# Ultimate Strength of a Rectangular Plate under Thrust†
# – with Consideration of the Effects of Initial Imperfections due to Welding –

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**Abstract**

*Steel structures are in general constructed by welding, and the structural elements are always accompanied by welding residual stresses and distortion. One of the most commonly used welded structure is a box girder such as ship hulls or some type of bridges. In these structures, the deck plates are the main strength members, which are usually subjected to compressive and/or tensile loads. The deck plates are reinforced by a number of stiffeners, and are sub-divided into narrow strip plate elements. When the ultimate strength of a deck plate is considered, that of a strip plate element plays an important role. From these view points, the ultimate strength of a rectangular plate under thrust is investigated taking into account of the effects of the shape and the magnitude of initial deflection. For this purpose, two series of the elastic large deflection analysis by the analytical method and the elastic-plastic large deflection analysis by the finite element method are carried out. From the results of the investigation, it was found that the minimum ultimate strength is attained when the aspect ratio of one half-wave at collapse is about 0.7 for small initial deflection, which decreases as the magnitude of the initial deflection increases.*

**KEY WORDS:** (Ultimate Strength) (Rectangular Plate) (Welded Structures) (Initial Imperfections) (Deck Plate of Box Girder)

## 1. Introduction

When steel structures of box girder type such as ship hulls or some type of bridges are subjected to lateral loads, compressive or tensile forces are acting in their upper deck plates due to longitudinal bending. These upper deck plates are the main strength members of the box girders, and local collapse of the deck plates may lead to the overall collapse of the box girders.

In general, the upper deck plates of the box girders are reinforced by a number of longitudinal and transverse stiffeners to prevent the overall buckling of the deck plates so that the panel plates may buckle locally between stiffeners under thrust. Usually, these stiffeners are attatched to the deck plate by welding, and consequently the panel plates are accompanied by welding residual stresses and distortions. These initial imperfection due to welding often reduce the strength and the rigidity of the panel plates, and this may sometimes lead to the overall collapse of the deck plates. Therfore, for the sake of safety of the structures, it is very important to eveluate the compressive strength of a rectangular plate containing the initial imperfections due to welding.

Concerning the effect of the welding residual stresses, Yoshiki *et al.*[1)], Fujita *et al.*[2)], Ueda *et al.*[3),4)] and Ueda[5)] dealt with the local buckling strength of I-section plate column or box plate column which are composed by welding. Ueda *et al.*[6 ~ 10)], Komatsu *et al.*[11)] and Fujita *et al.*[12),13)] also studied on the effects of the welding residual stresses and initial deflection on the ultimate strength of square plates with or without stiffeners, both theoretically and experimentally. However, on the ultimate strength of a rectangular plate, only the experimental works have been reported.[14 ~ 18)]

In this paper, the ultimate strength of a rectangular plate under thrust is investigated theoretically, considering the effects of the initial deflection. First, the elastic large deflection of a rectangular plate with initial deflection is calculated using the analytical method, and its behavior is investigated. Then, the effects of the shape and the magnitude of the initial deflection are discussed. Based on the results of the study, a concept of the critical combination of the components of initial deflection is introduced to discuss the most significant component on

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the behavior. Then, the ultimate strength of a rectangular plate with initial deflection is calculated by the elastic-plastic large deflection analysis using the finite element method. Based on the results of calculation, the minimum ultimate strength of a rectangular plate and the effect of the initial deflection on the ultimate strength are investigated.

## 2. Basic Formulation for Analysis

### 2.1 Elastic large deflection analysis

Here, a rectangular plate will be studied, being subjected to thrust. As shown in Fig. 1, the plate is simply-

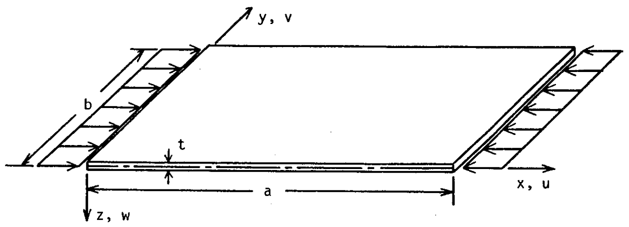



**Fig. 1** Rectangular plate under thrust

supported along all edges. This supporting condition is such that there are no out-of-plane deflections along their four edges. As to the in-plane deflections, the loaded edges and the sides can move, keeping a straight line. These boundary conditions may be supposed to be appropriate to post-buckling analysis of one panel of a multi-stiffened plate such as the deck plates under thrust. Adopting a coordinate system as shown in Fig. 1, and denoting the length, breadth and thickness of the plate by a, b and t, respectively, these boundary conditions may be expressed as

$$
\begin{aligned}
&w=\partial w/\partial y = \partial^2 w/\partial x^2 + \nu\partial^2 w/\partial y^2 = 0 && \text{at } x = 0 \text{ and } a\\
&w=\partial w/\partial x = \partial^2 w/\partial y^2 + \nu\partial^2 w/\partial x^2 = 0 && \text{at } y = 0 \text{ and } b\\
&u = \text{constant} && \text{at } x = 0 \text{ and } a\\
&v = \text{constant} && \text{at } y = 0 \text{ and } b
\end{aligned}
\tag{1}
$$

To clarify the large deflection behavior of the plate, the second order terms of the strain components due to large deflection must be taken into account. In this case, the compatibility equation and the equilibrium equation for the plate are

$$
\begin{aligned}
&\partial^4 F/\partial x^4 + 2\partial^4 F/\partial x^2 \partial y^2 + \partial^4 F/\partial y^4\\
&\quad = E\,[(\partial^2 w/\partial x\partial y)^2 - (\partial^2 w/\partial x^2)(\partial^2 w/\partial y^2)\\
&\qquad - (\partial^2 w_0/\partial x\partial y)^2 + (\partial^2 w_0/\partial x^2)(\partial^2 w_0/\partial y^2)]
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
&D[\partial^4 (w-w_0)/\partial x^4 + 2\partial^4 (w-w_0)/\partial x^2 \partial y^2\\
&\quad + \partial^4 (w-w_0)/\partial y^4]\\
&\quad = t\,[(\partial^2 F/\partial y^2)(\partial^2 w/\partial x^2)\\
&\qquad - 2(\partial^2 F/\partial x\partial y)(\partial^2 w/\partial x\partial y)]\\
&\qquad + (\partial^2 F/\partial x^2)(\partial^2 w/\partial y^2)]
\end{aligned}
\tag{3}
$$

where

$D = Et^3/12(1-\nu^2)$ ; flexual rigidity of the plate

$E$ ; Young's modulus

$\nu$ ; Poisson's ratio

$w_0$ ; Initial deflection of the plate

$F$ ; Airy's stress function

The stress components are expressed by Airy's stress function, F, as

$$\sigma_x = \partial^2 F/\partial y^2, \quad \sigma_y = \partial^2 F/\partial x^2, \quad \tau_{xy} = -\partial^2 F/\partial x\partial y \tag{4}$$

In general, the deflection mode of the plate under external load may be represented in the form,

$$w = \sum_{i=1}^{r} \sum_{j=1}^{s} A_{ij} \cdot \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \tag{5}$$

When a rectangular plate simply supported along its four edges is subjected to thrust as shown in Fig. 1, the deflection mode may be such that the summation of a number of different half-waves in the loading derection but one half-wave perpendicular to the loading direction. In this paper, two terms in Eq. (5) are adopted such that i=m, n and j=1, that is,

$$w = (A_m \sin\frac{m\pi x}{a} + A_n \sin\frac{n\pi x}{a}) \sin\frac{\pi y}{b} \tag{6}$$

This deflection satisfies the boundary conditions expressed by Eq. (1), and may be a good approximation below the secondary buckling strength, if m or n is taken as the number of half-waves at the primary buckling. The same mode is assumed as the initial deflection, which can be written as

$$w_0 = (A_{0m} \sin\frac{m\pi x}{a} + A_{0n} \sin\frac{n\pi x}{a}) \sin\frac{\pi y}{b} \tag{7}$$

Substituting Eqs. (6) and (7) into Eq. (2), the stress function, F, is derived. Then Eqs. (6), (7) and F are substituted into Eq. (3), and the equilibrium equation is obtained in terms of x, y, $A_{0m}$, $A_{0n}$, $A_m$ and $A_n$. An approximate solusion to this equation may be obtained using a Ritz-Galerkin technique. Application of the Ritz-Galerkin approximation leads to the following two cubic simultaneous equations with respect to $A_m$, $A_n$ and the mean compressive stress, $\sigma$.

$$\frac{a^2}{16m^4}\left[\left(\frac{m^4}{a^4}+\frac{1}{b^4}\right)(A_m^2-A_{0m}^2)+\frac{m^2n^2}{a^4}(A_n^2-A_{0n}^2)\right]A_m+\frac{4a^2}{m^2}\left[\frac{1}{16b^4}+\frac{1}{64}\{(m-n)^2\alpha_1+(m+n)^2\alpha_2\}\right](A_mA_n-A_{0m}A_{0n})A_n+\frac{a^2t^2}{12(1-\nu^2)}\left(\frac{m^2}{a^2}+\frac{1}{b^2}\right)^2(A_m-A_{0m})-\frac{1}{\pi^2}\sigma A_m=0 \quad (8)$$

$$\frac{a^2}{16n^2}\left[\left(\frac{n^4}{a^4}+\frac{1}{b^4}\right)(A_n^2-A_{0n}^2)+\frac{m^2n^2}{a^4}(A_m^2-A_{0m}^2)\right]A_n+\frac{4a^2}{n^2}\left[\frac{1}{16b^4}+\frac{1}{64}\{(m-n)^2\alpha_1+(m+n)^2\alpha_2\}\right](A_mA_n-A_{0m}A_{0n})A_m+\frac{a^2t^2}{12(1-\nu^2)}\left(\frac{n^2}{a^2}+\frac{1}{b^2}\right)^2(A_n-A_{0n})-\frac{1}{\pi^2}\sigma A_n=0 \quad (9)$$

where

$$\alpha_1=(m-n)^2/\{(m+n)^2b^2+4a^2\}^2$$
$$\alpha_2=(m+n)^2/\{(m-n)^2b^2+4a^2\}^2 \quad (10)$$

Elastic large deflection behavior of a rectangular plate with initial deflection under thrust can be clarified by solving Eqs. (8) and (9).

### 2.2 Elastic-plastic large deflection analysis

It is required to perform an analysis considering both material and geometrical nonlinearities to clarify the collapse state and to evaluate the ultimate strength. This analysis is very complicated, and it is almost impossible to obtain solutions by the analytical method. Therefore, some numerical technique must be employed such as the finite element method or the finite strip method to obtain accurate solusions for the ultimate strength. In this paper, elastic-plastic large deflection analysis is performed using the finite element method. The basic formulation of the finite element method for the elastic-plastic large deflection analysis is derived by one of the authors applying the incremental method, which is described in Refs. 6) and 19). According to these references, the final form of the equation to be solved can be written in the matrix form as

$$\{L\}+\{dF\}=[K]\{du\} \quad (11)$$

where

[K] ; Stiffness matrix of the plate
{dF} ; Increments of load
{du} ; Increments of displacement
{L} ; Load correction to improve the solusion

## 3. Elastic Large Deflection of a Rectangular Plate under Thrust

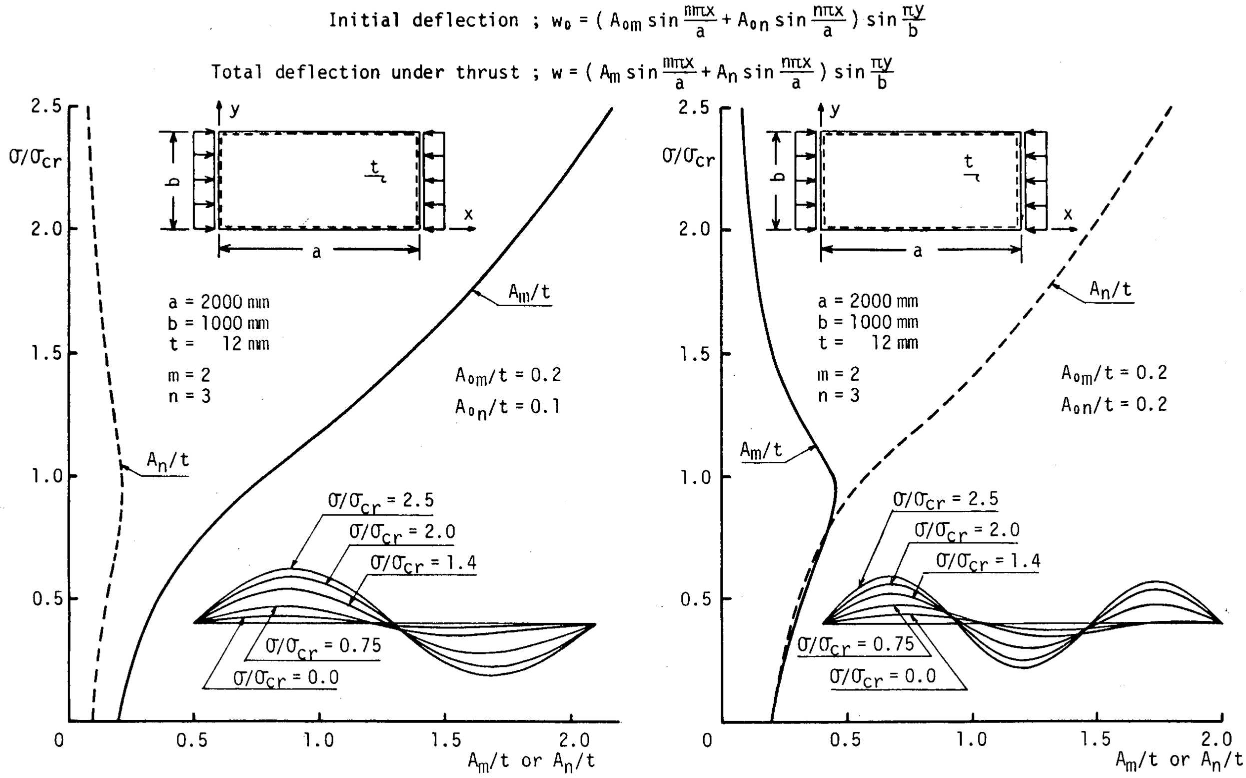
It is well known that an initially flat plate undergoes a primary buckling from an initially flat equilibrium state under external in-plane load. After the primary buckling occures, lateral deflection of the primary buckling mode stably increases with an increase of the load. In case of thin plate, it has been reported that a deformed plate after primary buckling snaps to another equilibrium state of a different deflection mode.[14),15)] This phenomenon is called the secondary buckling, and some theoretical investigations have been performed to predict this secondary buckling behavior and clarify their mechanisms.[20 ~ 22)] However, the secondary buckling strength of a rectangular plate under thrust is fairly high and more than two times the primary buckling strength when the aspect ratio of the plate is large.[22)] The panels of the deck plates of actual girders are so thick that plastification occures before the secondary buckling takes place. For this reason, the secondary buckling is not considered in this paper.

A rectangular plate in the actual welded structures is inevitablly accompanied by initial deflection, and the primary buckling does not occure in a strict sense. In this case, the effects of the shape and the magnitude of the initial deflection on the plate behavior should be discussed. For this purpose, a series of the two-term elastic large deflection analysis is carried out using Eqs. (8) and (9) in Section 2.2.

First, a rectangular plate of which aspect ratio, a/b, being 2.0 is considered. For this plate, the deflection mode of two half-waves corresponds to the primary buckling mode. Taking m=2 and n=3 in Eqs. (6) and (7), a series of elastic large deflection analysis is carried out for various combination of the coefficients of initial deflection, $A_{02}$ and $A_{03}$. Load-deflection coefficient curves for two typical cases are illustrated in Fig. 2 (a) and (b), of which ($A_{02}/t$, $A_{03}/t$) are (0.2, 0.1) and (0.2, 0.2), respectively. In the former case, the deflection mode of two half-waves is finally stable and the plate will collapse in two half-waves in the loading direction. Contrary to this, that of three half-waves is finally stable and the plate will collapse in three half-waves in the latter case. It is interesting that the collapse mode does not necessarily coincide with the buckling mode, depending upon the combination of the components of initial deflection.

The critical combinations of the initial deflection components, $A_{02}/t$ and $A_{03}/t$, are calculated, which govern the final deflection mode for stable deformation, and the results are plotted in Fig. 3. For the combinations of $A_{02}/t$ and $A_{03}/t$ above the curve in Fig. 3, the deflection mode in three half-waves may be finally stable, and for those below the curve, that of two half-waves may be finally stable. These phenomena are discussed also in Ref. 21).

For various aspect reatios of a rectangular plate, critical ratios of $A_{0m}/t$ and $A_{0n}/t$ are calculated. Figure 4 shows




(a) Load-deflection coefficient curves for ( $A_{0m}/t$, $A_{0n}/t$ ) = ( 0.2, 0.1 )

(b) Load-deflection coefficient curves for ( $A_{0m}/t$, $A_{0n}/t$ ) = ( 0.2, 0.2 )

**Fig. 2** Elastic large deflection behavior of a rectangular plate with initial deflection

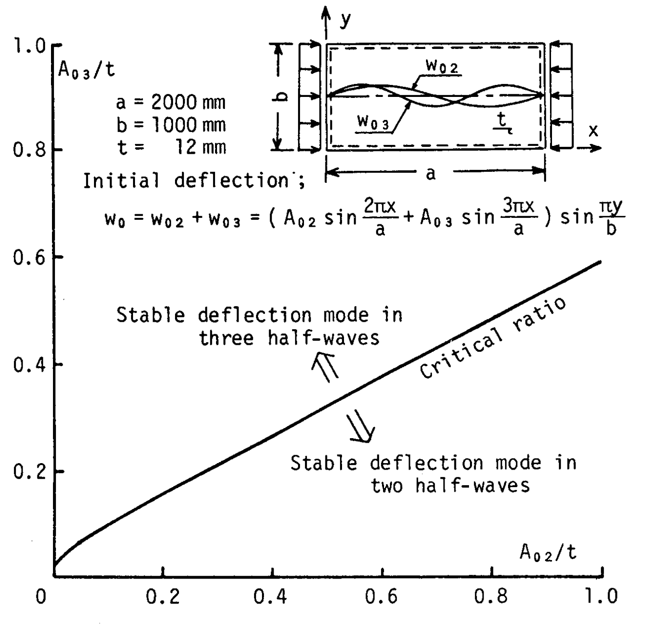



**Fig. 3** Critical combination of the components of initial deflection

special cases when the aspect ratio is an integer up to 6. In Fig. 4, m is taken as the number of half-waves of the

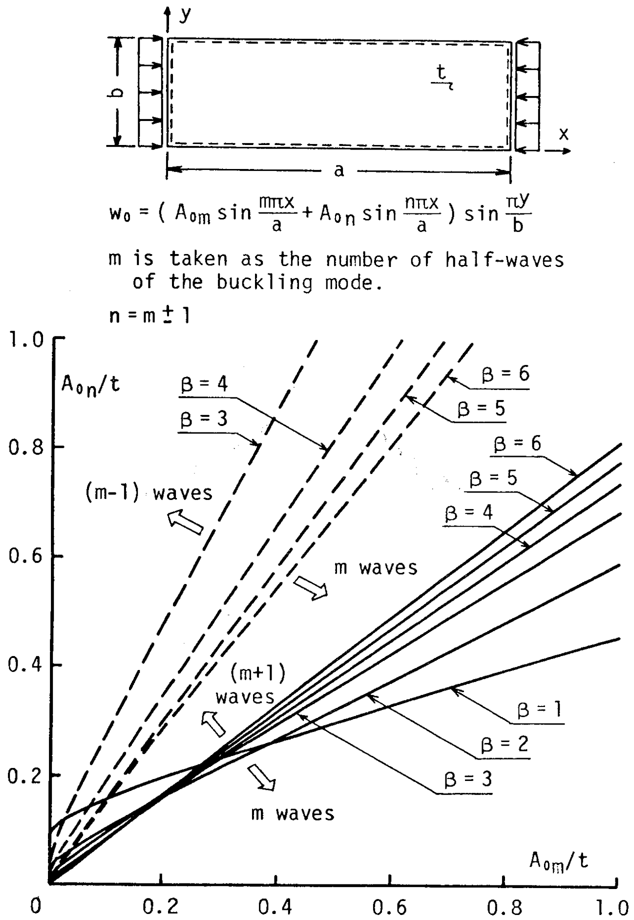



**Fig. 4** Critical combination of the components of initial deflection

primary buckling mode, and n=m ± 1.

The initial deflection of a rectangular plate in the actual structures may be very complicated, and can not be represented in such a simple form as Eq. (7). Furthermore, plastification often takes place before the stable wave form is attained. Therefore, the above mentioned results of the simple two-term elastic large deflection analysis may not be applied directly to the actual cases. However, it may be said that the stable deflection mode of a rectangular plate under thrust is not always the same as that of the primary buckling. Judging from the results shown in Fig. 4, the stable deflection mode is sometimes that of the shorter one half-wave length than that of buckling, which is important as will be discussed in the following chapter when the ultimate strength is considered.

## 4. Ultimate Strength of a Rectangular Plate under Thrust

A series of elastic-plastic large deflection analysis is carried out using the finite element method to evaluate the ultimate strength of a rectangular plate under thrust. The plate breadth, b, thickness, t, and the yield stress of the material, $\sigma_Y$, are taken as 1000 mm, 12 mm and 28 kg/mm$^2$, respectively. In the analysis, the initial deflection of one half-wave is assumed for the plate of which aspect ratio, a/b, is up to 1.2, that is

$$w_0 = A_{01}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{12}$$

The results of calculation of the ultimate strength are plotted against the aspect ratio of the plate in Fig. 5, by

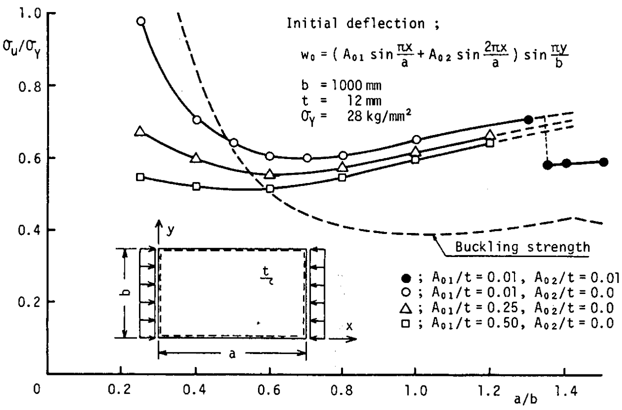



**Fig. 5** Ultimate strength of a rectangular plate under thrust

○, △ and □ for $A_{01}/t$ being 0.01, 0.25 and 0.5, respectively. For a plate with the aspect ratio being more than 1.3, the initial deflection expressed by Eq. (7) is assumed, taking m=1, n=2, $A_{01}/t$=0.01 and $A_{02}/t$=0.01. The results are plotted by ● in Fig. 5. It is very interesting that the aspect ratio of one half-wave which gives the minimum ultimate strength does not coincides with that for buckling strength. When the magnitude of the initial deflection is small, the aspect ratio which gives the minimum ultimate strength is about 0.7. As the magnitude of the initial deflection increases, the aspect ratio for the minimum ultimate strength decreases.

When the initial deflection of $A_{01}/t=A_{02}/t$=0.01 is assumed, the finally stable deflection mode is that of two half-waves in the loading direction for the plate of the aspect ratio being more than 1.36, which is also confirmed by the two-term elastic large deflection analysis. In this case, the calculated ultimate strength is almost the same with that of the plate of a half length of which initial deflection is expressed by Eq. (11). However, if a plate with the aspect ratio being more than 1.36 is assumed to collapse in one half-waves in the loading direction, the ultimate strength may be such as those represented by dashed lines, which are the extrapolations of the solid lines.

If it is assumed that the plate collapses in the same mode is complex, and such any idealized plate does not actually exists. Therefore, it might be conservative to regard the lowest possible ultimate strength for various

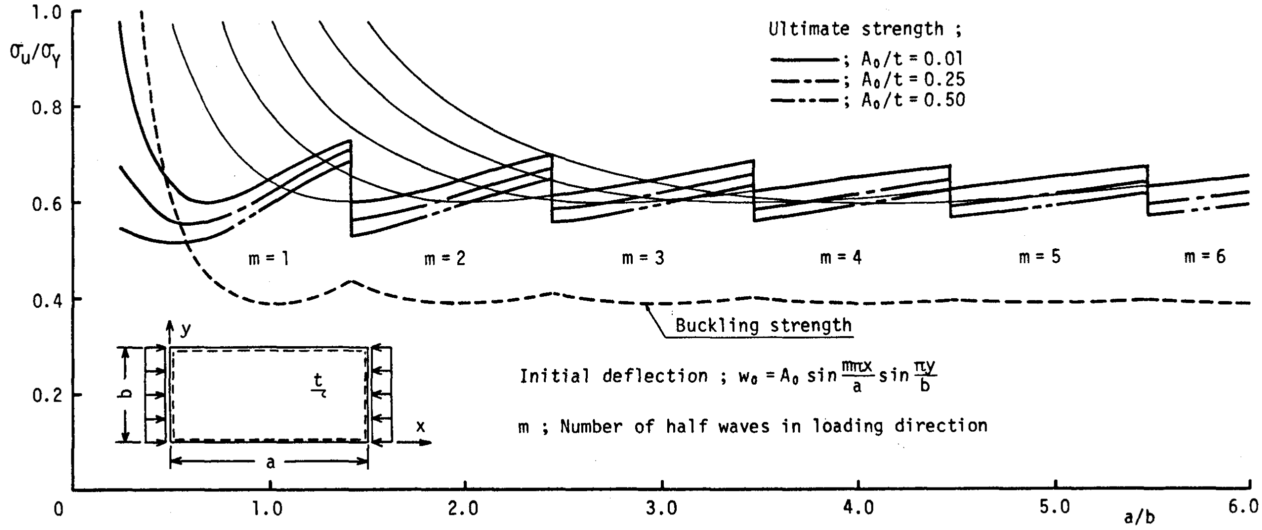



**Fig. 6** Ultimate strength of a rectangular plate under thrust (when collapse mode is same as buckling mode)

deflection mode as the primary buckling mode, the ultimate strength of a rectangular plate under thrust may be such as the solid and chain lines in Fig. 6, which are derived from the ultimate strength curves in Fig. 5. The ultimate strength curves show a saw-tooth appearance with respect to the aspect ratio of the plate, and there exist abrupt changes in the ultimate strength at the aspect ratios which terminate the buckling mode from one to the other. This might be correct when the plate is initially flat or accompanied by initial deflection of its buckling mode. However, the plate elements in actual structures are always accompanied by the initial deflection of which collapse mode as the the ultimate strength of a rectangular plate under thrust. Such ultimate strength curves are illustrated in Fig. 7 together with the ultimate strengths obtained by the finite element method.

To confirm this assumption of the conservative ultimate strength, further finite element analysis is performed for the plate of which aspect ratio is 2.0. In the analysis, the initial deflection of the form,

$$w_0 = (A_{01} \sin\frac{\pi x}{a} + A_{02} \sin\frac{2\pi x}{a} + A_{03} \sin\frac{3\pi x}{a}) \sin\frac{\pi y}{b} \tag{13}$$

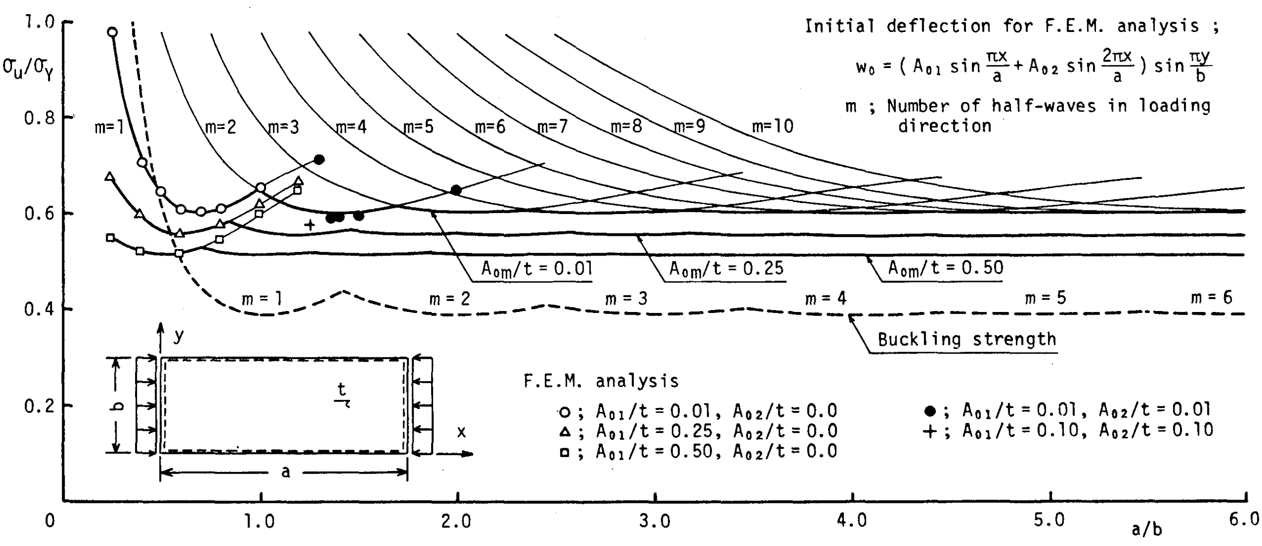



**Fig. 7** Ultimate strength of a rectangular plate with initial deflection under thrust

is assumed. Four cases are analyzed taking ($A_{01}/t$, $A_{02}/t$, $A_{03}/t$) being (0.01, 0.01, 0.01), (0.01, 0.2, 0.3), (0.01, 0.4, 0.3) and (0.01, 0.6, 0.3), and the results are plotted in Fig. 8 against the maximum magnitude of the initial deflection. The solid line, the dashed line and the chain line in Fig. 8 represent the ultimate strengths when the plate collapses in two half-waves, three half-waves and four half-waves in the loading direction, respectively, which are derived from the curves in Fig. 5. The ultimate strengths of the plate with various magnitude and shape of initial deflection are above the lowest ultimate strength curve of which collapse mode is in three or four half-waves in this case. It is observed that the ultimate strength can be estimated by the lowest ultimate strength with a small margin of safety.

Up to this time, the ultimate strength of a rectangular plate is regarded as to be nearly equal to that of a square plate if the aspect ratio of the plate is large. However, this is not a good approximation and sometimes gives a dangerous prediction of the ultimate strength of a rectangular plate with initial deflection. As the conservative estimation of the ultimate strength of a rectangular plate under thrust, such curves those shown in Fig. 7 should be used.

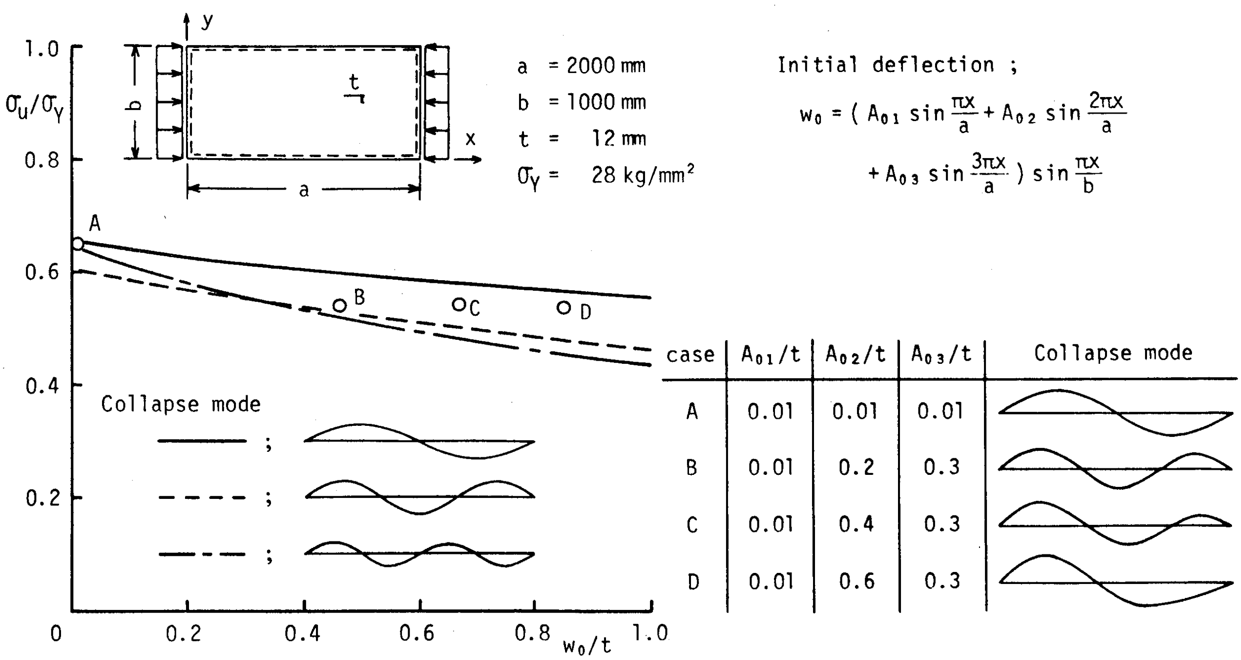



Fig. 8 Effects of the shape and the magnitude of initial deflection on the ultimate strength

## 5. Conclusions

Two series of the elastic large deflection analysis by the analytical method and the elastic-plastic large deflection analysis by the finite element method are carried out to clarify the large deflection behavior and evaluate the ultimate strength of a simply-supported rectangular plate under thrust. The results obtained here lead to the following conclusions.

(1) There exists some critical combination of the components of initial deflection which deflection mode is finally stable. The final mode of the stable deformation does not necessarily coincide with that of the primary buckling, depending on the shape and magnitude of initial deflection.

(2) The ultimate strength of a rectangular plate with very small initial deflection (including a flat plate) shows a saw-tooth appearance with respect to the aspect ratio of the plate, and there exist the abrupt changes in the ultimate strength at the aspect ratio where the primary buckling mode of the plate changes.

(3) The aspect ratio of one half-wave which gives the minimum ultimate strength does not coincide with that of the buckling strength. For the minimum ultimate strength, the aspect ratio is about 0.7 when the magnitude of the initial deflection is small, but decreases as the magnitude of the initial deflection increases.

(4) The ultimate strength of a rectangular plate with initial deflection under thrust may be estimated by the lowest possible ultimate strength for various collapse mode with a small margin of safety.

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