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A SEQUENCE IN THE CLASSICAL SCHOTTKY SPACE

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1. Introduction

Let $\mathbb{M}$ be the topological group of all linear fractional transformations. Its multiplication is the composition of mappings and its topology is the uniform convergence topology on the extended complex plane $\widehat{\mathbb{C}}$.

Let r be a positive integer. We denote the free group with basis $\{1, \dots, r\}$ by F_r. The mapping from $\theta \in \operatorname{Hom}(F_r, \mathbb{M})$ to $(\theta(1), \dots, \theta(r)) \in \mathbb{M}^r$ is bijective. We give $\operatorname{Hom}(F_r, \mathbb{M})$ the topology such that this bijection is a homeomorphism. When $\theta \in \operatorname{Hom}(F_r, \mathbb{M})$ is a monomorphism, θ^{-1} is the inverse of the isomorphism θ whose range is restricted to $\operatorname{Im}\theta$. For $\varphi \in \mathbb{M}$ and $\theta \in \operatorname{Hom}(F_r, \mathbb{M})$, we define $\varphi\theta \in \operatorname{Hom}(F_r, \mathbb{M})$ to be $(\varphi\theta)(x) = \varphi \circ \theta(x) \circ \varphi^{-1}$ for every x in F_r. In this way, $\mathbb{M}$ acts on $\operatorname{Hom}(F_r, \mathbb{M})$.

Let r be a positive integer greater than one. Define the *Schottky space* $\mathbb{S}_r$ *of rank* r to be

$$\mathbb{S}_r = \{\theta \in \operatorname{Hom}(F_r, \mathbb{M}) \mid \operatorname{Im}\theta \text{ is a Schottky group of rank } r\}.$$

$\mathbb{S}_r$ is $\mathbb{M}$-invariant. The Schottky space of rank r defined in Chuckrow [2] is $\mathbb{S}_r/\mathbb{M}$. But the results of Chuckrow [2] which we use also hold for the Schottky space in our sense. We denote by $\partial\mathbb{S}_r$ the boundary of $\mathbb{S}_r$ in $\operatorname{Hom}(F_r, \mathbb{M})$. An element of $\partial\mathbb{S}_r$ is called a *cusp* if its image has parabolic transformations. The following results are shown in Chuckrow [2]:

(1) $\mathbb{S}_r$ is open and connected in $\operatorname{Hom}(F_r, \mathbb{M})$ (Chuckrow [2, Lemma 5]).

(2) Every element of $\partial\mathbb{S}_r$ is a monomorphism and has an image without elliptic transformations (Chuckrow [2, Theorem 4]).

(3) If $\theta \in \partial\mathbb{S}_r$ is not a cusp, then $\operatorname{Im}\theta$ does not act discontinuously on any open subset of $\widehat{\mathbb{C}}$ (Chuckrow [2, Theorem 5]).

Define the *classical Schottky space* $\mathbb{S}_r^0$ *of rank* r to be

$$\mathbb{S}_r^0 = \{\theta \in \operatorname{Hom}(F_r, \mathbb{M}) \mid \operatorname{Im}\theta \text{ is a classical Schottky group of rank } r\}.$$

Let $\overline{\mathbb{S}_r^0}$ be the closure of $\mathbb{S}_r^0$ in $\operatorname{Hom}(F_r, \mathbb{M})$. If θ belongs to $\partial\mathbb{S}_r \cap \overline{\mathbb{S}_r^0}$, then $\operatorname{Im}\theta$ acts

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discontinuously on some open subset of $\widehat{\mathbb{C}}$ (Marden [4, Proposition 3.1]). Thus every element of $\partial\mathbb{S}_r \cap \overline{\mathbb{S}_r^0}$ is a cusp.

For each loxodromic transformation f, we denote the multiplier of f by $\lambda(f)$ ($|\lambda(f)| > 1$). The main result of this paper is as follows:

Theorem. *Let r be an integer greater than one. If a sequence $\{\theta_n\}_{n=1}^{+\infty}$ in $\mathbb{S}_r^0$ converges to θ in $\partial\mathbb{S}_r \cap \overline{\mathbb{S}_r^0}$ as n tends to $+\infty$, then for each parabolic transformation φ of* $\operatorname{Im}\theta$, *$\lambda(\theta_n \circ \theta^{-1}(\varphi))$ converges to* 1 *conically as n tends to $+\infty$. Namely, $\lambda(\theta_n \circ \theta^{-1}(\varphi))$ converges to* 1 *and*

$$\left\{ \frac{|\lambda(\theta_n \circ \theta^{-1}(\varphi)) - 1|}{|\lambda(\theta_n \circ \theta^{-1}(\varphi))| - 1} \right\}_{n=1}^{+\infty}$$

is bounded.

Using McMullen [7, Theorem 7.3], we obtain the following:

Corollary. *Let r be an integer greater than one. If a sequence $\{\theta_n\}_{n=1}^{+\infty}$ in $\mathbb{S}_r^0$ converges to θ in $\partial\mathbb{S}_r \cap \overline{\mathbb{S}_r^0}$ as n tends to $+\infty$, then*

(1) $\operatorname{Im}\theta_n$ *converges to* $\operatorname{Im}\theta$ *geometrically*;

(2) *the limit set of* $\operatorname{Im}\theta_n$ *converges to the limit set of* $\operatorname{Im}\theta$ *in the sense of Hausdorff convergence*;

(3) *the Patterson-Sullivan measure of* $\operatorname{Im}\theta_n$ *converges to the measure of* $\operatorname{Im}\theta$ *weakly*;

(4) *the critical exponent of* $\operatorname{Im}\theta_n$ *converges to the critical exponent of* $\operatorname{Im}\theta$,

as n tends to $+\infty$.

In section 2, we will recall the definition of a Schottky group, and we will also prove a lemma. In section 3, we will prove our theorem. In section 4, we will show that $\mathbb{S}_r^0$ *in our theorem cannot be replaced with $\mathbb{S}_r$ even if θ belongs to $\partial\mathbb{S}_r \cap \overline{\mathbb{S}_r^0}$.*

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2. Schottky Groups

Let r be an integer greater than one. A subgroup G of $\mathbb{M}$ is a *Schottky group of rank r* if there exist a set of generators $h_1, \dots, h_r$ of G and $2r$ mutually disjoint Jordan curves $C_1, C_{-1}, \dots, C_r, C_{-r}$ on $\widehat{\mathbb{C}}$ which satisfy the following conditions:

(1) $C_1, C_{-1}, \dots, C_r, C_{-r}$ bound a $2r$-ply connected region R.

(2) For each i in $\{1, \dots, r\}$, h_i maps C_i onto C_{-i}.

(3) For each i in $\{1, \dots, r\}$, $h_i(R)$ and R are mutually disjoint.

In the above definition, if Jordan curves can be replaced with circles, then G is called a *classical Schottky group of rank* r. A Schottky group of rank r is free of rank r, purely loxodromic and acts discontinuously on some open subset of $\widehat{\mathbb{C}}$.

Example (cf. McMullen [6, Theorem 3.1]). For each positive integer n, let $C_{1\,n}, \dots, C_{r+1\,n}$ be circles on $\widehat{\mathbb{C}}$ which bound an $(r+1)$-ply connected region ($r \geq 2$). Suppose that $C_{1\,n}, \dots, C_{r+1\,n}$ converge to circles $C_1, \dots, C_{r+1}$ as n tends to $+\infty$, respectively; $C_1, \dots, C_{r+1}$ may be tangent but cannot intersect. Define $\theta_n, \theta \in \mathrm{Hom}(F_r, \mathbb{M})$ to be

$$\theta_n(i) = \rho_{r+1\,n} \circ \rho_{i\,n}, \quad \theta(i) = \rho_{r+1} \circ \rho_i \quad \text{for every } i \text{ in } \{1, \dots, r\},$$

respectively, where $\rho_{j\,n}$ and ρ_j are the reflections in $C_{j\,n}$ and C_j on $\widehat{\mathbb{C}}$, respectively ($j = 1, \dots, r+1$). It is shown that $\{\theta_n\}_{n=1}^{+\infty}$ is contained in $\mathbb{S}_r^0$ and converges to θ as n tends to $+\infty$. If $\varphi \in \mathrm{Im}\,\theta$ is parabolic, then there exist $k, l \in \{1, \dots, r+1\}$ such that φ and $\rho_k \circ \rho_l$ are conjugate in the group generated by $\rho_1, \dots, \rho_{r+1}$ (in this case, C_k and C_l are tangent). Since the composite of two reflections in two mutually disjoint circles is hyperbolic, $\lambda(\theta_n \circ \theta^{-1}(\varphi))$ is real for every n. Therefore, $\lambda(\theta_n \circ \theta^{-1}(\varphi))$ converges to 1 conically as n tends to $+\infty$: this is a special case of our theorem.

We notice the following:

Lemma 1 (Marden [4, Lemma 4.1]). *Suppose that G is a Schottky group and that u, v and w are three distinct limit points of G. Fix a region R as in the above definition of a Schottky group. Then there exists one and only one $\varphi \in G$ such that u, v and w belong to three distinct components of $\widehat{\mathbb{C}} - \varphi(R)$.*

In order to prove our theorem, we will prove the following lemma.

Lemma 2. *Let G be a classical Schottky group. Suppose that f and g belong to G and have no common fixed points. Let u, v and w be the repulsive fixed point of f, the attractive fixed point of f and the attractive fixed point of g, respectively. Then there exist two closed disks P and Q in $\widehat{\mathbb{C}}$ which have the following properties*:
(1) *P and Q contain u and w, respectively and they do not intersect each other.*
(2) *$f(P)$ contains P and Q and it does not contain v.*
(3) *Q contains at least one of $g(u)$ and $g(v)$.*

Proof. Let r be the rank of G. Suppose that R is a region as in the above definition of a Schottky group. Since G is classical, we may assume that every component of ∂R is a circle. Note that u, v and w are limit points of G. By Lemma 1, there exists $\varphi \in G$ such that u, v and w belong to three distinct components of $\widehat{\mathbb{C}} - \varphi(R)$. Let U, V and W be components of $\widehat{\mathbb{C}} - \varphi(R)$ which contain u, v and w, respectively. By the definitions of U and V, we can show that $f(U)$ contains $\widehat{\mathbb{C}} - V$ and does not contain v. In particular, $f(U)$ contains U and W. If the repulsive fixed point of g does not belong to U (or V), then $g(u)$ (or $g(v)$) belongs to W. Thus we can put $P = U$ and $Q = W$. $\square$

3. Proof of Theorem

Choose a loxodromic transformation ψ of $\mathrm{Im}\,\theta$ which does not fix the fixed point of φ. We define $\varphi_n = \theta_n \circ \theta^{-1}(\varphi)$ and $\psi_n = \theta_n \circ \theta^{-1}(\psi)$ for each n. Note that φ_n and ψ_n have no common fixed points. Let p_n and q_n be the repulsive fixed point of φ_n and the attractive fixed point of φ_n, respectively. We write k_n for $\lambda(\varphi_n)$. Clearly, k_n converges to 1.

Choose an element γ of $\mathbb{M}$ such that $\gamma \circ \varphi \circ \gamma^{-1}(z) = z/(z+1)$. Both $\gamma(p_n)$ and $\gamma(q_n)$ converge to 0 as n tends to $+\infty$. We assume that n is sufficiently large such that neither $\gamma(p_n)$ nor $\gamma(q_n)$ is ∞. For each n, define $\gamma_n \in \mathbb{M}$ to be

$$\gamma_n(z) = \frac{1 - k_n}{\gamma(p_n) - \gamma(q_n)}(\gamma(z) - \gamma(q_n)).$$

We write

$$\gamma \circ \varphi_n \circ \gamma^{-1}(z) = \frac{a_n z + b_n}{c_n z + d_n}, \quad (a_n d_n - b_n c_n = 1),$$

for each n. Note that $c_n \neq 0$ and that c_n^2 converges to 1. Since $\gamma(p_n)$ and $\gamma(q_n)$ are the solutions of the quadratic equation $c_n x^2 - (a_n - d_n)x - b_n = 0$,

$$(\gamma(p_n) - \gamma(q_n))^2 = (\gamma(p_n) + \gamma(q_n))^2 - 4\gamma(p_n)\gamma(q_n) = \frac{(a_n + d_n)^2 - 4}{c_n^2}.$$

Using $(a_n + d_n)^2 = k_n + k_n^{-1} + 2$, we have

$$(\gamma(p_n) - \gamma(q_n))^2 = \frac{(k_n - 1)^2}{k_n c_n^2}.$$

Since both k_n and c_n^2 converge to 1,

$$\left(\frac{1 - k_n}{\gamma(p_n) - \gamma(q_n)}\right)^2 = k_n c_n^2 \longrightarrow 1 \quad (n \longrightarrow +\infty).$$

Thus γ_n converges to γ, or some subsequence of $\{\gamma_n\}$ converges to $-\gamma$, where $(-\gamma)(z) = -(\gamma(z))$. Considering fixed points and multipliers, we can show $\gamma_n \circ \varphi_n \circ \gamma_n^{-1}(z) = z/(z + k_n)$. Since $\gamma \circ \varphi \circ \gamma^{-1}(z) = z/(z+1)$ and k_n converges to 1, γ_n converges to γ as n tends to $+\infty$.

Let $\sigma \in \mathbb{M}$ map z to $1/z$. Define f_n and f to be

$$f_n = \sigma \circ \gamma_n \circ \varphi_n \circ \gamma_n^{-1} \circ \sigma^{-1} \text{ and } f = \sigma \circ \gamma \circ \varphi \circ \gamma^{-1} \circ \sigma^{-1},$$

respectively. Then $f_n(z) = k_n z + 1$ and $f(z) = z + 1$. Note that $1/(1 - k_n)$ is the repulsive fixed point of f_n. Define g_n and g to be

$$g_n = \sigma \circ \gamma_n \circ \psi_n \circ \gamma_n^{-1} \circ \sigma^{-1} \text{ and } g = \sigma \circ \gamma \circ \psi \circ \gamma^{-1} \circ \sigma^{-1},$$

respectively. Clearly, g_n converges to g as n tends to $+\infty$. Let w_n and w be the attractive fixed points of g_n and g, respectively. Note that neither w_n nor w is ∞. By Lemma 2, there exist two closed disks P_n and Q_n in $\widehat{\mathbb{C}}$ for each n which have the following properties:

(1) P_n and Q_n contain $1/(1 - k_n)$ and w_n, respectively and they do not intersect each other.

(2) $f_n(P_n)$ contains P_n and Q_n and it does not contain ∞.

(3) Q_n contains at least one of $g_n(\infty)$ and $g_n(1/(1 - k_n))$.

From (2), both P_n and Q_n are contained in $\mathbb{C}$. We put

$$P_n = \{z \in \mathbb{C} \mid |z - \alpha_n| \le \rho_n\}.$$

We easily obtain

$$f_n(P_n) = \{z \in \mathbb{C} \mid |z - (k_n\alpha_n + 1)| \le \rho_n|k_n|\}.$$

From $P_n \subset f_n(P_n)$, we deduce that

$$|\alpha_n(k_n - 1) + 1| \le \rho_n(|k_n| - 1).$$

Let l_n be the ray which has α_n as its initial point and which passes through the center (in the Euclidean sense) of Q_n. Suppose that l_n crosses ∂P_n at u_n', ∂Q_n at u_n and v_n, and $f_n(\partial P_n)$ at v_n' (u_n lies between u_n' and v_n). Under this condition,

$$|u_n - v_n| \le |u_n' - v_n'| = |v_n' - \alpha_n| - \rho_n \le |\alpha_n(k_n - 1) + 1| + \rho_n|k_n| - \rho_n.$$

Using $|\alpha_n(k_n - 1) + 1| \le \rho_n(|k_n| - 1)$, we have

$$|u_n - v_n| \le 2\rho_n(|k_n| - 1).$$

We assume that n is sufficiently large such that the following inequalities are satisfied:

$$|w - w_n| < \frac{|w - g(\infty)|}{4}.$$

$$|g(\infty) - g_n(\infty)| < \frac{|w - g(\infty)|}{4}.$$

$$\left|g(\infty) - g_n\left(\frac{1}{1-k_n}\right)\right| < \frac{|w - g(\infty)|}{4}.$$

From these inequalities, we obtain

$$\frac{|w - g(\infty)|}{2} < |w_n - g_n(\infty)|.$$
$$\frac{|w - g(\infty)|}{2} < \left|w_n - g_n\left(\frac{1}{1-k_n}\right)\right|.$$

Since Q_n contains w_n and at least one of $g_n(\infty)$ and $g_n(1/(1-k_n))$, and $|u_n - v_n|$ is the diameter (in the Euclidean sense) of Q_n,

$$\frac{|w - g(\infty)|}{2} < |u_n - v_n|.$$

Since $|u_n - v_n| \leq 2\rho_n(|k_n| - 1)$,

$$|w - g(\infty)| < 4\rho_n(|k_n| - 1).$$

Using this inequality and $|\alpha_n(k_n - 1) + 1| \leq \rho_n(|k_n| - 1)$, we have

$$\begin{aligned} 1 &\geq \frac{|\alpha_n(k_n - 1) + 1|}{\rho_n(|k_n| - 1)} \\ &\geq \frac{|\alpha_n|}{\rho_n}\frac{|k_n - 1|}{|k_n| - 1} - \frac{1}{\rho_n(|k_n| - 1)} \\ &> \frac{|\alpha_n|}{\rho_n}\frac{|k_n - 1|}{|k_n| - 1} - \frac{4}{|w - g(\infty)|}. \end{aligned}$$

Since w_n does not belong to P_n,

$$1 < \frac{|w_n - \alpha_n|}{\rho_n} \leq \frac{|w_n|}{\rho_n} + \frac{|\alpha_n|}{\rho_n} < \frac{4|w_n|(|k_n| - 1)}{|w - g(\infty)|} + \frac{|\alpha_n|}{\rho_n}.$$

Since $|w_n|(|k_n| - 1)$ converges to 0 as n tends to $+\infty$, $|\alpha_n|/\rho_n$ is greater than $1/2$ for sufficiently large n. Therefore,

$$\frac{|k_n - 1|}{|k_n| - 1} < \frac{\rho_n}{|\alpha_n|}\left(1 + \frac{4}{|w - g(\infty)|}\right) < 2\left(1 + \frac{4}{|w - g(\infty)|}\right)$$

for sufficiently large n. This completes the proof.

4. Convergence of critical exponents

Let B^3 be the unit ball model of three-dimensional hyperbolic space, and let ∂B^3 be the sphere at infinity of B^3. $\mathbb{M}$ acts naturally on both of B^3 and ∂B^3. A discrete subgroup of $\mathbb{M}$ acts on B^3 discontinuously. A discrete subgroup of $\mathbb{M}$ is called *geometrically finite* if there exists a finite-sided fundamental polyhedron for its action on B^3 and *geometrically infinite* otherwise. A Schottky group is geometrically finite.

Let G be a discrete subgroup of $\mathbb{M}$. Define the *critical exponent* $\delta(G)$ *of* G to be

$$\delta(G) = \inf\left\{\alpha \geq 0 \;\middle|\; \sum_{g \in G} \exp(-\alpha\rho(\mathbf{o}, g(\mathbf{o}))) < +\infty\right\},$$

where $\mathbf{o} = (0, 0, 0)$ and $\rho(\mathbf{o}, g(\mathbf{o}))$ is the hyperbolic distance between $\mathbf{o}$ and $g(\mathbf{o})$. Furthermore, suppose that G is geometrically finite. Then, there exists one and only one Borel probability measure μ on ∂B^3 such that it is supported on the limit set of G and that for every g in G and every Borel subset E of ∂B^3, the following equality holds:

$$\mu(g(E)) = \int_E |g'(x)|^{\delta(G)} d\mu(x),$$

where $|g'(x)|$ is the linear distortion of g at x in the spherical metric on ∂B^3 (Sullivan [8, Theorem 1]). We call this μ the *Patterson-Sullivan measure of* G.

Let r be an integer greater than one. For every θ in $\partial\mathbb{S}_r$, $\mathrm{Im}\,\theta$ is discrete (Marden [4, Lemma 2.2]). Using McMullen [7, Theorem 7.3], we obtain the following:

Proposition. *Suppose that a sequence* $\{\theta_n\}_{n=1}^{+\infty}$ *in* $\mathbb{S}_r$ *converges to a cusp* θ *as* n *tends to* $+\infty$ *and that* $\mathrm{Im}\,\theta$ *is geometrically finite. If for each parabolic transformation* φ *of* $\mathrm{Im}\,\theta$, $\lambda(\theta_n \circ \theta^{-1}(\varphi))$ *converges to* 1 *conically as* n *tends to* $+\infty$, *then*
(1) $\mathrm{Im}\,\theta_n$ *converges to* $\mathrm{Im}\,\theta$ *geometrically*;
(2) *the limit set of* $\mathrm{Im}\,\theta_n$ *converges to the limit set of* $\mathrm{Im}\,\theta$ *in the sense of Hausdorff convergence*;
(3) *the Patterson-Sullivan measure of* $\mathrm{Im}\,\theta_n$ *converges to the measure of* $\mathrm{Im}\,\theta$ *weakly*;
(4) *the critical exponent of* $\mathrm{Im}\,\theta_n$ *converges to the critical exponent of* $\mathrm{Im}\,\theta$,
as n *tends to* $+\infty$.

For every θ in $\partial\mathbb{S}_r \cap \overline{\mathbb{S}_r^0}$, $\mathrm{Im}\,\theta$ is geometrically finite (Jørgensen, Marden and Maskit [3]). Hence from this we obtain the corollary stated in the introduction.

Finally, we will show that $\mathbb{S}_r^0$ in our theorem cannot be replaced with $\mathbb{S}_r$. *If* $\theta \in \partial\mathbb{S}_r$ *is not a cusp, then* $\mathrm{Im}\,\theta$ *is geometrically infinite*. Using Mostow rigidity, we can prove this claim (see, for example, Matsuzaki and Taniguchi [5, Theorem 4.25]). If a sequence $\{\eta_n\}_{n=1}^{+\infty}$ in $\mathbb{S}_r$ converges to η and if $\mathrm{Im}\,\eta$ is geometrically infinite, then $\delta(\mathrm{Im}\,\eta_n)$ converges to 2 as n tends to $+\infty$ (Bishop and Jones [1, Theorem 6.2]). It is essentially proved in Chuckrow [2] that $\partial\mathbb{S}_r$ removed all cusps is dense in $\partial\mathbb{S}_r$. Consequently, by diagonal method, for each θ in $\partial\mathbb{S}_r$, there exists a sequence $\{\theta_n\}_{n=1}^{+\infty}$ in $\mathbb{S}_r$ such that θ_n converges to θ and $\delta(\mathrm{Im}\,\theta_n)$ converges to 2 as n tends to $+\infty$. On the other hand, if a discrete subgroup G of $\mathbb{M}$ is geometrically finite and if the limit set of G does not coincide with $\widehat{\mathbb{C}}$, then $\delta(G)$ is less than 2 (Sullivan [8, Theorem 1]). Therefore, $\mathbb{S}_r^0$ in our theorem cannot be replaced with $\mathbb{S}_r$ even if θ belongs to $\partial\mathbb{S}_r \cap \overline{\mathbb{S}_r^0}$.

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