ON EMBEDDED PRIMARY COMPONENTS

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Introduction

Throughout this paper all rings will be commutative with identities and R will always denote a Noetherian local domain with maximal ideal M.

In section one, we assume that depth $R=1$, (Krull) dim $R>1$ and the integral closure of R is a finite R-module. It is well known that a non-zero principal ideal aR ($\neq R$) has an embedded prime divisor M. Also, see [2, §5]. More generally, we consider the reason of the occurrence of an embedded primary component.

In section two, we assume that depth $R=d<\dim R$ and R is a Nagata local domain satisfying the demension formula. In treating this case, we can reduce to the case that depth $R=1$, using the theory of Rees rings. Hence we will study an embedded primary component in this manner.

Our general reference for undefined terminology is [4].

1. The case of Rings of depth one

Throughout this section, (R, M) denotes a Noetherian local domian such that depth $R=1$, dim $R<1$ and the integral closure $\bar{R}$ is a finite R-module. For an element α of the quotient field of R, we put $I_\alpha=\{x\in R/\alpha x\in R\}$. Moreover, we put

$$A = \{\alpha\in\bar{R}/I_\alpha\supset M^l \text{ for some positive integer } l\} \ .$$

From [1, 3.24], it follows immediately that depth $R=1$ if and only if $I_\alpha=M$ for some element α of the quotient field of R. From [3, Exercise 3, p. 12] and dim $R>1$, we have $\alpha\in\bar{R}$. Hence $\alpha\in A$ and $\alpha\notin R$. Thus $A\neq R$. Also it follows that A is an intermediate ring between R and $\bar{R}$. In fact, for any α, $\beta\in A$, there exist positive integers l and k such that $I_\alpha\supset M^l$ and $I_\beta\supset M^k$. Since $I_{\alpha+\beta}\supset I_\alpha\cdot I_\beta$ and $I_{\alpha\beta}\supset I_\alpha\cdot I_\beta$, we have $I_{\alpha+\beta}\supset M^{l+k}$ and $I_{\alpha\beta}\supset M^{l+k}$. Hence $\alpha+\beta\in A$ and $\alpha\beta\in A$. Moreover, the conductor ideal $\mathfrak{c}(A/R)=R:A$ is an M-primary ideal and A is the largest ring among the set $\{B/B$ is an intermediate ring between R and $\bar{R}$ such that $\mathfrak{c}(B/R)$ is M-primary$\}$. For, since $A=R\alpha_1+\cdots+R\alpha_n$ for some elements $\alpha_1, \cdots, \alpha_n$, there exist natural numbers l_i $(1\leq i\leq n)$

such that $I_{a_i} \supset M^{l_i}$. Put $l=l_1+\cdots+l_n$. We have $M'A \subset R$, that is, $M' \subset \mathfrak{c}(A/R)$. Hence $\mathfrak{c}(A/R)$ is M-primary. Let B be an intermediate ring between R and $\bar{R}$ and $\mathfrak{c}(B/R)$ be M-primary. Since $M'B \subset R$ for some integer l, we have $I_b \supset M^l$ for any element b of B. From the definition of A, it follows that $b \in A$, that is, $B \subset A$.

First we recall the following definitions.

DEFINITION. (1) Let I be an ideal of R. I is called *contractible* if $J \cap R = I$ for some intermediate ring $B\,(\neq R)$ between R and A and some ideal J of B.
(2) Let I be an ideal of R. Put $R(I)=\{\alpha \in A / \alpha I \subset I\}$.
This ring $R(I)$ is called the *coefficient ring of I*.
(3) Put $I_R^{-1}=\{\alpha \in A / \alpha I \subset R\}$.

REMARK. Let $I\,(\neq R)$ be an ideal of R. Then $I_R^{-1} \supsetneqq R$. In fact, since $A \neq R$, there exists an element $\alpha \in A$ such that $I_\alpha = M$. Hence $\alpha I \subset R$.

Lemma 1. *Let I be an ideal of R. Then $I = J \cap R$ for some ideal J of A if and only if $IA \cap R = I$. Moreover, if these conditions are satisfied, $I_R^{-1} = R(I)$. (Consequently, I_R^{-1} is an intermediate ring between R and A.)*

Proof. The first statement is easy and so the second remains to be proved. We assume that $IA \cap R = I$. Take any element α of I_R^{-1}. Then $\alpha I \subset IA \cap R = I$. Hence $\alpha \in R(I)$. Thus $I_R^{-1} \subset R(I)$. Clearly $R(I) \subset I_R^{-1}$, which implies $I_R^{-1} = R(I)$.

Proposition 2. *Let $I\,(\neq R)$ be an ideal of R. Then I is not contractible if and only if $R(I) = R$.*

Proof. First, we prove the "only if" part. Put $B = R(I)$. Suppose that $B \supsetneqq R$. Since I is also an ideal of B, we have $IB \cap R = I$. Thus I is contractible. This contradicts the assumption.
Conversely, suppose that I is contractible. So there exists an intermediate ring $B\,(\neq R)$ between R and A such that $J \cap R = I$ for some ideal J in B. Clearly $IB \cap R = I$. Put $C = \{\beta \in B / \beta I \subset R\}$. Then $R \subsetneqq C \subset R(I)$. In fact, there exists an element $\alpha \in B - R$. Since I_α is an M-primary ideal, there exists some element a of R such that $M = I_\alpha : aR = I_{a\alpha}$ and so we can take $a\alpha$ instead of α. Since $I \subset M$, $\alpha \in C$ and $\alpha \notin R$. Since $CI \subset IB \cap R = I$, $C \subset R(I)$. Thus $R \subsetneqq C \subset R(I)$. The proof is complete.

Proposition 3. *Let I be an ideal of R and let $I = Q_1 \cap \cdots \cap Q_t$ be an irredundant primary decomposition of I where Q_i is a P_i-primary ideal for $i = 1, \cdots, t$. If $P_i \subsetneqq M$ for every $i\,(1 \leq i \leq t)$, then $IA \cap R = I$.*

Proof. It is clear that $I \subset IA \cap R$. We shall prove that $IA \cap R \subset I$. Since $P_i \subsetneqq M$, we see that $P_i \not\supset \mathfrak{c}(A/R)$. Hence $R_{P_i} = A_{P_i}$ for $1 \leq i \leq t$. Thus

$(IA \cap R)_{P_i} = IR_{P_i} \subset Q_i R_{P_i}$ and so $IA \cap R \subset Q_i$ for $1 \leq i \leq t$. Consequently $IA \cap R \subset I$.

Theorem 4. *Let I be an ideal of R with* height $I < \dim R$. *If* $R(I) = R$, *then I has an embedded M-primary component.*

Proof. Suppose that I has no embedded M-primary components. From Proposition 3, we have $IA \cap R = I$. By Lemma 1, we have $I_R^{-1} = R(I)$. Since $I_R^{-1} \supsetneqq R$ by Remark, it contradicts the assumption. The proof is complete.

More precisely, Theorem 4 can be stated as follows:

Theorem 5. *Let I be an ideal of R with* height $I < \dim R$. *Also, let* $I = Q_1 \cap Q_2 \cap \cdots \cap Q_t \cap Q$ *be an irredundant primary decomposition, where* Q_i *is* P_i-*primary* $(i = 1, \cdots, t)$ *and* $P_i \neq M$ $(i = 1, \cdots, t)$. *If* $R(I) = R$, *then* Q *is an M-primary ideal such that* $R(Q) = R$.

Proof. By Theorem 4, an M-primary component Q must occur in the primary decomposition. Put $J = Q_1 \cap \cdots \cap Q_t$. By Proposition 3, we have $JA \cap R = J$. So $J_R^{-1} = R(J)$ by Lemma 1. Suppose that $R(Q) \supsetneqq R$. Then we claim that there exists an element $\alpha \in R(Q) - R$ such that $I_\alpha = M$. Since I_α is M-primary, there exists some element a of R such that $M = I_\alpha : aR$. On the other hand, $I_\alpha : aR = I_{a\alpha}$ and so we can take $a\alpha$ instead of α. By this claim, we see that $I_\alpha \supset J$ and so $\alpha J \subset R$. Thus $\alpha \in J_R^{-1} = R(J)$. Since $\alpha \in R(J) \cap R(Q) \subset R(I)$, it follows that $R(I) \supsetneqq R$. This contradicts the assumption. Hence $R(Q) = R$.

Remark. We can give another proof of the following well-known result:

Let $a \neq 0$ be a non-unit element of R. Then aR has M as an embedded prime divisor. In fact, since $R(aR) = R$ and height $(aR) \leq 1$, it follows from Theorem 4.

3. The Rees Rings and embedded primary components

Throughout this section, (R, M) denotes a Nagata local domain satisfying the dimension formula and depth $R = d < \dim R = n$.

We recall the following two definitions:

Definition. A Noetherian domain R satisfies the *dimension formula* if for any finitely generated extension domain T of R, and for any $Q \in \operatorname{Spec} T$ with $P = Q \cap R$, we have height $P + \operatorname{tr.deg}_R T =$ height $Q + \operatorname{tr.deg}_{R/P}(T/Q)$. Here $\operatorname{tr.deg}_A B$ is the transcendence degree of the quotient field of a domain B over that of a subdomain A of B.

Definition (cf. [4, (31.A)]). A ring B is a *Nagata ring* if it is Noetherian

and if, for any finite extension L of the quotient field of B/P, the integral closure of B/P in L is a finite B/P-module for every $P \in \operatorname{Spec} B$.

Let $a_1, \cdots, a_d$ be a maximal regular sequence of elements in R and write $I=(a_1, \cdots, a_d)$. Then $\operatorname{Ass}_R(R/I) = \operatorname{Ass}_T(R/I^l)$ for all $l > 0$ (cf. [3, p. 103, Exercise 13])· Since $M \in \operatorname{Ass}_R(R/I)$, we put $\operatorname{Ass}_R(R/I) = \{p_1, \cdots, p_u, M\}$. Let $I^l = q_{1,l} \cap q_{2,l} \cap \cdots \cap q_{u,l} \cap Q_l$ be an irredundant primary decomposition where the $q_{1,l}$ is p_i-primary and Q_l is M-primary. Put $J_l = q_{1,l} \cap q_{2,l} \cap \cdots \cap q_{u,l}$. J_l is independent of the irredundant primary decomposition of I^l. In fact, $J_l = I^l \cap R[1/a] \cup R$ for some $a \in M - \cup_{i=1}^u p_i$. Thus $J_l J_m \subset J_{l+m}$. Let A be the Rees ring of R with respect to I, that is, the ring $A = R[t^{-1}, It]$ with an indeterminate t. Put $\bar{A} = R[t^{-1}] \oplus (\oplus_{l>0} J_l t^l)$. $\bar{A}$ is a $\mathbb{Z}$-graded ring containing A. Since R is a Nagata ring, A and $\bar{A}$ are also Nagata rings by [4, (31. H)]. In the following let $A, \bar{A}, I, I^l$ and J_l be as above.

Proposition 6. *$\bar{A}$ is integral over A.*

Proof. Let $\tilde{A}$ be the integral closure of A. Since $\tilde{A}$ is a Krull domain, we have $\tilde{A} = \cap \tilde{A}_{\bar{P}}$, the intersection being taken over all $\bar{P} \in \operatorname{Ht}_1(\tilde{A})$ where $\operatorname{Ht}_1(\tilde{A})$ denotes the set of all prime ideals of height one in $\tilde{A}$. Put $P = \bar{P} \cap A$ for $\bar{P} \in \operatorname{Ht}_1(\tilde{A})$. Since R satisfies the dimension formula and A is a finitely generated R-algebra, it follows that A satisfies the dimension formula. Hence $P \in \operatorname{Ht}_1(A)$. Put $P \cap R = p$. We shall prove that $\bar{A} \subset \tilde{A}_{\bar{P}}$ for any $\bar{P} \in \operatorname{Ht}_1(\tilde{A})$. First, we consider the case that $t^{-1} \in P$. Using the dimension formula, we have height $p = \operatorname{tr.deg}_{R/p}(A/P)$. Since $t^{-1} \in P$, it follows that $P \supset I = (a_1, \cdots, a_d)$. Hence $p \supset I$. Thus height $p \geq$ height $I = d$. Since $I = (a_1, \cdots, a_d)$ and $a_1, \cdots, a_d$ is a regular sequence, it follows that $\oplus_{i \geq 0} I^i/I^{i+1} \cong (R/I) [X_1, \cdots, X_d]$, where $X_1, \cdots, X_d$ are indeterminates over R/I. We see that the canonical homomorphism $A/t^{-1}A = \oplus_{i \geq 0} I^i/I^{i+1} \to A/P$ is surjective, and so height $p = \operatorname{tr.deg}_{R/p} A/P \leq \operatorname{tr.deg}_{R/p}(R/p) [X_1, \cdots, X_d] = d$. Hence height $p = d$. Since height $M = n > d$, we see that $M \supsetneq p$. Therefore $(I^l)_p = (J_l)_p$. Since $A_p = R[t^{-1}]_p \oplus (\oplus_{l>0} (I^l)_p t^l) = R[t^{-1}]_p \oplus (\oplus_{l>0} (J_l)_p t^l) = \bar{A}_p$, we have $\tilde{A}_{\bar{P}} \supset \bar{A}_p$. Next, we consider the case that $t^{-1} \notin P$. Since $\bar{A} = R[t^{-1}] \oplus (\oplus_{l>0} J_l t^l)$ by definition, $R_p[t, t^{-1}] \supset \bar{A}$. Since $t^{-1} \notin P$, we have $A_P \supset R_p [t, t^{-1}]$. Thus $\bar{A} \subset A_P \subset \tilde{A}_{\bar{P}}$. Hence $\bar{A} \subset \cap_{\bar{P} \in \operatorname{Ht}_1(\tilde{A})} \tilde{A}_{\bar{P}} = \tilde{A}$. Therefore $\bar{A}$ is integral over A. The proof is complete.

Put $\bar{A}_R = \bar{A} \cap R[t, t^{-1}]$.

Lemma 7. *$\bar{A} = \{\alpha \in \bar{A}_R / M^l \alpha \subset A \text{ for some } l > 0\}$.*

Proof. Put $A' = \{\alpha \in \bar{A}_R / M^l \alpha \subset A \text{ for some } l > 0\}$. First we shall prove that $\bar{A} \subset A'$. Take a homogeneouse elment at^n $(a \in J_n)$. Then there exists a positive integer l such that $J_n M^l \subset I^n$. Hence $M^l(at^n) \subset A$. Thus $\bar{A} \cap A'$. Next,

we shall prove that $A' \subset \tilde{A}$. Take an element α of A'. Since A is a graded ring over R, we can assume that α is a homogeneous element. Let $\alpha = at^n$ where $a \in R$. It is obvious that $\alpha \in \tilde{A}$ in the case that $n \leq 0$. We suppose that $n > 0$. Since $M'\alpha \subset A$, we have $M'a \subset I^n$. Hence $a \in (I^n)_{p_i} \cap R \subset q_{i,n}$. Thus $a \in \cap_{i=1}^u q_{i,n} = J_n$. Therefore $\alpha \in \tilde{A}$. Thus we prove that $A' \subset \tilde{A}$. The proof is complete.

Lemma 8. $\mathrm{Ass}_R (\tilde{A}/A) = \{M\}$.

Proof. It is enough to prove that "if $P \in \mathrm{Ass}_A (\tilde{A}/A)$, then $P \cap R = M$" (cf. [4, p. 57, 9. A]). Since $\tilde{A}$ and A are graded rings, there exists $\alpha = at^n (a \in J_n)$ such that $P = A : \alpha$. Hence $P \cap R = I^n : a$. Since $a \in J_n$, it follows that $I^n : a \supset Q_n$. Therefore $I^n : a$ is an M-primary ideal. Thus $P \cap R = M$. The proof is complete.

Now, we consider the problem when M is a prime divisor of an ideal N containing I. We recall the definition:

$$R_{\tilde{A}}(IA) = \{\alpha \in \tilde{A} / \alpha IA \subset IA\}.$$

Theorem 9. *Let (R, M) be a Nagara local domain satisfying the dimension formula and* $\mathrm{depth}\, R = d < \dim R = n$. *Let N be an ideal of R containing I. If* $\mathrm{height}\, N < n$ *and* $R_{\tilde{A}}(NA) = A$, *then M is an embedded prime divisor of N.*

Proof. First, we shall prove that "if M is not a prime divisor of N then $N\tilde{A} \cap A = NA$". For this, it is enough to prove that $N\tilde{A} \cap A \subset NA$, that is, $NJ_n \cap I^n \subset NI^n$ for any $n > 0$. Take an element α of $NJ_n \cap I^n$,

$$\alpha = \sum x_{i_1, \dots, i_d} a_1^{i_1} \cdots a_d^{i_d},$$

the sum being taken over the integers $i_1, \dots, i_d$ such that $i_1 + i_2 + \cdots + i_d = n$. We claim that $x_{i_1, \dots, i_d} \in N$. Let $N = q_1 \cap \cdots \cap q_s$ be an irredundant primary decomposition of N. Let $p_i' = \mathrm{rad}\,(q_i)$ where $\mathrm{rad}\,(q_i)$ denotes the radical of q_i. It follows that $p_i' \subsetneqq M$ by the assumption. Put $p = p_i'$. Then $(J_n)_p = (I^n)_p$ (cf. The proof in Proposition 6). Since $\alpha \in (NJ_n)_p = (NI^n)_p$, it follows that

$$\alpha = \sum y_{i_1, \dots, i_d} a_1^{i_1} \cdots a_d^{i_d}$$

where $y_{i_1, \dots, i_d} \in N_p$. Since $\alpha \in (I^n)_p$, we have

$$\bar{\alpha} \in I_p^n / I_p^{n+1} \subset \bigoplus_{i \geq 0} I_p^i / I_p^{i+1} \cong (R_p/I_p) [X_1, \cdots, X_d].$$

Therefore

$$\bar{\alpha} = \sum \bar{y}_{i_1, \dots, i_d} a_1^{i_1} \cdots a_d^{i_d} = \sum \bar{x}_{i_1, \dots, i_d} a_1^{i_1} \cdots a_d^{i_d}.$$

Thus $y_{i_1, \dots, i_d} \equiv x_{i_1, \dots, i_d} \pmod{I_p}$, that is,

$$x_{i_1, \cdots, i_d} = y_{i_1, \cdots, i_d} + z_{i_1, \cdots, i_d} \quad \text{for some} \quad z_{i_1, \cdots, i_d} \in I_p \,.$$

Since $y_{i_1, \cdots, i_d} \in N_p$ and $z_{i_1, \cdots, i_d} \in I_p \subset N_p$, we see that $x_{i_1, \cdots, i_d} \in N_p \cap R \subset q_i$. Therefore $N\bar{A} \cap A = NA$.

Next, we shall prove that $R_{\bar{A}}(NA) = (NA)^{-1}_{\bar{A}} \gneqq A$. We recall the definition:

$$(NA)^{-1}_{\bar{A}} = \{\alpha \in \bar{A} / \alpha NA \subset A\} \,.$$

It is clear that $R_{\bar{A}}(NA) \subset (NA)^{-1}_{\bar{A}}$ and so we prove that $(NA)^{-1}_{\bar{A}} \subset R_{\bar{A}}(NA)$. Take any element θ of $(NA)^{-1}_{\bar{A}}$. Then $\theta \in \bar{A}$ and $\theta NA \subset A$. Since $N\bar{A} \cap A = NA$, we have $\theta(NA) \subset N\bar{A} \cap A = NA$. Thus $\theta \in R_{\bar{A}}(NA)$. Hence $R_{\bar{A}}(NA) = (NA)^{-1}_{\bar{A}}$. Now, we shall prove that $(NA)^{-1}_{\bar{A}} \gneqq A$. From Lemma 8, there exists some $\alpha \in \bar{A} - A$ such that $M = A :_R \alpha$. Since $N \subset M$, it follws that $\alpha N \subset A$, that is, $\alpha \in (NA)^{-1}_{\bar{A}}$. Hence $R_{\bar{A}}(NA) \gneqq A$. This is a contradiction.

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