\partial y}{\partial \varsigma}\right)\right|+\left|\frac{d}{dx}\left(\frac{\partial z}{\partial \varsigma}-1\right)\right|\leqq 2M_1\left(\left|\frac{\partial y}{\partial \varsigma}\right|+\left|\frac{\partial z}{\partial \varsigma}-1\right|+1\right)$$

for $|x-a|<L_2$. Hence[7]

$$\left|\frac{\partial y}{\partial \eta}-1\right|+\left|\frac{\partial z}{\partial \eta}\right|\leqq \exp(2M_1|x-x_1|)-1$$
$$\left|\frac{\partial y}{\partial \varsigma}\right|+\left|\frac{\partial z}{\partial \varsigma}-1\right|\leqq \exp(2M_1|x-x_1|)-1$$

for $|x-a|<L_2$, as $\partial y/\partial\eta-1$, $\partial z/\partial\eta$, $\partial y/\partial\varsigma$, $\partial z/\partial\varsigma-1$ vanish at $x=x_1$. As $|x_2-a|<L_2$, the above inequalities subsist for $x=x_2$. Hence as $|x_1-a|<L_2$, $|x_2-a|<L_2$ and by (5) $2-\exp(4M_1L_2)>0$,

$$\begin{cases} \left|\dfrac{\partial z}{\partial \eta}\right|, \left|\dfrac{\partial y}{\partial \varsigma}\right| \leqq \exp(4M_1L_2)-1 \\ 0<2-\exp(4M_1L_2)\leqq \dfrac{\partial z}{\partial \varsigma}, \dfrac{\partial y}{\partial \eta}\leqq \exp(4M_1L_2) \end{cases} \tag{7}$$

for $x=x_2$.

By the way of the construction of Q_2, the segment T of straight line on the plane P_{x_1}:

$$\varsigma-z(x_1, y_1)=t(\eta-y_1) \qquad y_1\leqq\eta\leqq\bar{y}_1$$

where
$$t=\frac{z(x_1, \bar{y}_1)-z(x_1, y_1)}{\bar{y}_1-y_1}$$

which joins the points $\{y_1, z(x_1, y_1)\}$ and $\{\bar{y}_1, z(x_1, \bar{y}_1)\}$ is totally contained in the domain $|\eta-b|<L_2$, $|\varsigma-z(a, b)|<L_3$ on the plane P_{x_1}. By (6)

$$|t|>2N. \tag{8}$$

We denote by T' the image of T on the plane P_{x_2} by the mapping A_{x_2}. T' is represented by

$$y=\varphi\{x_2, x_1, \eta, z(x_1, y_1)+t(\eta-y_1), Q_1\}=\lambda(\eta)$$
$$z=\psi\{x_2, x_1, \eta, z(x_1, y_1)+t(\eta-y_1), Q_1\}=\mu(\eta)$$
$$\bar{y}_1\geqq\eta\geqq y_1 \qquad (\eta \text{ is taken as parameter})$$

and $y_2=\lambda(y_1)$, $z_2=\mu(y_1)$, $\bar{y}_2=\lambda(\bar{y}_1)$, $\bar{z}_2=\mu(\bar{y}_1)$. As it can be shown

7) Cf. Kamke, § 17, Nr. 85, Hilfssatz 3.

easily, $d\lambda/d\eta$, $d\mu/d\eta$ exist and are continuous, and by (7), (8), (5)

$$t \neq 0, \left|\frac{1}{t}\frac{d\mu}{d\eta}\right| = \left|\frac{\partial z}{\partial\eta}\frac{1}{t}+\frac{\partial z}{\partial\varsigma}\right| \geqq \left|\frac{\partial z}{\partial\varsigma}\right| - \left|\frac{1}{t}\frac{\partial z}{\partial\eta}\right| \geqq 2-\exp(4M_1L_2)$$

$$-\frac{\exp(4M_1L_2)-1}{2N} = \frac{2N\{2-\exp(4M_1L_2)\}-\{\exp(4M_1L_2)-1\}}{2N} > 0,$$

$$\left|\frac{1}{t}\frac{d\lambda}{d\eta}\right| = \left|\frac{\partial y}{\partial\eta}\frac{1}{t}+\frac{\partial y}{\partial\varsigma}\right| \leqq \frac{\exp(4M_1L_2)}{2N}+\exp(4M_1L_2)-1$$

$$= \frac{\exp(4M_1L_2)+2N\{\exp(4M_1L_2)-1\}}{2N}.$$

Hence by (5)

$$\frac{d\mu}{d\eta} \neq 0, \left|\frac{d\lambda}{d\eta}\Big/\frac{d\mu}{d\eta}\right|$$

$$\leqq \frac{\exp(4M_1L_2)+2N\{\exp(4M_1L_2)-1\}}{2N\{2-\exp(4M_1L_2)\}-\{\exp(4M_1L_2)-1\}} \leqq \frac{2}{3N}$$

Therefore we can represent T' as

$$y = \gamma(z) \qquad z_2 \geqq z \geqq \bar{z}_2 \quad \text{or} \quad \bar{z}_2 \geqq z \geqq z_2$$

($z_2 \neq \bar{z}_2$ as $d\mu/d\eta \neq 0$) and $\gamma(z)$ satisfies following conditions:
$y_2 = \gamma(z_2)$, $\bar{y}_2 = \gamma(\bar{z}_2)$, continuous $d\gamma/dz$ exist and

$$|d\gamma/dz| \leqq 2/(3N) \quad \text{for} \quad z_2 \geqq z \geqq \bar{z}_2 \quad \text{or} \quad \bar{z}_2 \geqq z \geqq z_2.$$

Hence

$$\left|\frac{\bar{y}_2-y_2}{\bar{z}_2-z_2}\right| \leqq \frac{2}{3N} \qquad \bar{z}_2 \neq z_2.$$

As it is proved before, $\bar{z}_2 = z(x_2, \bar{y}_2)$ and $z_2 = z(x_2, y_2)$. So

$$\left|\frac{z(x_2, \bar{y}_2)-z(x_2, y_2)}{\bar{y}_2-y_2}\right| \geqq \frac{3N}{2}, \qquad \bar{y}_2 \neq y_2.$$

But $(x_2, \bar{y}_2)$, $(x_2, y_2) \in Q$ and either $(x_2, \bar{y}_2) \in F\cdot Q$ or $(x_2, y_2) \in F\cdot Q$. This contradicts (4).

Thus we have proved

$$|z(x_1, \bar{y}_1)-z(x_1, y_1)| \leqq 2N|\bar{y}_1-y_1| \tag{9}$$

for any pair of points $(x_1, \bar{y}_1)$, (x_1, y_1) in Q_4 with the same x coordinate.

5. **Domains** Q_5, Q_6. We now consider the following ordinary differential equation whose right side is defined and continuous on G,

$$\frac{dy}{dx} = f\{x, y, z(x, y)\}. \tag{10}$$

$f\{x, y, z(x, y)\}$ is defined and continuous on $\bar{Q}_4 \subset \bar{Q} \subset G$, so there is a positive M such that

$$|f\{x, y, z(x, y)\}| < M \qquad \text{in } Q_4. \tag{11}$$

In Q_1, $|f_y|$, $|f_z| < M_1$, so $|f(x, \bar{y}, \bar{z}) - f(x, y, z)| \leqq M_1(|\bar{y} - y| + |\bar{z} - z|)$ if $(x, \bar{y}, \bar{z})$, $(x, y, z) \in Q_1$. On the other hand $\{x, y, z(x, y)\} \in Q_2 \subset Q_1$, If $(x, y) \in Q_4 \subset Q_3$. Therefore by (9) if (x_1, y_1), $(x_1, \bar{y}_1) \in Q_4$,

$$|f\{x_1, \bar{y}_1, z(x_1, \bar{y}_1)\} - f\{x_1, y_1, z(x_1, y_1)\}| \leqq M_1(|\bar{y}_1 - y_1| + |z(x_1, \bar{y}_1) - z(x_1, y_1)|) \leqq M_1(1 + 2N)|\bar{y}_1 - y_1|. \tag{12}$$

Hence the right side of (10) satisfies Lipschitz condition on Q_4. Let us write $l = L_4/(M+1)$. We denote by η_1 any number such that $|\eta_1 - b_1| \leqq l$. Then $\eta_1 + lM \leqq b_1 + L_4$, $\eta_1 - lM \geqq b_1 - L_4$. Thus for any η_1 there exists a unique solution of (10) defined for $|x - a_1| < l$ which passes through (a_1, η_1) and lies in Q_4.[8] We denote it by $y = \chi(x, \eta_1)$. Hence if we denote by Q_5 the domain defined by:

$$\chi(x, b_1 - l) < y < \chi(x, b_1 + l) \qquad |x - a_1| < l,$$

the curves $y = \chi(x, \eta_1)$ fill up Q_5 simple-fold, when η_1 takes all values in the open interval $|\eta_1 - b_1| < l$,[9] and $(a_1, b_1) \in Q_5 \subset Q_4$.

By (11), (12), for any two η_1, $\bar{\eta}$ in the interval $|\eta_1 - b_1| < l$ and any x in the interval $|x - a_1| < l$,[10]

$$\begin{aligned} |\bar{\eta}_1 - \eta_1| &\leqq |\chi(x, \bar{\eta}_1) - \chi(x, \eta_1)| \exp\{M_1(1+2N)|x - a_1|\} \\ &\leqq |\chi(x, \bar{\eta}_1) - \chi(x, \eta_1)| \exp\{M_1(1+2N)l\} \end{aligned} \tag{13}$$

We denote by Q_6 the open square: $|\xi_1 - a_1| < l$, $|\eta_1 - b_1| < l$ in the (ξ_1, η_1)-plane. We denote by A the one to one mapping of Q_6 onto Q_5 defined by

$$x = \xi_1 \qquad y = \chi(\xi_1, \eta_1).$$

Then A is bicontinuous[11] and by (13) we can easily conclude that A_1^{-1} maps any null set in Q_5 to a null set in Q_6.

6. *$z(x, y)$ in the domain Q_5.* We take any pair of points (x_3, y_3), (x_4, y_4) belonging to Q_5. Then $\chi(x_4, \eta_4) = y_4$ for an η_4 in the open interval $|\eta_1 - b_1| < l$. Now we denote by (x_5, y_5):

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8), 9) Cf. Kamke, § 6, Nr. 30, Satz 1, § 10, Nr. 47, Satz 4, and § 12, Nr. 54, Satz 3.

10) Cf. Kamke, § 11, Nr. 51, Satz 1.

11) Cf. Kamke, § 10, Nr. 47, Satz 4.

Case I. The nearest point of F to (x_4, y_4) on the portion of the continuous curve $y=\chi(x, \eta_4)$ for $x_3 \leqq x \leqq x_4$ or $x_4 \leqq x \leqq x_3$, if it contains some points of F (such (x_5, y_5) exists in this case, as $F \cdot Q$ is closed in Q),

Case II. The point $x_5 = x_3$, $y_5 = \chi(x_3, \eta_4)$, if that portion contains no point of F.

The characteristic curve $C\{x_4, y_4, z(x_4, y_4), Q_1\}$ is defined for $|x-a_1|<l$, as $(x_4, y_4) \in Q_3$ and the interval $|x-a_1|<l$ is contained in the interval $|x-a|<L_2$. We shall prove that in both Cases the portion of the curve $y=\chi(x, \eta_4)$ for the interval $x_4 \leqq x \leqq x_5$ or $x_5 \leqq x \leqq x_4$ is contained in the curve $y=\varphi\{x, x_4, y_4, z(x_4, y_4), Q_1\}$ and the portion of $C\{x_4, y_4, z(x_4, y_4), Q_1\}$ for the interval $x_4 \leqq x \leqq x_5$ or $x_5 \leqq x \leqq x_4$ is contained S. If $(x_4, y_4) \in F \cdot Q_5$, then $x_5 = x_4$, so the proposition is obvious. Hence we assume that $(x_4, y_4) \notin F \cdot Q_5$.

Suppose, if possible, that the proposition were false. We denote by x_6 the nearest point to x_5 among the points ξ in the interval $x_5 \leqq x \leqq x_4$ or $x_4 \leqq x \leqq x_5$ such that: the portion of the curve $y=\chi(x, \eta_4)$ for the interval $x_4 \leqq x \leqq \xi$ or $\xi \leqq x \leqq x_4$ is contained in the curve $y=\varphi\{x, x_4, y_4, z(x_4, y_4), Q_1\}$ and the portion of $C\{x_4, y_4, z(x_4, y_4), Q_1\}$ for the same interval $x_4 \leqq x \leqq \xi$ or $\xi \leqq x \leqq x_4$ is contained in S. Such x_6 exists by the continuity of $\chi(x, \eta_4)$, $\varphi\{x, x_4, y_4, z(x_4, y_4), Q_1\}$, $\psi\{x, x_4, y_4, z(x_4, y_4), Q_1\}$ and $z(x, y)$. We denote $\chi(x_6, \eta_4)$ by y_6. Evidently $x_5 \neq x_6$, as the above proposition is supposed false. By the definition of (x_5, y_5), (x_6, y_6), $C\{x_4, y_4, z(x_4, y_4)\}$ passes through the point $\{x_6, y_6, z(x_6, y_6)\}$ and $(x_6, y_6) \notin F$. Hence $C\{x_4, y_4, z(x_4, y_4)\}$ is contained in S in some neighbourhood of the point $\{x_6, y_6, z(x_6, y_6)\}$. Also from this, the curve $y=\varphi\{x, x_4, y_4, z(x_4, y_4), Q_1\}$ passes through (x_6, y_6) and satisfies (10) in some neighbourhood of $x=x_6$. Hence, the curve $y=\chi(x, \eta_4)$ is contained in the curve $y=\varphi\{x, x_4, y_4, z(x_4, y_4), Q_1\}$ in some neighbourhood of (x_6, y_6), by the uniqueness of the solution in Q_4 of (10) passing through (x_6, y_6). These are inconsistent with the definition of (x_6, y_6). The above proposition is thus established.

By the above proposition, the curve $y=\chi(x, \eta_4)$ $z=z\{x, \chi(x, \eta_4)\}$ satisfies (2), (3) and is contained in Q_1 for the interval $x_4 \leqq x \leqq x_5$ or $x_5 \leqq x \leqq x_4$ (if $x_4 \neq x_5$). On $\bar{Q}_1$, $g(x, y, z)$ is defined and continuous. Hence there is a positive M_2 such that $|g(x, y, z)| \leqq M_2$ on Q_1. Therefore

$$\frac{dz\{x, \chi(x, \eta_4)\}}{dx} = |g[x, \chi(x, \eta_4), z\{x, \chi(x, \eta_4)\}]| \leqq M_2$$

for $x_5 \leqq x \leqq x_4$ or $x_4 \leqq x \leqq x_5$.

Thus $$|z(x_5, y_5) - z(x_4, y_4)| \leqq M_2 |x_5 - x_4| \leqq M_2 |x_3 - x_4| \tag{14}$$

(if $x_4 = x_5$, this is obvious).
Now $y = \chi(x, \eta_4)$ is a solution of (10) contained in Q_4 and $|f\{x, y, z(x, y)\}| < M$ on Q_4.

Thus $|y_4 - y_5| = |\chi(x_4, \eta_4) - \chi(x_5, \eta_4)| \leqq M|x_4 - x_5| \leqq M|x_3 - x_4|$.
Hence $|y_3 - y_5| \leqq |y_3 - y_4| + |y_4 - y_5| \leqq |y_3 - y_4| + M|x_3 - x_4|$. (15)

We have

$$|z(x_3, y_5) - z(x_5, y_5)| \leqq N|x_3 - x_5| \leqq N|x_3 - x_4| \tag{16}$$

in Case I, by (4) and as $(x_5, y_5) \in F\cdot Q$, $(x_3, y_5) \in Q$, and in Case II, simply as $x_3 = x_5$. Also we have by (9)

$$|z(x_3, y_3) - z(x_3, y_5)| \leqq 2N|y_3 - y_5| \tag{17}$$

as (x_3, y_3), $(x_3, y_5) \in Q_4$.

By (14), (15), (16), (17),

$$\begin{aligned}|z(x_3, y_3) - z(x_4, y_4)| &\leqq |z(x_3, y_3) - z(x_3, y_5)| + |z(x_3, y_5) - z(x_5, y_5)| \\ &\quad + |z(x_5, y_5) - z(x_4, y_4)| \leqq 2N|y_3 - y_5| + N|x_3 - x_4| + M_2|x_3 - x_4| \\ &\leqq 2N|y_3 - y_4| + (2NM + N + M_2)|x_3 - x_4| \\ &\leqq (2N + 2NM + M_2)(|y_3 - y_4| + |x_3 - x_4|).\end{aligned}$$

Hence if we denote $2NM + 2N + M_2$ by M_3,

$$\limsup_{(x,y)\to(x_3, y_3)} \frac{|z(x, y) - z(x_3, y_3)|}{|x - x_3| + |y - y_3|} \leqq M_3 \tag{18}$$

whenever $(x_3, y_3) \in Q_5$.

7. *Completion of the proof*. From (18), $z(x, y)$ is totally differentiable almost everywhere in Q_5, by Stepanoff's theorem on almost everywhere total differentiability.[12] Moreover $z(x, y)$ fulfills (1) almost everywhere in G and, as we have seen in section 5, A_1^{-1} maps any null set in Q_5 to a null set in Q_6. Hence if we write $\zeta_1(\xi_1, \eta_1) = z\{\xi_1, \chi(\xi_1, \eta_1)\}$ for $(\xi_1, \eta_1) \in Q_6$,

$$\left\{\begin{aligned} &\frac{\partial}{\partial \xi_1} \zeta_1(\xi_1, \eta_1) = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial \chi}{\partial \xi_1} \\ &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} f\{\xi_1, \chi(\xi_1, \eta_1), \zeta_1(\xi_1, \eta_1)\} = g\{(\xi_1, \chi(\xi_1, \eta_1), \zeta_1(\xi_1, \eta_1)\}\end{aligned}\right. \tag{19}$$

almost everywhere in Q_6.

12) Cf. Saks [3], pp. 238-239.

parallelepiped R_2: $|x-x_1|<r_2$, $|y-y_1|<r_2$, $|z-z_1|<r_3$ such that

i) $r_2, r_3 \leqq r_1$, that is, $R_2 \subset R_1$,

ii) $|z(x, y)-z(x, y_1)|<r_3$ for $|x-x_1|<r_2$, $|y-y_1|<r_2$,

iii) $C(x_0, y_0, z_0, R_1)$ is defined for the interval $|x-x_1|<r_2$ whenever $(x_0, y_0, z_0) \in R_2$.

i), iii) can be realized if we take r_2, r_3 sufficiently small (iii) by the boundedness of f, g in R_1) and ii) can be realized if we take r_2 still smaller. We denote by R_3 the open square: $|x-x_1|<r_2$, $|y-y_1|<r_2$.

If we take any point (x_2, y_2) which belongs to R_3 and denote $z(x_2, y_2)$ by z_2, then by ii) and iii), $(x_2, y_2, z_2) \in R_2$ and $C(x_2, y_2, z_2, R_1)$ is defined for $|x-x_1|<r_2$. We denote $\varphi(x_1, x_2, y_2, z_2, R_1)$, $\psi(x_1, x_2, y_2, z_2, R_1)$ by y_3, z_3 respectively. Then $z_3 = z(x_1, y_3)$, since $C(x_2, y_2, z_2, R_1)$ is totally contained in S by Theorem 1. By the continuity of $z(x, y)$ and of $\varphi(x, x_0, y_0, z_0, R_1)$, $\psi(x, x_0, y_0, z_0, R_1)$ with respect to all the arguments x, x_0, y_0, z_0,[13]

$$\begin{cases} \varphi(x, x_2, y_2, z_2, R_1) \longrightarrow \varphi(x_1, x_1, y_1, z_1, R_1) = y_1 \\ \psi(x, x_2, y_2, z_2, R_1) \longrightarrow \psi(x_1, x_1, y_1, z_1, R_1) = z_1 \\ \text{as } x \rightarrow x_1, \ x_2 \rightarrow x_1, \ y_2 \rightarrow y_1. \end{cases} \tag{23}$$

Hence by the continuity of $f(x, y, z)$, $g(x, y, z)$,

$$f\{x, \varphi(x, x_2, y_2, z_2, R_1), \psi(x, x_2, y_2, z_2, R_1)\} \longrightarrow f(x_1, y_1, z_1),$$
$$g\{x, \varphi(x, x_2, y_2, z_2, R_1), \psi(x, x_2, y_2, z_2, R_1)\} \longrightarrow g(x_1, y_1, z_1)$$
$$\text{as } x \rightarrow x_1, \ x_2 \rightarrow x_1, \ y_2 \rightarrow y_1.$$

On the other hand, by (2), (3)

$$y_2 - y_3 = \int_{x_1}^{x_2} f\{x, \varphi(x, x_2, y_2, z_2, R_1), \psi(x, x_2, y_2, z_2, R_1)\}\, dx$$

$$z_2 - z_3 = \int_{x_1}^{x_2} g\{x, \varphi(x, x_2, y_2, z_2, R_1), \psi(x, x_2, y_2, z_2, R_1)\}\, dx.$$

Therefore we have

$$\begin{cases} y_2 - y_3 = (x_2 - x_1)\{f(x_1, y_1, z_1) + \rho_1(x_2, y_2)\} \\ z_2 - z_3 = (x_2 - x_1)\{g(x_1, y_1, z_1) + \rho_2(x_2, y_2)\} \\ \rho_1(x_2, y_2), \ \rho_2(x_2, y_2) \rightarrow 0 \text{ as } (x_2, y_2) \rightarrow (x_1, y_1). \end{cases} \tag{24}$$

By the assumption, $z(x, y)$ has $\partial z/\partial y$ at (x_1, y_1) and by (23) $y_3 = \varphi(x_1, x_2, y_2, z_2, R_1) \rightarrow y_1$ as $(x_2, y_2) \rightarrow (x_1, y_1)$. Hence we have

13) Cf. Kamke [1], § 17, Nr. 84, Satz 3.

Also by (18) (if we write $x_3 = \xi_3,\ y_3 = X(\xi_3, \eta_3)$)

$$\left\{\begin{aligned} &\limsup_{\xi_1 \to \xi_3} \frac{|\varsigma_1(\xi_1, \eta_3) - \varsigma_1(\xi_3, \eta_3)|}{|\xi_1 - \xi_3|} \leqq \left(\limsup_{(x, y) \to (x_3, y_3)} \frac{|z(x, y) - z(x_3, y_3)|}{|x - x_3| + |y - y_3|}\right) \\ &\times \left(\limsup_{\xi_1 \to \xi_3} \frac{|\xi_1 - \xi_3| + |X(\xi_1, \eta_3) - X(\xi_3, \eta_3)|}{|\xi_1 - \xi_3|}\right) \leqq M_3\left(1 + \left|\frac{\partial X}{\partial \xi_1}(\xi_3, \eta_3)\right|\right) \\ &= M_3\,[1 + |f\{x_3, y_3, z(x_3, y_3)\}|] \leqq M_3\,(1 + M) \end{aligned}\right. \tag{20}$$

for any $(\xi_3, \eta_3) \in Q_6$. Therefore by Fubini's theorem $\varsigma_1(\xi_1, \eta_1)$ satisfies (19) almost everywhere in the interval $|\xi_1 - a_1| < l$, as a function of ξ_1, for almost all η_1 in the interval $|\eta_1 - b_1| < l$ and by (20) $\varsigma_1(\xi_1, \eta_1)$ is absolutely continuous as a function of ξ_1 in the interval $|\xi_1 - a_1| < l$ for all η_1 in the interval $|\eta_1 - b_1| < l$. Hence for any ξ_1 in the interval $|\xi_1 - a_1| < l$,

$$\varsigma_1(\xi_1, \eta_1) - \varsigma_1(a_1, \eta_1) = \int_{a_1}^{\xi_1} g\{\xi_1, X(\xi_1, \eta_1), \varsigma_1(\xi_1, \eta_1)\}\, d\xi_1 \tag{21}$$

for almost all η_1 in the interval $|\eta_1 - b_1| < l$. By the continuity of $z(x, y)$, $g(x, y, z)$, $X(\xi_1, \eta_1)$, accordingly of $\varsigma_1(\xi_1, \eta_1)$, $g\{\xi_1, X(\xi_1, \eta_1), \varsigma_1(\xi_1, \eta_1)\}$, (21) is established for any $(\xi_1, \eta_1) \in Q_6$. Hence by the continuity of $g\{\xi_1, X(\xi_1, \eta_1), \varsigma_1(\xi_1, \eta_1)\}$,

$$\frac{\partial \varsigma_1(\xi_1, \eta_1)}{\partial \xi_1} = g\{\xi_1, X(\xi_1, \eta_1), \varsigma_1(\xi_1, \eta_1)\} \tag{22}$$

for any $(\xi_1, \eta_1) \in Q_6$.

By the definition of $X(\xi_1, \eta_1)$, $\varsigma_1(\xi_1, \eta_1)$ and by (22), for any η_1 in the interval $|\eta_1 - b_1| < l$, the curve $y = X(x, \eta_1)$, $z = \varsigma_1(x, \eta_1)$ satisfies (2), (3) in the interval $|x - a_1| < l$, that is, is a characteristic curve in $D \cdot (Q_5 \times R)$, and is contained totally in S. On the other hand, the curves $y = X(x, \eta_1)$ fill up Q_5, when η_1 takes all values in the open interval $|\eta_1 - b_1| < l$. This is however excluded, since $(a_1, b_1) \in F \cdot Q_5 \neq 0$. We thus arrive at a contradiction and this completes the proof of Theorem 1.

§ 3. Proof of Theorem 2.

Now we shall prove Theorem 2 by the use of Theorem 1.

In this chaper the notations are the same as in the introduction and we assume that $z(x, y)$ satisfies the premises of Theorem 2.

8. We take an arbitrary but fixed point (x_1, y_1) which belongs to G. We denote $z(x_1, y_1)$ by z_1. We take an open cube R_1: $|x - x_1| < r_1$, $|y - y_1| < r_1$, $|z - z_1| < r_1$ such that $\bar{R}_1 \subset D$. Again we take an open

$$\begin{cases} z_3 - z_1 = z(x_1, y_3) - z(x_1, y_1) = (y_3 - y_1)\left\{\dfrac{\partial z(x_1, y_1)}{\partial y} + \rho_3(x_2, y_2)\right\} \\ \rho_3(x_2, y_2) \to 0 \text{ as } (x_2, y_2) \to (x_1, y_1). \end{cases} \tag{25}$$

By (24), (25) we have

$$\begin{aligned} z(x_2, y_2) - z(x_1, y_1) &= z_2 - z_1 = (z_2 - z_3) + (z_3 - z_1) \\ &= (x_2 - x_1)\{g(x_1, y_1, z_1) + \rho_2(x_2, y_2)\} + (y_3 - y_1)\left\{\frac{\partial z(x_1, y_1)}{\partial y} + \rho_3(x_2, y_2)\right\} \\ &= (x_2 - x_1)\left\{g(x_1, y_1, z_1) - f(x_1, y_1, z_1)\frac{\partial z(x_1, y_1)}{\partial y} + \rho_4(x_2, y_2)\right\} \\ &\quad + (y_2 - y_1)\left\{\frac{\partial z(x_1, y_1)}{\partial y} + \rho_3(x_2, y_2)\right\} \\ &\quad \rho_3(x_2, y_2),\ \rho_4(x_2, y_2) \to 0 \text{ as } (x_2, y_2) \to (x_1, y_1). \end{aligned}$$

Thus the total differentiability of $z(x, y)$ at any point (x_1, y_1) of G is proved. At the same time, we obtain, as the value of $\partial z/\partial x$ at (x_1, y_1),

$$g\{x_1, y_1, z(x_1, y_1)\} - f\{x_1, y_1, z(x_1, y_1)\}\frac{\partial z(x_1, y_1)}{\partial y}.$$

Hence

$$\frac{\partial z(x_1, y_1)}{\partial x} + f\{x_1, y_1, z(x_1, y_1)\}\frac{\partial z(x_1, y_1)}{\partial y} = g\{x_1, y_1, z(x_1, y_1)\}$$

at any point (x_1, y_1) of G.

This completes the proof of Theorem 2.

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