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## A SEQUENCE IN THE CLASSICAL SCHOTTKY SPACE

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### 1. Introduction

Let  $\mathbb{M}$  be the topological group of all linear fractional transformations. Its multiplication is the composition of mappings and its topology is the uniform convergence topology on the extended complex plane  $\widehat{\mathbb{C}}$ .

Let  $r$  be a positive integer. We denote the free group with basis  $\{1, \dots, r\}$  by  $F_r$ . The mapping from  $\theta \in \text{Hom}(F_r, \mathbb{M})$  to  $(\theta(1), \dots, \theta(r)) \in \mathbb{M}^r$  is bijective. We give  $\text{Hom}(F_r, \mathbb{M})$  the topology such that this bijection is a homeomorphism. When  $\theta \in \text{Hom}(F_r, \mathbb{M})$  is a monomorphism,  $\theta^{-1}$  is the inverse of the isomorphism  $\theta$  whose range is restricted to  $\text{Im } \theta$ . For  $\varphi \in \mathbb{M}$  and  $\theta \in \text{Hom}(F_r, \mathbb{M})$ , we define  $\varphi\theta \in \text{Hom}(F_r, \mathbb{M})$  to be  $(\varphi\theta)(x) = \varphi \circ \theta(x) \circ \varphi^{-1}$  for every  $x$  in  $F_r$ . In this way,  $\mathbb{M}$  acts on  $\text{Hom}(F_r, \mathbb{M})$ .

Let  $r$  be a positive integer greater than one. Define the *Schottky space*  $\mathbb{S}_r$  of rank  $r$  to be

$$\mathbb{S}_r = \{\theta \in \text{Hom}(F_r, \mathbb{M}) \mid \text{Im } \theta \text{ is a Schottky group of rank } r\}.$$

$\mathbb{S}_r$  is  $\mathbb{M}$ -invariant. The Schottky space of rank  $r$  defined in Chuckrow [2] is  $\mathbb{S}_r/\mathbb{M}$ . But the results of Chuckrow [2] which we use also hold for the Schottky space in our sense. We denote by  $\partial\mathbb{S}_r$  the boundary of  $\mathbb{S}_r$  in  $\text{Hom}(F_r, \mathbb{M})$ . An element of  $\partial\mathbb{S}_r$  is called a *cusp* if its image has parabolic transformations. The following results are shown in Chuckrow [2]:

- (1)  $\mathbb{S}_r$  is open and connected in  $\text{Hom}(F_r, \mathbb{M})$  (Chuckrow [2, Lemma 5]).
- (2) Every element of  $\partial\mathbb{S}_r$  is a monomorphism and has an image without elliptic transformations (Chuckrow [2, Theorem 4]).
- (3) If  $\theta \in \partial\mathbb{S}_r$  is not a cusp, then  $\text{Im } \theta$  does not act discontinuously on any open subset of  $\widehat{\mathbb{C}}$  (Chuckrow [2, Theorem 5]).

Define the *classical Schottky space*  $\mathbb{S}_r^0$  of rank  $r$  to be

$$\mathbb{S}_r^0 = \{\theta \in \text{Hom}(F_r, \mathbb{M}) \mid \text{Im } \theta \text{ is a classical Schottky group of rank } r\}.$$

Let  $\overline{\mathbb{S}_r^0}$  be the closure of  $\mathbb{S}_r^0$  in  $\text{Hom}(F_r, \mathbb{M})$ . If  $\theta$  belongs to  $\partial\mathbb{S}_r \cap \overline{\mathbb{S}_r^0}$ , then  $\text{Im } \theta$  acts

discontinuously on some open subset of  $\widehat{\mathbb{C}}$  (Marden [4, Proposition 3.1]). Thus every element of  $\partial\mathbb{S}_r \cap \overline{\mathbb{S}_r^0}$  is a cusp.

For each loxodromic transformation  $f$ , we denote the multiplier of  $f$  by  $\lambda(f)$  ( $|\lambda(f)| > 1$ ). The main result of this paper is as follows:

**Theorem.** *Let  $r$  be an integer greater than one. If a sequence  $\{\theta_n\}_{n=1}^{+\infty}$  in  $\mathbb{S}_r^0$  converges to  $\theta$  in  $\partial\mathbb{S}_r \cap \overline{\mathbb{S}_r^0}$  as  $n$  tends to  $+\infty$ , then for each parabolic transformation  $\varphi$  of  $\text{Im } \theta$ ,  $\lambda(\theta_n \circ \theta^{-1}(\varphi))$  converges to 1 conically as  $n$  tends to  $+\infty$ . Namely,  $\lambda(\theta_n \circ \theta^{-1}(\varphi))$  converges to 1 and*

$$\left\{ \frac{|\lambda(\theta_n \circ \theta^{-1}(\varphi)) - 1|}{|\lambda(\theta_n \circ \theta^{-1}(\varphi))| - 1} \right\}_{n=1}^{+\infty}$$

is bounded.

Using McMullen [7, Theorem 7.3], we obtain the following:

**Corollary.** *Let  $r$  be an integer greater than one. If a sequence  $\{\theta_n\}_{n=1}^{+\infty}$  in  $\mathbb{S}_r^0$  converges to  $\theta$  in  $\partial\mathbb{S}_r \cap \overline{\mathbb{S}_r^0}$  as  $n$  tends to  $+\infty$ , then*

- (1)  $\text{Im } \theta_n$  converges to  $\text{Im } \theta$  geometrically;
- (2) the limit set of  $\text{Im } \theta_n$  converges to the limit set of  $\text{Im } \theta$  in the sense of Hausdorff convergence;
- (3) the Patterson-Sullivan measure of  $\text{Im } \theta_n$  converges to the measure of  $\text{Im } \theta$  weakly;
- (4) the critical exponent of  $\text{Im } \theta_n$  converges to the critical exponent of  $\text{Im } \theta$ , as  $n$  tends to  $+\infty$ .

In section 2, we will recall the definition of a Schottky group, and we will also prove a lemma. In section 3, we will prove our theorem. In section 4, we will show that  $\mathbb{S}_r^0$  in our theorem cannot be replaced with  $\mathbb{S}_r$  even if  $\theta$  belongs to  $\partial\mathbb{S}_r \cap \overline{\mathbb{S}_r^0}$ .

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## 2. Schottky Groups

Let  $r$  be an integer greater than one. A subgroup  $G$  of  $\mathbb{M}$  is a *Schottky group of rank  $r$*  if there exist a set of generators  $h_1, \dots, h_r$  of  $G$  and  $2r$  mutually disjoint Jordan curves  $C_1, C_{-1}, \dots, C_r, C_{-r}$  on  $\widehat{\mathbb{C}}$  which satisfy the following conditions:

- (1)  $C_1, C_{-1}, \dots, C_r, C_{-r}$  bound a  $2r$ -ply connected region  $R$ .
- (2) For each  $i$  in  $\{1, \dots, r\}$ ,  $h_i$  maps  $C_i$  onto  $C_{-i}$ .
- (3) For each  $i$  in  $\{1, \dots, r\}$ ,  $h_i(R)$  and  $R$  are mutually disjoint.

In the above definition, if Jordan curves can be replaced with circles, then  $G$  is called a *classical Schottky group of rank  $r$* . A Schottky group of rank  $r$  is free of rank  $r$ , purely loxodromic and acts discontinuously on some open subset of  $\widehat{\mathbb{C}}$ .

EXAMPLE (cf. McMullen [6, Theorem 3.1]). For each positive integer  $n$ , let  $C_{1n}, \dots, C_{r+1n}$  be circles on  $\widehat{\mathbb{C}}$  which bound an  $(r+1)$ -ply connected region ( $r \geq 2$ ). Suppose that  $C_{1n}, \dots, C_{r+1n}$  converge to circles  $C_1, \dots, C_{r+1}$  as  $n$  tends to  $+\infty$ , respectively;  $C_1, \dots, C_{r+1}$  may be tangent but cannot intersect. Define  $\theta_n, \theta \in \text{Hom}(F_r, \mathbb{M})$  to be

$$\theta_n(i) = \rho_{r+1n} \circ \rho_{in}, \quad \theta(i) = \rho_{r+1} \circ \rho_i \quad \text{for every } i \text{ in } \{1, \dots, r\},$$

respectively, where  $\rho_{jn}$  and  $\rho_j$  are the reflections in  $C_{jn}$  and  $C_j$  on  $\widehat{\mathbb{C}}$ , respectively ( $j = 1, \dots, r+1$ ). It is shown that  $\{\theta_n\}_{n=1}^{+\infty}$  is contained in  $\mathbb{S}_r^0$  and converges to  $\theta$  as  $n$  tends to  $+\infty$ . If  $\varphi \in \text{Im } \theta$  is parabolic, then there exist  $k, l \in \{1, \dots, r+1\}$  such that  $\varphi$  and  $\rho_k \circ \rho_l$  are conjugate in the group generated by  $\rho_1, \dots, \rho_{r+1}$  (in this case,  $C_k$  and  $C_l$  are tangent). Since the composite of two reflections in two mutually disjoint circles is hyperbolic,  $\lambda(\theta_n \circ \theta^{-1}(\varphi))$  is real for every  $n$ . Therefore,  $\lambda(\theta_n \circ \theta^{-1}(\varphi))$  converges to 1 conically as  $n$  tends to  $+\infty$ : this is a special case of our theorem.

We notice the following:

**Lemma 1** (Marden [4, Lemma 4.1]). *Suppose that  $G$  is a Schottky group and that  $u, v$  and  $w$  are three distinct limit points of  $G$ . Fix a region  $R$  as in the above definition of a Schottky group. Then there exists one and only one  $\varphi \in G$  such that  $u, v$  and  $w$  belong to three distinct components of  $\widehat{\mathbb{C}} - \varphi(R)$ .*

In order to prove our theorem, we will prove the following lemma.

**Lemma 2.** *Let  $G$  be a classical Schottky group. Suppose that  $f$  and  $g$  belong to  $G$  and have no common fixed points. Let  $u, v$  and  $w$  be the repulsive fixed point of  $f$ , the attractive fixed point of  $f$  and the attractive fixed point of  $g$ , respectively. Then there exist two closed disks  $P$  and  $Q$  in  $\widehat{\mathbb{C}}$  which have the following properties:*

- (1)  $P$  and  $Q$  contain  $u$  and  $w$ , respectively and they do not intersect each other.
- (2)  $f(P)$  contains  $P$  and  $Q$  and it does not contain  $v$ .
- (3)  $Q$  contains at least one of  $g(u)$  and  $g(v)$ .

Proof. Let  $r$  be the rank of  $G$ . Suppose that  $R$  is a region as in the above definition of a Schottky group. Since  $G$  is classical, we may assume that every component of  $\partial R$  is a circle. Note that  $u, v$  and  $w$  are limit points of  $G$ . By Lemma 1, there exists  $\varphi \in G$  such that  $u, v$  and  $w$  belong to three distinct components of  $\widehat{\mathbb{C}} - \varphi(R)$ . Let  $U, V$  and  $W$  be components of  $\widehat{\mathbb{C}} - \varphi(R)$  which contain  $u, v$  and  $w$ , respectively. By

the definitions of  $U$  and  $V$ , we can show that  $f(U)$  contains  $\widehat{\mathbb{C}} - V$  and does not contain  $v$ . In particular,  $f(U)$  contains  $U$  and  $W$ . If the repulsive fixed point of  $g$  does not belong to  $U$  (or  $V$ ), then  $g(u)$  (or  $g(v)$ ) belongs to  $W$ . Thus we can put  $P = U$  and  $Q = W$ .  $\square$

### 3. Proof of Theorem

Choose a loxodromic transformation  $\psi$  of  $\text{Im } \theta$  which does not fix the fixed point of  $\varphi$ . We define  $\varphi_n = \theta_n \circ \theta^{-1}(\varphi)$  and  $\psi_n = \theta_n \circ \theta^{-1}(\psi)$  for each  $n$ . Note that  $\varphi_n$  and  $\psi_n$  have no common fixed points. Let  $p_n$  and  $q_n$  be the repulsive fixed point of  $\varphi_n$  and the attractive fixed point of  $\varphi_n$ , respectively. We write  $k_n$  for  $\lambda(\varphi_n)$ . Clearly,  $k_n$  converges to 1.

Choose an element  $\gamma$  of  $\mathbb{M}$  such that  $\gamma \circ \varphi \circ \gamma^{-1}(z) = z/(z+1)$ . Both  $\gamma(p_n)$  and  $\gamma(q_n)$  converge to 0 as  $n$  tends to  $+\infty$ . We assume that  $n$  is sufficiently large such that neither  $\gamma(p_n)$  nor  $\gamma(q_n)$  is  $\infty$ . For each  $n$ , define  $\gamma_n \in \mathbb{M}$  to be

$$\gamma_n(z) = \frac{1 - k_n}{\gamma(p_n) - \gamma(q_n)}(\gamma(z) - \gamma(q_n)).$$

We write

$$\gamma \circ \varphi_n \circ \gamma^{-1}(z) = \frac{a_n z + b_n}{c_n z + d_n}, \quad (a_n d_n - b_n c_n = 1),$$

for each  $n$ . Note that  $c_n \neq 0$  and that  $c_n^2$  converges to 1. Since  $\gamma(p_n)$  and  $\gamma(q_n)$  are the solutions of the quadratic equation  $c_n x^2 - (a_n - d_n)x - b_n = 0$ ,

$$(\gamma(p_n) - \gamma(q_n))^2 = (\gamma(p_n) + \gamma(q_n))^2 - 4\gamma(p_n)\gamma(q_n) = \frac{(a_n + d_n)^2 - 4}{c_n^2}.$$

Using  $(a_n + d_n)^2 = k_n + k_n^{-1} + 2$ , we have

$$(\gamma(p_n) - \gamma(q_n))^2 = \frac{(k_n - 1)^2}{k_n c_n^2}.$$

Since both  $k_n$  and  $c_n^2$  converge to 1,

$$\left( \frac{1 - k_n}{\gamma(p_n) - \gamma(q_n)} \right)^2 = k_n c_n^2 \longrightarrow 1 \quad (n \longrightarrow +\infty).$$

Thus  $\gamma_n$  converges to  $\gamma$ , or some subsequence of  $\{\gamma_n\}$  converges to  $-\gamma$ , where  $(-\gamma)(z) = -(\gamma(z))$ . Considering fixed points and multipliers, we can show  $\gamma_n \circ \varphi_n \circ \gamma_n^{-1}(z) = z/(z+k_n)$ . Since  $\gamma \circ \varphi \circ \gamma^{-1}(z) = z/(z+1)$  and  $k_n$  converges to 1,  $\gamma_n$  converges to  $\gamma$  as  $n$  tends to  $+\infty$ .

Let  $\sigma \in \mathbb{M}$  map  $z$  to  $1/z$ . Define  $f_n$  and  $f$  to be

$$f_n = \sigma \circ \gamma_n \circ \varphi_n \circ \gamma_n^{-1} \circ \sigma^{-1} \text{ and } f = \sigma \circ \gamma \circ \varphi \circ \gamma^{-1} \circ \sigma^{-1},$$

respectively. Then  $f_n(z) = k_n z + 1$  and  $f(z) = z + 1$ . Note that  $1/(1 - k_n)$  is the repulsive fixed point of  $f_n$ . Define  $g_n$  and  $g$  to be

$$g_n = \sigma \circ \gamma_n \circ \psi_n \circ \gamma_n^{-1} \circ \sigma^{-1} \text{ and } g = \sigma \circ \gamma \circ \psi \circ \gamma^{-1} \circ \sigma^{-1},$$

respectively. Clearly,  $g_n$  converges to  $g$  as  $n$  tends to  $+\infty$ . Let  $w_n$  and  $w$  be the attractive fixed points of  $g_n$  and  $g$ , respectively. Note that neither  $w_n$  nor  $w$  is  $\infty$ . By Lemma 2, there exist two closed disks  $P_n$  and  $Q_n$  in  $\widehat{\mathbb{C}}$  for each  $n$  which have the following properties:

- (1)  $P_n$  and  $Q_n$  contain  $1/(1 - k_n)$  and  $w_n$ , respectively and they do not intersect each other.
- (2)  $f_n(P_n)$  contains  $P_n$  and  $Q_n$  and it does not contain  $\infty$ .
- (3)  $Q_n$  contains at least one of  $g_n(\infty)$  and  $g_n(1/(1 - k_n))$ .

From (2), both  $P_n$  and  $Q_n$  are contained in  $\mathbb{C}$ . We put

$$P_n = \{z \in \mathbb{C} \mid |z - \alpha_n| \leq \rho_n\}.$$

We easily obtain

$$f_n(P_n) = \{z \in \mathbb{C} \mid |z - (k_n \alpha_n + 1)| \leq \rho_n |k_n|\}.$$

From  $P_n \subset f_n(P_n)$ , we deduce that

$$|\alpha_n(k_n - 1) + 1| \leq \rho_n(|k_n| - 1).$$

Let  $l_n$  be the ray which has  $\alpha_n$  as its initial point and which passes through the center (in the Euclidean sense) of  $Q_n$ . Suppose that  $l_n$  crosses  $\partial P_n$  at  $u'_n$ ,  $\partial Q_n$  at  $u_n$  and  $v_n$ , and  $f_n(\partial P_n)$  at  $v'_n$  ( $u_n$  lies between  $u'_n$  and  $v_n$ ). Under this condition,

$$|u_n - v_n| \leq |u'_n - v'_n| = |v'_n - \alpha_n| - \rho_n \leq |\alpha_n(k_n - 1) + 1| + \rho_n |k_n| - \rho_n.$$

Using  $|\alpha_n(k_n - 1) + 1| \leq \rho_n(|k_n| - 1)$ , we have

$$|u_n - v_n| \leq 2\rho_n(|k_n| - 1).$$

We assume that  $n$  is sufficiently large such that the following inequalities are satisfied:

$$\begin{aligned} |w - w_n| &< \frac{|w - g(\infty)|}{4}, \\ |g(\infty) - g_n(\infty)| &< \frac{|w - g(\infty)|}{4}. \end{aligned}$$

$$\left| g(\infty) - g_n \left( \frac{1}{1 - k_n} \right) \right| < \frac{|w - g(\infty)|}{4}.$$

From these inequalities, we obtain

$$\begin{aligned} \frac{|w - g(\infty)|}{2} &< |w_n - g_n(\infty)|, \\ \frac{|w - g(\infty)|}{2} &< \left| w_n - g_n \left( \frac{1}{1 - k_n} \right) \right|. \end{aligned}$$

Since  $Q_n$  contains  $w_n$  and at least one of  $g_n(\infty)$  and  $g_n(1/(1 - k_n))$ , and  $|u_n - v_n|$  is the diameter (in the Euclidean sense) of  $Q_n$ ,

$$\frac{|w - g(\infty)|}{2} < |u_n - v_n|.$$

Since  $|u_n - v_n| \leq 2\rho_n(|k_n| - 1)$ ,

$$|w - g(\infty)| < 4\rho_n(|k_n| - 1).$$

Using this inequality and  $|\alpha_n(k_n - 1) + 1| \leq \rho_n(|k_n| - 1)$ , we have

$$\begin{aligned} 1 &\geq \frac{|\alpha_n(k_n - 1) + 1|}{\rho_n(|k_n| - 1)} \\ &\geq \frac{|\alpha_n|}{\rho_n} \frac{|k_n| - 1}{|k_n| - 1} - \frac{1}{\rho_n(|k_n| - 1)} \\ &> \frac{|\alpha_n|}{\rho_n} \frac{|k_n| - 1}{|k_n| - 1} - \frac{4}{|w - g(\infty)|}. \end{aligned}$$

Since  $w_n$  does not belong to  $P_n$ ,

$$1 < \frac{|w_n - \alpha_n|}{\rho_n} \leq \frac{|w_n|}{\rho_n} + \frac{|\alpha_n|}{\rho_n} < \frac{4|w_n|(|k_n| - 1)}{|w - g(\infty)|} + \frac{|\alpha_n|}{\rho_n}.$$

Since  $|w_n|(|k_n| - 1)$  converges to 0 as  $n$  tends to  $+\infty$ ,  $|\alpha_n|/\rho_n$  is greater than  $1/2$  for sufficiently large  $n$ . Therefore,

$$\frac{|k_n - 1|}{|k_n| - 1} < \frac{\rho_n}{|\alpha_n|} \left( 1 + \frac{4}{|w - g(\infty)|} \right) < 2 \left( 1 + \frac{4}{|w - g(\infty)|} \right)$$

for sufficiently large  $n$ . This completes the proof.

#### 4. Convergence of critical exponents

Let  $B^3$  be the unit ball model of three-dimensional hyperbolic space, and let  $\partial B^3$  be the sphere at infinity of  $B^3$ .  $\mathbb{M}$  acts naturally on both of  $B^3$  and  $\partial B^3$ . A discrete

subgroup of  $\mathbb{M}$  acts on  $B^3$  discontinuously. A discrete subgroup of  $\mathbb{M}$  is called *geometrically finite* if there exists a finite-sided fundamental polyhedron for its action on  $B^3$  and *geometrically infinite* otherwise. A Schottky group is geometrically finite.

Let  $G$  be a discrete subgroup of  $\mathbb{M}$ . Define the *critical exponent*  $\delta(G)$  of  $G$  to be

$$\delta(G) = \inf \left\{ \alpha \geq 0 \mid \sum_{g \in G} \exp(-\alpha \rho(\mathbf{o}, g(\mathbf{o}))) < +\infty \right\},$$

where  $\mathbf{o} = (0, 0, 0)$  and  $\rho(\mathbf{o}, g(\mathbf{o}))$  is the hyperbolic distance between  $\mathbf{o}$  and  $g(\mathbf{o})$ . Furthermore, suppose that  $G$  is geometrically finite. Then, there exists one and only one Borel probability measure  $\mu$  on  $\partial B^3$  such that it is supported on the limit set of  $G$  and that for every  $g$  in  $G$  and every Borel subset  $E$  of  $\partial B^3$ , the following equality holds:

$$\mu(g(E)) = \int_E |g'(x)|^{\delta(G)} d\mu(x),$$

where  $|g'(x)|$  is the linear distortion of  $g$  at  $x$  in the spherical metric on  $\partial B^3$  (Sullivan [8, Theorem 1]). We call this  $\mu$  the *Patterson-Sullivan measure of  $G$* .

Let  $r$  be an integer greater than one. For every  $\theta$  in  $\partial \mathbb{S}_r$ ,  $\text{Im } \theta$  is discrete (Marden [4, Lemma 2.2]). Using McMullen [7, Theorem 7.3], we obtain the following:

**Proposition.** *Suppose that a sequence  $\{\theta_n\}_{n=1}^{+\infty}$  in  $\mathbb{S}_r$  converges to a cusp  $\theta$  as  $n$  tends to  $+\infty$  and that  $\text{Im } \theta$  is geometrically finite. If for each parabolic transformation  $\varphi$  of  $\text{Im } \theta$ ,  $\lambda(\theta_n \circ \theta^{-1}(\varphi))$  converges to 1 conically as  $n$  tends to  $+\infty$ , then*

- (1)  $\text{Im } \theta_n$  converges to  $\text{Im } \theta$  geometrically;
- (2) the limit set of  $\text{Im } \theta_n$  converges to the limit set of  $\text{Im } \theta$  in the sense of Hausdorff convergence;
- (3) the Patterson-Sullivan measure of  $\text{Im } \theta_n$  converges to the measure of  $\text{Im } \theta$  weakly;
- (4) the critical exponent of  $\text{Im } \theta_n$  converges to the critical exponent of  $\text{Im } \theta$ , as  $n$  tends to  $+\infty$ .

For every  $\theta$  in  $\partial \mathbb{S}_r \cap \overline{\mathbb{S}_r^0}$ ,  $\text{Im } \theta$  is geometrically finite (Jørgensen, Marden and Maskit [3]). Hence from this we obtain the corollary stated in the introduction.

Finally, we will show that  $\mathbb{S}_r^0$  in our theorem cannot be replaced with  $\mathbb{S}_r$ . If  $\theta \in \partial \mathbb{S}_r$  is not a cusp, then  $\text{Im } \theta$  is geometrically infinite. Using Mostow rigidity, we can prove this claim (see, for example, Matsuzaki and Taniguchi [5, Theorem 4.25]). If a sequence  $\{\eta_m\}_{m=1}^{+\infty}$  in  $\mathbb{S}_r$  converges to  $\eta$  and if  $\text{Im } \eta$  is geometrically infinite, then  $\delta(\text{Im } \eta_m)$  converges to 2 as  $m$  tends to  $+\infty$  (Bishop and Jones [1, Theorem 6.2]). It is essentially proved in Chuckrow [2] that  $\partial \mathbb{S}_r$  removed all cusps is dense in  $\partial \mathbb{S}_r$ . Consequently, by diagonal method, for each  $\theta$  in  $\partial \mathbb{S}_r$ , there exists a sequence  $\{\theta_n\}_{n=1}^{+\infty}$  in  $\mathbb{S}_r$  such that  $\theta_n$  converges to  $\theta$  and  $\delta(\text{Im } \theta_n)$  converges to 2 as  $n$  tends to  $+\infty$ . On the other hand, if a discrete subgroup  $G$  of  $\mathbb{M}$  is geometrically finite and if the limit



set of  $G$  does not coincide with  $\widehat{\mathbb{C}}$ , then  $\delta(G)$  is less than 2 (Sullivan [8, Theorem 1]). Therefore,  $\mathbb{S}_r^0$  in our theorem cannot be replaced with  $\mathbb{S}_r$  even if  $\theta$  belongs to  $\partial\mathbb{S}_r \cap \overline{\mathbb{S}_r^0}$ .

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