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ON SELF Δ-EQUIVALENCE OF BOUNDARY LINKS

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1. Introduction

An n-component oriented tame link $\ell = k_1 \cup \ldots \cup k_n$ in the oriented 3-space R^3 is called a boundary link if there are mutually disjoint oriented surfaces $F_1, \ldots, F_n$ in R^3 such that $\partial\mathcal{F} = \ell$, $\partial F_i = k_i$ for $\mathcal{F} = F_1 \cup \ldots \cup F_n$ and each $i = 1, \ldots, n$. Then $\mathcal{F}$ is called the spanning surface of ℓ.

It is known that boundary links are link-homotopic to a trivial link, [2].

For self #-equivalences (definition, see [11], [12], [13]) of boundary links, the followings are known:

1. Boundary links are self #-equivalent(I) to a trivial link, [11], [13].
2. Boundary links are self #-equivalent(II) to a trivial link if and only if the Arf invariant of each component is zero, [2].

In this paper, we consider another self local equivalence, called a self Δ-equivalence, of boundary links. Namely, for a link ℓ, let E^3 be a 3-ball such that $\ell \cap E^3$ is a tangle illustrated in Fig.1(a). The transformation from Fig.1(a) to 1(b) is called a Δ-move, [5].

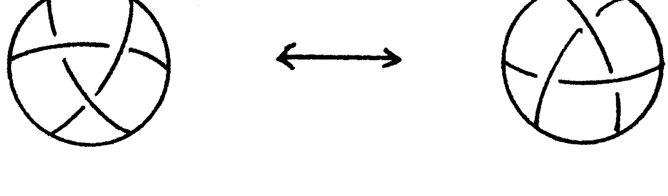

Fig.1

Especially if 3 arcs in Fig.1(a) are contained in a component of ℓ, it is called a self Δ-move. For two links ℓ and ℓ', if ℓ can be deformed into ℓ' by a finite sequence of (self) Δ-moves, ℓ is said to be (self) Δ-equivalent to ℓ'. It is known that self Δ-equivalence implies self #-equivalence(I) of links, [1], [6] and [12]. But the converse is not true, [7].

The aim of this paper is to give some partial answers to the follownig Conjecture.

Conjecture. *Let ℓ be an n-component boundary link. Then ℓ is self Δ-equivalent to the trivial link.*

A link $\ell = k_1 \cup ... \cup k_n$ is called a boundary link in the strong sense if there are mutually disjoint singular disks $D_1, ..., D_n$ in R^3 such that $\partial\mathcal{D} = \ell, \partial D_i = k_i$ for $\mathcal{D} = D_1 \cup ... \cup D_n$ and each $i = 1, ..., n$.

If ℓ is a boundary link in the strong sense, there are disks $D_1, ..., D_n$ satisfying the above. By the orientation preserving cut along each singularity of D_i, we obtain a spanning surface of ℓ. Namely ℓ is the boundary link.(But the converse is not true, namely there are links which are boundary links but not boundary links in the strong sense by Proposition 4.3.)

Theorem 4.2. *If ℓ is a boundary link in the strong sense, ℓ is self Δ-equivalent to the trivial link.*

A link ℓ is said to be p-trivial if there is a p-component sublink L of ℓ such that L is the trivial link.

Theorem 4.5. *Let ℓ be an n-component boundary link. If ℓ is $(n-1)$-trivial, ℓ is self Δ-equivalent to the trivial link.*

Lastly, we shall prove the Conjecture is true for $n = 2$ by using Theorem 4.5.

Theorem 4.6. *If $\ell = k_1 \cup k_2$ is a boundary link, ℓ is self Δ-equivalent to the trivial link.*

2. Ribbon Δ-cobordism of links.

To prove Theorems, we introduce the Δ-cobordism of links.

For two n-component links ℓ, L in $R^3[a]$, $R^3[b]$ respectively for $a < b$, ℓ is said to be Δ-cobordant to L if there are mutually disjoint annuli $A_1, ..., A_n$ in $R^3[a, b]$ satisfying the followings for $\mathcal{A} = A_1 \cup ... \cup A_n$, where $R^n[a, b] = \{(x_0, ..., x_n) \in R^{n+1} | a \leq x_n \leq b\}$ and $R^n[c] = \{(x_0, ..., x_n) \in R^{n+1} | x_n = c\}$: $\mathcal{A} \cap R^3[a] = \ell$ and $\mathcal{A} \cap R^3[b] = \tilde{L}$, the reflect inverse of L, and $A_i \cap R^3[a] \neq \phi$, $A_i \cap R^3[b] \neq \phi$ for each

$i = 1, ..., n$ and $\mathcal{A}$ are locally flat except finite points contained in the interior of $\mathcal{A}$, which are the singularity of $\mathcal{A}$, denoted by $\mathcal{S}(\mathcal{A})$, such that, for each point P of $\mathcal{S}(\mathcal{A})$, $(\partial N(P : R^3[a,b]), \partial N(P : \mathcal{A}))$ is the Borromean rings, Fig.2, where $N(x : X)$ means the regular neighborhood of x in X. The annuli $\mathcal{A}$ satisfying the above conditions are called Δ-annuli between ℓ and L. Especially, for Δ-annuli $\mathcal{A}$ between ℓ ($\subset R^3[a]$) and $L(\subset R^3[b])$ for $a < b$, if $\mathcal{A}$ do not have minimal points, [3], ℓ is said to be ribbon Δ-cobordant to L. Moreover if $\mathcal{S}(\mathcal{A}) = \phi$, ℓ is said to be ribbon cobordant to L. Therefore ℓ is a ribbon link, [15], if and only if ℓ is ribbon cobordant to the trivial link.

The following is proved in [15].

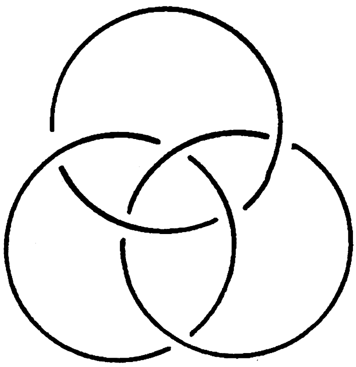

Fig.2

Lemma 2.1. *Any ribbon link is self Δ-equivalent to the trivial link.*

By using Lemma 2.1 and the similar proof to that of Lemma 1.19. in [10], we easily obtain Lemma 2.2 which is an extention of Lemma 2.1.

Lemma 2.2. *Suppose that ℓ is a link ribbon Δ-cobordant to the trivial link. Then ℓ is self Δ-equivalent to the trivial link.*

3. Local moves realizable by a finite sequence of (self) Δ-moves

In this section, we introduce some (self) local moves realizable by a finite sequence of (self) Δ-moves, which are used to prove Theorems.

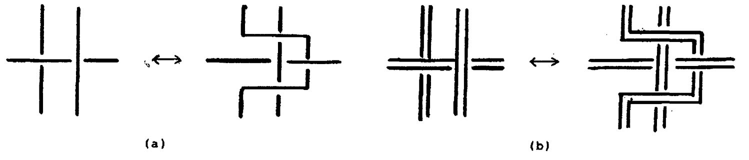

Fig.3

The following local moves of links are called a Π -move(Fig.3(a)) and a parallel Π -move(Fig.3(b)) respectively.

Lemma 3.1. *A* Π*-move and a parallel* Π*-move can be realized by a finite sequence of* Δ*-moves.*

Proof. A Π-move can be realized by a Δ-move by the following way, Fig.4

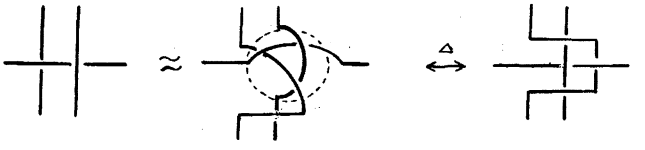

Fig.4

As a parallel Π-move can be realized by a finite sequence of Π-moves, it can be done by a finite sequence of Δ-moves. □

As a parallel Δ-move illustrated in Fig.5 can be realized by a parallel Π-move, we obtain Lemma 3.2 by Lemma 3.1.

Lemma 3.2. *A parallel* Δ*-move can be realized by a finite sequence of* Δ*-moves.*

Next we consider the following move, called a C_m-move , Fig.6.

Lemma 3.3. *A* C_m*-move can be realized by a finite sequence of* Δ*-moves.*

Proof. A C_m-move can be realized by a finite sequence of Π-moves, Fig.7. Hence we obtain Lemma 3.3 by Lemma 3.1. □

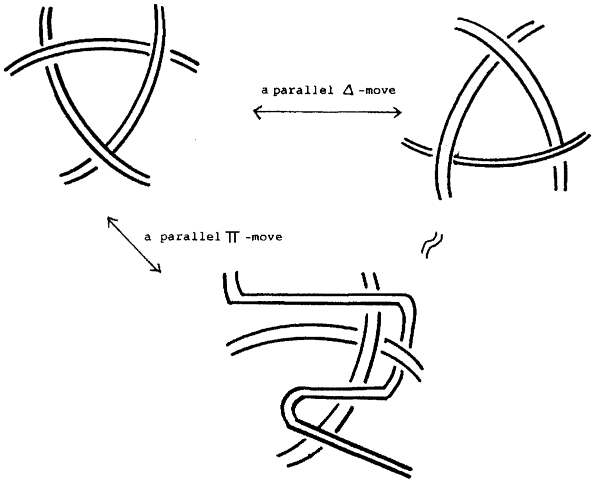

Fig.5

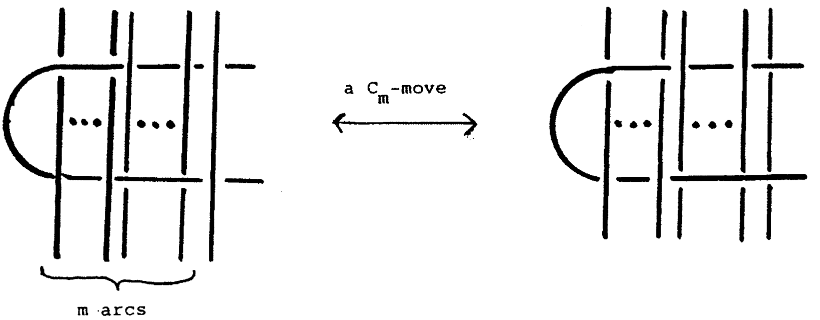

Fig.6

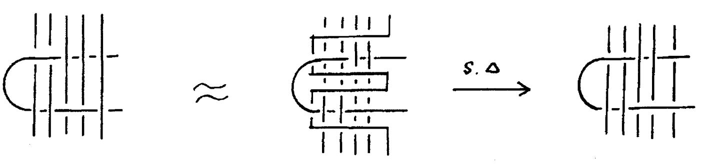

Fig.7

Lemma 3.4. *A parallel C_1-move illustrated in Fig.8 can be realized by a finite sequence of Δ-moves.*

Proof. A parallel C_1-move can be realized by 4-time C_2-moves illustrated in Fig.8. Hence we obtain Lemma 3.4 by Lemma 3.2. □

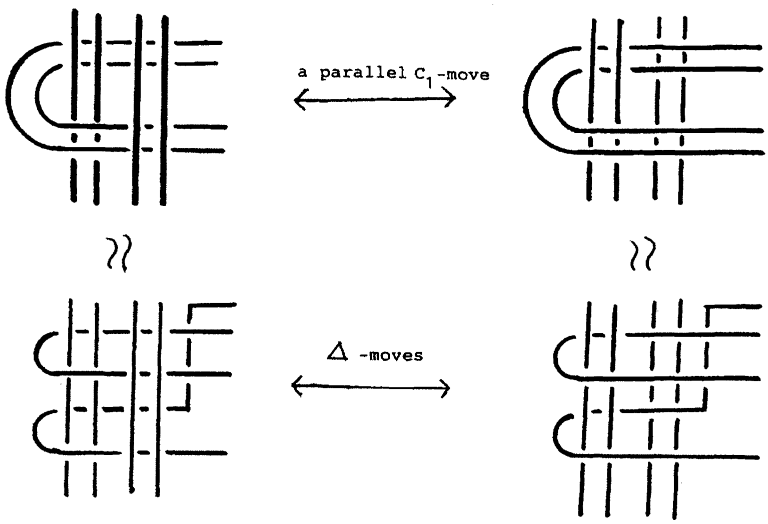

Fig.8

4. Proof of Theorems.

Now let us prove Theorems.

To prove Theorem 4.2, we define a link, called a double link, which is a kind of boudary link in the strong sence and prove Lemma 4.1.

Let $\mathcal{V} = V_1 \cup ... \cup V_n$ be a disjoint union of n solid tori $V_1, ..., V_n$ in R^3 and k_i a doubled knot in V_i, [16]. Then $\ell = k_1 \cup ... \cup k_n$ is called a doubled link (in $\mathcal{V}$).

Lemma 4.1. *Any doubled link is self Δ-equivalent to the trivial link.*

Proof. Let $\ell = k_1 \cup ... \cup k_n$ be a doubled link in $\mathcal{V} = V_1, \cup ... \cup, V_n$. Since the Δ-move is a kind of unknotting operations of knots, [5], we obtain a doubled link $\ell_1 = k_1' \cup (\ell - k_1)$ in $\mathcal{V}_1 = V_1' \cup (\mathcal{V} - V_1)$, $k_1' \subset V_1'$, such that V_1' is a trivial solid torus and ℓ_1 is self Δ-equivalent to ℓ by Lemma 3.2. Hence, by choosing ℓ_1 instead of ℓ

if necessary, we can assume that ℓ is a doubled link in $\mathcal{V} = V_1 \cup ... \cup V_n$ such that V_1 is trivial in R^3. Moreover, if k_1 is m-full twisted in V_1, we apply m-time self Δ-moves to k_1 in V_1 illustrated in Fig.9 and obtain the trivial knot, denoted by k_1 again, in V_1. Hence there is a disk D_1 with $\partial D_1 = k_1$, where D_1 is obtained by connecting 2 parallel disk, each of which is parallel to the disk C_1 with ∂C_1=(the longitude of ∂V_1), with a 1-full twisted band in V_1. Deform D_1 into $R^2[1]$ by an ambient isotopy φ_1 of R^3 and denote $\varphi_1(D_1), \varphi_1(\ell), \varphi_1(k_i), \varphi_1(V_i)$ and $\varphi_1(c_i)$ by D_1, ℓ, k_i, V_i and c_i respectively again, where c_i means the core of V_i.

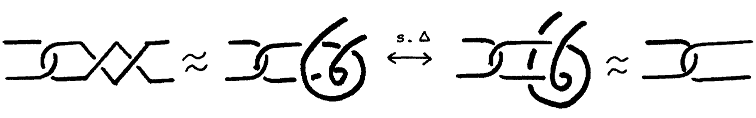

Fig.9

Suppose that $D_1 \cap V_i \neq \phi$ for some $i \geq 2$. As the intersection number of D_1 and c_i, denoted by $I(D_1, c_i)$, is equal to $Link(k_1, c_i)(= 0)$, there is the closure of connected component B_i^3 of $V_i - D_1$ such that $B_i^3 \cap D_1$ is contained is the positive, or negative side of D_1, denoted by D_1^+, D_1^- respectively, Fig.10(a).

Suppose that $D_1 \cap k_2 \neq \phi$. Then $D_1 \cap V_2 \neq \phi$ and there is B_2^3 satisfying the above, Fig.10(a). By deforming B_2^3 slightly with D_1 fixed, we may assume that $B_2^3 \subset R^2[1, \infty)$(or $R^2(-\infty, 1]$). Now let $B_2^3 \subset R^2[1, \infty)$. Moreover we can deform B_2^3 unknotted in $R^2[1, \infty)$ by Lemma 3.2.

Let E^2 be a non-singular disk in $R^2[1, \infty)$ with $E^2 \cap B_2^3 = \partial E^2 \cap \partial B_2^3$(={an arc}) and $E^2 \cap D_1 = \partial E^2 \cap \partial D_1$(={an arc}). If $k_2 \cap B_2^3$ does not contain the Hopf tangle, deform it into B_2^3 illustrated in Fig.10(a), (b).

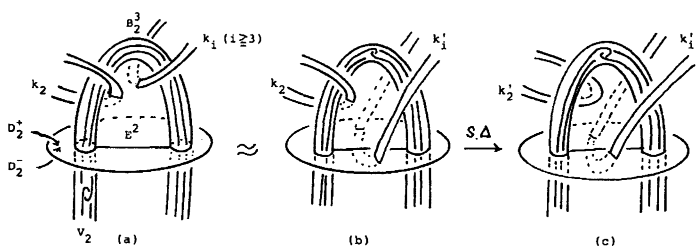

Fig.10

If $E^2 \cap k_i \neq \phi$ for some $i \geq 3$, deform $E^2 \cap k_i$ into $R^2(-\infty, 1]$ along E^2 by an ambient isotopy of R^3, Fig.10(a), (b). After this deformation, if $E^2 \cap k_2 \neq \phi$, we apply a finite sequence of self C_1-move to k_2, Fig.10(b), (c), and obtain a knot k_2' from k_2 such that $E^2 \cap k_2' = \phi$. By deforming B_2^3 along E^2 into $R^2(-\infty, 1]$, we obtain a doubled link $\ell' = k_1 \cup k_2' \cup ... \cup k_n'$ from ℓ such that ℓ' is a doubled link self Δ-equivalent to ℓ by Lemma 3.3 and #($k_2' \cap D_1$)= #($k_2 \cap D_1$) − 4, where #(X) means the number of points of X.

By performing the above discussion successively, we obtain a doubled link $L = k_1 \cup K_2 ... \cup K_n$ self Δ-equivalent to ℓ such that $K_2 \cap D_1 = \phi$

Next we apply the above discussion to K_2. If K_2 is not trivial, we can obtain a doubled link $L' = K_2' \cup (L - K_2)$ such that K_2' is trivial and $K_2' \cap D_1 = \phi$ by a finite sequence of self Δ-moves. Hence we write L' and K_2' by L and K_2 respectively again. There is a disk D_2 with $\partial D_2 = K_2$ by the same construction with that of D_1 such that $D_1 \cap D_2 = \phi$ and an ambient isotopy φ_2 of R^3 such that $\varphi_2(D_1) = D_1$ and $\varphi_2(D_2) \subset R^2[2]$. Let us denote $\varphi_2(D_2)$ and $\varphi_2(V_j)$ for $j \geq 3$ by D_2 and V_j respectively.

Suppose that $D_2 \cap V_3 \neq \phi$. As $I(D_2, c_3) = 0$, there is the closure of a connected component B_3^3 of $V_3 - D_1 - D_2$ such that $B_3^3 \cap D_p$ is contained in D_p^+ or D_p^- for $p = 1$ or 2 by the construction of D_1 and D_2. Hence we can apply the same discussion, which is used to remove $B_2^3 \cap D_1$, to $B_3^3 \cap D_p$. By doing the above successively, we obtain a doubled link $\mathcal{L} = k_1 \cup K_2 \cup \kappa_3 \cup ... \cup \kappa_n$ self Δ-equivalent to L such that κ_3 is trivial and $\kappa_3 \cap (D_1 \cup D_2) = \phi$, where $\partial D_1 = k_1$, $\partial D_2 = K_2$.

Performing the above in turn, we obtain the trivial link self Δ-equivalent to ℓ. □

Let us prove Theorem 4.2 by using Lemmas 2.2 and 4.1.

Theorem 4.2. *If ℓ is a boundary link in the strong sense, ℓ is self Δ-equivalent to the trivial link.*

Proof. Suppose that $\ell = k_1 \cup ... \cup k_n$ is a boundary link in the strong sense. Then there are mutually disjoint disks $D_1, ..., D_n$ with $\partial \mathcal{D} = \ell, \partial D_i = k_i$, where $\mathcal{D} = D_1 \cup ... \cup D_n$.

Any singularity of $\mathcal{D}$, denoted by $\mathcal{S}(\mathcal{D})$, can be deformed into simple (not self intersection) clasp singularity, Fig.11, which does not intersect with each other by deforming ℓ on $\mathcal{D}$ suitably, [9]. Hence we can assume that $\mathcal{S}(\mathcal{D})$ consists of mutually disjoint simple arcs of clasp type. For an arc α of $\mathcal{S}(D_i)$, the pre-images α^*,α'^* of α are said to be next on the pre-image D_i^* of D_i if there is an arc d of $\partial D_i^* - \alpha^* - \alpha'^*$ which does not contain a point of the pre-image of $\mathcal{S}(D_i)$, Fig.12. Suppose that α^*,α'^* are next on D_i^* and $d(\subset \partial D_i^*)$ an arc as above and let $\delta^* = \overline{\partial N(\alpha^* \cup \alpha'^* \cup d : D_i^*) - \partial D_i^*}$, Fig.12. Then $\delta^* \cap$ (the pre-image of $\mathcal{S}(D_i)$) $= \phi$ by the choice of α^* and α'^*. Performing the fission, [3], on D_i along δ, we obtain two disks D_{i0},D_{i1} on D_i such that ∂D_{i0} is a doubled knot and $\mathcal{S}(D_{i1}) = \mathcal{S}(D_i) - \alpha$,

Fig.12(b).

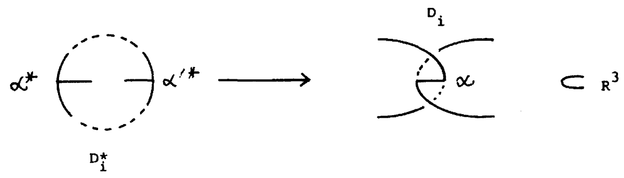

Fig.11

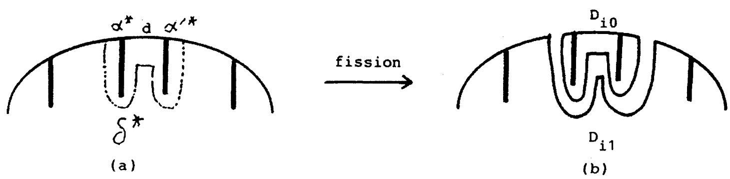

Fig.12

Next suppose that α^*, α'^* are not next on D_i^*, namely there are β^*,β'^*, the pre-image β of $\mathcal{S}(D_i)$, such that each of $\partial D_i^* - \alpha^* - \alpha'^*$ contains a point of β^* or one of β'^*. In this case, exchange α and β along a subarc of ∂D_i from Fig.13(a) to 13(b). This deformation can be accomplished by a finite sequence of self Δ-moves. We write the disk in Fig.13(b) by $\tilde{D}_i$. If the pre-images $\tilde{\alpha}^*$,$\tilde{\alpha'}^*$ of $\tilde{\alpha}$ of $\mathcal{S}(\tilde{D}_i)$, Fig.13(b), are next on $\tilde{D}_i^*$, we perform the above fission. If they are not next on $\tilde{D}_i^*$, we perform the deformations from Fig.13(a) to 13(b) again.

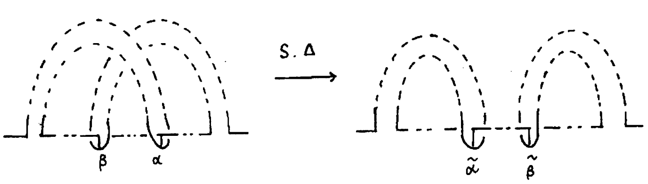

Fig.13

By doing the above discussions for each D_i successively, we obtain a doubled link L self Δ-equivalent to the trivial link by Lemma 4.1. Since L is obtained by a finite sequence of fissions of ℓ, ℓ is ribbon Δ-cobordant to the trivial link $\mathscr{O}$. Therefore ℓ is self Δ-equivalent to $\mathscr{O}$ by Lemma 2.2. □

By the following, there are links which are boundary links but not boundary links in the strong sense.

Proposition 4.3. *Let A be non-twisted annulus in R^3. If $\mathcal{S}(A) = \phi$, $\ell(= \partial A)$ is a boundary link. But if ℓ is not the trivial link, ℓ is not the boundary link in the strong sense.*

Proof. If A is non-twisted and $\mathcal{S}(A) = \phi$, i.e., A is non-singular, it is easily seen that $\ell = k_1 \cup k_2$ is a boundary link.

Suppose that ℓ is a boundary link in the strong sense. Namely there are mutually disjoint disks D_1, D_2 such that $\partial(D_1 \cup D_2) = \ell$, $\partial D_i = k_i$. Since $\mathcal{S}(A) = \phi$ and $D_1 \cap D_2 = \phi$, $A \cup D_1$ is a singular disk with $\partial(A \cup D_1) = k_2$ such that $k_2 \cap \mathcal{S}(A \cup D_1) = \phi$. Hence k_2 is the trivial knot,[4], [8], and so ℓ is the trivial link. Therefore, if ℓ is not the trivial link, ℓ is not the boundary link in the strong sense. □

Theorem 4.5. *Let ℓ be an n-component boundary link. If ℓ is $(n-1)$-trivial, ℓ is self Δ-equivalent to the trivial link.*

Proof. If $\ell = k_1 \cup ... \cup k_n$ is $(n-1)$-trivial, there is a knot, say k_1, such that $\ell - k_1$ is the trivial link. Hence there is an ambient isotopy φ of R^3 such that $\varphi(k_j) \subset R^2[j]$ for $j = 2, ..., n$.

As ℓ is a boudary link, $\varphi(\ell)$ is also a boudary link. We denote $\varphi(\ell)$,$\varphi(k_i)$ by ℓ,k_i again respectively and so there is a spanning surface $\mathcal{F} = F_1 \cup ... F_n$ of ℓ with $\partial F_i = k_i, i = 1, ..., n$. Let F_1 consist of a disk D_1 and a disjoint union of $2p$ bands $\mathscr{B}$ with $D_1 \cap \mathscr{B} = \partial D_1 \cap \partial\mathscr{B}$ =(4p arcs) for $p = g(F_1)$. Since k_j is trivial for $j = 2, ..., n$, we may assume that F_j is obtained by attaching tubes to a disk $D_j^*(\subset R^2[j])$ for $\partial D_j^* = k_j$. Let $B_j^3 = N(D_j^* : R^3)$. Then $B_2^3, ..., B_n^3$ are mutually disjoint 3-balls in R^3 such that $(\ell - k_1) \cap B_j^3 = k_j$. By the construction of F_j, we may assume that the connected component of $F_j \cap B_j^3$ whose boundary contains k_j is a perforated disk.

Since $F_1 \cap (\ell - K_1) = \phi$, we may assume that $F_1 \cap B_j^3 = \mathscr{B} \cap B_j^3$ by deforming F_1 suitably.

Let us denote $(\mathcal{F} - F_1) \cap \partial B_j^3$ by Γ_j. Then Γ_j consists of disjoint loops. If Γ_j contains a loop γ_0 such that there is a disk σ on ∂B_j^3 with $\partial\sigma = \gamma_0$ and $\sigma \cap \mathscr{B} = \phi$, we can eliminate γ_0 by attaching a 2-handle along to $\mathcal{F} - F_1$. Hence we may assume that, for each γ of Γ_j, each disk on ∂B_j^3 bounded by γ intersects with $\mathscr{B}$, Fig.14(a). Attach 2-handles to $\mathcal{F} - F_1$ along the disks on ∂B_j^3 bounded by loops of Γ_j, from an innermost loop of Γ_j in turn, illustrated in Fig.14(b). Then we obtain a non-singular disk in B_j^3 and spheres from $\mathcal{F} - F_1$ which are mutually disjoint.

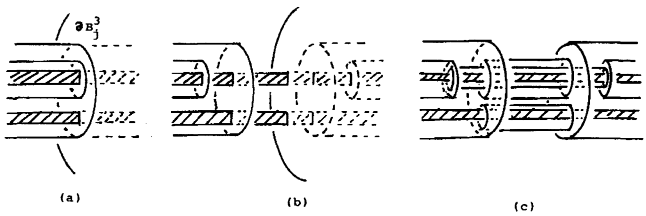

Fig.14

Now attach 1-handles $\mathcal{J}$ along the bands $\mathcal{B}$ such that, for each tube T of $\mathcal{J}$,$T \cap \partial B_j^3$ is a loop c_T and there is a disk σ_T of $\partial B_j^3 - c_T$ with $\sigma_T \cap \mathcal{B} = \{\text{an arc}\}$, Fig.14(c). As a result, we obtain a spanning surface $\mathcal{F}_0 = F_1 \cup F_{20} \cup ... \cup F_{n0}$ from $\mathcal{F} = F_1 \cup ... \cup F_n$ such that each F_{j0} consists of a perforated disk E_j in B_j^3 and tubes $\mathcal{J}_j(= F_{j0} \cap \mathcal{J})$.

Let D_j be the disk in B_j^3 with $\partial D_j = k_j$ and $D_j \supset E_j$ and $\mathcal{D} = D_2 \cup ... \cup D_n$ and T an outermost tube of $\mathcal{J}$, namely $T \cap \mathcal{D} = \partial T \cap \mathcal{D} = \{\text{two loops, say } \delta \text{ and } \delta'\}$. Suppose that $\partial T \subset D_2$. Let α be an arc on T which connects a point of δ and one of δ'. If α is knotted in $R^3 - D_2$, we may deform it unknotted by a finite sequence of self Δ-moves, for the Δ-move is an unknotting operation of knots, [5], and obtain an unknotted tube $\tilde{T}$ from T and a boundary link $\tilde{\ell}$ which is $(n-1)$-trivial and self Δ-equivalent to ℓ by Lemma 3.2. Therefore, by choosing $\tilde{T}$, $\tilde{\ell}$ instead of T, ℓ respectively, we may assume that T is unknotted. (If there is a tube T' of $\mathcal{J}$ which runs along the inside of T, we deform T' and obtain a tube $\tilde{T}'$ from T' such that $\tilde{T}'$ runs along the inside of $\tilde{T}$.)

Since T is an outermost and unknotted tube of $\mathcal{J}$, we may take a disk σ in R^3 such that $\sigma \cap T = \alpha$ and $\sigma \cap \mathcal{D} = \sigma \cap E_2 =\{\text{an arc}\}$. If $\sigma \cap (\mathcal{J} - T) = \phi$, we deform T along σ and eliminate T, Fig.15. Next assume that $\sigma \cap (\mathcal{J} - T) \neq \phi$. Then there is a tube T_0 of $\mathcal{J} - T$ such that $\sigma \cap T_0 \neq \phi$. Then there is a band B of $\mathcal{B}$ such that $\sigma \cap B \neq \phi$. Perform the fission, [3], of k_1 along $\sigma \cup T$ illustrated in Fig.16(b), we obtain a link $\ell_1 \cup \mathcal{L}_1$ from ℓ such that ℓ_1 is a boundary link and $(n-1)$-trivial and $\mathcal{L}_1$ is a 2-component trivial link, Fig.16(b). Next deform $\mathcal{L}_1$ towards to the foot of a band B_0 which runs through the inside of T, Fig.16(b),(c). As a result, we obtain a 2-component link $\tilde{\mathcal{L}}_1$ from $\mathcal{L}_1$. By doing the above fissions and deformations successively, we obtain a link $\ell_r \cup \tilde{\mathcal{L}}_1 \cup ... \cup \tilde{\mathcal{L}}_r$ from ℓ such that ℓ_r is a boundary link which is $(n-1)$-trivial with $\sigma \cap \ell_r = \phi$ and $\tilde{\mathcal{L}}_1 \cup ... \cup \tilde{\mathcal{L}}_r$ is a $2r$-component trivial link. Therefore, by deforming T along σ, we can eliminate T and so decrease $\#\mathcal{S}(\ell_r \cap \mathcal{D})(= \#\mathcal{S}(\ell \cap \mathcal{D}))$.

By performing the above discussion from an outermost tube of $\mathcal{J}$ in turn, we obtain a link $L = \mathcal{L} \cup k_2 \cup \ldots \cup k_n$ from ℓ such that $L - \mathcal{L}(= \ell - k_1)$ is the trivial link split from $\mathcal{L}$ and $\mathcal{L}$ is obtained by a finite sequence of fissions of k_1, Fig.17.

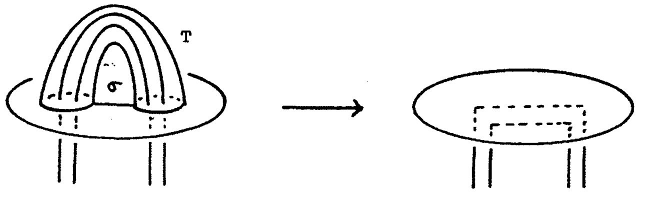

Fig.15

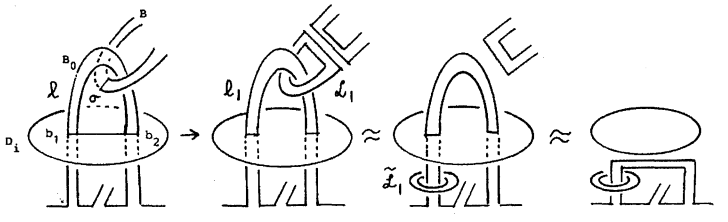

Fig.16

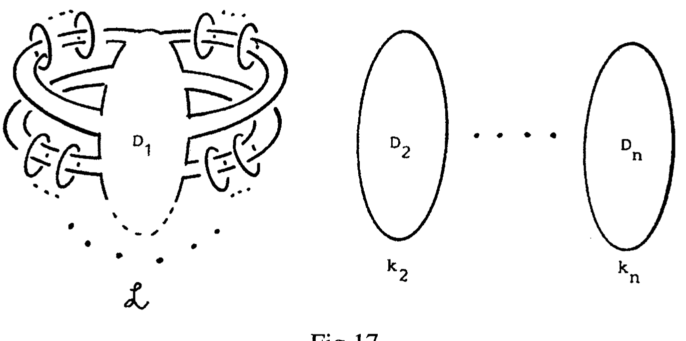

Fig.17

Since $\mathcal{L}$ is Δ-equivalent (not self) to the trivial link by [5], ℓ is ribbon Δ-cobordant to the trivial link. Hence ℓ is self Δ-equivalent to the trivial link by Lemma 2.2. □

Lastly, let us prove Theorem 4.6.

Theorem 4.6. *If $\ell = k_1 \cup k_2$ is a boundary link, ℓ is self Δ-equivalent to the trivial link.*

To prove it, it is enough to do the following by Theorem 4.5.

Lemma 4.7. *For any 2-component boundary link $\ell = k_1 \cup k_2$, ℓ is self Δ-equivalent to a boundary link $L = K_1 \cup K_2$ such that K_1 is the trivial knot.*

To prove Lemma 4.7, we prepare some Lemmas.

For an n-component boundary link $\tilde{\ell}$, a spanning surface $\mathcal{F} = F_1 \cup ... \cup F_n$ of $\tilde{\ell}$ is said to be normal if the bands of F_i are situated as illustrated in Fig.18 for each $i = 1, ..., n$.

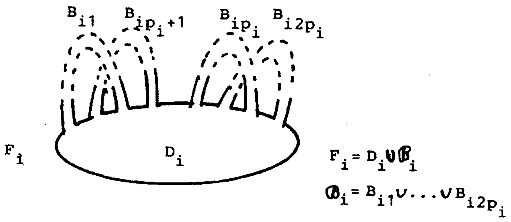

Fig.18

Since a Δ-move is a kind of unknotting operation of knots, [5], we obtain Lemma 4.8 by applying a finite sequence of self Δ-moves to each band B_{ij} of a normal surface $\mathcal{F}$ of ℓ and by using Lemma 3.2.

Lemma 4.8. *For a boundary link ℓ and a normal surface $\mathcal{F}$ of ℓ, there are a boundary link ℓ_0 self Δ-equivalent to ℓ and a normal surface $\mathcal{F}_0$ of ℓ_0 such that each band of $\mathcal{F}_0$ is unknotted.*

Since ℓ_0 is a boundary link self Δ-equivalent to ℓ, it is enough to prove Lemms 4.7 for ℓ_0.

Lemma 4.9. *Let ℓ_0, $\mathcal{F}_0$ be those of Lemma 4.8. Then there are a boundary link ℓ_1 self Δ-equivalent to ℓ_0 and a normal surface $\mathcal{F}_1$ of ℓ_1 such that each band of $\mathcal{F}_1$ is unknotted and non-twisted in R^3.*

Proof. Let $\mathcal{F}_0 = F_1 \cup F_2$ be a normal surface of ℓ_0 satisfying Lemma 4.8 and $B(= B_{1j})$, $B'(= B_{1p_1+j})$ a pair of bands of F_1 associated to a genus, Fig.19.

Suppose that B is twisted, Fig.19(a). Now we deform one full twist of B from Fig.19(a) to 19(b) and transform it along $\partial(B \cup B')$ from Fig.19(b) to 19(c),(d). During these transformations, a circle c can jump the other twists of $B \cup B'$ and subarcs of ∂B_{1i} by a finite sequence of self Δ-moves, Fig.3(a). We can eliminate the twist in

Fig.19(d) by a finite sequence of self Δ-moves from Fig.19(d) to 19(e),(f). The link in Fig.19(f) is ambient isotopic to that of Fig.19(h) whose spanning surface $\mathcal{F}_0' = F_1' \cup F_2'$ is normal. By these deformations, we can eliminate a full twist of B without increasing the number of full twists of the other bands of $\mathcal{F}_0'$. Moreover $\partial\mathcal{F}_0'$ is self Δ-equivalent to ℓ_0.

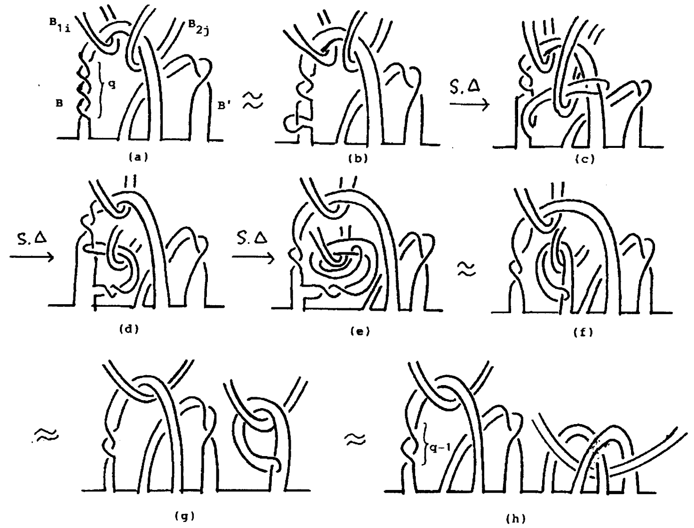

Fig.19

By repeating these deformations successively, we obtain a link ℓ_1 and a normal surface $\mathcal{F}_1$ of ℓ_1 satisfying Lemma 4.9. □

Let E^3 be a 3-ball and $\mathcal{L} = \kappa_1 \cup \kappa_2$ a 2-component link illustrated in Fig.20, where κ_i is the trivial knot and $\kappa_i \cap \partial E^3 (= \alpha_i)$ is an arc for $i = 1, 2$.

Lemma 4.10. *Let E^3,$\mathcal{L}$ and α_i be those of the above. Suppose that $L = K_1 \cup K_2$ be a 2-component link such that each K_i is the trivial knot with Link$(K_1, K_2) = q$ and $K_i \cap \partial E^3 = \alpha_i$. Then L can be deformed into $\mathcal{L}$ by a finite sequence of Δ-moves and an ambient isotopy of E^3 with $\alpha_1 \cup \alpha_2$ fixed.*

Proof. Since each K_i is trivial, there is a non-singular disk D_i in E^3 with $\partial D_i = K_i$ such that $\mathcal{S}(D_1 \cap D_2)$ consists of finite simple clasp singularities. The alternation of upper and under passes and the elimination of twists of bands, the change of order of clasps of $\mathcal{S}(D_1 \cap D_2)$ on D_1,D_2 can be realized by a finite sequenece of Δ-moves (not self Δ-moves) illustrated in Fig.21(a),(b) and (c) respectively. Since these Δ-moves can

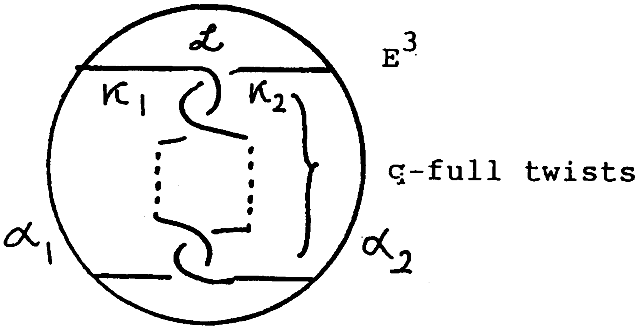

Fig.20

be done in the interior of E^3,$\alpha_1 \cup \alpha_2$ is fixed. Hence we obtain the link $\mathcal{L}$ illustrated in Fig.20 from L by a finite sequence of Δ-moves and ambient isotopy of E^3 with $\alpha_1 \cup \alpha_2$ fixed. □

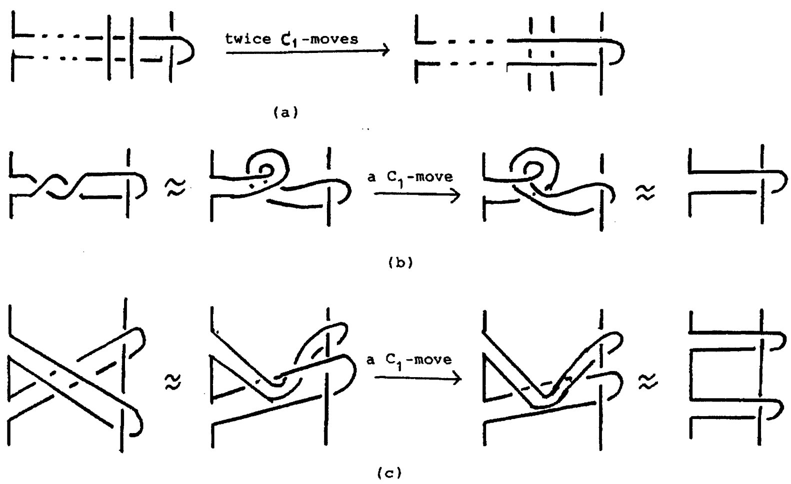

Fig.21

Let ℓ_1 and $\mathcal{F}_1(= F_1 \cup F_2)$ be those of Lemma 4.9. Two bands B_{ij},B_{ip_i+j} associated to a genus of $\mathcal{F}_1$ are said to be well-situated if B_{ij} and B_{ip_i+j} are situated illustrated in Fig.22. Moreover $\mathcal{F}_1$ is said to be well-situated if each pair B_{ij},B_{ip_i+j} of $\mathcal{F}_1$ is well-situated for $i = 1, 2$ and $j = 1, ..., p_i(= g(F_i))$.

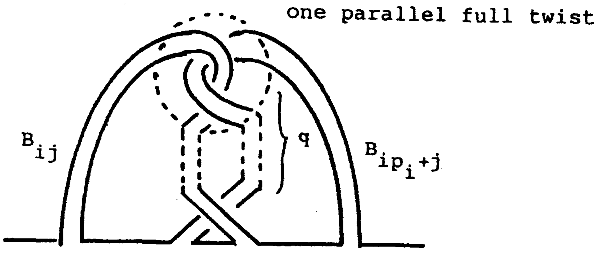

Fig.22

The following is easily obtained by Lemma 3.4 and 4.10.

Lemma 4.11. *Let ℓ_1 and $\mathcal{F}_1$ be those of Lemma 4.9. Then there are a boundary link ℓ_2 self Δ-equivalent to ℓ_1 and a normal surface $\mathcal{F}_2$ of ℓ_2 which is well-situated.*

Proof of Lemma 4.7. We write ℓ_2,$\mathcal{F}_2$ satisfying Lemma 4.11 by ℓ, $\mathcal{F}(= F_1 \cup F_2)$ again respectively and let F_i consist of a disk D_i and $2p_i$ bands B_{ij} of $\mathscr{B}_i$ illustrated in Fig.18.

Suppose that the bands B_{11},B_{1p_1+1} of $F_1(\subset \mathcal{F})$ have q parallel full twists, Fig.23 (a). Deform B_{1p_1+1} along B_{11} from Fig.23(a) to 23(b) and from 23(b) to 23(c) by an ambient isotopy φ of R^3 such that $\tilde{B}_{11} \subset D_1$, where $\tilde{B}_{ij} = \varphi(B_{ij})$. Let D_{11} be a disk of $\overline{D_1 - \tilde{B}_{11}}$ illustrated in Fig.23(c). If there is a band $\tilde{B}_{2i}$ such that $\tilde{B}_{2i} \cap D_{11} \neq \phi$, we deform $\tilde{B}_{2i}$ on $D_{11} \cup \tilde{B}_{1p_1+1}$ from Fig.23(c) to 23(d) and obtain $\tilde{B}'_{2i}$ from $\tilde{B}_{2i}$ satisfying that $\tilde{B}'_{2i} \cap D_{11} = \phi$. By repeating these deformations, we obtain a surface $\mathcal{F}' = F'_1 \cup F'_2$ from $\mathcal{F} = F_1 \cup F_2$ such that $F'_2 \cap D_{11} = \phi$. Then we can apply Lemma 3.3 to $\partial F'_1$ in Fig.23(d) and obtain the link ℓ' in Fig.23(e) by a finite sequence of self Δ-moves. After this, we deform $\tilde{B}_{11} \cap \tilde{B}'_{2i}$ into D_{11} along $\tilde{B}_{11}$ from Fig.23(e) to 23(f) and obtain a link ℓ'' which is ambient isotopic to ℓ' and so self Δ-equivalent to ℓ. Moreover we easily see that ℓ'' is the boundary link and ambient isotopic to the link $\ell^{(1)}$ illustrated in Fig.23(g). By applying the above deformations to B_{11}, B_{1p_1+1} successively, we obtain the bands $B_{11}^{(q)}$,$B_{1p_1+1}^{(q)}$ and the link $\ell^{(q)}$ illustrated in Fig.23(h) such that $\ell^{(q)}$ is the boudary link self Δ-equivalent to ℓ.

Now we denote the bands obtained by these deformations of $\mathscr{B}_i$ by $\mathscr{B}_i^{(q)}$. If the other band $B_{1s}^{(q)}(s \neq 1, p_1 + 1)$ of $\mathscr{B}_1^{(q)}$ does not link with $B_{11}^{(q)}$,$B_{11}^{(q)} \cup B_{1p_1+1}^{(q)}$ can be removed by an ambient isotopy of R^3, Fig.24(a),(b).

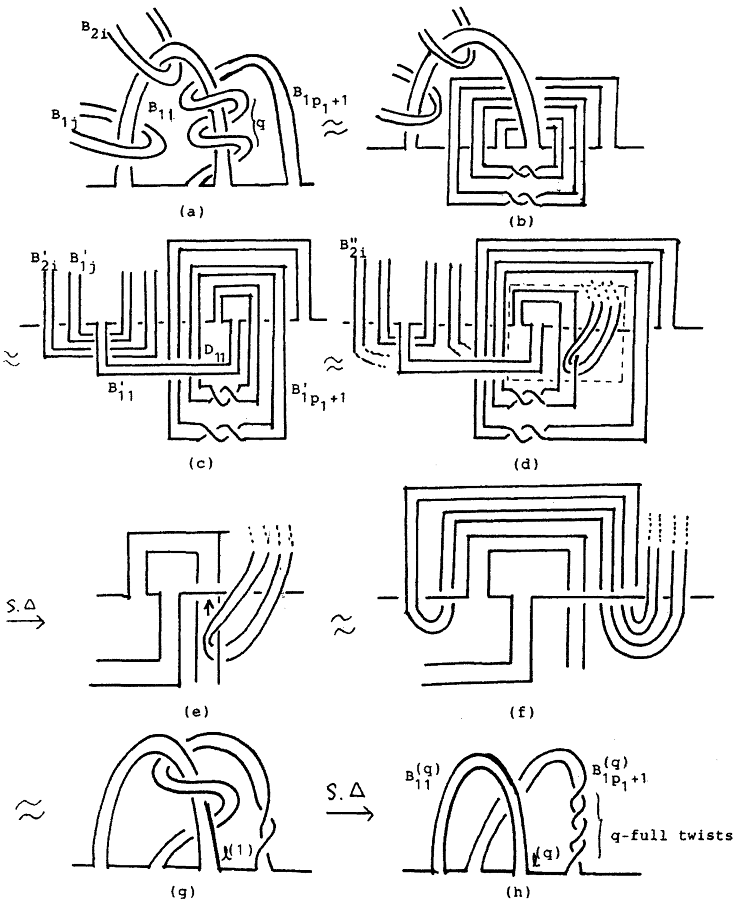

Fig.23

Next suppose that there is a band $B_{1s}^{(q)}(s \neq 1, p_1+1)$ which links with $B_{11}^{(q)}$. If there is a band $B_{2j}^{(q)}$ of $\mathscr{B}_2^{(q)}$ which links with $B_{11}^{(q)}$, deform $B_{2j}^{(q)}$ along $B_{1p_1+1}^{(q)}$ illustrated in Fig.25(a),(b).(If there is not a band $B_{2j}^{(q)}$ of $\mathscr{B}_2^{(q)}$ satisfying the above, it is not necessary to do the above deformation, Fig.25(a),(b).) After this deformation, we perform a finite sequence of self Δ-moves, Lemma 3.3, to $\mathscr{B}_1^{(q)}$ and obtain the bands $B_{11}^{(q)'}, B_{1p_1+1}^{(q)'}(\subset \mathscr{B}_1^{(q)'})$ from $B_{11}^{(q)}, B_{1p_1+1}^{(q)}(\subset \mathscr{B}_1^{(q)})$ such that $B_{1p_1+1}^{(q)'}$ is non-twisted and does not link with the other bands of $\mathscr{B}_1^{(q)'}$, Fig.25(b),(c) and deform $B_{2j}^{(q)'}$ along $B_{1p_1+1}^{(q)'}$ from Fig.25(c) to 25(d) and apply the deformation illustrated in Fig.24 to $B_{1p_1+1}^{(q)'}$ instead of $B_{11}^{(q)}$.

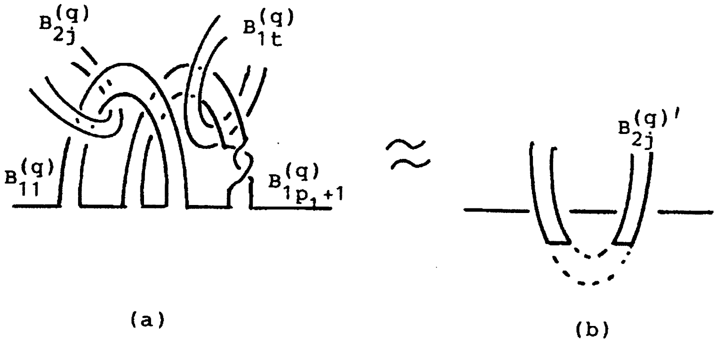

Fig.24

By doing the above discussions for each band of $\mathscr{B}_1$, we obtain a link $L = K_1 \cup K_2$ such that L is self Δ-equivalent to ℓ and K_1 is the trivial knot. Moreover we easily see that L is the boundary link by the above deformatinos.

Now the proof of Lemma 4.7 is complete. □

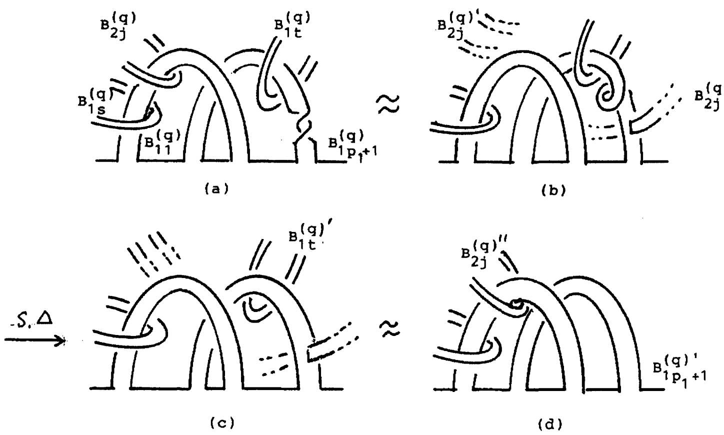

Fig.25

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