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ON THE GIBBS MEASURES OF COMMUTING ONE-SIDED SUBSHIFTS OF FINITE TYPE

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0. Introduction

This paper is concerned with the Gibbs measures on mixing one–sided subshifts of finite type. Let (X, S) be a mixing one–sided subshift of finite type and let $\varphi : X \to \mathbb{R}$ be a continuous function with summable variation. Then there exists a unique S–invariant probability measure $\mu_{S,\varphi}$ on X, called the *Gibbs measure* of the system (X, S, φ), which maximizes the measure theoretic pressure [6]. It is well known that if $\varphi, \psi : X \to \mathbb{R}$ are continuous functions on X with summable variation, then $\mu_{S,\varphi} = \mu_{S,\psi}$ if and only if there is a continuous function w on X such that

$$\partial_S\varphi - \partial_S\psi = \partial_S^2 w,$$

where the coboundary operator ∂_S is defined by $\partial_S f = f - f \circ S$ for any real-valued function f on X. In this case, if φ and ψ are Holder continuous, then w must be Holder continuous too. Moreover, it has recently been proved that if $T : X \to X$ is a positively expansive endomorphism and $S \circ T = T \circ S$, then (X, T) is also a mixing one–sided subshift of finite type [1, 3, 4, 5], and (X, S) and (X, T) have the same Parry measure, that is, $\mu_{S,0} = \mu_{T,0}$ [1, 3, 4]. In this paper, generalizing these results, we find a necessary and sufficient condition for two systems (X, S, φ) and (X, T, ψ) to have the same Gibbs measure(Theorem 2.2). Consequently, we prove that a cocycle admits an identical Gibbs measure(Theorem 2.3).

1. Preliminaries

Let us introduce some preliminaries. A dynamical system is a pair (X, S), where X is a compact metric space with metric d, and $S : X \to X$ is a continuous surjective mapping. A dynamical system (X, S) is called a *one–sided subshift* if there is a finite clopen partition $\mathcal{A}$, called an *alphabet* for (X, S), such that

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(S) if $x, y \in X$ and $\pi(S^i x) = \pi(S^i y)$ for all $i = 0, 1, 2, \ldots$, then $x = y$,

where $\pi = \pi_{\mathcal{A}} : X \to \mathcal{A}$ is the natural projection defined by $x \in \pi(x)$ for $x \in X$. Note that if (X, S) is a subshift, then $S : X \to X$ is positively expansive, i.e., there is a positive constant ϵ such that if $x, y \in X$ and $d(S^i x, S^i y) < \epsilon$ for all $i = 0, 1, 2, \ldots$, then $x = y$. In particular, $S : X \to X$ is locally one–to–one. If (X, S) is a positively expansive dynamical system, and $\varphi : X \to \mathbb{R}$ is continuous, then it is well known that there is an S–invariant Borel probability measure μ on X, called an *equilibrium state* of the system (X, S, φ), such that

$$P = h_\mu + \int_X \varphi d\mu,$$

where P is the topological pressure of (X, S, φ) and h_μ is the measure theoretic entropy of (X, S, μ) [2, p.65]. For a positively expansive dynamical system (X, S) and a continuous function $\varphi : X \to \mathbb{R}$, let $E(X, S, \varphi)$ denote the set of all equilibrium states of (X, S, φ).

Lemma 1.1. *Let (X, S) be a positively expansive dynamical system and let $\varphi : X \to \mathbb{R}$ be continuous. Then for all continuous functions $w : X \to \mathbb{R}$ we have*

$$E(X, S, \varphi) = E(X, S, \varphi + \partial_S w).$$

Proof. Suppose that μ is an equilibrium state of (X, S, φ) or $(X, S, \varphi + \partial_S w)$. Then, since μ is S–invariant, we have

$$\int_X \partial_S w d\mu = \int_X (w - w \circ S) d\mu = \int_X w d\mu - \int_X w \circ S d\mu = 0,$$

so that

$$h_\mu + \int_X \varphi d\mu = h_\mu + \int_X (\varphi + \partial_S) w d\mu.$$

On noting that the topological pressure is the supremum of all measure theoretic pressures, we prove our assertion. □

A subshift (X, S) is said to be of *finite type* if there is an alphabet $\mathcal{A}$ together with a 0–1, $\mathcal{A} \times \mathcal{A}$ matrix M, called the *transition matrix* of $(X, S, \mathcal{A})$, such that

(F) for $\langle a_i \rangle_{i=0}^{\infty} \in \mathcal{A}^{\mathbb{N}}$, there is a point $x \in X$ such that $S^i x \in a_i$ for all $i = 0, 1, 2, \ldots$ if and only if $M(a_i, a_{i+1}) = 1$ for all $i = 0, 1, 2, \ldots$.

Such an alphabet is called a *Markov* alphabet. It is well known and easy to show that a subshift (X, S) is of finite type if and only if $S : X \to X$ is a local homeomorphism. Let (X, S) be a subshift of finite type, and let $\mathcal{A}$ be an alphabet for (X, S) with the transition matrix M. If M is primitive, i.e., $M^N > 0$ for some positive integer N, then we say that

(X, S) is a *mixing* subshift of finite type. Note that this definition does not depend on the choice of an alphabet.

Let (X, S) be a subshift, and let $\mathcal{A}$ be an alphabet for (X, S). For a function $\varphi : X \to \mathbb{R}$ and $k = 0, 1, 2, \ldots$, we define

$$\text{var}_k \varphi = \sup\{|\varphi(x) - \varphi(y)| : x, y \in W, W \in \mathcal{A} \vee S^{-1}\mathcal{A} \vee \cdots \vee S^{-k}\mathcal{A}\}.$$

It is easy to see that φ is continuous if and only if $\text{var}_k \varphi \to 0$, as $k \to 0$. If $\sum_{k=0}^{\infty} \text{var}_k \varphi < \infty$, then φ is said to have *summable variation*, and we denote by $\mathcal{S}(X, S)$ the set of functions with summable variations. If there are $\alpha > 0$ and $\beta \in (0, 1)$ such that $\text{var}_k \varphi \leq \alpha\beta^k$ for $k = 0, 1, 2, \ldots$, then we say that φ is *Holder continuous* and denote by $\mathcal{H}(X, S)$ the set of Holder continuous functions. It should be noted that the definitions of variation summability and Holder continuity are irrelevant to the choice of an alphabet. The set of all continuous functions on X will be denoted by $\mathcal{C}(X)$.

Lemma 1.2. *Let (X, S) be a mixing subshift of finite type and let $w \in \mathcal{C}(X)$. If $\partial_S w \in \mathcal{H}(X, S)$, then $w \in \mathcal{H}(X, S)$.*

Proof. For simplicity, write $u = \partial_S w$. Let $\mathcal{A}$ be a Markov alphabet for (X, S). Since $u \in \mathcal{H}(X, S)$, there are $\alpha > 0$ and $\beta \in (0, 1)$ such that

$$\sup\{|u(x_1) - u(x_2)| : x_1, x_2 \in W\} \leq \alpha\beta^k \qquad (W \in \mathcal{A} \vee S^{-1}\mathcal{A} \vee \cdots \vee S^{-k}\mathcal{A}).$$

Since (X, S) is mixing, there is a point y whose forward orbit $\Gamma = \{y, Sy, S^2y, \ldots\}$ is dense in X. Fix a nonnegative integer k and let $W \in \mathcal{A} \vee S^{-1}\mathcal{A} \vee \cdots \vee S^{-k}\mathcal{A}$ be arbitrary. We will show that

$$|w(x_1) - w(x_2)| \leq \frac{\alpha\beta}{1 - \beta}\beta^k \qquad (x_1, x_2 \in W \cap \Gamma),$$

from which our assertion will follow.

Let $S^i y, S^j y \in W \cap \Gamma$, and assume that $i < j$. Then there is a point $z \in X$ such that $S^{j-i}z = z$ and

$$\pi_{\mathcal{A}}(S^{l+i}y) = \pi_{\mathcal{A}}(S^l z) \qquad (l = 0, 1, 2, \ldots, j - i + k).$$

Since $S^{j-i}z = z$, we have

$$u(z) + u(Sz) + \cdots + u(S^{j-i-1}z) = 0,$$

and hence

$$\begin{aligned}
&|w(S^i y) - w(S^j y)| \\
=&|u(S^i y) + u(S^{i+1} y) + \cdots + u(S^{j-1} y)| \\
\le&|u(S^i y) - u(z)| + |u(S^{i+1} y) - u(Sz)| + \cdots + |u(S^{j-1} y) - u(S^{j-i-1} z)| \\
\le&\alpha\beta^{j-i+k} + \alpha\beta^{j-i+k-1} + \cdots + \alpha\beta^{k+1} \\
\le&\frac{\alpha\beta}{1-\beta}\beta^k.
\end{aligned}$$

□

It is well known that $\mathcal{C}(X)$ is a Banach space with the supremum norm $\|\cdot\|$, and its dual is the space of all signed Borel measures on X. Let $M(X)$ denote the set of all Borel probability measures of X. In the case that (X, S) is a mixing subshift of finite type and $\varphi \in \mathcal{S}(X, S) + \partial_S \mathcal{C}(X)$, the system (X, S, φ) has one and only one equilibrium state $\mu_{S,\varphi}$, called the *Gibbs measure* of (X, S, φ). As we will see it in Theorem 1.4, the Gibbs measures are represented as products of eigen–functions and eigen–measures of the Ruelle operator which are defined as follows. Let $H : X \to X$ be a surjective local homeomorphism and let $\varphi \in \mathcal{C}(X)$. Then the *Ruelle operator* $\mathcal{L} = \mathcal{L}_{H,\varphi} : \mathcal{C}(X) \to \mathcal{C}(X)$ is defined by

$$\mathcal{L}f(x) = \sum_{y \in H^{-1}(x)} e^{\varphi(y)} f(y) \qquad (f \in \mathcal{C}(X), x \in X).$$

Note that $\mathcal{L}_{H,\varphi}$ is a well–defined bounded linear operator on $\mathcal{C}(X)$. From here on, the dual operator of $\mathcal{L}_{H,\varphi}$ will be denoted by $\mathcal{L}^*_{H,\varphi}$ or simply by $\mathcal{L}^*$.

Lemma 1.3. *Let $H : X \to X$ be a local homeomorphism and let $\varphi, \psi \in \mathcal{C}(X)$. If $\mathcal{L}_{H,\varphi} = \mathcal{L}_{H,\psi}$, then $\varphi = \psi$.*

Proof. Let $y_0 \in X$ be arbitrary. Since $H : X \to X$ is a local homeomorphism, there is a clopen neighborhood U of y_0 such that $H|_U$ is one–to–one. Then

$$\mathcal{L}_{H,\varphi}\chi_U(H(y_0)) = \sum_{y \in H^{-1}(H(y_0))} e^{\varphi(y)}\chi_U(y) = e^{\varphi(y_0)}.$$

Since the same identity holds for ψ, we conclude that $\varphi = \psi$. □

The following theorem provides a very useful representation of the Gibbs measures. The original form of the theorem was first proved by Ruelle, and Walters [6] proved it on the condition that $\varphi \in \mathcal{S}(X, S)$. We prove the result in a slightly generalized form, which is needed to prove Theorem 2.1.

Theorem 1.4 (Ruelle's Operator Theorem). *Let (X, S) be a mixing subshift of finite type, and let $\varphi \in \mathcal{S}(X, S) + \partial_S \mathcal{C}(X)$. Then there are a positive real number λ, $f \in \mathcal{C}(X)$, and $\mu \in M(X)$ such that*

(a) $\mathcal{L}f = \lambda f, \mathcal{L}^*\mu = \lambda\mu, f > 0$, *and* $\int_X f d\mu = 1$,

(b) *for all* $g \in \mathcal{C}(X)$, $\|\lambda^{-n}\mathcal{L}^n g - (\int_X g d\mu) f\| \to 0$ *as* $n \to \infty$, *and*

(c) *the Gibbs measure of the system* (X, S, φ) *is given by* $f d\mu$.

Here $\mathcal{L} = \mathcal{L}_{S,\varphi}$ *and* $\mathcal{L}^* = \mathcal{L}^*_{S,\varphi}$. *Moreover, the λ–eigenspaces of $\mathcal{L}$ and $\mathcal{L}^*$ are 1–dimensional, and the triple $(\lambda, f, d\mu)$ is uniquely determined by the properties* (a) *and* (b).

Proof. For the special case that $\varphi \in \mathcal{S}(X, S)$, see [6]. The general case follows from the special case and Lemma 1.1, because

$$\mathcal{L}_{S,\varphi+\partial_S w} = \mathcal{L}^{-1}_{I,w} \circ \mathcal{L}_{S,\varphi} \circ \mathcal{L}_{I,w} \qquad (\varphi, w \in \mathcal{C}(X)),$$

where $I : X \to X$ is the identity mapping. □

From here on, the triple $(\lambda, f, d\mu)$ in the above theorem will be called the *Ruelle triple* of (X, S, φ). It is clear that if $(\lambda, f, d\mu)$ is the Ruelle triple of (X, S, φ), then $(\lambda, e^{-w} f, e^{w} d\mu)$ is the Ruelle triple of $(X, S, \varphi + \partial_S w)$ for all $\varphi \in \mathcal{S}(X, S) + \partial_S \mathcal{C}(X)$ and $w \in \mathcal{C}(X)$. In particular, we note that the Gibbs measures of (X, S, φ) and $(X, S, \varphi + \partial_S w)$ are identical for all $\varphi \in \mathcal{S}(X, S) + \partial_S \mathcal{C}(X)$ and $w \in \mathcal{C}(X)$.

Lemma 1.5. *Let (X, S) be a mixing subshift of finite type, $\varphi \in \mathcal{S}(X, S) + \partial_S \mathcal{C}(X)$ and let $(\lambda, f, d\mu)$ be the Ruelle triple of (X, S, φ). If $g \in \mathcal{C}(X)$ satisfies*

$$\int_X g h d\mu = 0 \qquad (h \in \mathcal{C}(X)),$$

then $g \equiv 0$.

Proof. It is enough to show that

$$\int_X \chi_U d\mu \neq 0$$

for all nonempty clopen subsets U of X. Let $U \neq \emptyset$ be an arbitrary clopen subset of X. Since (X, S) is mixing, there is a positive integer N such that

$$\mathcal{L}^N_{S,\varphi}\chi_U(x) = \sum_{y \in S^{-N}(x)} \exp(\varphi(y) + \varphi(Sy) + \cdots + \varphi(S^{N-1}y))\chi_U(y) > 0 \qquad (x \in X).$$

Therefore

$$\int_X \chi_U d\mu = \lambda^{-N} \int_X \mathcal{L}^N_{S,\varphi}\chi_U d\mu > 0.$$ □

2. Main results

Before we prove the main results of the paper, we need one more lemma.

Lemma 2.1. *Let $S, T : X \to X$ be commuting local homeomorphisms on a compact metric space X, and let φ and ψ be continuous functions on X. Then the Ruelle operators $\mathcal{L}_{S,\varphi}$ and $\mathcal{L}_{T,\psi}$ commute if and only if $\partial_T \varphi = \partial_S \psi$.*

Proof. A straightforward calculation by the definition of the Ruelle operator yields that

$$\mathcal{L}_{S,\varphi} \circ \mathcal{L}_{T,\psi} = \mathcal{L}_{S \circ T, \psi + \varphi \circ T}.$$

Similarly we have

$$\mathcal{L}_{T,\psi} \circ \mathcal{L}_{S,\varphi} = \mathcal{L}_{T \circ S, \varphi + \psi \circ S}.$$

The assertion immediately follows from these two identities and Lemma 1.3. □

Let (X, S) and (X, T) be mixing subshifts of finite type, and assume that $ST = TS$. Then there is a common alphabet $\mathcal{A}$ for (X, S) and (X, T). Since S is positively expansive, a compactness argument shows that for each positive integer k there exists a positive integer l such that if x and y lie in the same member of $\mathcal{A} \vee S^{-1}\mathcal{A} \vee \cdots \vee S^{-l}\mathcal{A}$ then x and y lie in the same member of $\mathcal{A} \vee T^{-1}\mathcal{A} \vee \cdots \vee T^{-k}\mathcal{A}$. It also holds if we interchange S and T. Therefore we have $\mathcal{S}(X, S) = \mathcal{S}(X, T)$ and $\mathcal{H}(X, S) = \mathcal{H}(X, T)$. From here on, we will use the simpler notations $\mathcal{S}$ and $\mathcal{H}$ to denote $\mathcal{S}(X, S)$ and $\mathcal{H}(X, S)$, respectively.

Theorem 2.2. *Let $S, T : X \to X$ be commuting maps such that (X, S) and (X, T) are mixing one–sided subshifts of finite type, and assume that $\varphi, \psi : X \to \mathbb{R}$ are continuous functions with summable variation. Then $\mu_{S,\varphi} = \mu_{T,\psi}$ if and only if there is a continuous function w on X such that*

$$\partial_T \varphi - \partial_S \psi = \partial_S \partial_T w.$$

Moreover, if φ and ψ are Holder continuous, then w is Holder continuous.

Proof. If $\varphi, \psi \in \mathcal{H}$, $w \in \mathcal{C}(X)$, and $\partial_T \varphi - \partial_S \psi = \partial_S \partial_T w$, then $\partial_S \partial_T w \in \mathcal{H}$, and hence Lemma 1.2 implies that $w \in \mathcal{H}$. This proves the last statement of the theorem.

Now we prove the main assertion of the theorem. Assume that $\partial_T \varphi - \partial_S \psi = \partial_S \partial_T w$ for some $w \in \mathcal{C}(X)$. Then $\partial_T \varphi = \partial_S(\psi + \partial_T w)$. By Lemma 2.1, we see that the Ruelle operators $\mathcal{L}_{S,\varphi}$ and $\mathcal{L}_{T,\psi + \partial_T w}$ commute. Hence $\mathcal{L}_{S,\varphi}$ and $\mathcal{L}_{T,\psi+\partial_T w}$ have the same eigen–functions and eigen–measures, and so the Gibbs measures of (X, S, φ) and

$(X, T, \psi + \partial_T w)$ are identical. Since (X, T, ψ) and $(X, T, \psi + \partial_T w)$ have the same Gibbs measure by Lemma 1.1, the 'if' part of the assertion is proved.

We prove the converse. Now assume that we are given two functions $\varphi, \psi \in \mathcal{S}$. Let $(\lambda, f, d\mu)$ and $(\lambda', g, d\nu)$ denote the Ruelle triples of (X, S, φ) and (X, T, ψ), respectively. It is enough to prove that if $f d\mu = g d\nu$ then $\partial_T \varphi - \partial_S \psi = \partial_S \partial_T (\log g - \log f)$. First of all, we may assume that $\lambda = \lambda' = 1$ (if not, replace φ and ψ by $\varphi - \log \lambda$ and $\psi - \log \lambda'$, respectively). Noting that for any $h \in \mathcal{C}(X)$

$$\mathcal{L}_{T, \psi + \partial_T (\log g - \log f)} h = \frac{f}{g} \mathcal{L}_{T, \psi} \left(\frac{g}{f} h \right),$$

we have

$$\begin{aligned} & \int_X \mathcal{L}_{T, \psi + \partial_T (\log g - \log f)} h d\mu \\ = & \int_X \frac{f}{g} \mathcal{L}_{T, \psi} \left(\frac{g}{f} h \right) d\mu \\ = & \int_X \mathcal{L}_{T, \psi} \left(\frac{g}{f} h \right) d\nu \\ = & \int_X \frac{g}{f} h d\nu \\ = & \int_X h d\mu. \end{aligned}$$

Putting $\tilde{\psi} = \psi + \partial_T (\log g - \log f)$, we obtain

$$\mathcal{L}^*_{T, \tilde{\psi}} \mu = \mu.$$

Then we have

$$\begin{aligned} & \int_X \left(\mathcal{L}_{S, \varphi} \circ \mathcal{L}_{T, \tilde{\psi}} h \right) h' d\mu \\ = & \int_X \mathcal{L}_{S, \varphi} \circ \mathcal{L}_{T, \tilde{\psi}} \left(h \cdot (h' \circ S \circ T) \right) d\mu \\ = & \int_X h \cdot (h' \circ S \circ T) d\mu \\ = & \int_X h \cdot (h' \circ T \circ S) d\mu \\ = & \int_X \left(\mathcal{L}_{T, \tilde{\psi}} \circ \mathcal{L}_{S, \varphi} h \right) h' d\mu \end{aligned}$$

for all $h, h' \in \mathcal{C}(X)$, and hence Lemma 1.5 implies that

$$\mathcal{L}_{S, \varphi} \circ \mathcal{L}_{T, \tilde{\psi}} = \mathcal{L}_{T, \tilde{\psi}} \circ \mathcal{L}_{S, \varphi}.$$

By Lemma 2.1, we conclude that

$$\partial_T \varphi = \partial_S \tilde{\psi} = \partial_S \psi + \partial_S \partial_T (\log g - \log f),$$

which is what we desired. $\square$

Let $S, T : X \to X$ be commuting maps that (X, S) and (X, T) are mixing one–sided subshifts of finite type. Then it is obvious that $(X, S^i \circ T^j)$ is a mixing one–sided subshifts of finite type. Assume that $\varphi, \psi : X \to \mathbb{R}$ be continuous functions which satisfy the relation $\partial_T \varphi = \partial_S \psi$. Then we can define, for each pair of nonnegative integers i and j with $(i, j) \neq (0, 0)$, a real-valued continuous function $c_{(i,j)}$ by

$$c_{(i,j)} = \sum_{k=0}^{i-1} \varphi \circ S^k \circ T^j + \sum_{l=0}^{j-1} \psi \circ T^l$$

or equivalently

$$c_{(i,j)} = \sum_{l=0}^{j-1} \psi \circ T^l \circ S^i + \sum_{k=0}^{i-1} \varphi \circ S^k.$$

This c is called the *cocyle generated by φ and ψ*. By a similar argument to the one in the paragraph preceding Theorem 2.2, we find that if $\varphi \in \mathcal{S}(X, S)$ and $\psi \in \mathcal{S}(X, T)$ then $c_{(i,j)} \in \mathcal{S}(X, S^i \circ T^j)$. It can be easily checked that for all nonnegative integers i, j, m and k, not all zero,

$$c_{(i+m,j+n)} = c_{(i,j)} \circ S^m \circ T^n + c_{(m,n)}.$$

By interchanging (i, j) and (m, n) in the above identity, we have

$$\partial_{S^m \circ T^n} c_{(i,j)} = \partial_{S^i \circ T^j} c_{(m,n)}.$$

Applying Theorem 2.2, we conclude that $(X, S^i \circ T^j, c_{(i,j)})$ have the same Gibbs measure for all nonnegative integers i and j with $(i, j) \neq (0, 0)$. This proves the following theorem.

Theorem 2.3. *Let $S, T : X \to X$ be commuting maps such that (X, S) and (X, T) are mixing one–sided subshifts of finite type, and assume that $\varphi, \psi : X \to \mathbb{R}$ are continuous functions with summable variation, which satisfy $\partial_T \varphi = \partial_S \psi$. Let c be the cocyle generated by φ and ψ. Then, for any nonnegative integers i and j with $(i, j) \neq (0, 0)$, $(X, S^i \circ T^j)$ is a mixing subshift of finite type and the Gibbs measures of $(X, S^i \circ T^j, c_{(i,j)})$ are identical.*

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