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Osaka University
Alexander Polynomials as Isotopy Invariants, II

By Shin'ichi Kinoshita

Introduction

In this paper we shall consider the Alexander polynomials of linear graphs and closed surfaces, which may not be connected, in the 3-sphere $S^3$. The former have been already studied in [2] and in §1 the fact of §5 in [2] will be generalized. This result will be used in §§2-3. In §2 we shall define the Alexander polynomial, more explicitly a system of the Alexander polynomials, of a closed surface in $S^3$. This Alexander polynomial contains some arbitrary constants, and the number of it will be discussed in §3.

§1.

Let $L$ be a linear graph with integral coefficients in $S^3$. Suppose further that $\partial L = 0$. Let $\alpha_0$ and $\alpha_1$ be the number of vertices and edges of $|L|$ respectively. Then we have

$$\alpha_0 - \alpha_1 = \mu - \beta_1,$$

where $\mu$ is the number of components and $\beta_1$ is the 1-dimensional Betti number of $|L|$ respectively.

Now let $\rho$ be a normal projection of $|L|$ in a suitably chosen plane $E^2$. Further let $s$ be the number of crossing points of $\rho(|L|)$ and $r$ the number of regions of $E^2$ divided by $\rho(|L|)$. Then we have

$$(\alpha_0 - s) - \alpha_1 + r = 2.$$ 

From (1) and (2) it follows that

$$1 + \beta_1 - \mu = r - (s + 1).$$

The Alexander polynomial of $L$ is calculated from the matrix

$$\left( \frac{\partial R_i}{\partial x_j} \right)^{\psi \varphi},$$

where $R_i$ is a defining relation and $x_j$ is a generator of $F(S^3 - |L|)^\psi$.

1) See [2].
From a normal projection of $|L|$ given above, we can obtain the generators and defining relations of $F(S^3 - |L|)$. Actually they are seen to consist of $r$ generators and $s+1$ defining relations by the method of [2]. Then from (3) it follows that
\[ \Delta^{(d)}(t_1, \ldots, t_\mu) \quad \text{and} \quad \Delta^{(d)}(t) \]
are equal to 0, if $0 \leq d < r-(s+1)=1+p_{\mu}-\mu$. Thus we have the following

**Theorem 1.** Let $L$ be a linear graph with integral coefficients in $S^3$. Further suppose that $\partial L=0$. Let $\mu$ be the number of components of $|L|$ and $p_1$ the 1-dimensional Betti number of $|L|$. Then if $0 \leq d < 1+p_{\mu}-\mu$, $\Delta^{(d)}(t_1, \ldots, t_\mu)$ and $\Delta^{(d)}(t)$ are all equal to 0.

Hence it is natural to say that $\Delta^{(1+p_{\mu}-\mu)}(t_1, \ldots, t_\mu)$ and $\Delta^{(1+p_{\mu}-\mu)}(t)$ are Alexander polynomials of $L$. From now on we shall consider only Alexander polynomials of the type $\Delta^{(d)}(t)$.

§ 2.

New let $M$ be a closed surface in $S^3$ which may not connected. Further let $M_1, M_2, \ldots, M_\mu$ be components of $M$ and $g_i$ the genus of $M_i$ ($i=1, 2, \ldots, \mu$).

Put $g(M)=\sum_{i=1}^{\mu} g_i$. Then $M$ divides $S^3$ into $\mu+1$ regions $C_0, C_1, \ldots, C_\mu$.

For each $C_i$ we can define the Alexander polynomial as follows: Suppose that the boundary of $C_i$ consists of $M_{i_1}, \ldots, M_{i_{\nu_i}}$ and that $g_{i_1}, \ldots, g_{i_{\nu_i}}$ are genera of them respectively. Put $g^i=\sum_{j=1}^{\nu_i} g_{i_j}$. Then clearly $p_i(C_i)=g^i$.

Now we consider $F(C_i)$. If $\phi$ is a homomorphism of $F(C_i)/[F(C_i), F(C_i)]$ into the infinite cyclic group $Z$, then we have a sequence of homomorphisms
\[ X \rightarrow F(C_i) \rightarrow F(C_i)/[F(C_i), F(C_i)] \rightarrow Z. \]

From this we can define by the usual way the Alexander polynomial $\Delta^{(1+g^i-\nu_i)}(t_i)$. Since $\phi$ is arbitrary, we have actually a family of Alexander polynomials $\Delta^{(1+g^i-\nu_i)}(t)$. If $i$ moves from 0 to $\mu$, then we have a system of Alexander polynomials
\[ \{\Delta^{(1+g^i-\nu_i)}(t)\}. \quad (4) \]

From now on we shall say that (4) is the Alexander polynomial of $M$.

**Remark.** This definition of the Alexander polynomial of $M$ can be naturally extended to the case, where an $n$-dimensional manifold lies in the $(n+1)$-dimensional sphere $S^{n+1}$.

It is proved by R. H. Fox [1] that each $C_i$ is homeomorphic to a
complementary region of a suitably chosen linear graph $|L_i|$. The 1-dimensional homology group of $S^3 - |L_i|$ is a free abelian group with $p_1(|L_i|)$ generators. From this it is easy to see that the Alexander polynomial $\Delta_{\mathbb{C}_i}^{(1+g^i-v_i)}(t)$ of $C_i$ is a polynomial with at most $p_1(|L_i|) = g^i$ arbitrary constants.\(^2\) Thus the Alexander polynomial (4) of the closed surface $M$ has at most $2g(M)$ arbitrary constants. These illustrate also the way to calculate the Alexander polynomial of a given closed surface.

§ 3.

Using the notation of § 2, we shall now prove the following

**Theorem 2.** Let $M$ be a closed surface which may not be connected. Then the number of arbitrary constants of the system of Alexander polynomials of $M$ is at most $2g(M) - 1$ for every $g(M) \geq 1$.

Proof. It is proved by R. H. Fox [1] that a closed surface $M$ in $S^3$ can be deformed to a system of 2-spheres by a sequence of suitably chosen cuts, which are done along the disk $D$ whose interior $\text{int} D$ is disjoint from $N^3$ and whose boundary $\text{bd} D$ lies on a component, say $N_1$, of positive genus and is not homotopic to 0 on $N_1$. Our proof will be done by induction on the minimal number $n(M)$ of these cuts used for this purpose.

If $n = 1$, then our theorem is trivial. Now we assume that our theorem is true for $n \leq k - 1$. Suppose $n(M) = k$. Then $M$ can be deformed to a closed surface $N$ by a cut along a disk $D$ described above, where $n(N) = k - 1$. It occurs two cases.

The first case is that $\text{bd} D$ is homologous to 0 on $M$. In this case $g(M) = g(N)$. Suppose that $\text{bd} D$ lies on $M_i$, and that $M_i$ is the boundary of $C_0$ and $C_1$. Further suppose that $\text{int} D$ lies in $C_0$. Then $\text{int} D$ divides $C_0$ into two regions $C_{00}$ and $C_{01}$. Now let $\Delta_{\mathbb{C}_{00}}^{(1+g^{00} - \mu_{00})}(t)$, $\Delta_{\mathbb{C}_{01}}^{(1+g^{01} - \mu_{01})}(t)$ and $\Delta_{\mathbb{C}_i}^{(1+g^i - \mu^i)}(t)$ be Alexander polynomials of $C_{00}$, $C_{01}$ and $C_i$ respectively. Then it is easy to see that $g^{00} + g^{01} = g^i$ and $\mu_{00} + \mu_{01} = \mu_i + 1$. Furthermore it follows from the construction of $M$ and $N$ that

$$\Delta_{\mathbb{C}_{00}}^{(1+g^{00} - \mu_{00})}(t) \cdot \Delta_{\mathbb{C}_{01}}^{(1+g^{01} - \mu_{01})}(t) = \Delta_{\mathbb{C}_i}^{(1+g^i - \mu^i)}(t).$$

Thus the number of arbitrary constants of $\Delta_{\mathbb{C}_i}^{(1+g^i - \mu^i)}(t)$ is equal to the sum of that of $\Delta_{\mathbb{C}_{00}}^{(1+g^{00} - \mu_{00})}(t)$ and $\Delta_{\mathbb{C}_{01}}^{(1+g^{01} - \mu_{01})}(t)$.

Now we shall consider $C_i$. Let $E_i$ be a region of $S^3 - N$ which contains $C_i$. Let $\Delta_{\mathbb{C}_i}^{(1+g^i - \mu^i)}(t)$ and $\Delta_{E_i}^{(1+g^i - \mu^i)}(t)$ be Alexander polynomials of $C_i$.

\(^2\) Arbitrary constants are integers.

\(^3\) $N$ is a closed surface which appears while $M$ is deformed to a system of 2-spheres.
and $E_1$ respectively, where $g^1=h^1$ and $\mu_1=\nu_1-1$. From the construction of $M$ and $N$ it is easy to see that

$$\Delta_{E_1}^{(1+g^1-\nu_1)}(t) \equiv f(t) \cdot \Delta_{C_1}^{(1+g^1-\mu_1)}(t),$$

where $f(t)$ is a polynomial. Then the number of arbitrary constants of $\Delta_{C_1}^{(1+g^1-\mu_1)}(t)$ is equal to or smaller than that of $\Delta_{E_1}^{(1+g^1-\nu_1)}(t)$. Thus our proof of the first case is complete.

The second case is now that $\partial D$ is not homologous to 0 on $M$. In this case $g(M)=g(N)+1$. Suppose that $\partial D$ lies on $M_1$ and $\text{int} D$ lies in $C_0$. Then $C_0$ is homeomorphic to a complementary region of a linear graph which is the join $\rho$ of a circle and another linear graph whose complementary region is homeomorphic to $C_0-D$. Then we can see directly that the number of arbitrary constants of the Alexander polynomial of $C_0$ is at most $g^0-1$. Therefore the number of arbitrary constants of the Alexander polynomial of $M$ is at most $2g(M)-1$. Thus our proof is complete.

As an application of Theorem 2 we have the following fact. Let $M$ and $N$ be for instance two connected closed surfaces with the same genus $g$ in $S^3$ and $S^2$ respectively, and let $C$ be a complementary regions of $M$ and $E$ that of $N$ respectively. Further suppose that Alexander polynomials of $C$ and $E$ have $g$ arbitrary constants respectively. Then from our theorem 2 it follows that $C$ and $E$ do not make a 3-sphere by any identification of $M$ and $N$.

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References


4) Suppose that $A$ is a point on $S^2$ which lies in $S^3$. Let $|L_1|$ and $|L_2|$ be two linear graphs such that $|L_1| \cap S^3 = A$ and $|L_2| \cap S^3 = A$. Further let $|L_1| - A$ and $|L_2| - A$ be contained in the different components of $S^3 - S^2$. Then $|L_1| \cup |L_2|$ is said to be a join of $|L_1|$ and $|L_2|$. 