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INTEGRAL HOMOLOGY OF REAL ISOTROPIC AND ODD ORTHOGONAL GRASSMANNIANS

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Abstract

We obtain a combinatorial expression for the boundary map coefficients of real isotropic and odd orthogonal Grassmannians. It provides a natural generalization of the known formulas for Lagrangian and maximal isotropic Grassmannians. The results derive from the classification of Schubert cells into four types of covering pairs when identified with signed k-Grassmannian permutations. Our formulas show that the coefficients depend on the changed positions for each permutation pair type. We apply this to obtain an orientability criterion and compute the first and second homology groups for these Grassmannians. Furthermore, we exhibit an apparent symmetry of the boundary map coefficients.

1. Introduction

Let G be the odd orthogonal group $\mathrm{SO}(n, n+1)$ or the symplectic group $\mathrm{Sp}(n,\mathbb{R})$ with Lie algebra $\mathfrak{g}$ given by $\mathfrak{so}(n, n+1)$ or $\mathfrak{sp}(n,\mathbb{R})$, respectively. We denote the root system by $\Sigma = \{a_0,\ a_1 = \varepsilon_2 - \varepsilon_1\ ,\dots,\ a_{n-1} = \varepsilon_n - \varepsilon_{n-1}\}$ such that $a_0 = \varepsilon_1$ for type B and $a_0 = 2\varepsilon_1$ for type C. The Grassmannians are the minimal flag manifolds $G/P_{(k)}$, $0 \le k \le n-1$, where $P_{(k)}$ is the parabolic subgroup corresponding to the maximal proper subset $(k) = \Sigma - \{a_k\}$. If $G = \mathrm{SO}(n, n+1)$ then the odd orthogonal Grassmannian $\mathrm{OG}(n-k, 2n+1)$ is the set of $(n-k)$-dimensional isotropic subspaces in the vector space $V = \mathbb{R}^{2n+1}$ equipped with a nondegenerate symmetric bilinear form. If $G = \mathrm{Sp}(n,\mathbb{R})$ then the isotropic Grassmannian $\mathrm{IG}(n-k, 2n)$ is the set of $(n-k)$-dimensional isotropic subspaces in the symplectic vector space $V = \mathbb{R}^{2n}$.

The Schubert cells provide a cellular structure of these Grassmannians via Bruhat decomposition. In addition, the minimal representatives $\mathcal{W}_n^{(k)}$ of the Weyl group modulo the subgroup generated by reflections in (k) parametrize such Schubert cells. We use a permutation model to identify cells with the set of signed k-Grassmannians permutations, i.e., signed permutations of the form $w = u_1 \cdots u_k|\overline{\lambda_r} \cdots \overline{\lambda_1}\, v_1 \cdots v_{n-k-r}$, where $0 \leqslant r \leqslant n-k$, $0 < u_1 < \cdots < u_k, 0 < \lambda_1 < \cdots < \lambda_r$ and $0 < v_1 < \cdots < v_{n-k-r}$. This approach has been employed in related problems of Schubert calculus of such Grassmannians (see Pragacz-Ratajski [11], Buch-Kresch-Tamvakis [2] and Ikeda-Matsumura [6]). Our setting is based on the paper by both authors [9] where the covering pairs $w, w' \in \mathcal{W}_n^{(k)}$, i.e., those satisfying $w' \le w$ in the Bruhat-Chevalley order with $\ell(w) = \ell(w') + 1$, are classified into four types

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(B1, B2, B3 and B4). We associate with each permutation $w \in \mathcal{W}_n^{(k)}$ a double partition (α, λ) for which we have the corresponding half-shifted Young diagram (HSYD).

The cellular homology appears after computing the boundary map coefficients. In this paper, we provide an explicit formula for the coefficients of those real Grassmannians, generalizing the second author's results in [12] for the Lagrangian and maximal isotropic Grassmannians. This contributes to the study of the topology of real flag manifolds in comparison with the complex ones which have torsion-free homology groups. Our formulas are compatible with Burghelea-Hangan-Moscovici-Verona [3] which gave the first results about the topology of real flag manifolds in the 1970s.

San Martin and the second author have found in [13] that the set $\Pi_w = \Pi^+ \cap w\Pi^-$ contains the data required to determine these coefficients. The inspiration for our approach came from Ikeda-Naruse [7] and Graham-Kreiman [5] where Π_w appears as the shifted Young diagrams in the context of the maximal isotropic Grassmannians. These ideas were generalized here for odd orthogonal and isotropic Grassmannians through the half-shifted Young diagram associated with double partition (α, λ) of w. We characterize Π_w as a set of inversions of w and assign to each inversion a single box in its half-shifted Young diagram (cf. Sections 2.3 and 3.1).

Once the inversion set Π_w is known and the covering pairs w, w' are described, it remains to compute $\sigma(w) - \sigma(w')$, where $\sigma(w)$ is the sum over the roots of Π_w. Since this is a multiple $\kappa(w, w')$ of a unique root, the boundary map coefficient $c(w, w')$ is either 0 or ± 2 depending on the parity of $\kappa(w, w')$. It is worth pointing out that our Theorem 3.10 implies that $c(w, w')$ is determined by a well-defined triple (P, T, Q) composed of altered entries in permutations w and w'. Besides its simplicity, it represents a new interpretation of these coefficients as presented in Theorem 3.12, which relates κ with the height of a given root.

As a direct consequence of our formulas, we obtain an orientability criterion (see Proposition 3.13). Furthermore, we remark that the incidence graph, whose vertices are the permutations of $\mathcal{W}_n^{(k)}$ and the edges correspond to the coefficients between them, admits an apparent symmetry (see Figure 8 and Proposition 3.14). We finish by computing the 1st and 2nd homology groups of isotropic and odd orthogonal Grassmannians (see Section 3.5). This provides a combinatorial parallel to the results of Patrão-San Martin-Santos-Seco [10] concerning the orientability and the results of del Barco-San Martin [4] regarding the second de Rham cohomology group.

The article is organized as follows. Section 2 describes the isotropic and odd orthogonal Grassmannian, some properties of covering relations, and the half-shifted Young diagrams. In Section 3, we outline the boundary maps of odd orthogonal and isotropic Grassmannian and the inversion sets of such Grassmannians. We also state the main result, state and prove the results about orientability and symmetry. Section 4 is devoted to proving the main result. We conclude with Section 5 where we provide some perspectives for future work.

2. Isotropic and odd orthogonal Grassmannians

For $n, m \in \mathbb{Z}$, where $n \leqslant m$, denote the set $[n, m] = \{n, n+1, \dots, m\}$. For $n \in \mathbb{N}$, denote $[n] = [1, n]$.

Let G be a non-compact split semi-simple Lie group. Denote by Π the set of roots related to the Lie algebra $\mathfrak{g}$, $\Pi^{\pm}$ the set of positive and negative roots. Fix a simple root system

$\Sigma \subset \Pi$. Let P be the corresponding minimal parabolic subgroup with Lie algebra $\mathfrak{p}$. We call $\mathbb{F} = G/P$ the maximal flag manifold of G. The Weyl group $\mathcal{W}$ of G is the group generated by simple reflections $s_i = s_{a_i}$ through simple roots $a_i \in \Sigma$. The length $\ell(w)$ of $w \in \mathcal{W}$ is the number of simple reflections in any reduced decomposition of w. There is a partial order in the Weyl group called the Bruhat-Chevalley order: we say that $w_1 \leqslant w_2$ if given a reduced decomposition $w_2 = s_{j_1} \cdots s_{j_r}$ then $w_1 = s_{j_{i_1}} \cdots s_{j_{i_k}}$ for some $1 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant r$. It is known that $\mathcal{W}$ has a maximum element w_0, which is an involution, i.e., $w_0^2 = 1$.

A subset of simple roots $\Theta \subset \Sigma$ is associated with a parabolic subgroup P_Θ in G, which contains P. The corresponding homogeneous space $\mathbb{F}_\Theta = G/P_\Theta$ is called a partial flag manifold of G. The subgroup $\mathcal{W}_\Theta$ is the subgroup of the Weyl group $\mathcal{W}$ generated by the reflections with respect to the roots $\alpha \in \Theta$. We also define the subset $\mathcal{W}^\Theta$ of $\mathcal{W}$ by $\mathcal{W}^\Theta = \{w \in \mathcal{W} \colon \ell(ws_a) = \ell(w) + 1\ ,\ \forall a \in \Theta\}$. Since there exists a unique element $w^\Theta \in \mathcal{W}^\Theta$ of minimal length in each coset $w\mathcal{W}_\Theta$, $\mathcal{W}^\Theta$ is called the subset of minimal representatives of the cosets of $\mathcal{W}_\Theta$ in $\mathcal{W}$.

The Bruhat decomposition presents the flag manifolds as disjoint union $\mathbb{F}_\Theta = \coprod_{w \in \mathcal{W}^\Theta} N \cdot wb_\Theta$ where N is the nilpotent subgroup in the Iwasawa decomposition of G. A Schubert variety is the closure of a Bruhat cell, i.e., $\mathcal{S}_w = \mathrm{cl}(N \cdot wb_\Theta)$. The choice of a minimal representative $w \in \mathcal{W}^\Theta$ gives $\dim(\mathcal{S}_w) = \ell(w)$ since we are in a split case. The Bruhat-Chevalley order defines an order between the Schubert varieties by $\mathcal{S}_{w_1} \subset \mathcal{S}_{w_2}$ if, and only if, $w_1 \leqslant w_2$.

First, consider the symplectic group $\mathrm{Sp}(n, \mathbb{R})$ with Lie algebra $\mathfrak{sp}(n, \mathbb{R})$ of type C. The root system of type C is realized as a set of vectors $\Pi = \{\pm\varepsilon_i \pm \varepsilon_j \colon 1 \leqslant i < j \leqslant n\} \cup \{\pm 2\varepsilon_i \colon 1 \leqslant i \leqslant n\}$ in the Euclidean space $\mathbb{R}^n = \oplus_{i=1}^n \mathbb{R}\varepsilon_i$. Denote the (positive) simple roots by $a_0 = 2\varepsilon_1$ and $a_i = \varepsilon_{i+1} - \varepsilon_i$ for $1 \leqslant i < n$. Then, the set of all positive roots is $\Pi^+ = \{\varepsilon_j \pm \varepsilon_i \colon 1 \leqslant i < j \leqslant n\} \cup \{2\varepsilon_i \colon 1 \leqslant i \leqslant n\}$. Given $k \in [0, n-1]$, the minimal flag manifold $\mathrm{IG}(n-k, 2n) = \mathrm{Sp}(n, \mathbb{R})/P_{(k)}$, for $(k) = \Sigma - \{a_k\}$, is called an *isotropic Grassmannian* since it parametrizes $(n-k)$-dimensional isotropic subspaces of a real $2n$-dimensional symplectic vector space.

Now, consider the orthogonal group $\mathrm{SO}(n, n+1)$ with Lie algebra $\mathfrak{so}(n, n+1)$ of type B. The root system of type B is realized as a set of vectors $\Pi = \{\pm\varepsilon_i \pm \varepsilon_j \colon 1 \leqslant i < j \leqslant n\} \cup \{\pm\varepsilon_i \colon 1 \leqslant i \leqslant n\}$ in the Euclidean space $\mathbb{R}^n = \oplus_{i=1}^n \mathbb{R}\varepsilon_i$. Denote the (positive) simple roots by $a_0 = \varepsilon_1$ and $a_i = \varepsilon_{i+1} - \varepsilon_i$ for $1 \leqslant i < n$. Then, the set of all positive roots is $\Pi^+ = \{\varepsilon_j \pm \varepsilon_i \colon 1 \leqslant i < j \leqslant n\} \cup \{\varepsilon_i \colon 1 \leqslant i \leqslant n\}$. Given $k \in [0, n-1]$, the minimal flag manifold $\mathrm{OG}(n-k, 2n+1) = \mathrm{SO}(n, n+1)/P_{(k)}$, for $(k) = \Sigma - \{a_k\}$, is called an *odd orthogonal Grassmannian* since it parametrizes $(n-k)$-dimensional isotropic subspaces of a real $(2n+1)$-dimensional vector space equipped with a nondegenerate symmetric bilinear form.

2.1. Permutation model. For both types of Grassmannians, we denote $s_i = s_{a_i}$, for $i \in [0, n-1]$, the simple reflection given by the simple root a_i. The Weyl group $\mathcal{W}_n$ for the root systems B_n and C_n, also called the *hyperoctahedral group*, is the semidirect product $S_n \ltimes \mathbb{Z}_2^n$. We also realize it as the set of permutations with a sign (plus or minus) attached to each entry. Write these elements as barred permutations, where the bar denotes a negative sign and $\overline{n} < \cdots < \overline{1} < 1 < \cdots < n$ as usual. Then, a permutation $w \in \mathcal{W}_n$, usually denoted in one-line notation $w = w(1)\, w(2) \cdots w(n)$, satisfies the relation $\overline{w(i)} = w(\overline{i})$. With respect to this realization, the length of $w \in \mathcal{W}_n$ is given by the following formula ([1], Equation (8.3))

(1) $$\ell(w) = \mathrm{inv}(w(1), \dots, w(n)) - \sum_{\{j \in [n]:\ w(j)<0\}} w(j)$$

where $\mathrm{inv}(w(1), \dots, w(n)) = \#\{(i, j) \colon 1 \le i < j \le n, w(i) > w(j)\}$.

The simple reflections $s_0, \dots, s_{n-1}$ act on the right of a permutation w in $\mathcal{W}_n$ by

$$w(1)\, w(2) \cdots w(n) \cdot s_0 = \overline{w(1)}\, w(2) \cdots w(n);$$
$$w(1) \cdots w(i) w(i+1) \cdots w(n) \cdot s_i = w(1) \cdots w(i+1) w(i) \cdots w(n)\ ,\ 1 \leqslant i < n.$$

The hyperoctahedral group $\mathcal{W}_n$ is also the Coxeter group of type B generated by s_i and subject to the relations: (i) $s_i^2 = 1$, for $i \ge 0$; (ii) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$; (iii) $s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i$, for $i \in [n-1]$; (iv) $s_i s_j = s_j s_i$, for $|i - j| \geqslant 2$.

As defined above, for $(k) = \Sigma - \{a_k\}$, the corresponding subgroup $\mathcal{W}_{(k)}$ is generated by s_i, with $i \neq k$. Notice that $\mathcal{W}_{(k)} \cong \mathcal{W}_k \times S_{n-k}$, where $\mathcal{W}_k$ is the subgroup generated by $\{s_0, s_1, \dots, s_k\}$. Define by $\mathcal{W}_n^{(k)} \subset \mathcal{W}_n$ the set of minimal length coset representatives of $\mathcal{W}_n/\mathcal{W}_{(k)}$, which parametrizes the Schubert varieties in $\mathrm{IG}(n-k, 2n)$ and $\mathrm{OG}(n-k, 2n+1)$. This indexing set $\mathcal{W}_n^{(k)}$ can be identified by a set of signed permutations of the form

(2) $$w = w_{u,\lambda} = u_1 \ \cdots \ u_k | \overline{\lambda_r} \ \cdots \ \overline{\lambda_1}\, v_1 \ \cdots \ v_{n-k-r}$$

where $0 \leqslant r \leqslant n-k$ and

(3) $$\begin{aligned} &0 < u_1 < \cdots < u_k\ , && u_i = w(i),\ \text{for } i \in [k];\\ &0 < \lambda_1 < \cdots < \lambda_r\ , && \overline{\lambda_i} = w(k+r-i+1),\ \text{for } i \in [r];\\ &0 < v_1 < \cdots < v_{n-k-r}\ , && v_i = w(k+r+i),\ \text{for } i \in [n-k-r]. \end{aligned}$$

They are called signed k-Grassmannian permutations. The longest element $w_0^k \in \mathcal{W}_n^{(k)}$ is the k-Grassmannian permutation given by

(4) $$w_0^k = 1\, 2 \ \cdots \ k | \overline{n}\, \overline{n-1} \ \cdots \ \overline{k+1}.$$

2.2. Bruhat order and covering relations. We recall some results of [9] about the Bruhat-Chevalley order of $\mathcal{W}_n^{(k)}$ in the permutation model. Let $w, w' \in \mathcal{W}_n$ with $w' \leqslant w$ and $\ell(w) = \ell(w') + 1$. We say that it is a covering relation where w covers w'. By [9], there is a classification of pairs w, w' where w covers w' in $\mathcal{W}_n^{(k)}$. Suppose that w and w' are permutations in $\mathcal{W}_n^{(k)}$ according to Equation (2).

We say that w, w' is *a pair of type B1* if they are written as follows:

$$w = \cdots | \cdots \overline{1} \cdots \qquad \text{and} \qquad w' = \cdots | \cdots 1 \cdots .$$

In other words, if w is such that $\lambda_1 = 1$, then w' is obtained from w by removing the negative sign from $\overline{1}$.

We say that w, w' is *a pair of type B2* if they are written as follows:

$$w = \cdots | \cdots \overline{a} \cdots (a-1) \cdots \qquad \text{and} \qquad w' = \cdots | \cdots \overline{a-1} \cdots a \cdots ,$$

where $a > 0$. In other words, there are $t \in [r]$ and $q \in [n-k-r]$ such that $\lambda_t = a$ and $v_q = a - 1$, and w' is obtained from w by switching v_q and λ_t.

We say that w, w' is *a pair of type B3* if they are written as follows:

$$w = \cdots a \cdots | \cdots (a-x) \cdots \qquad \text{and} \qquad w' = \cdots (a-x) \cdots | \cdots a \cdots ,$$

where $a > x > 0$. In other words, there are $p \in [k]$ and $q \in [n-k-r]$ such that $u_p = a$ and $v_q = a - x$. The permutation w' is obtained from w by switching u_p and v_q.

Finally, we say that w, w' is *a pair of type B4* if they are written as follows:

$$w = \cdots (a-x) \cdots | \cdots \overline{a} \cdots \quad \text{and} \quad w' = \cdots a \cdots | \cdots \overline{a-x} \cdots ,$$

where $a > x > 0$. In other words, there are $p \in [k]$ and $t \in [r]$ such that $u_p = a - x$ and $\lambda_t = a$. The permutation w' is obtained from w by switching u_p and λ_t.

We will denote the type of a pair by $\mathsf{Type}(w, w')$. For each type of pair w, w', also consider the integers $P = P(w, w')$, $T = T(w, w')$, and $Q = Q(w, w')$ which correspond to the positions where w changes when compared to w'. They can be represented as a right action in the complete notation as follows:

$$\text{(5)} \qquad \begin{array}{ll} \text{(B1)} & w = w' \cdot (T, \overline{T}); \\ \text{(B2)} & w = w' \cdot (\overline{T}, Q)(\overline{Q}, T); \\ \text{(B3)} & w = w' \cdot (P, Q)(\overline{P}, \overline{Q}); \\ \text{(B4)} & w = w' \cdot (\overline{P}, T)(\overline{T}, P). \end{array}$$

Precisely, the integers P, T, and Q are given by Table 1.

Table 1. Integers P, T, and Q for a pair w, w'

$\mathsf{Type}(w, w')$	P	T	Q
B1		$w^{-1}(\overline{1}) = k + r$	
B2		$w^{-1}(\overline{\lambda_t}) = k + r - t + 1$	$w^{-1}(v_q) = k + r + q$
B3	$w^{-1}(u_p) = p$		$w^{-1}(v_q) = k + r + q$
B4	$w^{-1}(u_p) = p$	$w^{-1}(\overline{\lambda_t}) = k + r - t + 1$	

There is a straight relationship between types of pairs and covering relations.

Theorem 2.1 ([9], Theorem 5)**.** *Let $w, w' \in \mathcal{W}_n^{(k)}$. Then w covers w' if, and only if, $\mathsf{Type}(w, w')$ is* B1*,* B2*,* B3 *or* B4.

REMARK 2.2. In Ikeda [6] (Lemmas 3.1 and 3.2) there is a description of the covering relation with respect to the weak Bruhat order. The above result refines it to the strong Bruhat order. For another approach of the covering relation in terms of k-strict partitions we refer to Tamvakis-Wilson [14].

EXAMPLE 1. Consider $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4$ where $n = 7$ and $k = 2$. Theorem 2.1 guarantees that there are five permutations covered by w:

(B1) $w = 2\,6|\overline{7}\,\overline{5}\,\overline{\mathbf{1}}\,3\,4$, $w'_1 = 2\,6|\overline{7}\,\overline{5}\,\mathbf{1}\,3\,4$ and $T = 5$;

(B2) $w = 2\,6|\overline{7}\,\overline{\mathbf{5}}\,\overline{1}\,3\,\mathbf{4}$, $w'_2 = 2\,6|\overline{7}\,\overline{\mathbf{4}}\,\overline{1}\,3\,\mathbf{5}$ and $(T, Q) = (4, 7)$;

(B3) $w = 2\,\mathbf{6}|\overline{7}\,\overline{5}\,\overline{1}\,3\,\mathbf{4}$, $w'_3 = 2\,\mathbf{4}|\overline{7}\,\overline{5}\,\overline{1}\,3\,\mathbf{6}$ and $(P, Q) = (2, 7)$;

(B4) $w = 2\,\mathbf{6}|\overline{\mathbf{7}}\,\overline{5}\,\overline{1}\,3\,4$, $w'_4 = 2\,\mathbf{7}|\overline{\mathbf{6}}\,\overline{5}\,\overline{1}\,3\,4$ and $(P, T) = (2, 3)$;

(B4) $w = \mathbf{2}\,6|\overline{7}\,\overline{\mathbf{5}}\,\overline{1}\,3\,4$, $w'_5 = \mathbf{5}\,6|\overline{7}\,\overline{\mathbf{2}}\,\overline{1}\,3\,4$ and $(P, T) = (1, 4)$.

2.3. Double partitions and half-shifted Young diagrams. In this section, we will understand how a k-Grassmannian permutation gives rise to a double partition. In this way, there is a bijection between the set of all double partitions and $\mathcal{W}_n^{(k)}$. Moreover, we will show how each double partition may be represented by a half-shifted Young diagram.

Given n, k integers such that $0 \leqslant k < n$, we say that $\Lambda = (\alpha, \lambda)$ is a double partition when $\alpha = (0 \leqslant \alpha_1 \leq \cdots \leq \alpha_k \leqslant n-k)$ is a partition and $\lambda = (0 < \lambda_1 < \cdots < \lambda_r \leqslant n)$ is a strict partition (if $k = 0$ then α is represented as an empty set; an empty λ is represented by $r = 0$).

We define the Young diagram associated to the partition α by

$$\mathcal{D}_\alpha = \{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq k\,,\ 1 \leq j \leq \alpha_i\} \tag{6}$$

and the shifted Young diagram associated to the strict partition λ by

$$\mathcal{SD}_\lambda = \{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq r\ ,\ i \leq j \leq i - 1 + \lambda_{r+1-i}\}. \tag{7}$$

We can insert the diagram $\mathcal{D}_\alpha$ into a $k \times (n-k)$ rectangle while the shifted Young diagram $\mathcal{SD}_\lambda$ fits inside a stair shaped triangle with n rows. Let us denote by $\mathcal{D}_{k,n}$ the set of all partitions whose respective Young diagrams are inside a $k \times (n-k)$ rectangle and by $\mathcal{SD}_n$ the set of all strict partitions whose respective shifted Young diagrams are inside a stair shaped diagram of length n. We denote by $\mathcal{P}(k, n)$ the set of the pairs (α, λ) with $\alpha \in \mathcal{D}_{k,n}$ and $\lambda \in \mathcal{SD}_n$ with the property $\ell(\lambda) \leq \alpha_1$, i.e.,

$$\begin{aligned} 0 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq n-k\,; \\ 0 < \lambda_1 < \cdots < \lambda_r \leqslant n\,; \\ \ell(\lambda) \leq \alpha_1. \end{aligned} \tag{8}$$

Notice in this definition that the strict partition λ is empty when $\alpha_1 = 0$.

A half-shifted Young Diagram (HSYD) of the double partition $\Lambda = (\alpha, \lambda) \in \mathcal{P}(k, n)$ is obtained by the juxtaposition of the diagrams $\mathcal{D}_\alpha$ and $\mathcal{SD}_\lambda$ such that $\mathcal{D}_\alpha$ is above $\mathcal{SD}_\lambda$. We say that $\mathcal{D}_\alpha$ is the top diagram and that $\mathcal{SD}_\lambda$ is the bottom diagram. We refer to the rows and columns of the top (bottom) diagram as the top (bottom) rows and top (bottom) columns, respectively. The condition $\alpha_1 \geq \ell(\lambda)$ is equivalent to say that the number of bottom rows is at most the number of fulfilled top columns. Figure 1 shows the HSYD for the pair $\alpha = (3, 5)$ and $\lambda = (1, 5, 7)$.

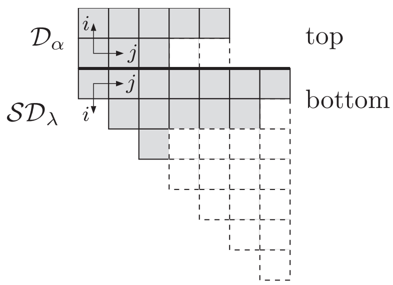


Fig. 1. A model of a HSYD obtained as a juxtaposition of the diagrams $\mathcal{D}_\alpha$ and $\mathcal{SD}_\lambda$.

Now, given a k-Grassmannian permutation in one-line notation as in (2), let us define its corresponding pair of double partitions α and λ. The strict partition is given by the negative part of w, i.e., $\lambda = (\lambda_r > \cdots > \lambda_1 > 0)$. The partition α is defined by $\alpha_i = u_i - i + d_i$, for $i \in [k]$, where $d_i = \#\{\lambda_j \mid \lambda_j > u_i\}$. As shown in Equation (2.10) of [9],

$$\alpha_i = n - k - \mu_i, \tag{9}$$

where $\mu_i = \mu_i(w) = \#\{v_j \mid v_j > u_i\}$, for $i \in [k]$. Since μ_i can be written as $\mu_i = n-k-r-\#\{v_j \mid v_j < u_i\}$, it follows that

$$\alpha_i = r + \#\{v_j \mid v_j < u_i\} \tag{10}$$

and α satisfies $n - k \geq \alpha_k \geq \alpha_{k-1} \geq \cdots \geq \alpha_1 \geq 0$.

The partition α counts the number of inversions of the first k entries of w while the strict partition λ the number of remaining inversions given by the negative entries of w. Hence, given $w \in \mathcal{W}_n^{(k)}$, the length $\ell(w)$ of w is the sum of entries of the pair (α, λ), i.e., $\ell(w) = |\alpha|+|\lambda|$.

Lemma 2.3 ([11], Lemma 1.2)**.** *There is a bijection between $\mathcal{W}_n^{(k)}$ and $\mathcal{P}(k, n)$.*

Given a partition $\alpha = (0 \leqslant \alpha_1 \leq \cdots \leq \alpha_k \leqslant n-k)$, the *conjugate partition* α^* of α is the partition defined by $\alpha_i^* = \#\{\alpha_j \colon \alpha_j \geqslant i\}$, for all $i \in [n-k]$ satisfying $k \geqslant \alpha_1^* \geq \cdots \geq \alpha_{n-k}^* \geqslant 0$. In other word, the conjugate α^* is given by the number of boxes in each column of the Young diagram $\mathcal{D}_\alpha$ of α.

If we denote $\mu_i^* = \#\{u_j \colon u_j > v_i\}$ for $i \in [n-k-r]$, then we have an explicit formula for α^*.

Lemma 2.4. *The conjugate partition is given by*

$$\alpha_i^* = \begin{cases} k & \text{if } 1 \leqslant i \leqslant r; \\ \mu_{i-r}^* & \text{if } r+1 \leqslant i \leqslant n-k. \end{cases} \tag{11}$$

Proof. Suppose that $1 \leqslant i \leqslant r$. We clearly have $\alpha_i^* = k$ since $\alpha_j \geqslant r$ for every j. Now, suppose that $r+1 \leq i \leq n-k$. By (10), $\#\{v_l \colon v_l < u_j\} = \alpha_j - r$ for $1 \leqslant j \leqslant k$, which is equivalent to $v_1 < v_2 < \cdots < v_{\alpha_j - r} < u_j$. Hence, $\alpha_j \geqslant i$ if, and only if, $v_{i-r} < u_j$. We conclude that $\alpha_i^* = \#\{\alpha_j \colon \alpha_j \geq i\} = \#\{u_j \colon v_{i-r} < u_j\} = \mu_{i-r}^*$. □

We now present an adaptation of a method introduced by [2] which provides the bijection in Lemma 2.3. We can label each bottom column as follows: given $t \in [n]$, the t-th bottom column is

- h-related if there exists $i \in [k]$ such that $t = \alpha_i + i$;
- v-related if there exists $j \in [n-k]$ such that $t = k + j - \alpha_j^*$. If such j exists then it must be unique.

There is a geometric interpretation in the HSYD: choose some bottom column and draw a 45-degree northwest line from the center of the first box in this column. If the line hits the last box in a top row, then the bottom column is h-related. Otherwise, it is v-related.

For instance, consider the permutation $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4$ in Example 1. Figure 2 exhibits the corresponding double partition $\alpha = (3, 5)$ and $\lambda = (1, 5, 7)$ for w as well how the h-related columns are in bijection with top rows and how the v-related columns are in bijection with top columns.[1] These bijections are established in the next proposition.

[1]The terms h-related (short for "horizontally" related) and v-related (short for "vertically" related) were motivated by the bijection with the rows and column, resp.

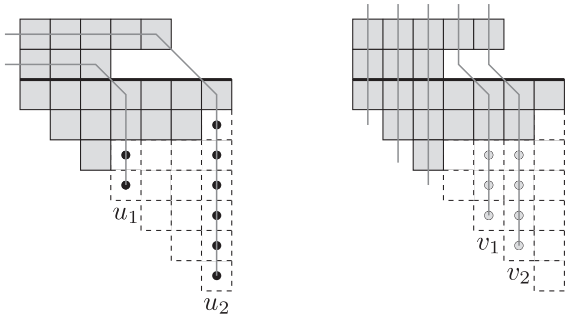


Fig. 2. On the left, we represent the h-related columns for $\alpha = (3, 5)$ and $\lambda = (1, 5, 7)$. On the right, we represent the v-related columns. The vacant length of a bottom column is the number of dots in the respective column.

Proposition 2.5.

(i) *There is a bijection between h-related columns and top rows;*

(ii) *There is a bijection between v-related columns and top columns;*

(iii) *Any bottom column is either h-related or v-related.*

Proof. For statement (i), suppose that there are two rows $i < i'$ related to the same h-related column, i.e., $\alpha_i + i = \alpha_{i'} + i'$. Then, $0 < i' - i = \alpha_i - \alpha_{i'} \leqslant 0$, which is impossible. Hence, different top rows are related to different h-related columns. Similarly, we have statement (ii).

For statement (iii), suppose that the t-th bottom column is, simultaneously, h-related and v-related. Then, there are i and j such that $t = \alpha_i + i = k + j - \alpha^*_j$, i.e., $\alpha_i - j = k - i - \alpha^*_j$. If $\alpha_i \geqslant j$ then $\alpha^*_j \leqslant k - i$ and $\alpha^*_j = \#\{l\colon \alpha_l \geqslant j\} \geqslant k - i + 1$, a contradiction. On the other hand, if $\alpha_i < j$ then $\alpha^*_j > k - i$ and $\alpha^*_j = \#\{l\colon \alpha_l \geqslant j\} < k - i$, also a contradiction. Hence, no bottom column can be, simultaneously, h-related and v-related. Since the number of h-related and v-related bottom columns is respectively k and $n - k$, a bottom column is either h-related or v-related. □

The *vacant length of a bottom column* is the number of empty boxes below the boxes of λ in the staircase $n \times n$ shape. Explicitly, the vacant length of the j-th bottom column is $j - \#\{i\colon \lambda_i + i > j\}$.

We may recover the permutation associated with such diagram by taking the vacant length of the h-related and v-related bottom columns. Namely, the permutation element for $\Lambda = (\alpha, \lambda)$ is defined by $w_{u,\lambda}$ in the equation (2), where $0 < u_1 < \cdots < u_k$ are the vacant length of the h-related columns, and $0 < v_1 < \cdots < v_{n-k-\ell(\lambda)}$ are the vacant length of the v-related columns. For instance, Figure 2 presents the vacant length as the number of dots in the respective h-related and v-related columns for $\alpha = (3, 5)$ and $\lambda = (1, 5, 7)$.

2.4. HSYD's and covering types. We now present how the diagrams illustrate covering relations.

We recall the following proposition about the covering relation of double partitions.

Proposition 2.6 ([9], Proposition 4.2)**.** *Let $w, w' \in \mathcal{W}^{(k)}_n$. Denote by $\Lambda = (\alpha, \lambda)$ and $\Lambda' = (\alpha', \lambda')$ the double partitions of w and w', respectively. Then,*

(i) $\mathsf{Type}(w,w') = B1$ *if, and only if, for any* $i \in [k]$ *and* $j \in [r-1]$ *we have* $\alpha'_i = \alpha_i$ *and* $\lambda'_j = \lambda_{j+1}$*;*

(ii) $\mathsf{Type}(w,w') = B2$ *if, and only if, for any* $i \in [k]$ *and* $j \in [r]$ *we have*

$$\alpha'_i = \alpha_i \quad \text{and} \quad \lambda'_j = \begin{cases} \lambda_j - 1 & \text{if } j = t \\ \lambda_j & \text{if } j \neq t \end{cases},$$

for some $t \in [r]$*;*

(iii) $\mathsf{Type}(w,w') = B3$ *if, and only if, for any* $i \in [k]$ *and* $j \in [r]$ *we have*

$$\alpha'_i = \begin{cases} \alpha_i - 1 & \text{if } i = p \\ \alpha_i & \text{if } i \neq p \end{cases} \quad \text{and} \quad \lambda'_j = \lambda_j,$$

for some $p \in [k]$*;*

(iv) $\mathsf{Type}(w,w') = B4$ *if, and only if, for any* $i \in [k]$ *and* $j \in [r]$ *we have*

$$\alpha'_i = \begin{cases} \alpha_i + x - 1 & \text{if } i = p \\ \alpha_i & \text{if } i \neq p \end{cases} \quad \text{and} \quad \lambda'_j = \begin{cases} \lambda_j - x & \text{if } j = t \\ \lambda_j & \text{if } j \neq t \end{cases},$$

for some $p \in [k]$ *and* $t \in [r]$.

Remark 2.7. By Proposition 2.6, the covering relations in $\mathcal{P}(k,n)$ are the same as the covering relations for $\mathcal{D}_{k,n}$ and $\mathcal{SD}_n$, except in the case of a pair of type B4 when $x \neq 1$. This is exactly the step from the weak to strong Bruhat order. It also reflects the non-fully commutativeness of the k-Grassmannian permutations when $k \neq 0$.

The results of Proposition 2.6 may be understood as a process of removing boxes of the HSYD. There are two types of boxes that can be removed: corners and middle boxes.

A *corner* is a box of the diagram that produces a new diagram when it is removed. By Proposition 2.6, we see that:

- if $\mathsf{Type}(w,w') = \text{B1}$, then the diagram of Λ' is obtained by removing a corner in the diagonal of the bottom diagram;
- if $\mathsf{Type}(w,w') = \text{B2}$, then the diagram of Λ' is obtained from the diagram of Λ by removing a corner of the bottom diagram that belongs to a v-related column;
- if $\mathsf{Type}(w,w') = \text{B3}$, then the diagram of Λ' is obtained from the diagram of Λ by removing a corner of the top diagram.

A *middle (bottom) box* is a box (which is not a corner) of the bottom diagram that produces a new diagram when it is removed, followed by a displacement of boxes at its right. A middle box is neither a corner nor a diagonal. It lies in an h-related column and all boxes to the right of it lie in a v-related column. If $\mathsf{Type}(w,w') = \text{B4}$, by Proposition 2.6, the diagram of Λ' is obtained from the diagram of Λ by removing a box of the bottom diagram that belongs to an h-related column which is either a corner or a middle box, respectively, when $x = 1$ or $x \neq 1$. Notice that if $x \neq 1$ then we move all the $(x-1)$ boxes at the right of the removed box in the bottom diagram to the top diagram.

In Example 1, the permutation $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4$ covers five different elements in $\mathcal{W}_7^{(k)}$. Figure 3 illustrates these covering pairs in terms of removing corners or middle boxes.

Remark 2.8. There is an easy way to get a reduced decomposition of a permutation w_Λ from the HSYD. Fill in the HSYD as described below:

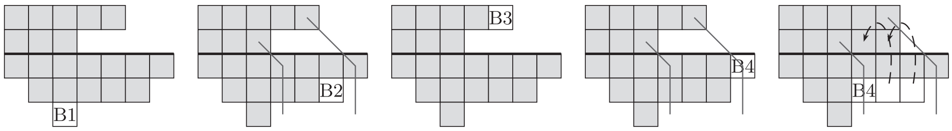


Fig. 3. The five covering pairs of $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4$ obtained by removing boxes in its HSYD according to the type of the pair.

- In the top diagram, assign a simple reflection consecutively to each box of the Young diagram from left to right and upwards, starting from s_1 in the bottom leftmost box;
- In the bottom diagram, assign a simple reflection consecutively to each box of the strict Young diagram from left to right, starting from s_0 in leftmost box of each bottom row.

A reduced decomposition of w_Λ is the word obtained by reading each row in the diagram from right to left and rows from the bottom to top. This is called the row-reading of w_Λ. Such decomposition is due to [11] and follows both the constructions given in [7] for Grassmannians of type A (Section 4.2) and for maximal isotropic Grassmannians of type B (Section 7).

For instance, in Figure 4, the row-reading of $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4$ is $w_{(3,5;1,5,7)} = s_0 \cdot s_4 s_3 s_2 s_1 s_0 \cdot s_6 s_5 s_4 s_3 s_2 s_1 s_0 \cdot s_3 s_2 s_1 \cdot s_6 s_5 s_4 s_3 s_2$. We have that w covers $w' = 5\,6|\overline{7}\,\overline{2}\,\overline{1}\,3\,4$. In Figure 4, the row-reading of w' is $w'_{(5,5;1,2,7)} = s_0 \cdot s_1 s_0 \cdot s_6 s_5 s_4 s_3 s_2 s_1 s_0 \cdot s_5 s_4 s_3 s_2 s_1 \cdot s_6 s_5 s_4 s_3 s_2$. The reduced decomposition of w' by the Bruhat-Chevalley order is ${w_{(3,5;1,5,7)}}' = s_0 \cdot s_4 s_3 \widehat{s_2} s_1 s_0 \cdot s_6 s_5 s_4 s_3 s_2 s_1 s_0 \cdot s_3 s_2 s_1 \cdot s_6 s_5 s_4 s_3 s_2$, i.e., a s_2 is deleted from the row-reading of w reflecting the fact that Λ' is obtained from Λ by removing a middle box. Notice that both reduced decompositions ${w_{(3,5;1,5,7)}}'$ and $w'_{(5,5;1,2,7)}$ are equivalent with respect to the Coxeter relations.

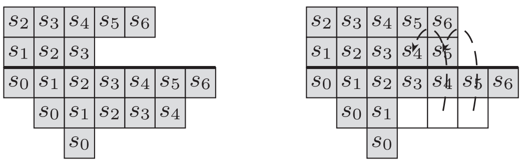


Fig. 4. Row-reading of $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4$ on the left and row-reading of $w' = 5\,6|\overline{7}\,\overline{2}\,\overline{1}\,3\,4$ on the right.

3. Boundary map and integral homology

In the previous section, we presented the permutation model together with some combinatorial properties of the Weyl group of type B. In this section, we go into the details of computing the homology groups.

We now summarize the main results of [13]. Let G be either an odd orthogonal or a symplectic group. The flag manifold $\mathbb{F}_{(k)}$ with respect to the set $(k) = \Sigma - \{a_k\}$ is, respectively, the isotropic Grassmannian $\mathrm{IG}(n-k, 2n)$ or the odd orthogonal Grassmannian $\mathrm{OG}(n-k, 2n+1)$.

The Bruhat decomposition $\mathbb{F}_{(k)} = \coprod_{w \in \mathcal{W}_n^{(k)}} N \cdot wb_{(k)}$ provides a CW-complex structure for $\mathbb{F}_{(k)}$ where the cells are the Schubert varieties $\mathcal{S}_w = \mathrm{cl}(N \cdot wb_{(k)})$. This induces a cellular chain complex $(\mathcal{C}, \partial)$ as follows: Let $\mathcal{C}$ be the $\mathbb{Z}$-module freely generated by $\mathcal{S}_w$, for every element w of the set of minimal representatives $\mathcal{W}_n^{(k)}$. The boundary map $\partial\colon \mathcal{C} \to \mathcal{C}$ is defined by

$$\partial \mathcal{S}_w = \sum_{w' \leqslant w} c(w, w') \mathcal{S}_{w'}, \tag{12}$$

where the coefficient $c(w, w') \in \mathbb{Z}$ satisfies:

- If $\dim \mathcal{S}_w - \dim \mathcal{S}_{w'} \neq 1$ then $c(w, w') = 0$;
- If $\dim \mathcal{S}_w - \dim \mathcal{S}_{w'} = 1$ then $c(w, w') = \deg\left(\phi_{w,w'} : S^{\ell(w)} \to S^{\ell(w')}\right)$ is degree of a certain map $\phi_{w,w'}$ between spheres.

This cellular chain complex provides the cellular homology groups $H_m(\mathbb{F}_{(k)}, \mathbb{Z})$, $m \geqslant 0$. Then, the cellular homology can be obtained once we get an explicit formula for $c(w, w')$.

In general, by [13] Theorem 2.2, $c(w, w')$ is either 0 or ± 2, since it is the sum of the degrees of two sphere homeomorphisms of degree ± 1.

We may go further and provide a formula to compute $c(w, w')$. For $w \in \mathcal{W}_n^{(k)}$, define $\Pi_w = \Pi^+ \cap w\Pi^-$, the set of positive roots sent to negative roots by w^{-1}. Let $\sigma(w)$ be the sum of roots in Π_w, i.e.,

$$\sigma(w) = \sum_{\beta \in \Pi_w} \beta.$$

Proposition 3.1 ([13], Proposition 2.7)**.** *Let γ be the unique root (not necessarily simple) such that $w = s_\gamma w'$. Then*

$$\sigma(w) - \sigma(w') = \kappa \cdot \gamma \tag{13}$$

for some integer $\kappa = \kappa(w, w')$.

Theorem 3.2 ([13], Theorem 2.8, [8], Theorem 1.1.4)**.** *Suppose that w covers w'. Then the coefficient $c(w, w')$ is given as follows:*

$$c(w, w') = \pm(1 + (-1)^\kappa) = \begin{cases} 0 & \text{if } \kappa \text{ is odd;} \\ \pm 2 & \text{if } \kappa \text{ is even.} \end{cases}$$

The signs on $c(w, w')$ can be chosen so that $\partial^2 = 0$ and the homology of $(\mathcal{C}, \partial)$ is the integral homology of $\mathbb{F}_{(k)}$.

REMARK 3.3. The formula for $c(w, w')$ obtained in [13] offers a choice of the signs defined in terms of the reduced decompositions for the Weyl group elements.

We use Theorem 3.2 to compute the boundary coefficients of the isotropic and odd orthogonal Grassmannians. This process will provide us both κ and γ. Our main strategy is based on a bijective correspondence between the roots of Π_w with the half-shifted Young diagram of w as follows.

3.1. Inversions. In this section, the HSYD's are used to describe Π_w as the union of two distinguished sets given by the top and bottom diagrams of w parametrizing the inversions of the corresponding partitions α and λ respectively. For each $w \in \mathcal{W}_n^{(k)}$, we define

$$\mathrm{Inv}^+(w) = \{(i, j) \in [n]^2 \mid i < j \text{ and } w(i) > w(j)\}; \tag{14}$$

$$\mathrm{Inv}^-(w) = \{(i, j) \in [n]^2 \mid i \leqslant j \text{ and } -w(i) > w(j)\}. \tag{15}$$

It follows that $|\mathrm{Inv}^+(w)| + |\mathrm{Inv}^-(w)| = |\alpha| + |\lambda| = \ell(w)$ by the equation (1) .

Recall that the root system of type C is realized in the Euclidean space $\mathbb{R}^n = \oplus_{i=1}^n \mathbb{R}\varepsilon_i$ as a set of vectors $\Pi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leqslant i < j \leqslant n\} \cup \{\pm 2\varepsilon_i : 1 \leqslant i \leqslant n\}$ and the root system of type B is realized as a set of vectors $\Pi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leqslant i < j \leqslant n\} \cup \{\pm\varepsilon_i : 1 \leqslant i \leqslant n\}$. Denote by $\varepsilon_{-i} = -\varepsilon_i$.

If $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced decomposition of w then it is known that $|\Pi_w| = \ell(w)$ and $\Pi_w = \{a_{i_1},\ s_{i_1}(a_{i_2}),\ s_{i_1}s_{i_2}(a_{i_3}), \ldots,\ s_{i_1}\cdots s_{i_{\ell-1}}(a_{i_\ell})\}$. Consider $\beta^+_{i,j}$ and $\beta^-_{i,j}$ defined, resp., for inversions (i, j) in $\mathrm{Inv}^+(w)$ and $\mathrm{Inv}^-(w)$ as follows.

- For $(i, j) \in \mathrm{Inv}^+(w)$,

$$\beta^+_{i,j}(w) = \varepsilon_{w(i)} - \varepsilon_{w(j)};$$

- For $(i, j) \in \mathrm{Inv}^-(w)$,

$$\beta^-_{i,j}(w) = \begin{cases} -\varepsilon_{w(i)} - \varepsilon_{w(j)} & \text{if type C;} \\ 2^{-\delta_{ij}}(-\varepsilon_{w(i)} - \varepsilon_{w(j)}) & \text{if type B;} \end{cases}$$

  where δ_{ij} is the Kronecker delta.

Let us denote $\beta^+ = \{\beta^+_{i,j} : (i, j) \in \mathrm{Inv}^+(w)\}$ and $\beta^- = \{\beta^-_{i,j} : (i, j) \in \mathrm{Inv}^-(w)\}$.

Proposition 3.4. *Let $w \in \mathcal{W}_n^{(k)}$. Then,*

(i) *$\beta^+_{i,j}$ is a positive root in Π_w for every $(i, j) \in \mathrm{Inv}^+(w)$;*
(ii) *$\beta^-_{i,j}$ is a positive root in Π_w for every $(i, j) \in \mathrm{Inv}^-(w)$;*
(iii) *Π_w is the disjoint union of β^+ and β^-.*

Proof. For $(i, j) \in \mathrm{Inv}^+(w)$ with $i < j$, the root $\varepsilon_i - \varepsilon_j$ is a negative root, whereas $w(\varepsilon_i - \varepsilon_j) = \varepsilon_{w(i)} - \varepsilon_{w(j)} = \beta^+_{i,j}$ is a positive root. Then, $\beta^+_{i,j} \in \Pi_w$. For $(i, j) \in \mathrm{Inv}^-(w)$ such that G is either the symplectic group (type C) with $i \leqslant j$ or the odd orthogonal group (type B) with $i < j$, the root $-\varepsilon_i - \varepsilon_j$ is a negative root, whereas $w(-\varepsilon_i - \varepsilon_j) = -\varepsilon_{w(i)} - \varepsilon_{w(j)} = \beta^-_{i,j}$ is a positive root. Then, $\beta^-_{i,j} \in \Pi_w$. For $(i, i) \in \mathrm{Inv}^-(w)$ such that G an odd orthogonal group (type B), the root $-\varepsilon_i$ is a negative root, whereas $w(-\varepsilon_i) = -\varepsilon_{w(i)} = \frac{1}{2}(-\varepsilon_{w(i)} - \varepsilon_{w(i)}) = \beta^-_{i,j}$ is a positive root. Then, $\beta^-_{i,j} \in \Pi_w$.

Since $|\Pi_w| = |\mathrm{Inv}^+(w)| + |\mathrm{Inv}^-(w)|$ and, furthermore, all roots β^+ and β^- are different to each other, the sets Π_w and $\beta^+ \cup \beta^-$ coincide. □

Proposition 3.5. *Given $w \in \mathcal{W}_n^{(k)}$, we have that* $\mathrm{Inv}^+(w) = \{(i, j) : i \in [k]$ *and* $j \in [k + 1, k + \alpha_i]\}$.

Proof. Recall that $w \in \mathcal{W}_n^{(k)}$ is given by $w = u_1 \cdots u_k|\overline{\lambda_r} \cdots \overline{\lambda_1}\, v_1 \cdots v_{n-k-r}$ and it satisfies (3).

Consider $I_1 = \{(i, j) : i \in [k]$ and $j \in [k+1, k+\alpha_i]\}$. Since $|I_1| = \sum_{i=1}^k |\alpha_i| = |\alpha| = |\mathrm{Inv}^+(w)|$, we only need to prove that $I_1 \subset \mathrm{Inv}^+(w)$. Suppose that $(i, j) \in I_1$. Since $\alpha_i \geqslant r$, we can split in two cases: if $j \in [k+1, k+r]$ then $w(i) = u_i$ and $w(j) = \overline{\lambda}_{k+r-i+1}$, which clearly implies that $w(i) > w(j)$; if $j \in [k+r+1, k+\alpha_i]$ then, by (10), we conclude that $1 \leqslant j-k-r \leqslant \#\{v_l : v_l < u_i\}$, i.e., $w(j) = v_{j-k-r} < u_i = w(i)$. □

The description of $\mathrm{Inv}^-(w)$ is based on a relationship between w and its corresponding Lagrangian permutation $\widetilde{w}$ through a "translation" map.

Let $w \in \mathcal{W}_n^{(k)}$ be a permutation in the form (2). Define $\widetilde{w} \in \mathcal{W}_n^{(0)}$ by $\widetilde{w} = \overline{\lambda_r} \cdots \overline{\lambda_1}\widetilde{v}_1 \cdots \widetilde{v}_{n-r}$, where $0 < \widetilde{v}_1 < \cdots < \widetilde{v}_{n-r}$ is the reordering of the entries u's and v's of w. The next lemma allows us to keep track the position of the $w(i)$'s after this process. Indeed, if $w(i) = j$ then $w^{-1}(j) = i$, so that the position of the value j in the permutation w is equal to $w^{-1}(j)$.

Lemma 3.6. *Let $w \in \mathcal{W}_n^{(k)}$. The position of the entry $w(i)$, for $i \in [n]$, in the Lagrangian permutation $\widetilde{w} \in \mathcal{W}_n^{(0)}$ is given by the formula*

$$\widetilde{w}^{-1}(w(i)) = \begin{cases} i + \alpha_i & \text{if } 1 \leqslant i \leqslant k; \\ i - \alpha^*_{i-k} & \text{if } k+1 \leqslant i \leqslant n. \end{cases}$$

Proof. If $1 \leqslant i \leqslant k$ then, by (10), $\widetilde{w}^{-1}(w(i)) = i+r+\#\{v_j : v_j < u_i\} = i+\alpha_i$. If $k+1 \leqslant i \leqslant k+r$ then, by Lemma 2.4, we have $\widetilde{w}^{-1}(w(i)) = i - k = i - \alpha^*_{i-k}$. If $k + r + 1 \leqslant i \leqslant n$ then, by Lemma 2.4, we have $\widetilde{w}^{-1}(w(i)) = i - \#\{u_j : u_j > v_{i-k-r}\} = i - \mu^*_{i-k-r} = i - \alpha^*_{i-k}$. □

Proposition 3.7. *Given $w \in \mathcal{W}_n^{(k)}$ and $\widetilde{w}$ the respective Lagrangian permutation in $\mathcal{W}_n^{(0)}$, we have that* $\mathrm{Inv}^-(\widetilde{w}) = \{(i, j) : i \in [r]$ *and* $j \in [i, i - 1 + \lambda_{r-i+1}]\}$.

Proof. Consider $I_2 = \{(i, j) : i \in [r]$ and $j \in [i, i - 1 + \lambda_{r-i+1}]\}$. Observe that $\lambda_l \geqslant l$ for every $l \in [r]$ because λ is a strict partition. Then, $i - 1 + \lambda_{r-i+1} \geqslant r$. Since $|I_2| = \sum_{i=1}^r |\lambda_i| = \ell(\widetilde{w}) = |\mathrm{Inv}^-(\widetilde{w})|$, we only need to prove that $I_2 \subset \mathrm{Inv}^-(\widetilde{w})$. Suppose that $(i, j) \in I_2$ with $i \leqslant j$. If $j \in [r]$ then $w(i) = \overline{\lambda}_{r-i+1}$ and $w(j) = \overline{\lambda}_{r-j+1}$ so that $-w(i) > w(j)$. If $j \in [r+1, i-1+\lambda_{r-i+1}]$ then $w(j) = \widetilde{v}_{j-r} \leqslant \widetilde{v}_{\lambda_{r-i+1}-(r-i+1)}$. Since $\widetilde{v}_{\lambda_l - l} < \lambda_l$ for every $l \in [r]$, we conclude that $w(j) < \lambda_{r-i+1} = -w(i)$. □

Finally, a description of $\mathrm{Inv}^-(\widetilde{w})$ is quite enough for us since there is a bijection between $\mathrm{Inv}^-(w)$ and $\mathrm{Inv}^-(\widetilde{w})$, as shown in the next lemma.

Lemma 3.8. *Given $w \in \mathcal{W}_n^{(k)}$, then $(i, j) \in \mathrm{Inv}^-(w)$ if, and only if, $(i_0, j_0) \in \mathrm{Inv}^-(\widetilde{w})$, where $w(i) = \widetilde{w}(i_0)$ and $w(j) = \widetilde{w}(j_0)$. Moreover, $\beta^-_{i,j}(w) = \beta^-_{i_0,j_0}(\widetilde{w})$.*

Proof. It follows directly from the definition of inversions in $\mathrm{Inv}^-(w)$ and by Lemma 3.6. □

In general, the desired correspondence between the Π_w and the HSYD of w according to the results above is done as follows:

- Top diagram: for each $(i, j) \in \mathrm{Inv}^+(w)$ with $i \in [k]$ and $j \in [k + 1, k + \alpha_i]$, we place the root $\beta^+_{i,j} \in \Pi_w$ at the position $(i, j - k)$ in the corresponding top diagram $\mathcal{D}_\alpha$;
- Bottom diagram: for each $(i, j) \in \mathrm{Inv}^-(\widetilde{w})$ with $i \in [r]$ and $j \in [r+1, r+\lambda_i]$, we place the root $\beta^-_{i,j} \in \Pi_w$ at the position (i, j) in the corresponding bottom diagram $\mathcal{SD}_\lambda$.

Figure 5 illustrates this procedure for $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4 \in \mathrm{OG}(5, 15)$.

REMARK 3.9. Propositions 3.4, 3.5, 3.7 and Lemma 3.8 generalize Lemma 10 of [7] in types B and C for any type of Grassmannian beyond the maximal ones. Notice that in Figure 5 we have both the top and bottom diagrams.

3.2. Results. The integral homology groups $H_m(\mathbb{F}_{(k)}, \mathbb{Z})$, $m \geqslant 0$, can be computed after we determine the coefficients of the boundary map according to the formula given in Theorem

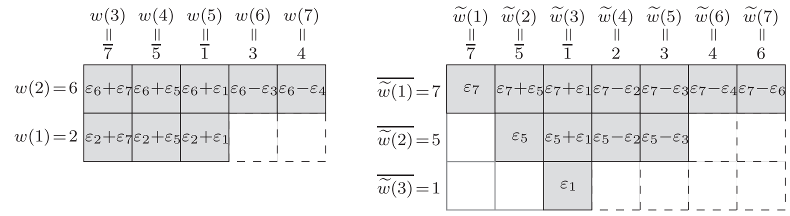


Fig. 5. The roots of Π_w inside the HSYD of $w = 2\,6|\overline{7}\,\overline{5}\,\overline{1}\,3\,4 \in \mathrm{OG}(5, 15)$.

3.2 and make a choice of signs. We provide an explicit expression of κ for each type of covering pairs along with a interpretation into the HSYD.

Recall that P, T, and Q are the positions changed in w and w' as in the equation (5).

Theorem 3.10. *Let w, w' be in $\mathcal{W}_n^{(k)}$ such that w covers w'. Then, κ as in the equation* (13) *depends on the type of the pair w, w' as follows:*

- *For* $\mathrm{IG}(n-k, 2n)$*:*

$$\kappa(w, w') = \begin{cases} T & \text{if } \mathrm{type}(w, w') = B1; \\ T + Q & \text{if } \mathrm{type}(w, w') = B2; \\ Q - P & \text{if } \mathrm{type}(w, w') = B3; \\ P + T & \text{if } \mathrm{type}(w, w') = B4; \end{cases}$$

- *For* $\mathrm{OG}(n-k, 2n+1)$*:*

$$\kappa(w, w') = \begin{cases} 2T - 1 & \text{if } \mathrm{type}(w, w') = B1; \\ T + Q - 1 & \text{if } \mathrm{type}(w, w') = B2; \\ Q - P & \text{if } \mathrm{type}(w, w') = B3; \\ P + T - 1 & \text{if } \mathrm{type}(w, w') = B4. \end{cases}$$

REMARK 3.11. The coefficients for the classical real Grassmannians $\mathrm{Gr}(k, n)$ – those of type A – occurs as a particular case when w, w' is of type B3. The Schubert varieties are parametrized by permutations of the symmetric group with a descent at k-th position. They correspond to the partitions α and poset $\mathcal{D}_{k,n}$ of the Young diagrams. Theorem 3.10 shows that if w, w' is a covering pair and $w = w' \cdot (P, Q)$ then $\kappa(w, w') = Q - P$.

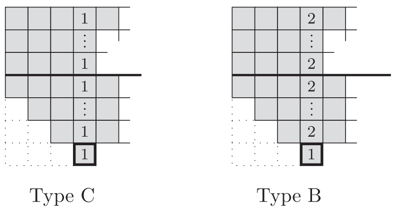
The proof will be postponed to Section 4 (Propositions 4.1, 4.4, 4.5 and 4.6).

We may also visualize κ for each covering pair through the HSYD. We start by filling in each diagram of w and w' with the corresponding inversions of Π_w and $\Pi_{w'}$, remembering that the diagram of w' is obtained from the diagram of w by removing either a corner or a middle box (cf. Section 2.4). The distinction between these diagrams may be displayed inside the diagram of w as follows:

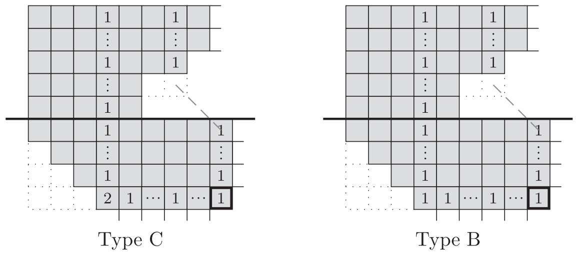
- fill in the removed box (r.b.) with 1;
- do not fill the boxes if the corresponding roots of Π_w and $\Pi_{w'}$ are the same;
- fill in with either ± 1 or 2 (as it will be clarified below) if the corresponding roots of Π_w and $\Pi_{w'}$ are different.

The computations in Section 4 will show that $\kappa(w, w')$ is given as the sum of such numbers in the diagram. According to the changes in the permutations, the difference between the roots occurs along specific rows and columns depending on the type of the pair.

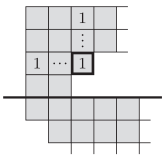
- If type(w, w') = B1 then we fill in the column above the r.b. in the diagonal of the bottom diagram according to Figure 6a;
- If type(w, w') = B2 then the r.b. belongs to a v-related column in the bottom diagram. We fill in the hook defined by the diagonal box contained in the row of the r.b., and the column above the r.b. in the bottom diagram together with its related top column according to Figure 6b;
- If type(w, w') = B3 then the r.b. belongs to the top diagram. We fill in the boxes to the left and above the r.b. according to Figure 6c;
- If type(w, w') = B4 then the r.b. belongs to an h-related column in the bottom diagram. We fill in the hook defined by the diagonal box contained in the row of the r.b., and the column above the r.b. in the bottom diagram together with its related top row according to Figure 6d.

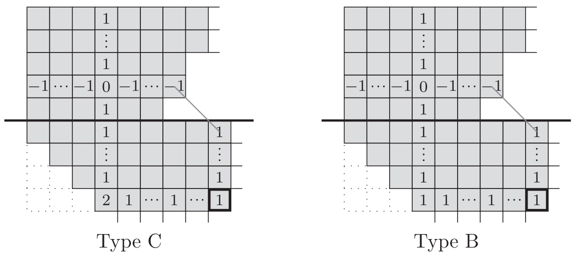


(A) Type$(w, w') = \mathrm{B1}$.



(B) Type$(w, w') = \mathrm{B2}$.


(C) Type$(w, w') = \mathrm{B3}$.



(D) Type$(w, w') = \mathrm{B4}$.

Fig. 6. Filling in the diagrams for each type of pair.

Example 2. Let $w = 2\,6|\bar{7}\,\bar{5}\,\bar{1}\,3\,4 \in \mathrm{OG}(5, 15)$, where $n = 7$, $k = 2$. There are five covering pairs w, w'_i, $i = 1, \ldots, 5$, according to Example 1. We can compute κ of each pair w, w'_i as follows:

(B1) $w, w'_1 : T = 5$ and $\kappa = 2T - 1 = 9$;

(B2) $w, w'_2 : (T, Q) = (4, 7)$ and $\kappa = T + Q - 1 = 10$;

(B3) $w, w'_3 : (P, Q) = (2, 7)$ and $\kappa = Q - P = 5$;

(B4) $w, w'_4 : (P, T) = (2, 3)$ and $\kappa = P + T - 1 = 4$;

(B4) $w, w'_5 : (P, T) = (1, 4)$ and $\kappa = P + T - 1 = 4$.

Therefore, by Theorem 3.10, $\partial\mathcal{S}_{2\,6|\bar{7}\,\bar{5}\,\bar{1}\,3\,4} = \pm 2\mathcal{S}_{2\,6|\bar{7}\,\bar{4}\,\bar{1}\,3\,5} \pm 2\mathcal{S}_{2\,7|\bar{6}\,\bar{5}\,\bar{1}\,3\,4} \pm 2\mathcal{S}_{5\,6|\bar{7}\,\bar{2}\,\bar{1}\,3\,4}$.

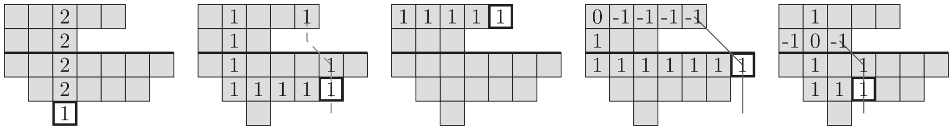


Fig. 7. The sum of the number in each diagram gives the value of $\kappa(w, w'_i)$.

The comparison between the formulas for $\kappa(w, w')$ and the description of the covering relations by the right action has revealed a hidden phenomenon that will be generalized in a forthcoming paper.

Let us define the coroot of a root $\delta \in \Pi$ by $\delta^\vee = \dfrac{2\delta}{\langle \delta, \delta \rangle}$. The set of coroots Π^* is also a root system which is called the dual root system. In particular, the root systems of types B and C are dual to each other.

If $\delta \in \Pi$ is given in terms of the system Σ of simple roots as $\delta = \sum_{\xi \in \Sigma} d_\xi \xi$, the height of the root δ is the sum $\mathrm{ht}(\delta) = \sum_{\xi \in \Sigma} d_\xi$.

Theorem 3.12. *Let w, w' be in $\mathcal{W}_n^{(k)}$ such that w covers w' and δ be the root for which $w = w' \cdot s_\delta$. Then*

$$\kappa(w, w') = \mathrm{ht}(\delta^\vee).$$

Proof. Recall that the simple roots of type B are defined by $a_0 = \varepsilon_1$ and $a_i = \varepsilon_{i+1} - \varepsilon_i$ for $1 \leqslant i < n$. The positive roots Π^+ of type B are given by ε_j $(j \geq 1)$, $\varepsilon_i - \varepsilon_j$ $(i > j)$ and $\varepsilon_i + \varepsilon_j$ $(i > j)$. The height of the corresponding roots and coroots are given by

$$\begin{aligned} \mathrm{ht}(\varepsilon_j) &= j, \text{ for } j \geqslant 1; \\ \mathrm{ht}(\varepsilon_i - \varepsilon_j) &= i - j, \text{ for } i > j; \\ \mathrm{ht}(\varepsilon_i + \varepsilon_j) &= i + j, \text{ for } i > j. \end{aligned}$$

Now, recall that the simple roots of type C are defined by $a_0 = 2\varepsilon_1$ and $a_i = \varepsilon_{i+1} - \varepsilon_i$ for $1 \leqslant i < n$. The positive roots Π^+ of type C are given by $2\varepsilon_j$ $(j \geq 1)$, $\varepsilon_i - \varepsilon_j$ $(i > j)$ and $\varepsilon_i + \varepsilon_j$ $(i > j)$. The height of the corresponding roots and coroots are given by

$$\begin{aligned} \mathrm{ht}(2\varepsilon_j) &= 2j - 1, \text{ for } j \geqslant 1; \\ \mathrm{ht}(\varepsilon_i - \varepsilon_j) &= i - j, \text{ for } i > j; \\ \mathrm{ht}(\varepsilon_i + \varepsilon_j) &= i + j - 1, \text{ for } i > j. \end{aligned}$$

Furthermore, both root systems are dual to each other.

- For the type B root ε_j, its coroot is the type C root $2\varepsilon_j$;
- For the type B root $\varepsilon_i \pm \varepsilon_j$, its coroot is the type C root $\varepsilon_i \pm \varepsilon_j$.

Considering $w = w' \cdot s_\delta$, the equation (5), and Theorem 3.10, we have the following:

- For type C:
  - If $\mathsf{Type}(w, w') = \mathrm{B1}$ then $\delta = 2\varepsilon_T$ and $\kappa(w, w') = T = \mathrm{ht}(\delta^\vee)$;
  - If $\mathsf{Type}(w, w') = \mathrm{B2}$ then $\delta = \varepsilon_Q + \varepsilon_T$ and $\kappa(w, w') = T + Q = \mathrm{ht}(\delta^\vee)$;
  - If $\mathsf{Type}(w, w') = \mathrm{B3}$ then $\delta = \varepsilon_Q - \varepsilon_P$ and $\kappa(w, w') = Q - P = \mathrm{ht}(\delta^\vee)$;
  - If $\mathsf{Type}(w, w') = \mathrm{B4}$ then $\delta = \varepsilon_T + \varepsilon_P$ and $\kappa(w, w') = P + T = \mathrm{ht}(\delta^\vee)$;
- For type B:

- If $\mathsf{Type}(w, w') = \mathrm{B1}$ then $\delta = \varepsilon_T$ and $\kappa(w, w') = 2T - 1 = \mathrm{ht}(\delta^\vee)$;
- If $\mathsf{Type}(w, w') = \mathrm{B2}$ then $\delta = \varepsilon_Q + \varepsilon_T$ and $\kappa(w, w') = T + Q - 1 = \mathrm{ht}(\delta^\vee)$;
- If $\mathsf{Type}(w, w') = \mathrm{B3}$ then $\delta = \varepsilon_Q - \varepsilon_P$ and $\kappa(w, w') = Q - P = \mathrm{ht}(\delta^\vee)$;
- If $\mathsf{Type}(w, w') = \mathrm{B4}$ then $\delta = \varepsilon_T + \varepsilon_P$ and $\kappa(w, w') = P + T - 1 = \mathrm{ht}(\delta^\vee)$.

□

3.3. Orientability. We provide a general criteria of orientability.

Proposition 3.13.

(i) $\mathrm{IG}(n-k, 2n)$ *is orientable if, and only if,* $n - k$ *is odd;*
(ii) $\mathrm{OG}(n, 2n+1)$ *is orientable for every* n*;*
(iii) $\mathrm{OG}(n-k, 2n+1)$ *is orientable if, and only if,* $k > 0$ *and* $n - k$ *is even.*

Proof. Consider the Schubert variety $\mathcal{S}_{w_0^k}$ where $w_0^k \in \mathcal{W}_n^{(k)}$ is the longest element given by (4). Being orientable is equivalent to $\partial \mathcal{S}_{w_0^k} = 0$.

If $k = 0$ then the only possible choice is $w_0^k = |\overline{n}\,\overline{n-1} \cdots \overline{1}$ and $w' = |n\,\overline{n-1} \cdots 1$, which is a pair of type B1. In this case, $T = n$ and $P = Q = 0$, implying that

$$c(w_0^k, w') = \begin{cases} \pm(1 + (-1)^n) & \text{for } \mathrm{IG}(n, 2n); \\ 0 & \text{for } \mathrm{OG}(n, 2n+1). \end{cases}$$

Therefore, $\mathrm{IG}(n, 2n)$ is orientable if, and only if, n is odd, and $\mathrm{OG}(n, 2n+1)$ is orientable for every n. If $k > 0$ then there is only one possible choice of w' such that w_0^k cover w', namely, $w_0^k = 1\,2 \cdots (k-1)\,k|\overline{n}\,\overline{n-1} \cdots \overline{k+1}$ and $w' = 1\,2 \cdots (k-1)\,(k+1)|\overline{n}\,\overline{n-1} \cdots \overline{k}$, which is a pair of type B4. Then, $P = k$, $T = n$, and $Q = 0$, implying that

$$c(w_0^k, w') = \begin{cases} \pm(1 + (-1)^{k+n}) & \text{for } \mathrm{IG}(n-k, 2n); \\ \pm(1 - (-1)^{k+n}) & \text{for } \mathrm{OG}(n-k, 2n+1). \end{cases}$$

Therefore, $\mathrm{IG}(n-k, 2n)$ is orientable if, and only if, $k + n \equiv n - k \bmod 2$ is odd, and $\mathrm{OG}(n-k, 2n+1)$ is orientable if, and only if, $k + n \equiv n - k \bmod 2$ is even. □

3.4. Duality. Given $w \in \mathcal{W}_n^{(k)}$, define $w^\vee = w w_0^k$ the *dual permutation of* w. Write w as in equation (2), the one-line notation of the dual permutation of w is

$$w^\vee = u_1 \cdots u_k|\overline{v_{n-k-r}} \cdots \overline{v_1}\,\lambda_1 \cdots \lambda_r.$$

The length of $w^\vee$ is $\ell(w^\vee) = \ell(w_0^k) - \ell(w)$. The next proposition states that the duality of a permutation also implies a duality over the covering pairs.

Proposition 3.14 ([9]). *Let* w, w *be permutations in* $\mathcal{W}_n^{(k)}$*. Then,* w *covers* w' *if, and only if,* $(w')^\vee$ *covers* $w^\vee$*. Moreover,*

(1) $\mathsf{Type}(w, w') = \mathrm{B1}$ *if, and only if,* $\mathsf{Type}((w')^\vee, w^\vee) = \mathrm{B1}$*;*
(2) $\mathsf{Type}(w, w') = \mathrm{B2}$ *if, and only if,* $\mathsf{Type}((w')^\vee, w^\vee) = \mathrm{B2}$*;*
(3) $\mathsf{Type}(w, w') = \mathrm{B3}$ *if, and only if,* $\mathsf{Type}((w')^\vee, w^\vee) = \mathrm{B4}$.

The next proposition shows that $c((w')^\vee, w^\vee)$ can be obtained, according to $c(w, w')$ and the type of the pair w, w'.

Proposition 3.15. *Let w, w' be in $\mathcal{W}_n^{(k)}$ such that w covers w'. Then, $|c(w,w')| = |c((w')^\vee, w^\vee)|$ if, and only if, one of the following happens:*

- *For* $\mathrm{IG}(n-k, 2n)$*:*
  1. $\mathsf{Type}(w,w') = \mathrm{B1}$ *and $n-k$ is odd;*
  2. $\mathsf{Type}(w,w') = \mathrm{B2}$*;*
  3. $\mathsf{Type}(w,w') = \mathrm{B3}$ *or* $\mathrm{B4}$*, and $n-k$ is odd;*
- *For* $\mathrm{OG}(n-k, 2n+1)$*:*
  1. $\mathsf{Type}(w,w') = \mathrm{B1}$ *or* $\mathrm{B2}$*;*
  2. $\mathsf{Type}(w,w') = \mathrm{B3}$ *or* $\mathrm{B4}$*, and $n-k$ is even.*

Proof. Denote by $P^\vee$, $T^\vee$, and $Q^\vee$ the integers of Table 1 for the dual pair $(w')^\vee, w^\vee$. We will describe such integers in terms of P, T, and Q case-by-case. Assume the flag manifold is $\mathrm{IG}(n-k, 2n)$.

If $\mathsf{Type}(w,w') = \mathrm{B1}$ then $T^\vee = n-r+1 = (n-k+1-2r)+T$. Thus, $(-1)^{T^\vee} = (-1)^{n-k+1}(-1)^T$ and we conclude that both coefficients coincide if, and only if, $n-k+1$ is even.

If $\mathsf{Type}(w,w') = \mathrm{B2}$ then $T^\vee = n-r-q+1 = (n-k-2q-2r+1)+Q$ and $Q^\vee = n-r+t = (n-k-2r+2t-1)+T$. Thus, $(-1)^{T^\vee+Q^\vee} = (-1)^{T+Q}$ and we conclude that both coefficients coincide.

If $\mathsf{Type}(w,w') = \mathrm{B3}$ then $P^\vee = p = P$ and $T^\vee = n-r-q+1 = (n-k-2q-2r+1)+Q$. Thus, $(-1)^{P^\vee+T^\vee} = (-1)^{n-k+1}(-1)^{Q-P}$ and we conclude that both coefficients coincide if, and only if, $n-k+1$ is even.

If $\mathsf{Type}(w,w') = \mathrm{B4}$ then $P^\vee = p = P$ and $Q^\vee = n-r+t = (n-k-2r+2t-1)+T$. Thus, $(-1)^{Q^\vee-P^\vee} = (-1)^{n-k+1}(-1)^{P+T}$ and we conclude that both coefficients coincide if, and only if, $n-k+1$ is even.

The proof for $\mathrm{OG}(n-k, 2n+1)$ is analogous. □

We define the *incidence graph of a Grassmannian* as the graph whose vertices are the permutations of $\mathcal{W}_n^{(k)}$ and the edges "→" and "⇒" are given by the covering relations as follows: if w covers w' and $c(w,w') = 0$ then $w \to w'$; if w covers w' and $c(w,w') = \pm 2$ then $w \Rightarrow w'$.

EXAMPLE 3. The incidence graphs for the odd orthogonal Grassmannian $\mathrm{OG}(2,9)$ and isotropic Grassmannian $\mathrm{IG}(2,8)$, where $n = 4$ and $k = 2$, are given in Figure 8. The duality of permutations and pairs can be seen as the symmetry through the horizontal dashed line.

(1) For $\mathbb{F} = \mathrm{IG}(2,8)$, by Proposition 3.15, only dual pairs w, w' of type B2 satisfy $|c(w,w')| = |c((w')^\vee, w^\vee)|$. In the incidence graph in Figure 8, this means that the edge of a pair of type B1, B3, or B4 should be different from the edge associated with its dual. Therefore, the homology groups are

$$\begin{array}{llll} H_{11}(\mathbb{F},\mathbb{Z}) = 0, & H_8(\mathbb{F},\mathbb{Z}) = \mathbb{Z}_2, & H_5(\mathbb{F},\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}_2, & H_2(\mathbb{F},\mathbb{Z}) = \mathbb{Z}_2, \\ H_{10}(\mathbb{F},\mathbb{Z}) = \mathbb{Z}_2, & H_7(\mathbb{F},\mathbb{Z}) = \mathbb{Z}_2, & H_4(\mathbb{F},\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}_2, & H_1(\mathbb{F},\mathbb{Z}) = \mathbb{Z}_2, \\ H_9(\mathbb{F},\mathbb{Z}) = \mathbb{Z}, & H_6(\mathbb{F},\mathbb{Z}) = (\mathbb{Z}_2)^2, & H_3(\mathbb{F},\mathbb{Z}) = \mathbb{Z}_2, & H_0(\mathbb{F},\mathbb{Z}) = \mathbb{Z}. \end{array}$$

(2) For $\mathbb{F} = \mathrm{OG}(2,9)$, by Proposition 3.15, the coefficients and the corresponding edges in Figure 8 are the same for all dual pairs. Therefore, the homology groups are

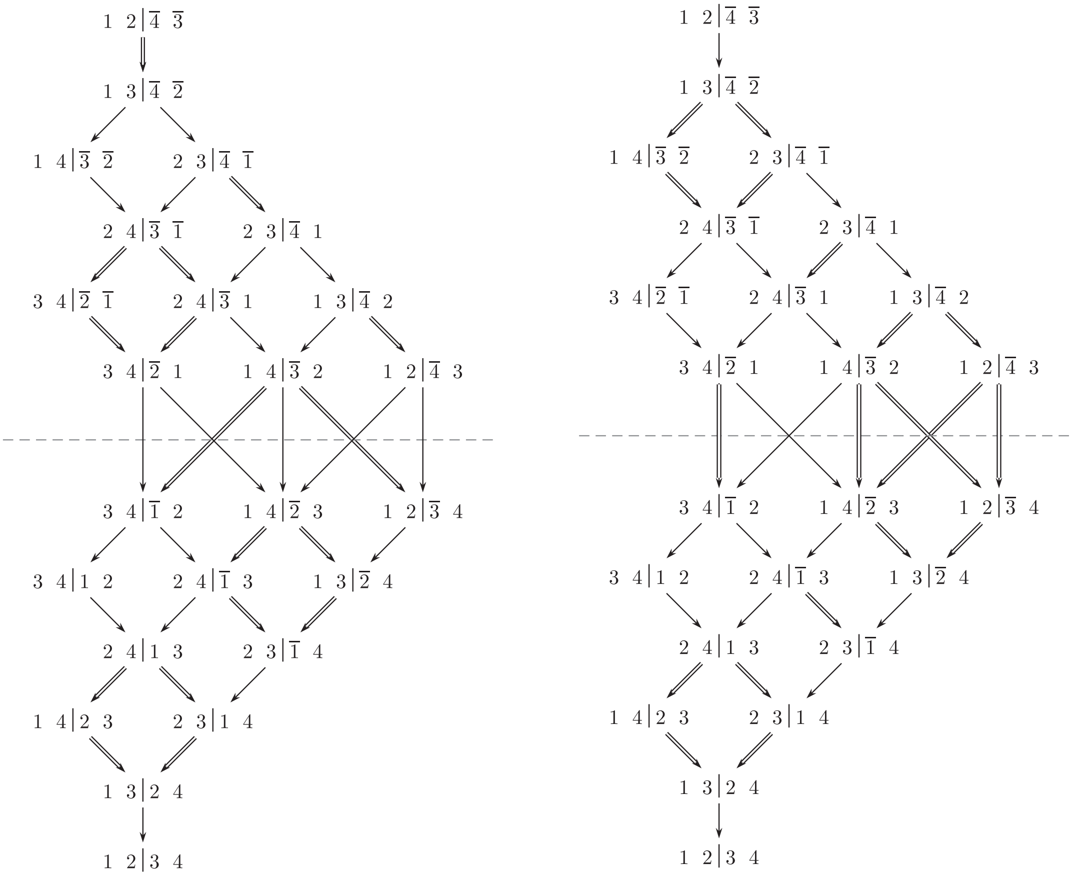


Fig. 8. The incidence graph of IG(2, 8) on the left and OG(2, 9) on the right.

$$
\begin{array}{llll}
H_{11}(\mathbb{F},\mathbb{Z}) = \mathbb{Z}, & H_8(\mathbb{F},\mathbb{Z}) = \mathbb{Z}_2, & H_5(\mathbb{F},\mathbb{Z}) = (\mathbb{Z}_2)^2, & H_2(\mathbb{F},\mathbb{Z}) = \mathbb{Z}_2, \\
H_{10}(\mathbb{F},\mathbb{Z}) = 0, & H_7(\mathbb{F},\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}_2, & H_4(\mathbb{F},\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}_2, & H_1(\mathbb{F},\mathbb{Z}) = \mathbb{Z}_2, \\
H_9(\mathbb{F},\mathbb{Z}) = \mathbb{Z}_2, & H_6(\mathbb{F},\mathbb{Z}) = \mathbb{Z}_2, & H_3(\mathbb{F},\mathbb{Z}) = \mathbb{Z}_2, & H_0(\mathbb{F},\mathbb{Z}) = \mathbb{Z}.
\end{array}
$$

3.5. First and second homology groups. Finally, as a by-product of our methods to compute the coefficients for the boundary map, it is not difficult to obtain results about the 1st and 2nd homology groups of odd orthogonal Grassmannians and isotropic Grassmannians.

Proposition 3.16.

(i) *If* $\mathbb{F} = \mathrm{IG}(n-k, 2n)$ *then*

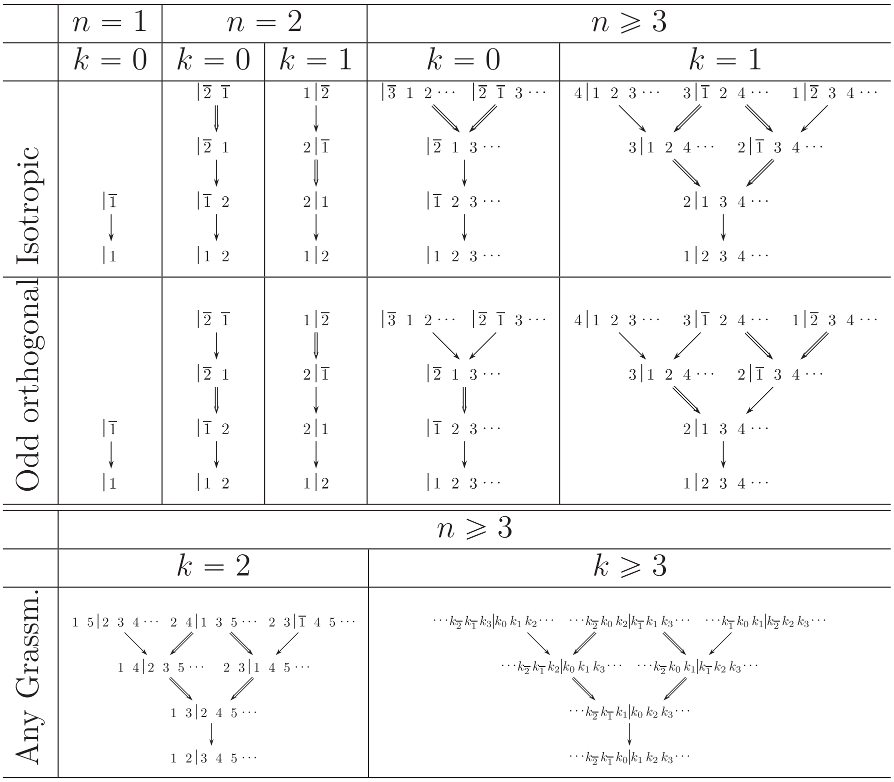
$$
H_1(\mathbb{F},\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0; \\ \mathbb{Z}_2 & \text{otherwise;} \end{cases} \qquad H_2(\mathbb{F},\mathbb{Z}) = \begin{cases} 0 & \text{if } n = 2, k = 1; \\ \mathbb{Z}_2 & \text{otherwise;} \end{cases}
$$

(ii) *If* $\mathbb{F} = \mathrm{OG}(n-k, 2n+1)$ *then*

$$
H_1(\mathbb{F},\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 1, k = 0; \\ \mathbb{Z} & \text{if } n = 2, k = 1; \\ \mathbb{Z}_2 & \text{otherwise;} \end{cases} \qquad H_2(\mathbb{F},\mathbb{Z}) = \begin{cases} 0 & \text{if } k = 0; \\ \mathbb{Z}_2 & \text{otherwise.} \end{cases}
$$

Proof. To compute $H_1(\mathbb{F}, \mathbb{Z})$ and $H_2(\mathbb{F}, \mathbb{Z})$, we only require to know the boundary maps ∂_1, ∂_2, and ∂_3, which depend on k and n. Table 2 shows the incidence graph up to 3-dimensional cells for different arrangements of n and k. In the last graph, we denote $k_a = k + a$.

Table 2. Bruhat graphs to compute 1st and 2nd homology



The 1st and 2nd homology groups can be computed using such diagrams. □

4. Proof of the main results

In this section, we compute $\sigma(w) - \sigma(w')$ for each type of covering pairs w, w'. Remember that $\mathrm{Inv}^+(w)$ and $\mathrm{Inv}^-(w)$ are given by the equations (14) and (15).

Let us denote $(\beta')^+_{i,j} = \beta^+_{i,j}(w')$ where $(i, j) \in \mathrm{Inv}^+(w')$, $\widetilde{\beta}^-_{i,j} = \beta^-_{i,j}(\widetilde{w})$ where $(i, j) \in \mathrm{Inv}^-(\widetilde{w})$, and $(\widetilde{\beta}')^-_{i,j} = \beta^-_{i,j}(\widetilde{w}')$ where $(i, j) \in \mathrm{Inv}^-(\widetilde{w}')$. It will be useful to write $\sigma(w) - \sigma(w') = S^+ + S^-$ with

$$S^+ = \sum_{(i,j) \in \mathrm{Inv}^+(w)} \beta^+_{i,j} - \sum_{(i,j) \in \mathrm{Inv}^+(w')} (\beta')^+_{i,j} \quad \text{and} \quad S^- = \sum_{(i,j) \in \mathrm{Inv}^-(\widetilde{w})} \widetilde{\beta}^-_{i,j} - \sum_{(i,j) \in \mathrm{Inv}^-(\widetilde{w}')} (\widetilde{\beta}')^-_{i,j}.$$

We can see in Theorem 3.10 that κ depends on the type of G. We can merge both types in a single formula after adopting the following notation: let isB be the variable that indicates whether the Grassmannian is odd orthogonal or not, i.e.,

$$\mathsf{isB} = \begin{cases} 1 & \text{for } \mathrm{OG}(n-k, 2n+1); \\ 0 & \text{for } \mathrm{IG}(n-k, 2n). \end{cases}$$

Then, κ in Theorem 3.10 can be given as follows:

- Type B1: $\kappa = (1 + \mathsf{isB})T - \mathsf{isB}$;
- Type B2: $\kappa = T + Q - \mathsf{isB}$;
- Type B3: $\kappa = Q - P$;
- Type B4: $\kappa = P + T - \mathsf{isB}$.

Proposition 4.1. *Let w, w' be in $\mathcal{W}_n^{(k)}$ such that* $\mathsf{Type}(w, w') = B1$*. Denote $T = w^{-1}(\overline{\lambda_1}) = k + r$. Then, $\kappa = (\mathsf{isB} + 1)T - \mathsf{isB}$ and $\gamma = 2^{1-\mathsf{isB}}\, \varepsilon_1$.*

Proof. Observe that $w(T) = \overline{\lambda_1} = \overline{1}$, $w'(T) = \lambda_1 = 1$, and $w(i) = w'(i)$ whenever $i \neq T$. By Propositions 2.6(i) and 3.5, $\mathrm{Inv}^+(w) = \mathrm{Inv}^+(w') = \{(i, j) \colon i \in [k] \text{ and } j \in [k + 1, k + \alpha_i]\}$.

Clearly, $T \in [k + 1, k + \alpha_i]$ for every $i \in [k]$. For $(i, j) \in \mathrm{Inv}^+(w)$,

$$\beta^+_{i,j} = \begin{cases} \varepsilon_{w(i)} + \varepsilon_1 & \text{if } j = T; \\ \varepsilon_{w(i)} - \varepsilon_{w(j)} & \text{if } j \neq T; \end{cases} \qquad (\beta')^+_{i,j} = \begin{cases} \varepsilon_{w(i)} - \varepsilon_1 & \text{if } j = T; \\ \varepsilon_{w(i)} - \varepsilon_{w(j)} & \text{if } j \neq T. \end{cases}$$

Then, S^+ can be rearranged as

$$S^+ = \sum_{i=1}^{k}\left(\sum_{j=k+1}^{k+\alpha_i}(\beta^+_{i,j} - (\beta')^+_{i,j})\right) = \sum_{i=1}^{k}(\beta^+_{i,T} - (\beta')^+_{i,T}) = 2k\varepsilon_1 = (2^{\mathsf{isB}}k)(2^{1-\mathsf{isB}}\varepsilon_1).$$

To compute S^-, we know that $\lambda'_t = \lambda_{t+1}$ for $1 \leqslant t \leqslant r - 1 = r'$ by Proposition 2.6(i). Using Lemma 3.6 yields the position of $\overline{\lambda_1} = \overline{1}$ in $\widetilde{w}$ and the position of $\lambda_1 = 1$ in $\widetilde{w}'$ are equal to r. In other words, $\widetilde{w}(r) = \overline{1}$, $\widetilde{w}'(r) = 1$, and $\widetilde{w}(i) = \widetilde{w}'(i)$ whenever $i \neq r$.

For $i = r$, we have that $[i, i - 1 + \lambda_{r-i+1}] = [r, r]$. By Proposition 3.7, $\mathrm{Inv}^-(\widetilde{w}) = \{(i, j) \colon i \in [r] \text{ and } j \in [i, i - 1 + \lambda_{r-i+1}]\}$ and $\mathrm{Inv}^-(\widetilde{w}') = \mathrm{Inv}^-(\widetilde{w}) - \{(r, r)\}$. Assume that $(\widetilde{\beta}')^-_{r,r} = 0$ to simplify the notation.

For $(i, j) \in \mathrm{Inv}^-(\widetilde{w})$,

$$\widetilde{\beta}^-_{i,j} = \begin{cases} 2^{1-\mathsf{isB}}\varepsilon_1 & \text{if } i = r; \\ -\varepsilon_{\widetilde{w}(i)} + \varepsilon_1 & \text{if } i < r, j = r; \\ 2^{-\mathsf{isB}\cdot\delta_{ij}}(-\varepsilon_{\widetilde{w}(i)} - \varepsilon_{\widetilde{w}(j)}) & \text{otherwise;} \end{cases}$$

$$(\widetilde{\beta}')^-_{i,j} = \begin{cases} 0 & \text{if } i = r; \\ -\varepsilon_{\widetilde{w}(i)} - \varepsilon_1 & \text{if } i < r, j = r; \\ 2^{-\mathsf{isB}\cdot\delta_{ij}}(-\varepsilon_{\widetilde{w}(i)} - \varepsilon_{\widetilde{w}(j)}) & \text{otherwise.} \end{cases}$$

Then, S^- can be rearranged as

$$\begin{aligned} S^- &= \sum_{i=1}^{r}\left(\sum_{j=i}^{i-1+\lambda_{r-i+1}}(\widetilde{\beta}^-_{i,j} - (\widetilde{\beta}')^-_{i,j})\right) = \sum_{i=1}^{r-1}(\widetilde{\beta}^-_{i,r} - (\widetilde{\beta}')^-_{i,r}) + (\widetilde{\beta}^-_{r,r} - (\widetilde{\beta}')^-_{r,r}) \\ &= (2^{\mathsf{isB}}r - 2^{\mathsf{isB}} + 1)(2^{1-\mathsf{isB}}\varepsilon_1). \end{aligned}$$

We can easily observe that $2^{\mathsf{isB}} = \mathsf{isB} + 1$. Therefore,

$$\sigma(w) - \sigma(w') = (2^{\mathsf{isB}}k + 2^{\mathsf{isB}}r - 2^{\mathsf{isB}} + 1)(2^{1-\mathsf{isB}}\varepsilon_1) = ((\mathsf{isB}{+})T - \mathsf{isB})(2^{1-\mathsf{isB}}\varepsilon_1). \qquad \square$$

The next two lemmas are required for the next proofs. Consider $w \in \mathcal{W}_n^{(k)}$, with double partition $\Lambda = (\alpha, \lambda)$ and $r = \ell(\lambda)$.

Lemma 4.2. *If $1 \leqslant i < j \leqslant r$ then $j + \lambda_{r-j+1} \leqslant i + \lambda_{r-i+1}$.*

Proof. For every $i \in [r-1]$, we have $\lambda_{r-i+1} \geqslant 1 + \lambda_{r-i}$ since λ is a strict partition. Furthermore, if $l \in [r-i]$ then $\lambda_{r-i+1} \geqslant l + \lambda_{r-i+1-l}$. Hence, given $i < j$, take $l = j - i$. □

Lemma 4.3. *Let i and j be integers such that $i \in [k]$ and $j \in [n-k]$. Then, $j \leqslant \alpha_i$ if, and only if, $k - i + 1 \leqslant \alpha^*_j$.*

Proof. Notice that $j \leqslant \alpha_i$ if, and only if, $\{\alpha_l : \alpha_l \geq j\} \supseteq \{\alpha_i, \dots, \alpha_k\}$. The result follows from the definition of α^*. □

Proposition 4.4. *Let w, w' be in $\mathcal{W}_n^{(k)}$ such that $\mathsf{Type}(w, w') = B2$. Denote $T = w^{-1}(\overline{\lambda_t}) = k + r - t + 1$ and $Q = w^{-1}(v_q) = k + r + q$. Then, $\kappa = T + Q - \mathsf{isB}$ and $\gamma = \varepsilon_{\lambda_t} - \varepsilon_{v_q}$.*

Proof. Observe that the indexes T and Q were chosen such that $v_q = \lambda_t - 1$, $w(T) = \overline{\lambda_t}$, $w(Q) = v_q$, $w'(T) = \overline{v_q}$, $w'(Q) = \lambda_t$, and $w(i) = w'(i)$ whenever $i \neq T$ and $i \neq Q$. By Propositions 2.6(ii) and 3.5, $\mathrm{Inv}^+(w) = \mathrm{Inv}^+(w') = \{(i, j) : i \in [k] \text{ and } j \in [k+1, k+\alpha_i]\}$.

Notice that $T \in [k+1, k+r] \subset [k+1, k+\alpha_i]$ and $Q \notin [1, k+r]$. For $(i, j) \in \mathrm{Inv}^+(w)$,

$$\beta^+_{i,j} = \begin{cases} \varepsilon_{w(i)} + \varepsilon_{\lambda_t} & \text{if } j = T; \\ \varepsilon_{w(i)} - \varepsilon_{v_q} & \text{if } j = Q; \\ \varepsilon_{w(i)} - \varepsilon_{w(j)} & \text{otherwise;} \end{cases} \qquad (\beta')^+_{i,j} = \begin{cases} \varepsilon_{w(i)} + \varepsilon_{v_q} & \text{if } j = T; \\ \varepsilon_{w(i)} - \varepsilon_{\lambda_t} & \text{if } j = Q; \\ \varepsilon_{w(i)} - \varepsilon_{w(j)} & \text{otherwise.} \end{cases}$$

The summation S^+ can be rearranged as follows

$$S^+ = \sum_{i=1}^{k} \left(\sum_{j=k+1}^{k+\alpha_i} \left(\beta^+_{i,j} - (\beta')^+_{i,j} \right) \right).$$

Applying Lemma 4.3 gives $Q \leqslant k + \alpha_i$ if, and only if, $k - \alpha^*_{Q-k} + 1 \leqslant i$. Keeping in mind that T always lies in the interval $[k+1, k+\alpha_i]$ for any i, we can split the summation S^+ over i in the following parts:

(i) If $i \in [k - \alpha^*_{Q-k}]$ then $Q \notin [k+1, k+\alpha_i]$ and $\sum_{j=k+1}^{k+\alpha_i} (\beta^+_{i,j} - (\beta')^+_{i,j}) = (\beta^+_{i,T} - (\beta')^+_{i,T}) = \varepsilon_{\lambda_t} - \varepsilon_{v_q}$.

(ii) If $i \in [k - \alpha^*_{Q-k} + 1, k]$ then $Q \in [k+1, k+\alpha_i]$ and $\sum_{j=k+1}^{k+\alpha_i} (\beta^+_{i,j} - (\beta')^+_{i,j}) = (\beta^+_{i,T} - (\beta')^+_{i,T}) + (\beta^+_{i,Q} - (\beta')^+_{i,Q}) = 2(\varepsilon_{\lambda_t} - \varepsilon_{v_q})$.

Hence,

$$\begin{aligned} S^+ &= \sum_{i=1}^{k-\alpha^*_{Q-k}} \left(\sum_{j=k+1}^{k+\alpha_i} (\beta^+_{i,j} - (\beta')^+_{i,j}) \right) + \sum_{i=k-\alpha^*_{Q-k}+1}^{k} \left(\sum_{j=k+r+1}^{k+\alpha_i} (\beta^+_{i,j} - (\beta')^+_{i,j}) \right) \\ &= (k + \alpha^*_{Q-k})(\varepsilon_{\lambda_t} - \varepsilon_{v_q}). \end{aligned}$$

To compute S^-, we know that $\lambda'_t = \lambda_t - 1$ and $\lambda'_i = \lambda_i$ for $i \neq t$ by Proposition 2.6(ii). Using Lemma 3.6 yields the position of $\overline{\lambda_t}$ in $\widetilde{w}$ is $\widetilde{T} := \widetilde{w}^{-1}(\overline{\lambda_t}) = T - k = r - t + 1$, and the position of v_q in $\widetilde{w}$ is $\widetilde{Q} := \widetilde{w}^{-1}(v_q) = Q - \alpha^*_{Q-k}$. In other words, $\widetilde{w}(\widetilde{T}) = w(T) = \overline{\lambda_t}$, $\widetilde{w}'(\widetilde{T}) = w'(T) = \overline{v_q}$, $\widetilde{w}(\widetilde{Q}) = w(Q) = v_q$, $\widetilde{w}'(\widetilde{Q}) = w'(Q) = \lambda_t$, and $\widetilde{w}(i) = \widetilde{w}'(i)$ whenever $i \neq \widetilde{T}$ and $i \neq \widetilde{Q}$. By Proposition 3.7, $\mathrm{Inv}^-(\widetilde{w}) = \{(i, j) : i \in [r] \text{ and } j \in [i, i - 1 + \lambda_{r-i+1}]\}$ and $\mathrm{Inv}^-(\widetilde{w}') = \mathrm{Inv}^-(\widetilde{w}) - \{(\widetilde{T}, \widetilde{Q})\}$.

Notice that $\widetilde{T} \in [1,r]$ and $\widetilde{Q} \notin [1,r]$. It remains to determine whether $\widetilde{T}$ and $\widetilde{Q}$ lie in the interval $[i, i-1+\lambda_{r-i+1}]$. Applying Lemma 2.4 gives $\widetilde{Q} = Q - \alpha^*_{Q-k} = k + r + q - \mu^*_q$. The number μ^*_q can be computed as follows: $\mu^*_q = \#\{u_l \colon u_l > v_q\} = n - v_q - \#\{\lambda_l \colon \lambda_l > v_q\} - \#\{v_l \colon v_l > v_q\} = n - (\lambda_t - 1) - (r - t + 1) - (n - k - r - q) = k + t + q - \lambda_t$. Hence, $\widetilde{Q} = r - t + \lambda_t = \widetilde{T} - 1 + \lambda_{r-\widetilde{T}+1}$.

By Lemma 4.2, given $i \in [1,r]$, we can say whether $\widetilde{T}$ and $\widetilde{Q}$ belong to $[i, i-1+\lambda_{r-i+1}]$ according to how i compares to $\widetilde{T}$. Namely,

$$\widetilde{T}, \widetilde{Q} \in [i, i-1+\lambda_{r-i+1}], \text{ for } 1 \leqslant i < \widetilde{T}; \tag{16}$$

$$[\widetilde{T}, \widetilde{Q}] = [i, i-1+\lambda_{r-i+1}], \text{ for } i = \widetilde{T}; \tag{17}$$

$$\widetilde{T}, \widetilde{Q} \notin [i, i-1+\lambda_{r-i+1}], \text{ for } \widetilde{T} < i \leqslant r. \tag{18}$$

Since $(\widetilde{T}, \widetilde{Q}) \notin \mathrm{Inv}^-(\widetilde{w}')$, we will assume $(\widetilde{\beta}')^-_{\widetilde{T},\widetilde{Q}} = 0$ to simplify the notation. For $(i,j) \in \mathrm{Inv}^-(w)$,

$$\widetilde{\beta}^-_{i,j} = \begin{cases} \varepsilon_{\lambda_t} - \varepsilon_{v_q} & \text{if } i = \widetilde{T}, j = \widetilde{Q}; \\ 2^{1-\mathsf{isB}}\varepsilon_{\lambda_t} & \text{if } i = \widetilde{T}, j = \widetilde{T}; \\ \varepsilon_{\lambda_t} - \varepsilon_{\widetilde{w}(j)} & \text{if } i = \widetilde{T}, \widetilde{T} < j < \widetilde{Q}; \\ -\varepsilon_{\widetilde{w}(i)} + \varepsilon_{\lambda_t} & \text{if } i < \widetilde{T}, j = \widetilde{T}; \\ -\varepsilon_{\widetilde{w}(i)} - \varepsilon_{v_q} & \text{if } i < \widetilde{T}, j = \widetilde{Q}; \\ 2^{-\mathsf{isB}\cdot\delta_{ij}}(-\varepsilon_{\widetilde{w}(i)} - \varepsilon_{\widetilde{w}(j)}) & \text{otherwise;} \end{cases}$$

$$(\widetilde{\beta}')^-_{i,j} = \begin{cases} 0 & \text{if } i = \widetilde{T}, j = \widetilde{Q}; \\ 2^{1-\mathsf{isB}}\varepsilon_{v_q} & \text{if } i = \widetilde{T}, j = \widetilde{T}; \\ \varepsilon_{v_q} - \varepsilon_{\widetilde{w}(j)} & \text{if } i = \widetilde{T}, \widetilde{T} < j < \widetilde{Q}; \\ -\varepsilon_{\widetilde{w}(i)} + \varepsilon_{v_q} & \text{if } i < \widetilde{T}, j = \widetilde{T}; \\ -\varepsilon_{\widetilde{w}(i)} - \varepsilon_{\lambda_t} & \text{if } i < \widetilde{T}, j = \widetilde{Q}; \\ 2^{-\mathsf{isB}\cdot\delta_{ij}}(-\varepsilon_{\widetilde{w}(i)} - \varepsilon_{\widetilde{w}(j)}) & \text{otherwise.} \end{cases}$$

The summation S^- can be rearranged as follows

$$S^- = \sum_{i=1}^{r} \left(\sum_{j=i}^{i-1+\lambda_{r-i+1}} \left(\widetilde{\beta}^-_{i,j} - (\widetilde{\beta}')^-_{i,j} \right) \right).$$

We can split the above summation over i in the following parts:

(i) If $i \in [\widetilde{T} - 1]$ then, by (16),

$$\sum_{j=i}^{i-1+\lambda_{r-i+1}} (\widetilde{\beta}^-_{i,j} - (\widetilde{\beta}')^-_{i,j}) = (\widetilde{\beta}^-_{i,\widetilde{T}} - (\widetilde{\beta}')^-_{i,\widetilde{T}}) + (\widetilde{\beta}^-_{i,\widetilde{Q}} - (\widetilde{\beta}')^-_{i,\widetilde{Q}}) = 2(\varepsilon_{\lambda_t} - \varepsilon_{v_q}).$$

(ii) If $i = \widetilde{T}$ then, by (17),

$$\begin{aligned} \sum_{j=\widetilde{T}}^{\widetilde{Q}} (\widetilde{\beta}^-_{\widetilde{T},j} - (\widetilde{\beta}')^-_{\widetilde{T},j}) &= (\widetilde{\beta}^-_{\widetilde{T},\widetilde{T}} - (\widetilde{\beta}')^-_{\widetilde{T},\widetilde{T}}) + \sum_{j=\widetilde{T}+1}^{\widetilde{Q}-1} (\widetilde{\beta}^-_{\widetilde{T},j} - (\widetilde{\beta}')^-_{\widetilde{T},j}) + (\widetilde{\beta}^-_{\widetilde{T},\widetilde{Q}} - (\widetilde{\beta}')^-_{\widetilde{T},\widetilde{Q}}) \\ &= (2^{1-\mathsf{isB}} + \widetilde{Q} - \widetilde{T})(\varepsilon_{\lambda_t} - \varepsilon_{v_q}). \end{aligned}$$

(iii) If $i \in [\widetilde{T}+1, r]$ then, by (18), $\displaystyle\sum_{j=i}^{i-1+\lambda_{r-i+1}} (\widetilde{\beta}^-_{i,j} - (\widetilde{\beta}')^-_{i,j}) = 0.$

We can easily observe that $2^{1-\mathsf{isB}} - 2 = -\mathsf{isB}$. Thus,

$$S^- = \sum_{i=1}^{\widetilde{T}-1}\left(\sum_{j=i}^{i-1+\lambda_{r-i+1}} (\widetilde{\beta}^-_{i,j} - (\widetilde{\beta}')^-_{i,j})\right) + \sum_{j=\widetilde{T}}^{\widetilde{Q}} (\widetilde{\beta}^-_{\widetilde{T},j} - (\widetilde{\beta}')^-_{\widetilde{T},j}) + \sum_{i=\widetilde{T}+1}^{r}\left(\sum_{j=i}^{i-1+\lambda_{r-i+1}} (\widetilde{\beta}^-_{i,j} - (\widetilde{\beta}')^-_{i,j})\right)$$
$$= (-\mathsf{isB} + T + Q - k - \alpha^*_{Q-k})(\varepsilon_{\lambda_t} - \varepsilon_{v_q}).$$

Therefore,

$$\sigma(w) - \sigma(w') = (T + Q - \mathsf{isB})(\varepsilon_{\lambda_t} - \varepsilon_{v_q}). \qquad \square$$

Proposition 4.5. *Let w, w' be in $\mathcal{W}_n^{(k)}$ such that* $\mathsf{Type}(w,w') = B3$*. Denote $P = w^{-1}(u_p) = p$ and $Q = w^{-1}(v_q) = k + r + q$. Then, $\kappa = Q - P$ and $\gamma = \varepsilon_{u_p} - \varepsilon_{v_q}$.*

Proof. Observe that the indexes P and Q were chosen such that $v_q = u_p - x$, $w(P) = u_p$, $w(Q) = v_q$, $w'(P) = v_q$, $w'(Q) = u_p$, and $w(i) = w'(i)$ whenever $i \neq P$ and $i \neq Q$. Proposition 2.6(iii), recall that $\alpha'_P = \alpha_P - 1$ and $\alpha'_i = \alpha_i$ for $i \neq P$. Then, by Proposition 3.5, $\mathrm{Inv}^+(w) = \{(i,j)\colon 1 \leqslant i \leqslant k \text{ and } k+1 \leqslant j \leqslant k+\alpha_i\}$ and $\mathrm{Inv}^+(w') = \mathrm{Inv}^+(w) - \{(P,Q)\}$.

Clearly $P \in [1,k]$ and $Q \notin [1,k]$. Since $(P,Q) \notin \mathrm{Inv}^-(\widetilde{w}')$, we will assume $(\widetilde{\beta}'_{P,Q}) = 0$. For $(i,j) \in \mathrm{Inv}^+(w)$,

$$\beta^+_{i,j} = \begin{cases} \varepsilon_{u_p} - \varepsilon_{v_q} & \text{if } i = P, j = Q;\\ \varepsilon_{u_p} - \varepsilon_{w(j)} & \text{if } i = P, j \neq Q;\\ \varepsilon_{w(i)} - \varepsilon_{v_q} & \text{if } i \neq P, j = Q;\\ \varepsilon_{w(i)} - \varepsilon_{w(j)} & \text{otherwise;}\end{cases} \qquad (\beta')^+_{i,j} = \begin{cases} 0 & \text{if } i = P, j = Q;\\ \varepsilon_{v_q} - \varepsilon_{w(j)} & \text{if } i = P, j \neq Q;\\ \varepsilon_{w(i)} - \varepsilon_{u_p} & \text{if } i \neq P, j = Q;\\ \varepsilon_{w(i)} - \varepsilon_{w(j)} & \text{otherwise.}\end{cases}$$

Then, S^+ can be rearranged as follows

$$S^+ = \sum_{i=1}^{k}\left(\sum_{j=k+1}^{k+\alpha_i} (\beta^+_{i,j} - (\beta')^+_{i,j})\right).$$

To compute this summation, we need to check when $Q \leqslant k + \alpha_i$. Notice that if $i = P$ then, by (10), $\alpha_P = r + \#\{v_l\colon v_l < u_P\} = r + q$ which implies that $Q = k + \alpha_P$. Applying Lemma 4.3 gives $Q \leqslant k+\alpha_i$ if, and only if, $k - \alpha^*_{Q-k} + 1 \leqslant i$. But, $\alpha^*_{Q-k} = \alpha^*_{\alpha_P} = \#\{\alpha_l\colon \alpha_l > \alpha_P\} = k - P + 1$. Hence, $Q \leqslant k + \alpha_i$ if, and only if, $i \geqslant P$.

We can split the summation of S^+ over i in the following parts:

(i) If $1 \leqslant i < P$ then $Q \notin [k+1, k+\alpha_i]$ and $\displaystyle\sum_{j=k+1}^{k+\alpha_i} (\beta^+_{i,j} - (\beta')^+_{i,j}) = 0.$

(ii) If $i = P$ then $[k+1, k+\alpha_P] = [k+1, Q]$ and $\displaystyle\sum_{j=k+1}^{Q} (\beta^+_{P,j} - (\beta')^+_{P,j}) = (\beta^+_{P,Q} - (\beta')^+_{P,Q}) + \sum_{j=k+1}^{Q-1} (\beta^+_{P,j} - (\beta')^+_{P,j}) = (Q-k)(\varepsilon_{u_p} - \varepsilon_{v_q}).$

(iii) If $P < i \leqslant k$ then $Q \in [k+1, k+\alpha_i]$ and $\displaystyle\sum_{j=k+1}^{k+\alpha_i} (\beta^+_{i,j} - (\beta')^+_{i,j}) = (\beta^+_{i,Q} - (\beta')^+_{i,Q}) = \varepsilon_{u_p} - \varepsilon_{v_q}.$

Hence,

$$S^+ = \sum_{i=1}^{P-1}\left(\sum_{j=k+1}^{k+\alpha_i}(\beta^+_{i,j} - (\beta')^+_{i,j})\right) + \sum_{j=k+1}^{Q}(\beta^+_{P,j} - (\beta')^+_{P,j}) + \sum_{i=P+1}^{k}\left(\sum_{j=k+1}^{k+\alpha_i}(\beta^+_{i,j} - (\beta')^+_{i,j})\right)$$
$$= (Q - P)(\varepsilon_{u_p} - \varepsilon_{v_q}).$$

To compute S^-, it is clear that both $\widetilde{w}$ and $\widetilde{w}'$ coincide. Then, $\beta^- = (\beta')^-$ and the sum S^- is zero. Therefore,

$$\sigma(w) - \sigma(w') = (Q - P)(\varepsilon_{u_p} - \varepsilon_{v_q}). \qquad \square$$

Proposition 4.6. *Let w, w' be in $\mathcal{W}_n^{(k)}$ such that w, w' is a pair of type B4. Denote $P = w^{-1}(u_p) = p$ and $T = w^{-1}(\overline{\lambda_t}) = k + r - t + 1$. Then, $\kappa = P + T - \mathsf{isB}$ and $\gamma = \varepsilon_{\lambda_t} - \varepsilon_{u_p}$.*

Proof. Observe that the indexes $P = p$ and $T = k + r - t + 1$ were chosen such that $u_p = \lambda_t - x$, $w(P) = u_p$, $w(T) = \overline{\lambda_t}$, $w'(P) = \lambda_t$, $w'(T) = \overline{u_p}$, and $w(i) = w'(i)$ whenever $i \neq P$ and $i \neq Q$. By Proposition 2.6(iv), recall that $\alpha'_P = \alpha_P + x - 1$ and $\alpha'_i = \alpha_i$ for $i \neq P$. Then, by Proposition 3.5, $\mathrm{Inv}^+(w) = \{(i, j) \colon 1 \leqslant i \leqslant k \text{ and } k + 1 \leqslant j \leqslant k + \alpha_i\}$ and $\mathrm{Inv}^+(w') = \mathrm{Inv}^+(w) \cup A$, where A is the set given by: if $x > 1$ then $A = \{(P, k + \alpha_P + l) \colon 1 \leqslant l \leqslant x - 1\}$; if $x = 1$ then $A = \emptyset$.

Notice that $P \in [1, k]$ and $T \in [k + 1, k + \alpha_i]$ for every $i \in [1, k]$ since $T \leqslant k + r \leqslant k + \alpha_i$. Then, all roots $\beta^+_{i,j}$ and $(\beta')^+_{i,j}$ for $(i, j) \in \mathrm{Inv}^+(w)$ are

$$\beta^+_{i,j} = \begin{cases} \varepsilon_{u_p} + \varepsilon_{\lambda_t} & \text{if } i = P, j = T; \\ \varepsilon_{w(i)} + \varepsilon_{\lambda_t} & \text{if } i \neq P, j = T; \\ \varepsilon_{u_p} - \varepsilon_{w(j)} & \text{if } i = P, j \neq T; \\ \varepsilon_{w(i)} - \varepsilon_{w(j)} & \text{otherwise;} \end{cases} \qquad (\beta')^+_{i,j} = \begin{cases} \varepsilon_{\lambda_t} + \varepsilon_{u_p} & \text{if } i = P, j = T; \\ \varepsilon_{w(i)} + \varepsilon_{u_p} & \text{if } i \neq P, j = T; \\ \varepsilon_{\lambda_t} - \varepsilon_{w(j)} & \text{if } i = P, j \neq T; \\ \varepsilon_{w(i)} - \varepsilon_{w(j)} & \text{otherwise.} \end{cases}$$

For $(P, j) \in A$, i.e., $k + \alpha_P + 1 \leqslant j \leqslant k + \alpha_P + x - 1$, the additional roots $(\beta')^+_{P,j}$ of w' are $(\beta')^+_{P,j} = \varepsilon_{\lambda_t} - \varepsilon_{w(j)}$.

Then, S^+ can be rearranged as follows

$$S^+ = \sum_{i=1}^{k}\left(\sum_{j=k+1}^{k+\alpha_i}(\beta^+_{i,j} - (\beta')^+_{i,j})\right) - \sum_{j=k+\alpha_P+1}^{k+\alpha_P+x-1}(\beta')^+_{P,j}.$$

Keeping in mind that T always lies in the interval $[k + 1, k + \alpha_i]$ for any i, we can split the above summation over i in the following parts:

(i) If $1 \leqslant i \leqslant k$ and $i \neq P$ then $\displaystyle\sum_{j=k+1}^{k+\alpha_i}(\beta^+_{i,j} - (\beta')^+_{i,j}) = \beta^+_{i,T} - (\beta')^+_{i,T} = \varepsilon_{\lambda_t} - \varepsilon_{u_p}$.

(ii) If $i = P$ then $\displaystyle\sum_{j=k+1}^{k+\alpha_P}(\beta^+_{P,j} - (\beta')^+_{P,j}) = \sum_{j=k+1}^{T-1}(\beta^+_{P,j} - (\beta')^+_{P,j}) + (\beta^+_{P,T} - (\beta')^+_{P,T}) + \sum_{j=T+1}^{k+\alpha_P}(\beta^+_{P,j} - (\beta')^+_{P,j}) = (1 - \alpha_P)(\varepsilon_{\lambda_t} - \varepsilon_{u_p})$.

Hence,

$$S^+ = \sum_{\substack{i\in[1,k] \\ i\neq T}}(\varepsilon_{\lambda_t} - \varepsilon_{u_p}) + (1 - \alpha_P)(\varepsilon_{\lambda_t} - \varepsilon_{u_p}) - \sum_{j=k+\alpha_P+1}^{k+\alpha_P+x-1}(\varepsilon_{\lambda_t} - \varepsilon_{w(j)})$$

$$= (k - \alpha_P)(\varepsilon_{\lambda_t} - \varepsilon_{u_p}) - \sum_{j=k+\alpha_P+1}^{k+\alpha_P+x-1} (\varepsilon_{\lambda_t} - \varepsilon_{w(j)}).$$

To compute S^-, we know that $\lambda'_t = \lambda_t - x$ and $\lambda'_i = \lambda_i$ for $i \neq t$ by Proposition 2.6(iv). Using Lemma 3.6 yields the position of $\overline{\lambda_t}$ in $\widetilde{w}$ is $\widetilde{T} := \widetilde{w}^{-1}(w(T)) = T - k = r - t + 1$, and the position of u_p in $\widetilde{w}$ is $\widetilde{P} := \widetilde{w}^{-1}(w(P)) = P + \alpha_P$. In other words, $\widetilde{w}(\widetilde{P}) = w(P) = u_p$, $\widetilde{w}'(\widetilde{P}) = w'(P) = \lambda_t$, $\widetilde{w}(\widetilde{T}) = w(T) = \overline{\lambda_t}$, $\widetilde{w}'(\widetilde{T}) = w'(T) = \overline{u_p}$, and $\widetilde{w}(i) = \widetilde{w}'(i)$ whenever $i \neq \widetilde{T}$ and $i \neq \widetilde{P}$. By Proposition 3.7, $\mathrm{Inv}^-(\widetilde{w}) = \{(i, j) \colon 1 \leqslant i \leqslant r \text{ and } i \leqslant j \leqslant i - 1 + \lambda_{r-i+1}\}$ and $\mathrm{Inv}^-(\widetilde{w}') = \mathrm{Inv}^-(\widetilde{w}) - B$, where $B = \{(\widetilde{T}, \widetilde{P} + l) \colon 0 \leqslant l \leqslant x - 1\}$.

Notice that $\widetilde{T} \in [1, r]$, $\widetilde{P} \notin [1, r]$, and $\widetilde{P} = P + \alpha_P = u_p + \#\{\lambda_l \colon \lambda_l > u_p\} = (\lambda_t - x) + (r - t + 1) = \widetilde{T} - x + \lambda_{r-\widetilde{T}+1}$. By Lemma 4.2, given $i \in [1, r]$, we can say whether $\widetilde{P}$ and $\widetilde{T}$ belong to $[i, i - 1 + \lambda_{r-i+1}]$ according to how i compares to $\widetilde{T}$. Namely,

$$\widetilde{T}, \widetilde{P} \in [i, i - 1 + \lambda_{r-i+1}], \text{ for } 1 \leqslant i < \widetilde{T}; \tag{19}$$

$$[\widetilde{T}, \widetilde{P} + x - 1] = [i, i - 1 + \lambda_{r-i+1}], \text{ for } i = \widetilde{T}; \tag{20}$$

$$\widetilde{T}, \widetilde{P} \notin [i, i - 1 + \lambda_{r-i+1}], \text{ for } \widetilde{T} < i \leqslant r. \tag{21}$$

For $j \geqslant \widetilde{P}$, since $(\widetilde{T}, j) \notin \mathrm{Inv}^-(\widetilde{w}')$, we will assume that $(\widetilde{\beta}')^-_{\widetilde{T},j} = 0$ to simplify the notation. For $(i, j) \in \mathrm{Inv}^-(w)$,

$$\widetilde{\beta}^-_{i,j} = \begin{cases} \varepsilon_{\lambda_t} - \varepsilon_{u_p} & \text{if } i = \widetilde{T}, j = \widetilde{P}; \\ 2^{1-\mathsf{isB}} \varepsilon_{\lambda_t} & \text{if } i = \widetilde{T}, j = \widetilde{T}; \\ \varepsilon_{\lambda_t} - \varepsilon_{\widetilde{w}(j)} & \text{if } i = \widetilde{T}, j > \widetilde{T}, j \neq \widetilde{P}; \\ -\varepsilon_{\widetilde{w}(i)} + \varepsilon_{\lambda_t} & \text{if } i < \widetilde{T}, j = \widetilde{T}; \\ -\varepsilon_{\widetilde{w}(i)} - \varepsilon_{u_p} & \text{if } i < \widetilde{T}, j = \widetilde{P}; \\ 2^{-\mathsf{isB}\cdot\delta_{ij}}(-\varepsilon_{\widetilde{w}(i)} - \varepsilon_{\widetilde{w}(j)}) & \text{otherwise}; \end{cases}$$

$$(\widetilde{\beta}')^-_{i,j} = \begin{cases} 0 & \text{if } i = \widetilde{T}, j \geqslant \widetilde{P}; \\ 2^{1-\mathsf{isB}} \varepsilon_{u_p} & \text{if } i = \widetilde{T}, j = \widetilde{T}; \\ \varepsilon_{u_p} - \varepsilon_{\widetilde{w}(j)} & \text{if } i = \widetilde{T}, \widetilde{T} < j < \widetilde{P}; \\ -\varepsilon_{\widetilde{w}(i)} + \varepsilon_{u_p} & \text{if } i < \widetilde{T}, j = \widetilde{T}; \\ -\varepsilon_{\widetilde{w}(i)} - \varepsilon_{\lambda_t} & \text{if } i < \widetilde{T}, j = \widetilde{P}; \\ 2^{-\mathsf{isB}\cdot\delta_{ij}}(-\varepsilon_{\widetilde{w}(i)} - \varepsilon_{\widetilde{w}(j)}) & \text{otherwise}. \end{cases}$$

Then, S^- can be rearranged as follows

$$S^- = \sum_{i=1}^{r} \left(\sum_{j=i}^{i-1+\lambda_{r-i+1}} (\widetilde{\beta}^-_{i,j} - (\widetilde{\beta}')^-_{i,j}) \right).$$

We can split the above summation on i in the following parts:

(i) If $1 \leqslant i < \widetilde{T}$ then, by (19), $\sum_{j=i}^{i-1+\lambda_{r-i+1}} (\widetilde{\beta}^-_{i,j} - (\widetilde{\beta}')^-_{i,j}) = (\widetilde{\beta}^-_{i,\widetilde{T}} - (\widetilde{\beta}')^-_{i,\widetilde{T}}) + (\widetilde{\beta}^-_{i,\widetilde{P}} - (\widetilde{\beta}')^-_{i,\widetilde{P}}) = 2(\varepsilon_{\lambda_t} - \varepsilon_{u_p})$.

(ii) If $i = \widetilde{T}$ then, by (20), $\sum_{j=\widetilde{T}}^{\widetilde{P}+x-1} (\widetilde{\beta}^-_{\widetilde{T},j} - (\widetilde{\beta}')^-_{\widetilde{T},j}) = (\widetilde{\beta}^-_{\widetilde{T},\widetilde{T}} - (\widetilde{\beta}')^-_{\widetilde{T},\widetilde{T}}) + \sum_{j=\widetilde{T}+1}^{\widetilde{P}-1} (\widetilde{\beta}^-_{\widetilde{T},j} - (\widetilde{\beta}')^-_{\widetilde{T},j}) +$

$$(\widetilde{\beta}^-_{\widetilde{T},\widetilde{P}} - (\widetilde{\beta}')^-_{\widetilde{T},\widetilde{P}}) + \sum_{j=\widetilde{P}+1}^{\widetilde{P}+x-1} (\widetilde{\beta}^-_{\widetilde{T},j} - (\widetilde{\beta}')^-_{\widetilde{T},j}) = (2^{1-\mathsf{isB}} + \widetilde{P} - \widetilde{T})(\varepsilon_{\lambda_t} - \varepsilon_{u_p}) + \sum_{j=P+\alpha_P+1}^{P+\alpha_P+x-1} (\varepsilon_{\lambda_t} - \varepsilon_{\widetilde{w}(j)}).$$

(iii) If $\widetilde{T} < i \leqslant r$ then, by (21), $\sum_{j=i}^{i-1+\lambda_{r-i+1}} (\widetilde{\beta}^-_{i,j} - (\widetilde{\beta}')^-_{i,j}) = 0.$

Hence,

$$S^- = \sum_{i=1}^{\widetilde{T}-1} \left(\sum_{j=i}^{i-1+\lambda_{r-i+1}} (\widetilde{\beta}^-_{i,j} - (\widetilde{\beta}')^-_{i,j}) \right) + \sum_{j=\widetilde{T}}^{\widetilde{P}+x-1} (\widetilde{\beta}^-_{\widetilde{T},j} - (\widetilde{\beta}')^-_{\widetilde{T},j}) + \sum_{i=\widetilde{T}+1}^{r} \left(\sum_{j=i}^{i-1+\lambda_{r-i+1}} (\widetilde{\beta}^-_{i,j} - (\widetilde{\beta}')^-_{i,j}) \right)$$

$$= (P + T + \alpha_p - k - \mathsf{isB})(\varepsilon_{\lambda_t} - \varepsilon_{u_p}) + \sum_{j=P+\alpha_P+1}^{P+\alpha_P+x-1} (\varepsilon_{\lambda_t} - \varepsilon_{\widetilde{w}(j)}).$$

Finally, the remaining summation over S^- is supposed to cancel with the one in S^+. If we prove that $\widetilde{w}(j) = w(j + k - P)$ for $j \in [P + \alpha_P + 1, P + \alpha_P + x - 1]$, then we clearly have

$$\sum_{j=P+\alpha_{P+1}}^{P+\alpha_P+x-1} (\varepsilon_{\lambda_t} - \varepsilon_{\widetilde{w}(j)}) = \sum_{j=P+\alpha_{P+1}}^{P+\alpha_P+x-1} (\varepsilon_{\lambda_t} - \varepsilon_{w(j+k-P)}) = \sum_{j=k+\alpha_{P+1}}^{k+\alpha_P+x-1} (\varepsilon_{\lambda_t} - \varepsilon_{w(j)}).$$

Considering $l = j - P - \alpha_P$, the above assertion is equivalent to prove that $w(l + \alpha_P + k) = \widetilde{w}(l + P + \alpha_P)$, for $l \in [1, x - 1]$. Clearly, $l + \alpha_P + k > k$ and, by Lemma 3.6, $\widetilde{w}^{-1}(w(l + \alpha_P + k)) = l + \alpha_P + k - \alpha^*_{l+\alpha_P}$. We know that $\alpha_{P+1} = \alpha'_{P+1} \geqslant \alpha'_P = \alpha_P + x - 1$ since α' is also a partition. Then, by definition, $\alpha^*_{l+\alpha_P} = \#\{i : \alpha_i \geqslant l + \alpha_P\} = \#\{P + 1, \ldots, k\} = k - P$ for every $l \in [1, x - 1]$. Hence, $\widetilde{w}^{-1}(w(l + \alpha_P + k)) = l + P + \alpha_P$.

Therefore,

$$\sigma(w) - \sigma(w') = (P + T - \mathsf{isB})(\varepsilon_{\lambda_t} - \varepsilon_{v_q}). \qquad \square$$

5. Final comments and further directions

(1) We have realized that Theorem 3.12 is true for all flag manifolds of split real forms. This will appear in a forthcoming paper.

(2) Example 3 of OG(2, 9) has revealed how the choice of signs may be an obstacle to get the homology groups. We expect to apply the theory in [13] to find an algorithm that provides an appropriate choice of signs for covering pairs.

(3) If one obtains a complete characterization of the covering pairs for type D, we await to achieve similar results for the coefficients of even orthogonal Grassmannians.

(4) We look forward to get analogous excitations as defined by [7] and [5] inside the HSYD's for any odd orthogonal and isotropic Grassmannians.

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