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STOCHASTIC INTEGRALS IN ADDITIVE PROCESSES AND APPLICATION TO SEMI-LÉVY PROCESSES

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1. Introduction

A stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is called an additive process if it has independent increments and if it is continuous in probability with cadlag paths and $X_0 = 0$. It is called a Lévy process if, in addition, it has stationary increments. Path behaviors and distributional properties of Lévy processes are deeply analyzed (see [1], [19]). Concerning additive processes, the Lévy–Itô decomposition of paths is known in complete generality. But, in order to get further results, we have to restrict our study to some special classes. Examples are the class of selfsimilar additive processes introduced in [18] and the class of semi-selfsimilar additive processes in [12]. Another interesting class is that of semi-Lévy processes, that is, additive processes with semi-stationary (sometimes called periodically stationary) increments. In order to analyze distributional properties of processes of these classes, it is important to treat stochastic integrals (of nonrandom integrands) based on additive processes. Keeping in mind this application, we study in this paper stochastic integrals based on additive processes and their distributions.

Our study in this paper does not depend on the cadlag property. We define additive processes in law, Lévy processes in law, and semi-Lévy processes in law, dropping the cadlag requirement in their definitions but retaining the requirement of continuity in probability. We will call an additive process in law $\{X_t : t \geq 0\}$ *natural* if the location parameter γ_t in the generating triplet (A_t, ν_t, γ_t) of the distribution of X_t is locally of bounded variation in t. An additive process is natural if and only if it is, at the same time, a semimartingale. This fact is essentially given in Jacod and Shiryaev [5]. Thus we can consider stochastic integrals for natural additive processes as a special case of semimartingale integrals of Kunita and Watanabe [9]. But we will not rely on the theory of semimartingales, but directly define stochastic integrals (of nonrandom functions) and seek the representation of the characteristic functions of their distributions. This is in the same line as the study of independently scattered random measures by Urbanik and Woyczynski [23] and Rajput and Rosinski [15]. We show that a natural additive process in law on $\mathbb{R}^d$ induces an $\mathbb{R}^d$-valued independently scattered random measure, and vice versa. Thus our random measures are $\mathbb{R}^d$-valued, not $\mathbb{R}$-valued as in [23] and [15]. Further, we are interested in construction of random measures on the same probability space as the original additive process in law is defined.

For a natural additive process in law $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ we use a system of infinitely divisible distributions $\{\rho_s : s \geq 0\}$ and a measure σ on $[0, \infty)$ such that

$$Ee^{i\langle z, X_t\rangle} = \exp \int_0^t \log \widehat{\rho}_s(z)\sigma(ds) \quad \text{for } z \in \mathbb{R}^d,$$

where $\widehat{\rho}_s(z)$ is the characteristic function of ρ_s. We will call $(\{\rho_s\}, \sigma)$ a *factoring* of $\{X_t\}$. In fact, existence of a factoring is a necessary and sufficient condition for naturalness. There is a canonical one among such pairs $(\{\rho_s\}, \sigma)$, which we call the *canonical* factoring of $\{X_t\}$. For a class of $l \times d$ matrix-valued functions $F(s)$ including all locally bounded measurable functions, stochastic integrals $\int_B F(s)\, dX_s$ for bounded Borel sets B are defined and shown to satisfy

$$E \exp \left[i \left\langle z, \int_B F(s)\, dX_s \right\rangle \right] = \exp \int_B \log \widehat{\rho}_s(F(s)'z)\sigma(ds) \quad \text{for } z \in \mathbb{R}^l,$$

where $F(s)'$ is the transpose of $F(s)$. Based on this formula we will study properties of stochastic integrals. Then we will treat the problem of the existence of stochastic integrals $\int_0^\infty e^{-sQ}\, dX_s$, where Q is a $d \times d$ matrix all of whose eigenvalues have positive real parts and $\{X_t : t \geq 0\}$ is a semi-Lévy process in law. It will be shown that $\int_0^\infty e^{-sQ}\, dX_s$ exists if and only if $\{X_t\}$ has finite log-moment.

In a forthcoming paper joint with M. Maejima, these results will be applied to a study of the relationship of semi-Lévy processes, semi-selfsimilar additive processes, and semi-stationary Ornstein–Uhlenbeck type processes. This study will extend the theory of the representation of selfdecomposable distributions by Wolfe [24], Jurek and Vervaat [7], Sato and Yamazato [20], [21], Sato [18], and Jeanblanc, Pitman, and Yor [6] to the case of semi-selfdecomposable and (b, Q)-decomposable distributions.

Natural additive processes in law and factorings are discussed in Section 2. Their relations to independently scattered random measures are studied in Section 3. Then stochastic integrals are treated in Section 4. Finally Section 5 contains the study of $\int_0^\infty e^{-sQ}\, dX_t$ for semi-Lévy processes in law.

Our notation and definitions follow [19]. Besides, we use the following: $ID = ID(\mathbb{R}^d)$ is the class of infinitely divisible distributions on $\mathbb{R}^d$; $\mathcal{B}(\mathbb{R}^d)$ is the class of Borel sets in $\mathbb{R}^d$; $\mathcal{B}_0(\mathbb{R}^d)$ is the class of $B \in \mathcal{B}(\mathbb{R}^d)$ satisfying $\inf_{x \in B} |x| > 0$; $\mathcal{B}_J$ for an interval J is the class of Borel sets in J; $\mathcal{B}^0_{[0,\infty)}$ is the class of bounded Borel sets in $[0, \infty)$; p-lim stands for limit in probability; $\mathbf{S}_d^+$ is the class of $d \times d$ symmetric nonnegative-definite matrices; $\mathbf{M}_{l \times d}$ is the class of $l \times d$ real matrices; $\mathbf{M}_d = \mathbf{M}_{d \times d}$ is the class of $d \times d$ real matrices; $\operatorname{tr} A$ is the trace of $A \in \mathbf{S}_d^+$; $\mathbf{M}_d^+$ is the class of $Q \in \mathbf{M}_d$ all of whose eigenvalues have positive (> 0) real parts; I is the $d \times d$ identity matrix. Recall that an element of the Euclidean space $\mathbb{R}^d$ is understood to be a column vector with d components. For $F \in \mathbf{M}_{l \times d}$, F' denotes the transpose of F. The norm of $F \in \mathbf{M}_{l\times d}$ is $\|F\| = \sup_{|x|\le 1}|Fx|$. For $b > 0$ and $Q \in \mathbf{M}_d$, $b^Q = \sum_{n=0}^{\infty}(n!)^{-1}(\log b)^n Q^n \in \mathbf{M}_d$. The inner products in $\mathbb{R}^d$ and $\mathbb{R}^l$ are denoted by the same symbol $\langle\ \ \rangle$. Thus we have $\langle z, Fx\rangle = \langle F'z, x\rangle$ for $x \in \mathbb{R}^d$, $z \in \mathbb{R}^l$, and $F \in \mathbf{M}_{l\times d}$. A set or a function is called measurable if it is Borel measurable. The characteristic function of a distribution μ is denoted by $\widehat{\mu}(z)$. Denote by $\mathcal{L}(X)$ the distribution of a random element X. When $\mathcal{L}(X) = \mathcal{L}(Y)$, we write $X \stackrel{\mathrm{d}}{=} Y$. For two stochastic processes $\{X_t\}$ and $\{Y_t\}$, $\{X_t\} \stackrel{\mathrm{d}}{=} \{Y_t\}$ means that they have an identical distribution as infinite-dimensional random elements, that is, have an identical system of finite-dimensional distributions, while $X_t \stackrel{\mathrm{d}}{=} Y_t$ means that X_t and Y_t have an identical distribution for a fixed t. If the characteristic function $\widehat{\mu}(z)$ of a distribution μ on $\mathbb{R}^d$ vanishes nowhere, then there is a unique continuous function $f(z)$ on $\mathbb{R}^d$ such that $f(0) = 0$ and $\widehat{\mu}(z) = e^{f(z)}$. This $f(z)$ is called the distinguished logarithm of $\widehat{\mu}(z)$ and written as $f(z) = \log\widehat{\mu}(z)$ ([19] p. 33). The word *increase* is used in the wide sense allowing flatness.

2. Natural additive processes in law and factorings

When $\{X_t\colon t \ge 0\}$ is an additive process in law on $\mathbb{R}^d$, we write $\mu_t = \mathcal{L}(X_t) \in ID$. Let $c(x)$ be a real-valued bounded measurable function satisfying

$$c(x) = \begin{cases} 1 + o(|x|) & \text{as } |x| \to 0, \\ O(|x|^{-1}) & \text{as } |x| \to \infty. \end{cases} \tag{2.1}$$

Then we get the Lévy–Khintchine representation of μ_t of the form

$$\widehat{\mu}_t(z) = \exp\left[-\frac{1}{2}\langle z, A_t z\rangle + \int_{\mathbb{R}^d} g_c(z,x)\nu_t(dx) + i\langle z, \gamma_t\rangle\right] \tag{2.2}$$

with

$$g_c(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle c(x)\,. \tag{2.3}$$

Here $A_t \in \mathbf{S}_d^+$, ν_t is a measure on $\mathbb{R}^d$ satisfying $\nu_t(\{0\}) = 0$ and $\int(1\wedge|x|^2)\nu_t(dx) < \infty$, and $\gamma_t \in \mathbb{R}^d$. They are called Gaussian covariance, Lévy measure, and location parameter, respectively. The triplet of A_t, ν_t, and γ_t is denoted by $(A_t, \nu_t, \gamma_t)_c$. Here A_t and ν_t do not depend on the choice of $c(x)$. See [19] Theorem 8.1 and Remark 8.4. Standard choice of $c(x)$ is $1_{\{|x|\le 1\}}(x)$ or $(1+|x|^2)^{-1}$. The system $\{(A_t, \nu_t, \gamma_t)\colon t \ge 0\}$ satisfies the following:

(1) $A_0 = 0$, $\nu_0 = 0$, $\gamma_0 = 0$,

(2) $A_t - A_s \in \mathbf{S}_d^+$ and $\nu_t - \nu_s \ge 0$ for $s \le t$,

(3) $A_s \to A_t$, $\nu_s(B) \to \nu_t(B)$ for all $B \in \mathcal{B}_0(\mathbb{R}^d)$, and $\gamma_s \to \gamma_t$ as $s \to t$.

Conversely, any system satisfying (1), (2), and (3) induces, uniquely in law, an additive process in law (Theorem 9.8 of [19]) and it has a modification which is an additive process (Theorem 11.5 of [19]).

DEFINITION. An additive process in law $\{X_t\}$ on $\mathbb{R}^d$ is said to be *natural* if the location parameter γ_t is locally of bounded variation in t (that is, of bounded variation on any finite subinterval of $[0,\infty)$).

This definition does not depend on the choice of $c(x)$ by the following assertion.

Proposition 2.1. *Let $c_1(x)$ and $c_2(x)$ be real-valued bounded measurable functions satisfying* (2.1). *Let $\{X_t\}$ be an additive process in law on $\mathbb{R}^d$ with triplets $(A_t,\nu_t,\gamma_t^1)_{c_1}$ and $(A_t,\nu_t,\gamma_t^2)_{c_2}$. Then γ_t^1 is locally of bounded variation if and only if γ_t^2 is locally of bounded variation.*

Proof. We have $\gamma_t^2=\gamma_t^1+f(t)$ with $f(t)=\int_{\mathbb{R}^d}x(c_2(x)-c_1(x))\nu_t(dx)$. We can check that $f(t)$ is locally of bounded variation. □

For example, any Lévy process in law $\{X_t\}$ is a natural additive process in law, since $\gamma_t=t\gamma_1$.

Proposition 2.2. *If $\{X_t\}$ is an additive process in law on $\mathbb{R}^d$, then there is an $\mathbb{R}^d$-valued continuous function $a(t)$ on $[0,\infty)$ such that $\{X_t-a(t)\}$ is a natural additive process in law.*

Proof. Use an arbitrary $c(x)$ satisfying (2.1) and choose $a(t)=\gamma_t$. □

Proposition 2.3. *Let $\{X_t\}$ be an additive process in law on $\mathbb{R}^d$. Suppose that $\int_{|x|\le 1}|x|\nu_t(dx)<\infty$ for all t and let $\gamma_t^\sharp$ be the drift of μ_t. Then $\{X_t\}$ is natural if and only if $\gamma_t^\sharp$ is locally of bounded variation.*

See [19] p. 39 for the definition of the drift.

Proof. Note that $\gamma_t^\sharp=\gamma_t-\int_{|x|\le 1}x\nu_t(dx)$ and that $\int_{|x|\le 1}x\nu_t(dx)$ is locally of bounded variation in t. □

For example, any additive process $\{X_t\}$ on $\mathbb{R}$ with increasing paths is a natural additive process.

Henceforth we use

$$c(x)=(1+|x|^2)^{-1}, \tag{2.4}$$

unless mentioned otherwise. Thus the triplet (A,ν,γ) of an infinitely divisible distribution stands for $(A,\nu,\gamma)_c$ with $c(x)$ of (2.4). The following fact is basic.

Lemma 2.4. *Let μ and μ_n, $n = 1, 2, \dots$, be in $ID(\mathbb{R}^d)$ such that $\mu_n \to \mu$ as $n \to \infty$. Let (A, ν, γ) and (A_n, ν_n, γ_n) be the triplets of μ and μ_n, respectively. Then*

$$\operatorname{tr} A_n + \int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu_n(dx) \to \operatorname{tr} A + \int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu(dx) \tag{2.5}$$

and

$$\gamma_n \to \gamma\ . \tag{2.6}$$

Proof. Noting that $c(x)$ of (2.4) is bounded and continuous, use Theorem 8.7 of [19]. □

Lemma 2.5. *Let $\{X_t\}$ be an additive process in law on $\mathbb{R}^d$ with triplet (A_t, ν_t, γ_t) of $\mu_t = \mathcal{L}(X_t)$. Then, for every $B \in \mathcal{B}^0_{[0,\infty)}$, there are a unique $A_B \in \mathbf{S}^+_d$ and a unique measure ν_B on $\mathbb{R}^d$ such that A_B and $\nu_B(C)$ for any $C \in \mathcal{B}_0(\mathbb{R}^d)$ are countably additive in $B \in \mathcal{B}^0_{[0,\infty)}$ and that $A_{[0,t]} = A_t$ and $\nu_{[0,t]} = \nu_t$. The components $A_{jk}(B)$, $j, k = 1, \dots, d$, of A_B are absolutely continuous with respect to the measure $\operatorname{tr} A_B$ on $[0, t_0]$ for each t_0. If, moreover, $\{X_t\}$ is natural, then there is a unique γ_B such that γ_B is countably additive in $B \in \mathcal{B}^0_{[0,\infty)}$ and $\gamma_{[0,t]} = \gamma_t$.*

Proof. Since $\operatorname{tr} A_t$ is increasing and continuous in t, it induces an atomless measure σ_1 on $[0, \infty)$. Let $A_{jk}(t)$ and $\gamma_j(t)$ be the components of A_t and γ_t. Since $|A_{jk}(t) - A_{jk}(s)| \le \operatorname{tr} A_t - \operatorname{tr} A_s$ for $s \le t$, $A_{jk}(t)$ is locally of bounded variation and absolutely continuous with respect to σ_1. Thus $A_{jk}(t)$ induces a signed measure $A_{jk}(B)$. We have $\sum_{j,k} A_{jk}(B) z_j z_k \ge 0$, since it is true when B is an interval. Thus $A_B = (A_{jk}(B)) \in \mathbf{S}^+_d$. If $\{X_t\}$ is natural, then the assertion on γ_B is proved similarly. Concerning ν_B, there is a unique measure $\widetilde{\nu}$ on $[0, \infty) \times \mathbb{R}^d$ such that

$$\widetilde{\nu}([0, t] \times C) = \nu_t(C) \quad \text{for } C \in \mathcal{B}(\mathbb{R}^d) \text{ and } t \ge 0, \tag{2.7}$$

as in [19] Remark 9.9. Then it suffices to let $\nu_B(C) = \widetilde{\nu}(B \times C)$. □

DEFINITION. Let $\{X_t\}$ be an additive process in law on $\mathbb{R}^d$. A pair $(\{\rho_s : s \ge 0\}, \sigma)$ is called a *factoring* of $\{X_t\}$ if the following conditions are satisfied:
(1) σ is a locally finite measure on $[0, \infty)$, that is, a measure on $[0, \infty)$ such that $\sigma([0, t]) < \infty$ for all $t \in [0, \infty)$,
(2) σ is continuous (that is, atomless),
(3) $\rho_s \in ID(\mathbb{R}^d)$ for all $s \in [0, \infty)$,
(4) $\log \widehat{\rho}_s(z)$ is measurable in s for each $z \in \mathbb{R}^d$,
(5) $\int_0^t |\log \widehat{\rho}_s(z)| \sigma(ds) < \infty$ for all $t \in [0, \infty)$ and $z \in \mathbb{R}^d$,

(6) we have

$$\widehat{\mu}_t(z) = \exp \int_0^t \log \widehat{\rho}_s(z)\sigma(ds) \quad \text{for all } t \in [0,\infty) \text{ and } z \in \mathbb{R}^d. \tag{2.8}$$

For example, any Lévy process in law $\{X_t\}$ on $\mathbb{R}^d$ has a factoring given by $\rho_s = \mathcal{L}(X_1)$ for all s and by σ = Lebesgue. The following theorem is a main result of this section.

Theorem 2.6. *Let $\{X_t : t \geq 0\}$ be an additive process in law on $\mathbb{R}^d$. Then, $\{X_t\}$ is natural if and only if $\{X_t\}$ has a factoring.*

Denote by $(A_s^\rho, \nu_s^\rho, \gamma_s^\rho)$ the triplet of ρ_s.

Lemma 2.7. *If $(\{\rho_s : s \geq 0\}, \sigma)$ is a factoring of an additive process in law $\{X_t\}$ on $\mathbb{R}^d$, then*
(7) *A_s^ρ, γ_s^ρ, and $\nu_s^\rho(C)$ for any $C \in \mathcal{B}_0(\mathbb{R}^d)$ are measurable in s,*
(8) *we have*

$$\int_0^t \left(\mathrm{tr}(A_s^\rho) + \int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu_s^\rho(dx) + |\gamma_s^\rho| \right) \sigma(ds) < \infty \quad \text{for all } t \in [0,\infty), \tag{2.9}$$

(9) *we have*

$$A_t = \int_0^t A_s^\rho \sigma(ds), \quad \nu_t(B) = \int_0^t \nu_s^\rho(B)\sigma(ds), \quad \gamma_t = \int_0^t \gamma_s^\rho \sigma(ds) \tag{2.10}$$

for $B \in \mathcal{B}_0(\mathbb{R}^d)$,
(10) *$\int_0^t \log \widehat{\rho}_s(z)\,\sigma(ds) = \log \widehat{\mu}_t(z)$, the distinguished logarithm of $\widehat{\mu}_t(z)$.*

Proof. Since, for each s, we can express A_s^ρ, γ_s^ρ, and $\nu_s^\rho(C)$ by using $\log \widehat{\rho}_s(z)$ as in Section 8 of [19], assertion (7) is proved. To see (8), we use

$$|\widehat{\mu}_s(z)| = \exp \int_0^t \mathrm{Re}(\log \widehat{\rho}_s(z))\sigma(ds).$$

Assertions (9) and (10) follow from (8). Details are omitted. □

The "if" part of Theorem 2.6 is proved by this lemma. Indeed, if $\{X_t\}$ has a factoring $(\{\rho_s\}, \sigma)$, then γ_t is locally of bounded variation by the expression in (2.10) and hence $\{X_t\}$ is natural. The "only if" part of the theorem will be proved in the form of Proposition 2.8 after we introduce the notions of canonical measures and canonical factorings.

DEFINITION. Let $\{X_t\}$ be a natural additive process in law on $\mathbb{R}^d$ and let $|\gamma|_t$ be the variation function of γ_t. Use the notation in Lemma 2.5. Denote by $|\gamma|_B$ the measure such that $|\gamma|_{[0,t]} = |\gamma|_t$. Then a measure σ on $[0,\infty)$ defined by

$$\sigma(B) = \operatorname{tr} A_B + \int_{\mathbb{R}^d} (1 \wedge |x|^2)\, \nu_B(dx) + |\gamma|_B \tag{2.11}$$

is called the *canonical measure* of $\{X_t\}$. A pair $(\{\rho_s\},\sigma)$ is called a *canonical factoring* of $\{X_t\}$ if it is a factoring of $\{X_t\}$ and if σ is the canonical measure of $\{X_t\}$.

Proposition 2.8. *Let $\{X_t\}$ be a natural additive process in law on $\mathbb{R}^d$. Then there exists a canonical factoring of $\{X_t\}$. It is unique in the sense that, if $(\{\rho_s^1\},\sigma)$ and $(\{\rho_s^2\},\sigma)$ are canonical factorings of $\{X_t\}$, then $\rho_s^1 = \rho_s^2$ for σ-a. e. s. If $(\{\rho_s\},\sigma)$ is a canonical factoring of $\{X_t\}$, then*

$$\operatorname*{esssup}_{s\in[0,\infty)} \sup_{|z|\le a} |\log \widehat{\rho}_s(z)| < \infty \tag{2.12}$$

for any $a \in (0,\infty)$ and

$$\operatorname*{esssup}_{s\in[0,\infty)} \left(\operatorname{tr}(A_s^\rho) + \int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu_s^\rho(dx) + |\gamma_s^\rho| \right) < \infty\,, \tag{2.13}$$

where the essential supremums are with respect to σ.

Proof. Let σ_1, σ_2, and σ_3 be the measures defined by $\sigma_1(B) = \operatorname{tr} A_B$, $\sigma_2(B) = \int_{\mathbb{R}^d}(1\wedge|x|^2)\nu_B(dx)$, and $\sigma_3(B) = |\gamma|_B$. Let $h_l(t)$ be the Radon–Nikodým derivative of σ_l with respect to σ for $l = 1, 2, 3$. Let $A^\sharp_{jk}(t)$ and $\gamma^\sharp_j(t)$ be the Radon–Nikodým derivatives of $A_{jk}(B)$ and $\gamma_j(B)$ with respect to σ_1 and σ_3, respectively. For the measure $\widetilde{\nu}$ in the proof of Lemma 2.5, there are a measure $\sigma^\sharp$ on $[0,\infty)$ and measures $\nu_s^\sharp$ on $\mathbb{R}^d$ such that $\sigma^\sharp$ is continuous and locally finite, $\nu_s^\sharp(C)$ is measurable in $s \ge 0$ for each $C \in \mathcal{B}_0(\mathbb{R}^d)$, $\int_{\mathbb{R}^d}(1\wedge|x|^2)\nu_s^\sharp(dx) = 1$, $\nu_s^\sharp(\{0\}) = 0$, and

$$\widetilde{\nu}(B\times C) = \int_B \sigma^\sharp(ds)\int_C \nu_s^\sharp(dx) \quad \text{for } B \in \mathcal{B}_{[0,\infty)},\ C \in \mathcal{B}(\mathbb{R}^d)\,.$$

The argument is similar to the construction of conditional distributions. Letting $\nu_s^\rho(dx) = \nu_s^\sharp(dx)h_2(s)$, $A_s^\rho = (A^\rho_{jk}(s))$ with $A^\rho_{jk}(s) = A^\sharp_{jk}(s)h_1(s)$, and $\gamma_s^\rho = (\gamma_j^\rho(s))$ with $\gamma_j^\rho(s) = \gamma_j^\sharp(s)h_3(s)$, we can prove that $(\{\rho_s\},\sigma)$ is a factoring of $\{X_t\}$ and that (2.10) is satisfied. Properties (2.12) and (2.13) are proved easily. The uniqueness of a canonical factoring is proved by (9) and (10) in Lemma 2.7. □

Proposition 2.9. *Let $\{X_t\}$ be an additive process in law on $\mathbb{R}^d$. Then it is natural if and only if $\widehat{\mu}_t(z)$ is locally of bounded variation in t for each $z \in \mathbb{R}^d$.*

Proof. In order to see the "only if" part, notice that $\{X_t\}$ has a factoring by Theorem 2.6 and use (2.8). Let us prove the "if" part. Define $Y_t = X_t - \gamma_t$. Then $\{Y_t\}$ is a natural additive process in law and hence $Ee^{i\langle z, Y_t\rangle}$ is locally of bounded variation in t. We have $Ee^{i\langle z, Y_t\rangle} = e^{-i\langle z,\gamma_t\rangle}\widehat{\mu}_t(z)$. Since $\widehat{\mu}_t(z)$ is continuous, non-vanishing, and locally of bounded variation in t for each z, it follows that $e^{-i\langle z,\gamma_t\rangle}$ is locally of bounded variation in t for each z. Hence $\langle z, \gamma_t\rangle$ is locally of bounded variation in t for each z. Hence, so is γ_t. □

Proposition 2.10. *Let $\{X_t\}$ be an additive process on $\mathbb{R}^d$. Then $\{X_t\}$ is natural if and only if $\{X_t\}$ is a semimartingale.*

Proof. This is a consequence of Proposition 2.9 combined with Jacod and Shiryaev [5], Chapter II, Theorem 4.14. □

We add some facts on factorings.

Proposition 2.11. *If $\{\rho_s : s \geq 0\}$ and σ satisfy conditions* (1), (2), (3), *of the definition of a factoring and* (7), (8) *of* Lemma 2.7, *then $(\{\rho_s\}, \sigma)$ is a factoring of some additive process in law $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$.*

Proof. Define A_t, ν_t, and γ_t by (2.10). Then (A_t, ν_t, γ_t) is the triplet of some $\mu_t \in ID$ and satisfies conditions (1), (2), and (3) in the first paragraph of this section. Thus there is an additive process in law $\{X_t\}$ such that $\mathcal{L}(X_t) = \mu_t$. Conditions (4) and (5) of the definition of a factoring follow from (7) and (8) and we can see that $(\{\rho_s\}, \sigma)$ is a factoring of $\{X_t\}$. □

Proposition 2.12 (Time change). *Let $\{X_t : t \geq 0\}$ be a natural additive process in law on $\mathbb{R}^d$. Given an increasing continuous function $\tau(t)$ from $[0,\infty)$ into $[0,\infty)$ with $\tau(0) = 0$, define $Y_t = X_{\tau(t)}$. Then $\{Y_t : t \geq 0\}$ is a natural additive process in law on $\mathbb{R}^d$. If $(\{\rho_s\}, \sigma)$ is a factoring of $\{X_t\}$, then $(\{\widetilde{\rho}_s\}, \widetilde{\sigma})$ defined by*

$$\widetilde{\rho}_s = \rho_{\tau(s)} \quad \text{and} \quad \widetilde{\sigma}([0,s]) = \sigma([0,\tau(s)]). \tag{2.14}$$

gives a factoring of $\{Y_t\}$. If $(\{\rho_s\}, \sigma)$ is canonical, then $(\{\widetilde{\rho}_s\}, \widetilde{\sigma})$ is canonical.

Proof is elementary and omitted.

Let us study conditions for naturalness in some classes of additive processes. In analogy to definitions in [18] and [19] we give the following definition.

DEFINITION. Let $Q \in \mathbf{M}_d^+$. An $\mathbb{R}^d$-valued stochastic process $\{X_t : t \geq 0\}$ is called *Q-selfsimilar* if, for every $a > 1$ (hence for every $a > 0$),

$$\{X_{at} : t \geq 0\} \stackrel{\mathrm{d}}{=} \{a^Q X_t : t \geq 0\}. \tag{2.15}$$

It is called *Q-semi-selfsimilar* if (2.15) holds for some $a > 1$. In this case a is called an *epoch*.

A basic property we need of a general $Q \in \mathbf{M}_d^+$ is the following: there are positive constants $c_1, \ldots, c_4$ such that

$$c_4 e^{-c_2 s}|x| \leq |e^{-sQ}x| \leq c_3 e^{-c_1 s}|x| \quad \text{for } s \geq 0 \text{ and } x \in \mathbb{R}^d \tag{2.16}$$

(see [22] p. 139 or [17]). We have $c_1 \leq c_2$ and $c_4 \leq 1 \leq c_3$ automatically. It follows that

$$c_3^{-1} e^{c_1 s}|x| \leq |e^{sQ}x| \leq c_4^{-1} e^{c_2 s}|x| \quad \text{for } s \geq 0 \text{ and } x \in \mathbb{R}^d. \tag{2.17}$$

Theorem 2.13. *Let $Q \in \mathbf{M}_d^+$. Let $\{X_t\}$ be a Q-semi-selfsimilar additive process in law on $\mathbb{R}^d$ with epoch a. Then $\{X_t\}$ is natural if and only if γ_t is of bounded variation on $[1, a]$.*

Proof. The "only if" part is evident. Let us prove the "if" part. Assume that γ_t is of bounded variation on $[1, a]$. It follows from (2.15) that

$$\gamma_{at} = a^Q \gamma_t + \int_{\mathbb{R}^d} a^Q x \, r_a(x) \nu_t(dx) \tag{2.18}$$

with

$$r_a(x) = \frac{1}{1 + |a^Q x|^2} - \frac{1}{1 + |x|^2} = \frac{|x|^2 - |a^Q x|^2}{(1 + |a^Q x|^2)(1 + |x|^2)} .$$

Denote by $|\gamma|_{[a_1, a_2]}$ the variation of γ_t on $[a_1, a_2]$. Then

$$|\gamma|_{[a^{n+1}, a^{n+2}]} \leq \|a^Q\| \, |\gamma|_{[a^n, a^{n+1}]} + \int |a^Q x| \, |r_a(x)| (\nu_{a^{n+1}} - \nu_{a^n})(dx) .$$

Finiteness of the last integral follows from (2.17). Hence γ_t is locally of bounded variation on $[1, \infty)$. As $n \to \infty$, $\|a^{-nQ}\|^{1/n}$ tends to $\max_{1 \leq j \leq d} |a^{-q_j}|$, where $q_1, \ldots, q_d$ are the eigenvalues of Q ([8], p. 153). Since $Q \in \mathbf{M}_d^+$, this limit is less than 1. Thus we can choose an integer $m \geq 1$ such that $\|a^{-mQ}\| < 1$. Let $b = a^m$ and use b as an epoch of $\{X_t\}$. We get (2.18) with a replaced by b. It follows that

$$|\gamma|_{[b^{-n-1}, b^{-n}]} \leq \|b^{-Q}\| \, |\gamma|_{[b^{-n}, b^{-n+1}]} + \int |x| \, |r_b(x)| (\nu_{b^{-n}} - \nu_{b^{-n-1}})(dx).$$

Hence we obtain

$$(1-\|b^{-Q}\|)|\gamma|_{[0,1]} \le \|b^{-Q}\|\,|\gamma|_{[1,b]} + \int |x|\,|r_b(x)|\nu_1(dx).$$

Now we see that $|\gamma|_{[0,1]} < \infty$. □

Theorem 2.14. *Let $Q \in \mathbf{M}_d^+$. Let $\{X_t\}$ be a Q-selfsimilar additive process in law on $\mathbb{R}^d$. Then $\{X_t\}$ is natural.*

Proof. It is enough to show that γ_t is of bounded variation on $[1,2]$, since we can use the preceding theorem with $a=2$. Since we have $\gamma_t = t^Q\gamma_1 + \int t^Q x\, r_t(x)\nu_1(dx)$, we can prove that γ_t has continuous derivative in $t > 0$. It follows that γ_t is of bounded variation on $[1,2]$. □

REMARK. When $Q = cI$ with $c \in (0,\infty)$, the Q-semi-selfsimilarity is the c-semi-selfsimilarity studied in Maejima and Sato [12]. Theorems 7 and 10 of [12] show that, given a semi-selfdecomposable (see [19] Definition 15.1) distribution μ on $\mathbb{R}^d$ with span a^c, there is a wide variety of choice of $\mathcal{L}(X_t)$ for $1 < t < a$ in constructing a c-semi-selfsimilar additive process in law $\{X_t : t \ge 0\}$ with epoch a such that $\mathcal{L}(X_1) = \mu$. Thus we can find a non-natural c-semi-selfsimilar additive process in law $\{X_t : t \ge 0\}$ with epoch a satisfying $\mathcal{L}(X_1) = \mu$.

3. Independently scattered random measures

Following Urbanik and Woyczynski [23] and Rajput and Rosinski [15] and extending the notion from real-valued to $\mathbb{R}^d$-valued, we give the following definition.

DEFINITION. A family of $\mathbb{R}^d$-valued random variables $\{M(B) : B \in \mathcal{B}^0_{[0,\infty)}\}$ is called an $\mathbb{R}^d$-valued *independently scattered random measure* (i.s.r.m.) if the following conditions are satisfied:
(1) (countably additive) for any sequence $B_1, B_2, \ldots$ of disjoint sets in $\mathcal{B}^0_{[0,\infty)}$ with $\bigcup_{n=1}^\infty B_n \in \mathcal{B}^0_{[0,\infty)}$, $\sum_{n=1}^\infty M(B_n)$ converges a.s. and equals $M(\bigcup_{n=1}^\infty B_n)$ a.s.,
(2) (independent increments) for any finite sequence $B_1, \ldots, B_n$ of disjoint sets in $\mathcal{B}^0_{[0,\infty)}$, $M(B_1), \ldots, M(B_n)$ are independent,
(3) (atomless) $M(\{a\}) = 0$ a.s. for every one-point set $\{a\}$.

Note that, if $B_1, B_2, \ldots$ is an increasing sequence in $\mathcal{B}^0_{[0,\infty)}$ with $B = \bigcup_{n=1}^\infty B_n \in \mathcal{B}^0_{[0,\infty)}$, then $M(B_n) \to M(B)$ a.s. This follows from property (1). Note also that property (1) implies that $M(\emptyset) = 0$ a.s., where 0 is the origin of $\mathbb{R}^d$.

Lemma 3.1. *If $\{M(B)\}$ is an $\mathbb{R}^d$-valued i.s.r.m., then $\mathcal{L}(M(B)) \in ID$ for any $B \in \mathcal{B}^0_{[0,\infty)}$.*

Proof. Let $B \in \mathcal{B}^0_{[0,\infty)}$. Define $Y_t = M(B \cap [0,t])$. Then it follows from the defining properties of i. s. r. m. that $\{Y_t : t \geq 0\}$ is an additive process in law. Hence $\mathcal{L}(Y_t) \in ID$. Thus $\mathcal{L}(M(B)) \in ID$, since $M(B) = Y_t$ for large t. □

Here is our main result in this section.

Theorem 3.2. (i) *Let $\{M(B) : B \in \mathcal{B}^0_{[0,\infty)}\}$ be an $\mathbb{R}^d$-valued i. s. r. m. Define $\{X_t : t \geq 0\}$ by*

$$X_t = M([0,t]) \tag{3.1}$$

for $t \in [0,\infty)$. Then $\{X_t\}$ is a natural additive process in law.
(ii) *Let $\{X_t : t \geq 0\}$ be a natural additive process in law on $\mathbb{R}^d$. Then there is an $\mathbb{R}^d$-valued i. s. r. m. $\{M(B) : B \in \mathcal{B}^0_{[0,\infty)}\}$ such that* (3.1) *holds. This is unique in the sense that, if $\{M_1(B)\}$ and $\{M_2(B)\}$ both satisfy this condition, then $M_1(B) = M_2(B)$ a. s. for every $B \in \mathcal{B}^0_{[0,\infty)}$. Denote $\mu_B = \mathcal{L}(M(B))$. Then A_B, ν_B, and γ_B in the triplet of μ_B coincides with those of* Lemma 2.5. *For any factoring $(\{\rho_s\}, \sigma)$ of $\{X_t\}$ and any $B \in \mathcal{B}^0_{[0,\infty)}$,*

$$\log \widehat{\mu}_B(z) = \int_B \log \widehat{\rho}_s(z) \sigma(ds) \,. \tag{3.2}$$

Proof. (i) Using Lemma 3.1, denote the location parameter of $\mathcal{L}(M(B))$ by γ_B. Then γ_B is countably additive in $B \in \mathcal{B}^0_{[0,\infty)}$, which follows from countable additivity of $M(B)$ and (2.6) of Lemma 2.4. Hence $\gamma_t = \gamma_{[0,t]}$ is a function locally of bounded variation by Section 29 of [4].
(ii) Let $\{X_t\}$ be a natural additive process in law on $\mathbb{R}^d$ with a factoring $(\{\rho_s\}, \sigma)$. We will define $M(B)$ for $B \in \mathcal{B}^0_{[0,\infty)}$ in several steps.

STEP 1. If J is an empty set or a one-point set, then we define $M(J) = 0$. If J is a finite interval in $[0,\infty)$ with left end s and right end t, that is, $J = (s,t)$, $[s,t]$, $(s,t]$, or $[s,t)$, then we define $M(J) = X_t - X_s$. If $B = \bigcup_{j=1}^n J_j$ with disjoint finite intervals $J_1, \ldots, J_n$ in $[0,\infty)$, then we define $M(B) = \sum_{j=1}^n M(J_j)$. This definition does not depend on the expression of B. We see from (2.8) that (3.2) is true for this B. Finite additivity and independent increment property within the class of sets of this type are obvious.

STEP 2. Let G be a bounded open set in $[0,\infty)$. Then G is expressed uniquely (up to the order) as $G = \bigcup_j J_j$, where $J_1, J_2, \ldots$ (finite or infinite sequence) are disjoint open intervals (possibly of the form $[0,t)$). If it is a finite sequence, $M(G)$ is defined in Step 1. So we assume that it is an infinite sequence. Let $J_j = (s_j, t_j)$ and let $Y_n = \sum_{j=1}^n (X_{t_j} - X_{s_j})$. Then, for $m < n$,

$$E e^{i\langle z, Y_n - Y_m\rangle} = \exp \int_{\bigcup_{j=m+1}^n J_j} \log \widehat{\rho}_s(z) \sigma(ds) \to 1 \quad \text{as } m,n \to \infty.$$

In general, a sequence $\{Y_n\}$ of $\mathbb{R}^d$-valued random variables converges in probability if $Ee^{i\langle z, Y_n - Y_m\rangle} \to 1$ as $m, n \to \infty$ with $m < n$, because, for any $\varepsilon > 0$,

$$\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\left(1 - \operatorname{Re} Ee^{iz_j((Y_n)_j - (Y_m)_j)}\right) dz_j \geq P\left[|(Y_n)_j - (Y_m)_j| \geq \frac{\varepsilon}{2}\right] ,$$

where the subscript j denotes the jth component ([19] p. 430). This fact is often useful. Since the summands are independent, convergence in probability implies convergence a. s. Thus we define $M(G) = \sum_{j=1}^{\infty} M(J_j)$ a. s. We can prove that this definition does not depend on the order of summation.

STEP 3. Let K be a compact set and $K \subset [0, \infty)$. Choose t_0 such that $K \subset [0, t_0)$ and let $G = [0, t_0) \setminus K$. Then G is open in $[0, \infty)$. Define $M(K)$ by $M(K) = M([0, t_0)) - M(G)$. If K is an interval, then this definition is consistent with that of Step 1. Let $a = \sup_{x\in K} x$. Express $G = \bigcup_j J_j$, where $J_1, J_2, \ldots$ is a sequence of disjoint open intervals and $J_1 = (a, t_0)$. Let $G_n = \bigcup_{j=1}^{n} J_j$. Then $M(K) = \lim_{n\to\infty} M([0, t_0) \setminus G_n)$ a. s. We have an expression $[0, t_0) \setminus G_n = \bigcup_{l=1}^{k_n} [s_{n,l}, t_{n,l}]$, where $[s_{n,1}, t_{n,1}], \ldots, [s_{n,k_n}, t_{n,k_n}]$ are disjoint closed intervals (possibly one-point sets). It follows that

$$M(K) = \lim_{n\to\infty} \sum_{l=1}^{k_n} (X_{t_{n,l}} - X_{s_{n,l}}) \quad \text{a. s.} \tag{3.3}$$

Obviously (3.2) is true for $B = K$. Using the expression (3.3), we can prove that, if K_1 and K_2 are compact and $K_1 \supset K_2$, then

$$Ee^{i\langle z, M(K_1) - M(K_2)\rangle} = \exp \int_{K_1 \setminus K_2} \log \widehat{\rho}_s(z)\sigma(ds) . \tag{3.4}$$

We can also show that, if $K_1, \ldots, K_m$ are disjoint compact sets, then $M(K_1), \ldots, M(K_m)$ are independent and $M\left(\bigcup_{p=1}^{m} K_p\right) = \sum_{p=1}^{m} M(K_p)$ a. s.

STEP 4. Let $B \in \mathcal{B}^0_{[0,\infty)}$. By the regularity of the measure σ (see Section 52 of [4]), we can find an increasing sequence of compact sets $K_1, K_2, \ldots$ such that $K_p \subset B$ and $\lim_{p\to\infty} \sigma(K_p) = \sigma(B)$. As $p, q \to \infty$ with $q < p$,

$$Ee^{i\langle z, M(K_p) - M(K_q)\rangle} = \exp \int_{K_p \setminus K_q} \log \widehat{\rho}_s(z)\sigma(ds) \to 1$$

by (3.4). Hence $M(K_p)$ is convergent in probability as $p \to \infty$. We define $M(B) = \text{p-lim}_{p\to\infty} M(K_p)$. We can show that this definition does not depend on the choice of the sequence K_p.

STEP 5. It follows from the definition in Step 4 that M has properties (1) and (2) of the definition of i. s. r. m. and also (3.2).

STEP 6. Proof of uniqueness of M. Let M_1 and M_2 be $\mathbb{R}^d$-valued i. s. r. m. satisfying (3.1). Fix $t_0 > 0$. The class $\mathcal{S}$ of all $B \in \mathcal{B}_{[0,t_0]}$ satisfying $M_1(B) = M_2(B)$ is a

λ-system and contains $\emptyset$ and all intervals of the form $(s,t]$ in $[0,t_0]$. Hence $\mathcal{S}=\mathcal{B}_{[0,t_0]}$ by Dynkin's π-λ theorem in [2] p. 37. Hence $M_1(B)=M_2(B)$ a.s. for every B. Let (A_B,ν_B,γ_B) be its triplet. Then A_B, γ_B, and $\nu_B(C)$ for each $C\in\mathcal{B}_0(\mathbb{R}^d)$ are countably additive in $B\in\mathcal{B}^0_{[0,\infty)}$, which follows from countable additivity of $M(B)$, as in the proof of Theorem 9.8 of [19]. Hence we see that they coincide with those of Lemma 2.5. □

REMARK. Like in the proof of Proposition 2.1 (b) of [15], one can construct an $\mathbb{R}^d$-valued i.s.r.m. on the product measurable space $(\mathbb{R}^d)^{\mathcal{B}^0_{[0,\infty)}}$, using Kolmogorov's extension theorem. Namely, let $\omega=(\omega_B)_{B\in\mathcal{B}^0_{[0,\infty)}}$ be a general element of this space. Start with $\mu_B\in ID(\mathbb{R}^d)$ with triplet (A_B,ν_B,γ_B) in Lemma 2.5 and define $P((\omega_{B_1},\dots,\omega_{B_n})\in D)$, $D\in\mathcal{B}((\mathbb{R}^d)^n)$, as the product of $\mu_{B_1},\dots,\mu_{B_n}$ if $B_1,\dots,B_n$ are disjoint. If $B_1,\dots,B_n$ are not disjoint, then express $B_1,\dots,B_n$ as unions of some of disjoint sets $C_1\dots,C_m$ and define $P((\omega_{B_1},\dots,\omega_{B_n})\in D)$ in the form derived from the product of $\mu_{C_1},\dots,\mu_{C_m}$ and check the consistency. However, we cannot in this way construct $\{M(B)\}$ in the same probability space that the given $\{X_t\}$ is defined. (Added to the final version: Pedersen [13] observes that the construction of an i.s.r.m. from a given natural additive process can be done on the basis of the Lévy–Itô decomposition of the additive process.)

The case where the process $\{X_t\}$ in Theorem 3.2 is a Lévy process in law is important. An $\mathbb{R}^d$-valued i.s.r.m. $\{M(B)\}$ is called *homogeneous* if $M(B)\overset{\mathrm{d}}{=}M(B+a)$ a.s. for any $B\in\mathcal{B}^0_{[0,\infty)}$ and $a\ge 0$.

Proposition 3.3. *Let $\{M(B)\}$ be an $\mathbb{R}^d$-valued i.s.r.m. Then the following statements are equivalent*:
(1) *$\{M(B)\}$ is homogeneous,*
(2) *$M((s,t])\overset{\mathrm{d}}{=}M((s+a,t+a])$ a.s. if $0\le s<t<\infty$ and $a\ge 0$,*
(3) *the process $\{X_t\colon t\ge 0\}$ defined by $X_t=M([0,t])$ is a Lévy process in law.*

Proof is easy and omitted.

Let $\{M(B)\}$ be an $\mathbb{R}^d$-valued i.s.r.m. Then the canonical measure of the natural additive process in law $\{X_t\}$ defined by (3.1) is called the canonical measure of $\{M(B)\}$.

Proposition 3.4. *Let $\{M(B)\}$ be an $\mathbb{R}^d$-valued i.s.r.m. and σ its canonical measure. Then, $B\in\mathcal{B}^0_{[0,\infty)}$ satisfies $\sigma(B)=0$ if and only if*

(3.5) $$M(C)=0 \text{ a.s. for all Borel sets } C \text{ satisfying } C\subset B.$$

Proof. If $\sigma(B)=0$, then, for any Borel set $C\subset B$, $\sigma(C)=0$ and hence $M(C)=0$ a.s. by formula (3.2). Conversely, if (3.5) holds, then $A_C=0$, $\nu_C=0$, and $\gamma_C=0$ for all Borel sets $C\subset B$ and we have $\sigma(B)=0$ by formula (2.11). □

Proposition 3.5. *Let $\{M(B)\}$ be an $\mathbb{R}^d$-valued i.s.r.m. If $(\{\rho_s^0\},\sigma^0)$ is a factoring of the natural additive process in law $\{X_t\}$ defined by* (3.1), *then the canonical measure σ is absolutely continuous with respect to σ^0.*

Proof. By Theorem 3.2, (3.2) is true both for $(\{\rho_s\},\sigma)$ and for $(\{\rho_s^0\},\sigma^0)$. Let $B\in\mathcal{B}^0_{[0,\infty)}$. If $\sigma^0(B)=0$, then $\sigma^0(C)=0$ for all $C\subset B$ and thus $M(C)=0$ a.s. for all $C\subset B$, which implies $\sigma(B)=0$ by Proposition 3.4. □

The following useful result is by Urbanik and Woyczynski [23] when $d=1$.

Proposition 3.6. *For $n=1,2,\dots$, let $\{M_n(B)\}$ be $\mathbb{R}^d$-valued i.s.r.m. Suppose that for each $B\in\mathcal{B}^0_{[0,\infty)}$ there is an $\mathbb{R}^d$-valued random variable $M(B)$ such that*

$$\operatorname*{p-lim}_{n\to\infty} M_n(B)=M(B)\,. \tag{3.6}$$

Then, $\{M(B)\}$ is an i.s.r.m.

Proof. It is clear that $M(B)$ is finitely additive and satisfies (2) and (3) of the definition of i.s.r.m. Since $\mathcal{L}(M_n(B))\in ID$ and $\mathcal{L}(M_n(B))\to\mathcal{L}(M(B))$, we have $\mathcal{L}(M(B))\in ID$ for each B. Let $(A_B^n,\nu_B^n,\gamma_B^n)$ and (A_B,ν_B,γ_B) be the triplets of $\mathcal{L}(M_n(B))$ and $\mathcal{L}(M(B))$, respectively. Define $\tau_B^n=\operatorname{tr}(A_B^n)+\int_{\mathbb{R}^d}(1\wedge|x|^2)\nu_B^n(dx)$ and $\tau_B=\operatorname{tr}(A_B)+\int_{\mathbb{R}^d}(1\wedge|x|^2)\nu_B(dx)$. Then, for each B, $\tau_B^n\to\tau_B$ and $\gamma_B^n\to\gamma_B$ by Lemma 2.4. Hence by the Nikodým Theorem (see Dunford and Schwartz [3] p. 160) τ_B and γ_B are countably additive in B. We claim that, if $B_1,B_2,\dots$ is a decreasing sequence of bounded Borel sets with $\bigcap_{k=1}^\infty B_k=\emptyset$, then $\operatorname{p-lim}_{k\to\infty}M(B_k)=0$. Indeed, we have $\bigcup_{j=k}^\infty C_j=B_k$ for $C_j=B_j\setminus B_{j+1}$ and hence $\sum_{j=k}^\infty\gamma_{C_j}=\gamma_{B_k}$. This shows that $\gamma_{B_k}\to 0$. Similarly, $\tau_{B_k}\to 0$. Since $|g_c(z,x)|\le\text{const}\,(1\wedge|x|^2)$ for any fixed z, it follows that $Ee^{i\langle z,M(B_k)\rangle}\to 1$ as $k\to\infty$. That is, p-lim $M(B_k)=0$. It follows that M is countably additive. □

4. Stochastic integrals based on natural additive processes in law

In this section let $\{X_t\colon t\ge 0\}$ be a natural additive process in law on $\mathbb{R}^d$ and let $(\{\rho_s\},\sigma)$ be its canonical factoring. By Theorem 3.2 $\{X_t\}$ induces a unique $\mathbb{R}^d$-valued independently scattered random measure $\{M(B)\colon B\in\mathcal{B}^0_{[0,\infty)}\}$. We will define and study stochastic integrals of $\mathbf{M}_{l\times d}$-valued nonrandom functions based on this random measure. As is remarked in Proposition 2.10, the process $\{X_t\}$ is a semimartingale. Thus stochastic integrals based on $\{X_t\}$ are defined for some class of random integrands in L^2-theory through random localization; see Kunita and Watanabe [9] and Jacod and Shiryaev [5]. But we will define stochastic integrals of nonrandom integrands directly through convergence in probability and give a representation of the characteristic functions of the distributions of the integrals, as was done in [23] and [15]. In the case of Lévy processes, this was already done by Lukacs [10]; see also [16]. We prove some properties of the integrals including a Fubini type theorem.

An $\mathbf{M}_{l\times d}$-valued function $F(s)$ on $[0,\infty)$ is called a *simple function* if

$$F(s) = \sum_{j=1}^{n} 1_{B_j}(s) R_j \tag{4.1}$$

for some n, where $B_1,\ldots,B_n$ are disjoint Borel sets in $[0,\infty)$ and $R_0,\ldots,R_n \in \mathbf{M}_{l\times d}$. It is called a *step function* if, in addition, $B_1,\ldots,B_n$ are intervals or one-point sets. The following definition of integrals and Proposition 4.1 follow [23] of the case $d=1$.

DEFINITION. Let $F(s)$ be an $\mathbf{M}_{l\times d}$-valued simple function on $[0,\infty)$ in (4.1) and let $B \in \mathcal{B}^0_{[0,\infty)}$. Define

$$\int_B F(s)\,dX_s = \int_B F(s)M(ds) = \sum_{j=1}^{n} R_j\, M(B\cap B_j)\,. \tag{4.2}$$

We use $\int_B F(s)\,dX_s$ and $\int_B F(s)M(ds)$ in the same meaning.

The definition (4.2) does not depend (in the a. s. sense) on the choice of a representation (4.1) of $F(s)$.

DEFINITION. An $\mathbf{M}_{l\times d}$-valued function $F(s)$ on $[0,\infty)$ is said to be *M-integrable* or *$\{X_t\}$-integrable* if it is measurable and if there is a sequence of simple functions $F_n(s)$, $n=1,\,2,\ldots$, such that (1) $F_n(s)\to F(s)$ σ-a. e. and (2) for every $B\in\mathcal{B}^0_{[0,\infty)}$, the sequence $\int_B F_n(s)\,dX_s$ is convergent in probability as $n\to\infty$.

Proposition 4.1. *If $F(s)$ is M-integrable and if $F^1_n(s)$ and $F^2_n(s)$ are sequences satisfying* (1) *and* (2) *above, then*

$$\operatorname*{p-lim}_{n\to\infty}\int_B F^1_n(s)\,dX_s = \operatorname*{p-lim}_{n\to\infty}\int_B F^2_n(s)\,dX_s \quad \textit{a. s. for each } B\in\mathcal{B}^0_{[0,\infty)}\,. \tag{4.3}$$

DEFINITION. For any M-integrable $\mathbf{M}_{l\times d}$-valued function $F(s)$ on $[0,\infty)$, define

$$\int_B F(s)\,dX_s = \int_B F(s)M(ds) = \operatorname*{p-lim}_{n\to\infty}\int_B F_n(s)\,dX_s\,, \tag{4.4}$$

using the sequence $F_n(s)$ in the definition of M-integrability and using Proposition 4.1.

Proposition 4.2. *Let $F_1(s)$ and $F_2(s)$ be M-integrable $\mathbf{M}_{l\times d}$-valued functions on $[0,\infty)$. Then, for any $a_1,\ a_2\in\mathbb{R}$, $a_1F_1(s)+a_2F_2(s)$ is M-integrable and*

$$\int_B (a_1F_1(s)+a_2F_2(s))\,dX_s = a_1\int_B F_1(s)\,dX_s + a_2\int_B F_2(s)\,dX_s \quad a.\,s. \tag{4.5}$$

for $B\in\mathcal{B}^0_{[0,\infty)}$.

Proposition 4.3. *Let $F(s)$ be an M-integrable $\mathbf{M}_{l\times d}$-valued function on $[0,\infty)$. Let $\Lambda(B)=\int_B F(s)\,dX_s$ and $\lambda_B=\mathcal{L}(\Lambda(B))$ for $B\in\mathcal{B}^0_{[0,\infty)}$. Then $\{\Lambda(B)\colon B\in\mathcal{B}^0_{[0,\infty)}\}$ is an $\mathbb{R}^l$-valued i. s. r. m.,*

$$\int_0^t |\log\widehat{\rho}_s(F(s)'z)|\,\sigma(ds)<\infty \quad \text{for } t\in(0,\infty), \tag{4.6}$$

and

$$\log\widehat{\lambda}_B(z)=\int_B \log\widehat{\rho}_s(F(s)'z)\,\sigma(ds) \quad \text{for } B\in\mathcal{B}^0_{[0,\infty)}. \tag{4.7}$$

Here $\log\widehat{\rho}_s(F(s)'z)$ means $(\log\widehat{\rho}_s(w))_{w=F(s)'z}$.

Proof of Propositions 4.1, 4.2, and 4.3. If $F_1(s)$ and $F_2(s)$ are simple functions, then $a_1F_1(s)+a_2F_2(s)$ is simple and (4.5) is obvious. If $F(s)$ is a simple function, then the statements in Proposition 4.3 are easily shown. Indeed, in this case, it follows from (3.2) and (4.2) that

$$\begin{aligned} Ee^{i\langle z,\Lambda(B)\rangle} &= E\exp\left[i\left\langle z,\sum_{j=1}^n R_j\,M(B\cap B_j)\right\rangle\right]=\prod_{j=1}^n Ee^{i\langle z,R_jM(B\cap B_j)\rangle}\\ &=\prod_{j=1}^n \exp\int_{B\cap B_j}\log\widehat{\rho}_s(R_j'z)\sigma(ds)=\exp\int_B\log\widehat{\rho}_s\left(\sum_{j=1}^n 1_{B_j}(s)R_j'z\right)\sigma(ds), \end{aligned}$$

which gives (4.6) and (4.7).

Let $F_n^1(s)$ and $F_n^2(s)$ be the sequences in the statement of Proposition 4.1. Define $G_n(s)=F_n^1(s)-F_n^2(s)$, $\widetilde{\Lambda}_n(B)=\int_B G_n(s)\,dX_s$, and $\widetilde{\Lambda}(B)=\text{p-lim}_{n\to\infty}\widetilde{\Lambda}_n(B)$. Since $\widetilde{\Lambda}_n$ is an i. s. r. m., $\widetilde{\Lambda}$ is also i. s. r. m. by Proposition 3.6. By Egoroff's theorem ([4] p. 88), for any $t_0>0$, there are disjoint Borel sets $C_1,\ C_2,\dots$ in $[0,t_0]$ such that $\lim_{n\to\infty}\sup_{s\in C_l}\|G_n(s)\|=0$ for each l and $\sigma([0,t_0]\setminus C)=0$, where $C=\bigcup_{l=1}^\infty C_l$. Using (4.7) for $\widetilde{\Lambda}_n$ and noting (2.12), we see that $Ee^{i\langle z,\widetilde{\Lambda}_n(B\cap C_l)\rangle}\to 1$ as $n\to\infty$ for every $B\in\mathcal{B}^0_{[0,\infty)}$ and l. Hence $\widetilde{\Lambda}(B\cap C_l)=0$ a. s. Therefore, $\widetilde{\Lambda}(B\cap C)=\sum_{l=1}^\infty\widetilde{\Lambda}(B\cap$

$C_l) = 0$ a.s. Moreover, $\widetilde{\Lambda}(B \setminus C) = 0$ a.s. since $\widetilde{\Lambda}_n(B \setminus C) = 0$ a.s. by (4.6) for $\widetilde{\Lambda}_n$. It follows that $\widetilde{\Lambda}(B) = 0$ a.s. for all $B \in \mathcal{B}_{[0,t_0]}$ and thus for all $B \in \mathcal{B}^0_{[0,\infty)}$. This proves Proposition 4.1. Proposition 4.2 is now straightforward.

Turning to proof of Proposition 4.3, let $F_n(s)$ be simple functions in the definition of M-integrability and let $\Lambda_n(B) = \int_B F_n(s)\,dX_s$ and $\lambda^n_B = \mathcal{L}(\Lambda_n(B))$. This Λ_n is an i.s.r.m. It follows from $\text{p-lim}\,\Lambda_n(B) = \Lambda(B)$ that Λ is an i.s.r.m. by Proposition 3.6 and that $\log\widehat{\lambda}^n_B(z) \to \log\widehat{\lambda}_B(z)$ by Lemma 7.7 of [19]. Fix $t_0 > 0$. We see that $\log\widehat{\lambda}^n_B(z)$ is countably additive in $B \in \mathcal{B}_{[0,t_0]}$ and absolutely continuous with respect to σ, since it satisfies (4.7) with $F_n(s)$ replacing $F(s)$. Hence $\log\widehat{\lambda}_B(z)$ is countably additive in $B \in \mathcal{B}_{[0,t_0]}$ and absolutely continuous with respect to σ by the Vitali–Hahn–Saks theorem and the Nikodým theorem ([3] p. 158–160). Hence there is the Radon–Nikodým derivative $h(s,z)$ such that $\int_0^{t_0} |h(s,z)|\sigma(ds) < \infty$ and $\log\widehat{\lambda}_B(z) = \int_B h(s,z)\sigma(ds)$. On the other hand, fix z and find that $\log\widehat{\rho}_s(F_n(s)'z) \to \log\widehat{\rho}_s(F(s)'z)$ for σ-a.e. s, since $\log\widehat{\rho}_s(w)$ is continuous in w. A use of Egoroff's theorem as in the proof of Proposition 2.6 of [15] yields that $\log\widehat{\rho}_s(F(s)'z) = h(s,z)$ for σ-a.e. s in $[0,t_0]$. Hence (4.6) and (4.7) follow. □

Corollary 4.4. *Let* $(\{\rho^0_s\}, \sigma^0)$ *be a* (*not necessarily canonical*) *factoring of* $\{X_t\}$. *Then, in the situation of* Proposition 4.3,

$$\int_0^t |\log\widehat{\rho}^0_s(F(s)'z)|\,\sigma^0(ds) < \infty \quad \text{for } t \in (0,\infty),$$

$$\log\widehat{\lambda}_B(z) = \int_B \log\widehat{\rho}^0_s(F(s)'z)\,\sigma^0(ds) \quad \text{for } B \in \mathcal{B}^0_{[0,\infty)},$$

and the additive process in law $\{Y_t\}$ *defined by* $Y_t = \Lambda([0,t])$ *has a factoring* $(\{\rho^\sharp_s\}, \sigma^0)$, *where* $\widehat{\rho}^\sharp_s(z) = \widehat{\rho}^0_s(F(s)'z)$, $z \in \mathbb{R}^l$.

Proof. By Proposition 3.5, the canonical measure σ is absolutely continuous with respect to σ^0. Thus there is a measurable function $h(s) \geq 0$ such that $\sigma(ds) = h(s)\sigma^0(ds)$. Let $C = \{s\colon h(s) > 0\}$. We can prove that $\rho_s = (\rho^0_s)^{1/h(s)}$ for σ-a.e. s and that $\rho^0_s = \delta_0$ for σ^0-a.e. s in $[0,\infty) \setminus C$, where δ_0 is the unit mass at 0. Thus the assertion follows from (4.6) and (4.7). Details are omitted. □

Proposition 4.5. *Let* $(\{\rho^0_s\}, \sigma^0)$ *be a factoring of* $\{X_t\}$. *Let* $F(s)$ *be an* $\mathbf{M}_{l\times d}$*-valued measurable function locally bounded on* $[0,\infty)$. *Then* $F(s)$ *is* M*-integrable. If* $F_n(s)$ *is a sequence of simple functions on* $[0,\infty)$ *such that* $F_n(s) \to F(s)$ σ^0*-a.e. and, for any* $t_0 > 0$, $\|F_n(s)\|$ *is uniformly bounded on* $[0,t_0]$, *then*

$$\text{p-}\lim_{n\to\infty} \int_B F_n(s)\,dX_s = \int_B F(s)\,dX_s \quad \text{for } B \in \mathcal{B}^0_{[0,\infty)}.$$

Proof. We can find simple functions $F_n(s)$ such that $F_n(s) \to F(s)$ for all s and $\|F_n(s)\|$ is uniformly bounded on $[0, t_0]$ for any t_0. Then

$$E \exp\left[i \left\langle z, \int_B F_n(s)\, dX_s - \int_B F_m(s)\, dX_s \right\rangle\right] = \exp \int_B \log \widehat{\rho}_s\big((F_n(s) - F_m(s))' z\big) \sigma(ds),$$

which tends to 1 as n, $m \to \infty$, using (2.12). Hence, $\int_B F_n(s)\, dX_s$ is convergent in probability. Hence $F(s)$ is M-integrable. To prove the second half of the assertion we use Propositions 3.5, 4.1, and the argument above. □

Theorem 4.6. *Let $F(s)$ be an $\mathbf{M}_{l\times d}$-valued measurable function locally bounded on $[0, \infty)$. Define $\Lambda(B) = \int_B F(s)\, dX_s$ and $Y_t = \Lambda([0, t])$. Then, for any $\mathbf{M}_{m\times l}$-valued measurable function $G(s)$ locally bounded on $[0, \infty)$ and for any $B \in \mathcal{B}^0_{[0,\infty)}$,*

$$\int_B G(s)\, dY_s = \int_B G(s) F(s)\, dX_s \quad a.s. \tag{4.8}$$

Proof. Choose simple functions $F_n(s)$ and $G_k(s)$ such that $F_n(s) \to F(s)$ and $G_k(s) \to G(s)$ for all s and, for any $t_0 > 0$, $F_n(s)$ and $G_k(s)$ are uniformly bounded on $[0, t_0]$. Then $\int_B G_k(s)\, dY_s = \text{p-lim}_{n\to\infty} \int_B G_k(s) F_n(s)\, dX_s$ from the definitions and, using (4.7), we get $\int_B G_k(s) F_n(s)\, dX_s - \int_B G_k(s) F(s)\, dX_s \to 0$ in probability as $n \to \infty$. Then, letting $k \to \infty$, we get (4.8). □

REMARK. Let $F(s)$ be an M-integrable $\mathbf{M}_{l\times d}$-valued function on $[0, \infty)$. Sometimes we write

$$\int_s^t F(u)\, dX_u = \begin{cases} \int_{(s,t]} F(u)\, dX_u & \text{for } 0 \le s < t < \infty \\ 0 & \text{for } 0 \le s = t < \infty \\ -\int_{(t,s]} F(u)\, dX_u & \text{for } 0 \le t < s < \infty. \end{cases} \tag{4.9}$$

By Theorem 11.5 of [19], there is an additive process modification $\{\widetilde{Y}_t : t \ge 0\}$ of the additive process in law $\{Y_t : t \ge 0\}$ of Corollary 4.4. We understand $\int_s^t F(u)\, dX_u$ in the meaning that

$$\int_s^t F(u)\, dX_u = \widetilde{Y}_t - \widetilde{Y}_s\ , \tag{4.10}$$

without explicit mention.

Theorem 4.7. *Let $F(s)$ be $\mathbf{M}_{l\times d}$-valued and $G(s)$ be $\mathbf{M}_{m\times l}$-valued, both locally bounded, measurable on $[0, \infty)$. Then, for $0 \le t_0 < t_1 < \infty$,*

$$\int_{t_0}^{t_1} G(s) \left(\int_{t_0}^{s} F(u)\, dX_u \right) ds = \int_{t_0}^{t_1} \left(\int_u^{t_1} G(s)\, ds \right) F(u)\, dX_u \quad a.s. \tag{4.11}$$

Lemma 4.8. *If $G(s)$ is an $\mathbf{M}_{m\times l}$-valued bounded measurable function on $[t_0, t_1]$, then there is a sequence $G_n(s)$ of uniformly bounded step functions on $[t_0, t_1]$ such that $G_n(s) \to G(s)$ except on a set of Lebesgue measure 0.*

Proof. By Lusin's theorem ([4] p. 243), for each n, there is a closed set $B_n \subset [t_0, t_1]$ such that $[t_0, t_1] \setminus B_n$ has Lebesgue measure $< 2^{-n}$ and the restriction of $G(s)$ to B_n is continuous. Then, by Urysohn's theorem in general topology, there is an $\mathbf{M}_{m\times l}$-valued, uniformly bounded, continuous function G_n^0 on $[t_0, t_1]$ such that $G_n^0 = G$ on B_n. Now choose uniformly bounded step functions G_n on $[t_0, t_1]$ such that $\|G_n(s) - G_n^0(s)\| < 2^{-n}$. □

Outline of proof of Theorem 4.7. Define

$$Y = \int_{t_0}^{t_1} G(s) \left(\int_{t_0}^{s} F(u)\, dX_u \right) ds \, , \qquad Z = \int_{t_0}^{t_1} \left(\int_{u}^{t_1} G(s)\, ds \right) F(u)\, dX_u \, .$$

STEP 1. Show that

$$Ee^{i\langle z,Y\rangle} = Ee^{i\langle z,Z\rangle} = \exp \int_{t_0}^{t_1} \log \widehat{\rho}_u \left(F(u)' \int_{u}^{t_1} G(s)'\, ds\, z \right) \sigma(du). \tag{4.12}$$

The second equality in (4.12) is a consequence of (4.7). Calculation of $Ee^{i\langle z,Y\rangle}$ is done by approximation by $Y_n = \int_{t_0}^{t_1} G(s) \left(\int_{t_0}^{\tau_n(s)} F(u)\, dX_u \right) ds$, where $\tau_n(s) = t_{n,k}$ for $t_{n,k-1} < s \le t_{n,k}$ with $t_{n,k} = t_0 + k2^{-n}(t_1 - t_0)$.

STEP 2. Prove the identity

$$\int_{t_0}^{t_1} s\, dX_s = t_1 X_{t_1} - t_0 X_{t_0} - \int_{t_0}^{t_1} X_s\, ds \quad \text{a.s.},$$

by approximation of s by $\tau_n(s)$ in Step 1. Then, using this, we can show that $Y = Z$ a. s., under the assumption that $F(s)$ and $G(s)$ are step functions.

STEP 3. Let $G(s)$ be a step function. If there are step functions $F_n(s)$ such that $F_n(s) \to F(s)$ σ-a. e., then we can show that $Y = Z$ a. s. by using Step 2 for $F_n(s)$ and $G(s)$ and then, for convergence, using (4.12) with $F(u) - F_n(s)$ in place of $F(u)$. Thus $Y = Z$ a. s. is true if $F(s) = 1_B(s)R$ with B an open set and $R \in \mathbf{M}_{l\times d}$. Then the case where $F(s) = 1_B(s)R$ with B compact is treated and then the case with B Borel. Next we can show that $Y = Z$ a. s. when $F(s)$ is locally bounded and measurable.

STEP 4. Show that $Y = Z$ a. s. when $F(s)$ and $G(s)$ satisfy the conditions in the theorem, using Lemma 4.8 and the result in Step 3. □

Corollary 4.9 (Integration-by-parts formula). *Let $F(s)$ be an $\mathbf{M}_{l\times d}$-valued function of class C^1 on $[0,\infty)$. Then, for $0 \le t_0 < t_1 < \infty$,*

$$\int_{t_0}^{t_1} F(s)\,dX_s = F(t_1)X_{t_1} - F(t_0)X_{t_0} - \int_{t_0}^{t_1} \frac{dF(s)}{ds} X_s\,ds \quad a.s., \tag{4.13}$$

where X_s in the integrand of the last integral is understood to be a Lévy process modification.

Proof. Rewrite $\int_{t_0}^{t_1} F(s)\,dX_s$, using $F(s) = F(t_1) - \int_s^{t_1}(dF(u)/du)\,du$ and then apply Theorem 4.7. □

Theorem 4.10 (Time change). *Let $\tau(t)$ be an increasing continuous mapping from $[0,\infty)$ into $[0,\infty)$ with $\tau(0) = 0$. Define a natural additive process in law $\{Y_t : t \ge 0\}$ on $\mathbb{R}^d$ by $Y_t = X_{\tau(t)}$. Let $F(s)$ be an $\mathbf{M}_{l\times d}$-valued measurable function locally bounded on $[0,\infty)$. Then, for any Borel set B satisfying $B \subset [0,t_0]$ with some $t_0 < \tau(\infty)$,*

$$\int_{\tau^{-1}(B)} F(\tau(s))\,dY_s = \int_B F(s)\,dX_s \quad a.s. \tag{4.14}$$

Proof. The process $\{Y_t\}$ is a natural additive process in law by Proposition 2.12. Denote by $\{\Lambda(B)\}$ the $\mathbb{R}^d$-valued i.s.r.m. induced by $\{Y_t\}$. Then we can show that $\Lambda(\tau^{-1}(B)) = M(B)$ for any Borel set satisfying $B \subset [0,t_0]$ with some $t_0 < \tau(\infty)$. Thus we can show (4.14) whenever F is a simple function. Then we can extend it to F in the theorem, using Propositions 2.8, 2.12, and 4.5. □

5. Some stochastic integrals over unbounded sets

In the preceding section we defined stochastic integrals $\int_B F(s)\,dX_s$ only for bounded Borel sets B in $[0,\infty)$. Now we consider unbounded Borel sets B.

DEFINITION. Let $\{X_t : t \ge 0\}$ be a natural additive process in law on $\mathbb{R}^d$ and let M be the $\mathbb{R}^d$-valued independently scattered random measure induced by $\{X_t\}$. Let $F(s)$ be an M-integrable $\mathbf{M}_{l\times d}$-valued function. Let B be an unbounded Borel set in $[0,\infty)$. We define

$$\int_B F(s)\,dX_s = \int_B F(s)M(ds) = \operatorname*{p-lim}_{t\uparrow\infty} \int_{B\cap[0,t]} F(s)\,dX_s\ ,$$

whenever this limit in probability exists. In this case we say that $\int_B F(s)\,dX_s$ is *definable*. When $B = [t_0,\infty)$ and $\int_B F(s)\,dX_s$ is definable, we sometimes write $\int_{t_0}^{\infty} F(s)\,dX_s$ for $\int_B F(s)\,dX_s$.

When $\{X_t\}$ is a Lévy process on $\mathbb{R}^d$ and $F(s) = e^{-sQ}$ with $Q \in \mathbf{M}_d^+$, the following important facts are known (see [7], [16], [21], [24]). The integral $\int_0^\infty e^{-sQ}\, dX_s$ is definable if and only if $\{X_t\}$ has finite log-moment, that is, $E \log^+ |X_t| < \infty$ for all t. If the integral $\int_0^\infty e^{-sQ}\, dX_s$ is definable, then its distribution μ is Q-selfdecomposable, that is, for each $b \in (0, 1)$, there is a distribution (automatically infinitely divisible) ρ_b such that

$$\widehat{\mu}(z) = \widehat{\mu}(b^{Q'} z)\widehat{\rho}_b(z) .$$

Conversely, any Q-selfdecomposable distribution can be expressed as $\mathcal{L}\left(\int_0^\infty e^{-sQ}\, dX_s\right)$ with a unique (in law) Lévy process with finite log-moment.

We study a case where $\{X_t\}$ belongs to a class of additive processes in law more general than Lévy processes in law.

DEFINITION. A stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is called a *semi-Lévy process in law* or *additive process in law with semi-stationary increments* on $\mathbb{R}^d$ if it is an additive process in law on $\mathbb{R}^d$ such that, for some $p > 0$,

$$X_t - X_s \overset{\mathrm{d}}{=} X_{t+p} - X_{s+p} \quad \text{for any choice of } 0 \leq s < t < \infty . \tag{5.1}$$

This p is called a *period* of the semi-Lévy process in law. A cadlag modification of a semi-Lévy process in law is called a *semi-Lévy process*. An additive process in law $\{X_t\}$ on $\mathbb{R}^d$ is said to have *finite log-moment* if $E \log^+ |X_t| < \infty$ for all t.

REMARK. Let $\{X_t\}$ be a semi-Lévy process in law on $\mathbb{R}^d$ with period p and let (A_t, ν_t, γ_t) be the triplet of X_t. Then $\{X_t\}$ is natural if and only if γ_t is of bounded variation on $[0, p]$. There exist non-natural semi-Lévy processes in law on $\mathbb{R}^d$.

Proposition 5.1. *Let $\{X_t : t \geq 0\}$ be a natural additive process in law on $\mathbb{R}^d$. Then the following statements are equivalent*:
(1) *$\{X_t\}$ is a semi-Lévy process in law with period p,*
(2) *the canonical factoring $(\{\rho_s\}, \sigma)$ of $\{X_t\}$ is periodic with period p in the sense that $\rho_s = \rho_{s+p}$ for σ-a. e. s and $\sigma(B) = \sigma(B + p)$ for all $B \in \mathcal{B}([0, \infty))$,*
(3) *the i. s. r. m. $\{M(B)\}$ induced by $\{X_t\}$ is periodic with period p in the sense that $M(B) \overset{\mathrm{d}}{=} M(B + p)$ for all $B \in \mathcal{B}^0_{[0,\infty)}$.*

Using Proposition 2.8 and (2.11), proof of Proposition 5.1 is easy and omitted.

Let us recall some classes of distributions defined in [14]. Let $Q \in \mathbf{M}_d^+$ and $b \in (0, 1)$. A probability measure μ on $\mathbb{R}^d$ is said to be *(b, Q)-decomposable* if $\widehat{\mu}(z) = \widehat{\mu}(b^{Q'} z)\widehat{\rho}(z)$ with some $\rho \in ID(\mathbb{R}^d)$. The class of all such probability measures is denoted by $L_0(b, Q)$. In the terminology of [19], the class $L_0(b, cI)$ with $c > 0$ is the class of semi-selfdecomposable distributions with span b^{-c}.

Theorem 5.2. *Let $\{X_t\colon t \geq 0\}$ be a natural semi-Lévy process in law on $\mathbb{R}^d$ with period p. Suppose that it has finite log-moment, that is,*

$$E\log^+|X_p| < \infty. \tag{5.2}$$

Then, for any $Q \in \mathbf{M}_d^+$, the stochastic integral $\int_0^\infty e^{-sQ}\,dX_s$ is definable and its distribution μ belongs to $L_0(e^{-p}, Q)$. Moreover, for any $a \in (0,\infty)$,

$$\int_0^\infty \sup_{|z|\leq a} |\log\widehat{\rho}_s(e^{-sQ'}z)|\sigma(ds) < \infty \tag{5.3}$$

and

$$\log\widehat{\mu}(z) = \int_0^\infty \log\widehat{\rho}_s(e^{-sQ'}z)\sigma(ds)\ , \tag{5.4}$$

where $(\{\rho_s\},\sigma)$ is the periodic canonical factoring of $\{X_t\colon t\geq 0\}$.

In particular, when $Q = cI$ with $c > 0$, the stochastic integral $\int_0^\infty e^{-csI}\,dX_s$ is definable and has a semi-selfdecomposable distribution with span e^{cp}, if condition (5.2) is satisfied.

The case without finite log-moment will be treated in Theorem 5.4.

REMARK. In a forthcoming paper jointly written with M. Maejima, it will be proved that, for any $\mu \in L_0(e^{-p}, Q)$, there exists a natural semi-Lévy process in law $\{X_t\}$ with finite log-moment such that $\mathcal{L}\left(\int_0^\infty e^{-sQ}\,dX_s\right) = \mu$.

Proof of Theorem 5.2 uses the following lemma.

Lemma 5.3. *Let $\{X_t\colon t\geq 0\}$ be a semi-Lévy process in law on $\mathbb{R}^d$ with period p. Let ν_t be the Lévy measure of X_t and let $\widetilde{\nu}$ be the unique measure on $[0,\infty)\times\mathbb{R}^d$ satisfying* (2.7). *Then, there are a measure ν^* on $\mathbb{R}^d$ and measures σ_x^*, $x\in\mathbb{R}^d$, on $[0,\infty)$ satisfying the following conditions*:

(1) $\nu^*(\{0\}) = 0$ *and* $\int_{\mathbb{R}^d}(1\wedge|x|^2)\nu^*(dx) < \infty$,

(2) *for any* $x \in \mathbb{R}^d$, σ_x^* *is a periodic measure with period* p *and* $\sigma_x^*((0,p]) = \sigma_x^*([0,p]) = 1$,

(3) *for any* $B\in\mathcal{B}_{[0,\infty)}$, $\sigma_x^*(B)$ *is measurable in* x,

(4) *for any nonnegative function* $f(s,x)$ *measurable in* (s,x),

$$\int_{[0,\infty)\times\mathbb{R}^d} f(s,x)\widetilde{\nu}(d(s,x)) = \int_{\mathbb{R}^d}\nu^*(dx)\int_{[0,\infty)} f(s,x)\sigma_x^*(ds)\ . \tag{5.5}$$

If (ν^*,σ_x^*) and (ν^{**},σ_x^{**}) both satisfy these conditions, then $\nu^* = \nu^{**}$ and $\sigma_x^* = \sigma_x^{**}$ for ν^*-a.e. x.

When $\{X_t\}$ in the lemma has a factoring $(\{\rho_s\},\sigma)$, then

$$\int_0^\infty \sigma(ds)\int_{\mathbb{R}^d} f(s,x)\nu_s^\rho(dx) = \int_{[0,\infty)\times\mathbb{R}^d} f(s,x)\widetilde{\nu}(d(s,x)) \tag{5.6}$$

for all nonnegative measurable function $f(s,x)$. If $f(s,x) = 1_{[0,t]}(s)\,1_B(x)$, then (5.6) holds by (2.10) and (2.7). From this (5.6) follows in general. Comparing (5.5) and (5.6), we see that $(\{\nu_s^\rho\},\sigma)$ and $(\nu^*,\{\sigma_x^*\})$ are dual in a sense.

Proof of Lemma 5.3. We have $\widetilde{\nu}(\{t\}\times\mathbb{R}^d) = 0$ for all $t \geq 0$. Fix a positive integer N. Apply the conditional distribution theorem to the probability measure $m(C) = a\int_C (1\wedge|x|^2)\widetilde{\nu}(d(s,x))$ on $[0,Np]\times\mathbb{R}^d$, where a is a normalizing constant. Then we get a measure ν^* on $\mathbb{R}^d$ and measures σ_x^* on $[0,Np]$ such that $\nu^*(\{0\}) = 0$, $\int_{\mathbb{R}^d}(1\wedge|x|^2)\nu^*(dx) = (Na)^{-1}$, $\sigma_x^*([0,Np]) = N$, $\sigma_x^*(B)$ is measurable in x for each $B\in\mathcal{B}([0,Np])$ and

$$\int_{[0,Np]\times\mathbb{R}^d} f(s,x)(1\wedge|x|^2)\widetilde{\nu}(d(s,x)) = \int_{\mathbb{R}^d}(1\wedge|x|^2)\nu^*(dx)\int_{[0,Np]} f(s,x)\sigma_x^*(ds)$$

for all nonnegative measurable $f(s,x)$. Since $\nu_{p+s} = \nu_p + \nu_s$, we can show that

$$\int_B \nu^*(dx)\int_{(p,p+s]}\sigma_x^*(du) = \int_B \nu^*(dx)\int_{(0,s]}\sigma_x^*(du) \quad \text{for } B\in\mathcal{B}(\mathbb{R}^d).$$

Hence $\sigma_x^*((p,p+s]) = \sigma_x^*((0,s])$ for ν^*-a.e. x. By right-continuity in s, the exceptional set of x can be chosen to be independent of s. Thus we can choose σ_x^* satisfying property (2). By the uniqueness in the conditional distribution theorem, ν^* does not depend on N and σ_x^* can be extended to a periodic measure on $[0,\infty)$. $\square$

Proof of Theorem 5.2. Let $(\{\rho_s\},\sigma)$ be the periodic canonical factoring of $\{X_t\}$. Let us prove (5.3). First notice that

$$|g_c(z,x)| \leq C_z|x|^2(1+|x|^2)^{-1} \quad \text{with } C_z = (|z|^2+2|z|)\vee(4+|z|)\,.$$

Then,

$$\begin{aligned}|\log\widehat{\rho}_s(e^{-sQ'}z)| &\leq \frac{1}{2}(\operatorname{tr}A_s^\rho)|e^{-sQ'}z|^2 + |z||e^{-sQ}\gamma_s^\rho| \\ &\quad + C_z\int_{\mathbb{R}^d}\frac{|e^{-sQ}x|^2}{1+|e^{-sQ}x|^2}\nu_s^\rho(dx) + |z|\int_{\mathbb{R}^d}|e^{-sQ}x|\,|r_s(x)|\,\nu_s^\rho(dx)\,,\end{aligned}$$

where $r_s(x) = c(e^{-sQ}x) - c(x)$. Since the estimate (2.16) remains true if Q is replaced by Q' and since σ is periodic, $\int_0^\infty |e^{-sQ'}z|^2\sigma(ds) \leq \text{const}\,|z|^2$ and $\int_0^\infty |e^{-sQ'}z|\sigma(ds) \leq \text{const}\,|z|$. Note that $\operatorname{tr}A_s^\rho$ and $|\gamma_s^\rho|$ are σ-essentially bounded

(Proposition 2.8). Further, we will prove that

$$\int_0^\infty \sigma(ds)\int_{\mathbb{R}^d}\frac{|e^{-sQ}x|^2}{1+|e^{-sQ}x|^2}\nu_s^\rho(dx)<\infty\ , \tag{5.7}$$

$$\int_0^\infty \sigma(ds)\int_{\mathbb{R}^d}|e^{-sQ}x|\,|r_s(x)|\,\nu_s^\rho(dx)<\infty\ . \tag{5.8}$$

Write $f(\xi)=\xi^2/(1+\xi^2)$. Then, by (5.5), (5.6), and (2.16), the iterated integral in (5.7) is

$$\le \int_{\mathbb{R}^d}\nu^*(dx)\int_0^\infty f(c_3e^{-c_1s}|x|)\sigma_x^*(ds)=a,\ \text{ say.}$$

Notice that, by (2) of Lemma 5.3,

$$\int_{(np,(n+1)p]} f(c_3e^{-c_1s}|x|)\sigma_x^*(ds)\le f(c_3e^{-c_1np}|x|)\le\frac{1}{p}\int_{(n-1)p}^{np} f(c_3e^{-c_1s}|x|)\,ds.$$

Hence,

$$a\le\frac{1}{p}\int_{\mathbb{R}^d}\nu^*(dx)\int_{-p}^\infty f(c_3e^{-c_1s}|x|)\,ds=\frac{1}{2c_1p}\int_{\mathbb{R}^d}\log(1+c_3^2e^{2c_1p}|x|^2)\nu^*(dx)\ ,$$

which is finite by (1) of Lemma 5.3 and by $\int\log^+|x|\nu^*(dx) = \int\log^+|x|\nu_p(dx)$. Note that the condition (5.2) is equivalent to the condition that $\int\log^+|x|\nu_p(dx)<\infty$ by [19], Theorem 25.3 and Proposition 25.4. Thus we get (5.7). Proof of (5.8) is similar, since the iterated integral in (5.8) is

$$\le \text{const}\int_0^\infty\sigma(ds)\int_{\mathbb{R}^d}\frac{|e^{-sQ}x|\,|x|^2}{(1+|e^{-sQ}x|^2)(1+|x|^2)}\nu_s^\rho(dx)$$

and since $(1+\xi)/(1+\xi^2)\le 2$ for $\xi\ge 0$. This finishes a proof of (5.3).

Note that e^{-sQ} is M-integrable by Proposition 4.5. Let $t_1<t_2$. Then

$$E\exp\left[i\left\langle z,\int_{t_1}^{t_2}e^{-sQ}\,dX_s\right\rangle\right]=\exp\int_{t_1}^{t_2}\log\widehat{\rho}_s(e^{-sQ'}z)\sigma(ds)\to 1$$

as $t_1,t_2\to\infty$, by using (4.7) and (5.3). Hence $\int_0^t e^{-sQ}\,dX_s$ is convergent in probability as $t\to\infty$ by the remark in Step 2 of the proof of Theorem 3.2 (ii).

Let us prove that $\mu=\mathcal{L}\left(\int_0^\infty e^{-sQ}\,dX_s\right)$ is in $L_0(e^{-p},Q)$. Let

$$\mu_{(p)}=\mathcal{L}\left(\int_0^p e^{-sQ}\,dX_s\right).$$

Then $\mu_{(p)}\in ID$. Since $\int_0^p e^{-sQ}\,dX_s$ and $\int_p^\infty e^{-sQ}\,dX_s$ are independent and the latter has the same law as $e^{-pQ}\int_0^\infty e^{-sQ}\,dX_s$ by property (5.1), we get $\widehat{\mu}(z)=\widehat{\mu}_{(p)}(z)\widehat{\mu}(e^{-pQ'}z)$. That is, μ is in $L_0(e^{-p},Q)$. $\square$

Theorem 5.4. *Let $\{X_t : t \geq 0\}$ be a natural semi-Lévy process in law on $\mathbb{R}^d$ with period p. Assume that*

$$E \log^+ |X_p| = \infty. \tag{5.9}$$

Then, for any $Q \in \mathbf{M}_d^+$, $\int_0^\infty e^{-sQ} \, dX_s$ is not definable. Moreover, for any sequence $t_n \to \infty$, $\mathcal{L}\left(\int_0^{t_n} e^{-sQ} \, dX_s\right)$ does not converge to any probability measure.

Proof. Fix Q and a sequence $t_n \to \infty$. Denote $\mathcal{L}\left(\int_0^{t_n} e^{-sQ} \, dX_s\right) = \mu^{(n)}$. Suppose that $\mu^{(n)} \to \mu^{(\infty)}$ for some probability measure $\mu^{(\infty)}$. Then $\mu^{(\infty)} \in ID$, since $\mu^{(n)} \in ID$. Let $\nu^{(n)}$ and $\nu^{(\infty)}$ be the Lévy measures of $\mu^{(n)}$ and $\mu^{(\infty)}$, respectively. Then, by [19] Theorem 8.7,

$$\int f(x)\nu^{(n)}(dx) \to \int f(x)\nu^{(\infty)}(dx) \tag{5.10}$$

for all bounded continuous functions f vanishing on a neighborhood of 0. We have, by (2.16), (5.5), and (5.6),

$$\begin{aligned}\int_{|x|>1} \nu^{(n)}(dx) &= \int_0^{t_n} \sigma(ds) \int 1_{\{|e^{-sQ}x|>1\}} \nu_s^\rho(dx) \geq \int_{[0,t_n]\times\mathbb{R}^d} 1_{\{c_4 e^{-c_2 s}|x|>1\}} \widetilde{\nu}(d(s,x)) \\ &= \int_{\mathbb{R}^d} \nu^*(dx) \int_{(0,t_n]} 1_{\{|x|>c_4^{-1}e^{c_2 s}\}} \sigma_x^*(ds) \geq \int_{|x|>c_4^{-1}} \nu^*(dx) \int_{(0,mp]} 1_{\{s<c_2^{-1}\log c_4|x|\}} \sigma_x^*(ds),\end{aligned}$$

where m is an integer such that $mp \leq t_n < (m+1)p$. The inner integral is

$$= \sum_{j=0}^{m-1} \int_{(0,p]} 1_{\{jp+s<c_2^{-1}\log c_4|x|\}} \sigma_x^*(ds) \geq \sum_{j=0}^{m-1} 1_{\{(j+1)p<c_2^{-1}\log c_4|x|\}},$$

which is bounded from below by $((c_2 p)^{-1} \log c_4|x| - 1) \wedge m$. Since $\int \log^+ |x|\nu^*(dx) = \int \log^+ |x|\nu_p(dx) = \infty$ by (5.9), it follows that $\int_{|x|>1} \nu^{(n)}(dx) \to \infty$. This contradicts (5.10). $\square$

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