Shima, H. and Hao, J.-H.
Osaka J. Math.
37 (2000), 509–517

GEOMETRY ASSOCIATED WITH NORMAL DISTRIBUTIONS

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(Received June 26, 1998)

1. Introduction

It is known that manifolds of smooth families of probability distributions admit *dualistic structures*. S. Amari proposed *Information Geometry*, whose keywords are *dualistic connections* [1]. Among them the case of dual connections being flat is interesting. Many important families of probability distributions, e.g. exponential families admit *flat* dual connections.

The notion of flat dual connections is the same with *Hessian structures* which have being developed from a different view point [9]-[12].

In this paper, for a linear mapping ρ of a domain Ω into the space of positive definite symmetric matrices we construct an exponential family of probability distributions $\{p(x;\theta,\omega)\}$ on $\mathbf{R}^n$ parametrized by $\theta \in \mathbf{R}^n$, $\omega \in \Omega$, and study a Hessian structure on $\mathbf{R}^n \times \Omega$ given by the exponential family. Such families contain n-dimensional normal distributions (Example 1) and a family of constant negative curvature (Example 2).

In case of a Lie group acting on Ω, ρ is assumed to be equivariant. O.S. Rothaus and I. Satake studied such a linear mapping ρ for homogeneous convex cones [7][8]. Using ρ we introduce a Hessian structure on a vector bundle over a compact hyperbolic affine manifold and prove a certain vanishing theorem (Theorem 2).

2. Hessian structures

We first review some fundamental facts on Hessian structures needed in this paper [1][11].

Let U be an n-dimensional real vector space with canonical flat connection D. Let Ω be a domain in U with a convex function ψ, i.e. the Hessian $Dd\psi$ is positive definte on Ω. Then the metric $g = Dd\psi$ is called a *Hessian metric* and the pair (D, g) a *Hessian structure* on Ω. Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ and $\{\mathbf{u}^{*1}, \cdots, \mathbf{u}^{*n}\}$ be dual basis of U and U^* (the dual vector space of U) respectively. We denote by $\{x^1, \cdots, x^n\}$ (resp. $\{x_1^*, \cdots, x_n^*\}$) the linear coordinate system with respect to $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ (resp. $\{\mathbf{u}^{*1}, \cdots, \mathbf{u}^{*n}\}$). Let $\iota : \Omega \longrightarrow U^*$ be a mapping given by

$$x_i^* \circ \iota = -\frac{\partial \psi}{\partial x^i},$$

which will be called the *gradient mapping*. We define a falt affine connection D' on Ω by

$$\iota_*(D'_X Y) = D^*_X \iota_*(Y),$$

where the right hand side is the covariant differentiation along ι induced by the flat affine connection D^* on U^*. Then we have

$$D' = 2\nabla - D,$$
$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D'_X Z),$$

where ∇ is the Levi-Civita connection of g. Putting $x'_i = -(\partial\psi/\partial x^i)$ we have an affine coordinate system $\{x'_1, \cdots, x'_n\}$ with respect to D'. A function ψ_L on Ω defined by

$$\psi_L = -\sum_i x'_i x^i - \psi$$

is called the *Legendre transform* of ψ. Then we have

$$g = D' d\psi_L.$$

The Hessian structure $(D', g = D'd\psi_L)$ is said to be *the dual Hessian structure* of $(D, g = Dd\psi)$. The *divergence* $\mathbf{D}$ for the Hessian structure $(D, g = Dd\psi)$ is defined by

$$\mathbf{D}(p, q) = \psi(p) + \psi_L(q) - \sum_i x^i(p) x'_i(q) \quad (p, q \in \Omega).$$

3. Probability distributions and Hessian structures induced by ρ

Let Ω be a domain in a real vector space V^m of dimension m. Let ρ be an injective linear mapping of V^m into the space $\mathfrak{S}_n$ of $n \times n$ symmetric matrices such that

(A.1) $\rho(\omega) \in PD\mathfrak{S}_n$ for $\omega \in \Omega$,

where $PD\mathfrak{S}_n$ is the set of positive definite matrices in $\mathfrak{S}_n$. For column vectors $x, \mu \in \mathbf{R}^n$ and $\omega \in \Omega$ we define a density function of $x \in \mathbf{R}^n$ by

$$p(x; \mu, \omega) = (2\pi)^{-n/2}(\det \rho(\omega))^{1/2} \exp\left\{-\frac{1}{2}\,{}^t(x - \mu)\rho(\omega)(x - \mu)\right\}. \tag{1}$$

The family $\{p(x; \mu, \omega) \mid \mu \in \mathbf{R}^n, \omega \in \Omega\}$ parametrized by μ, ω is called the probability distributions induced by ρ.

Proposition 1. *The probability distributions induced by ρ is an exponential family parametrized by $(\theta, \omega) \in \mathbf{R}^n \times \Omega$ where $\theta = \rho(\omega)\mu$. The Fisher information metric* *coincides with the Hessian metric $g = Dd\psi$ where D is the canonical flat connection on $\mathbf{R}^n \times V^m$ and*

$$\psi(\theta, \omega) = \frac{1}{2}\{{}^t\theta\rho(\omega)^{-1}\theta - \log\det\rho(\omega)\}.$$

Proof. For $x = [x^i]$, $\mu = [\mu^i] \in \mathbf{R}^n$ and $\omega = \sum_\alpha \omega_\alpha \mathbf{v}^\alpha \in \Omega$ where $\{\mathbf{v}^1, \cdots, \mathbf{v}^m\}$ is a basis of V^m, we set

$$F^\alpha(x) = -\frac{1}{2}{}^t x\rho(\mathbf{v}^\alpha)x,$$
$$\theta = \rho(\omega)\mu.$$

Then we have

$$p(x;\mu,\omega) = p(x;\theta,\omega) = \exp\left\{\sum_j \theta_j x^j + \sum_\alpha \omega_\alpha F^\alpha(x) - \psi(\theta,\omega) - \frac{n}{2}\log 2\pi\right\}.$$

This implies that the family $\{p(x;\theta,\omega)\}$ of probability distributions parametrized by $(\theta,\omega) \in \mathbf{R}^n \times \Omega$ is an exponential family, and the Fisher information metric coincides with the Hessian metric $g = Dd\psi$. □

A straightforward calculation shows

$$\frac{\partial\psi}{\partial\theta_i} = {}^t\mathbf{e}^i\rho(\omega)^{-1}\theta, \tag{3}$$

$$\frac{\partial\psi}{\partial\omega_\alpha} = -\frac{1}{2}\{{}^t\theta\rho(\omega)^{-1}\rho(\mathbf{v}^\alpha)\rho(\omega)^{-1}\theta + \mathrm{Tr}\rho(\omega)^{-1}\rho(\mathbf{v}^\alpha)\}, \tag{4}$$

$$\left[\frac{\partial^2\psi}{\partial\theta_i\partial\theta_j}\right] = \rho(\omega)^{-1}, \tag{5}$$

$$\frac{\partial^2\psi}{\partial\theta_i\partial\omega_\alpha} = -{}^t\mathbf{e}^i\rho(\omega)^{-1}\rho(\mathbf{v}^\alpha)\rho(\omega)^{-1}\theta, \tag{6}$$

$$\frac{\partial^2\psi}{\partial\omega_\alpha\partial\omega_\beta} = {}^t\theta\rho(\omega)^{-1}\rho(\mathbf{v}^\alpha)\rho(\omega)^{-1}\rho(\mathbf{v}^\beta)\rho(\omega)^{-1}\theta + \frac{1}{2}\mathrm{Tr}\rho(\omega)^{-1}\rho(\mathbf{v}^\alpha)\rho(\omega)^{-1}\rho(\mathbf{v}^\beta), \tag{7}$$

where $\mathbf{e}^i$ is the vector in $\mathbf{R}^n$ whose j-th component is δ^{ij}(Kronecker's delta). The Legendre transform ψ_L of ψ is

$$\psi_L = \frac{1}{2}\log\det\rho(\omega). \tag{8}$$

Proposition 2. *The divergence* $\mathbf{D}$ *of the probability distributions* $\{p(x;\mu,\omega) \mid \mu\in\mathbf{R}^n, \omega\in\Omega\}$ *is given by*

$$\begin{aligned}\mathbf{D}(p,q) = \frac{1}{2}\{&{}^t(\mu(p)-\mu(q))\rho(\omega(p))(\mu(p)-\mu(q)) + \mathrm{Tr}(\rho(\omega(p))\rho(\omega(q))^{-1})\\ &-\log\det(\rho(\omega(p))\rho(\omega(q))^{-1}) - n\}.\end{aligned}$$

Proof. Using (3), (4) we have

$$\begin{aligned}\mathbf{D}(p,q) &= \sum_{i=1}^{n}(\theta_i(q)-\theta_i(p))\frac{\partial\varphi}{\partial\theta_i}(q) + \sum_{\alpha=1}^{m}(\omega_\alpha(q)-\omega_\alpha(p))\frac{\partial\varphi}{\partial\omega_\alpha}(q)\\ &\quad-(\varphi(q)-\varphi(p))\\ &= \{{}^t\theta(q)\rho(\omega(q))^{-1}\theta(q) - {}^t\theta(p)\rho(\omega(q))^{-1}\theta(q)\}\\ &\quad-\frac{1}{2}\{{}^t\theta(q)\rho(\omega(q))^{-1}\rho(\omega(q))\rho(\omega(q))^{-1}\theta(q) + \mathrm{Tr}\,\rho(\omega(q))^{-1}\rho(\omega(q))\\ &\quad-{}^t\theta(q)\rho(\omega(q))^{-1}\rho(\omega(p))\rho(\omega(q))^{-1}\theta(q) - \mathrm{Tr}\,\rho(\omega(q))^{-1}\rho(\omega(p))\}\\ &\quad-\frac{1}{2}\{{}^t\theta(q)\rho(\omega(q))^{-1}\theta(q) - \log\det\rho(\omega(q))\\ &\quad-{}^t\theta(p)\rho(\omega(p))^{-1}\theta(p) + \log\det\rho(\omega(p))\}\\ &= \frac{1}{2}\{{}^t(\mu(p)-\mu(q))\rho(\omega(p))(\mu(p)-\mu(q)) + \mathrm{Tr}(\rho(\omega(p))\rho(\omega(q))^{-1})\\ &\quad-\log\det(\rho(\omega(p))\rho(\omega(q))^{-1}) - n\}.\end{aligned}$$

□

EXAMPLE 1. Let Ω be the set of positive definite matrices in $\mathfrak{S}_n$, and let $\rho : \mathfrak{S}_n \longrightarrow \mathfrak{S}_n$ be the identity mapping. Then $\{p(x;\mu,\omega)\}$ is the family of n-dimensional normal distributions. Then we have

$$\psi = \frac{1}{2}({}^t\theta\omega^{-1}\theta - \log\det\omega),$$

for $\theta = [\theta_i]\in\mathbf{R}^n$, $\omega = [\omega_{ij}]\in\Omega$. Setting

$$\eta^i = -\frac{\partial\psi}{\partial\theta_i} = -{}^t\mathbf{e}^i\omega^{-1}\theta,\quad \xi^{ij} = -\frac{\partial\psi}{\partial\omega_{ij}} = \omega^{ij} + \frac{1}{2}\eta^i\eta^j$$

where $[\omega^{ij}] = [\omega_{ij}]^{-1}$, we obtain

$$\theta = -\left[\xi^{ij} - \frac{1}{2}\eta^i\eta^j\right]^{-1}\eta,\quad \omega = \left[\xi^{ij} - \frac{1}{2}\eta^i\eta^j\right]^{-1}.$$

The image of the gradient mapping is the set of $(\eta, \xi) \in \mathbf{R}^n \times \mathfrak{S}_n$ satisfying

$$\xi - \frac{1}{2}\eta\,{}^t\eta > 0.$$

This domain is an affine analogy of *Siegel domains*. Such domains have been treated in [2][14] in the course of the realization of homogeneous convex domains. The Legendre transform ψ_L of ψ is

$$\psi_L(\eta, \xi) = -\frac{1}{2}\log\det\left(\xi - \frac{1}{2}\eta\,{}^t\eta\right).$$

EXAMPLE 2. Let $\mathbf{R}^+$ be the set of positive numbers and let ρ be a linear mapping of $\mathbf{R}$ into $\mathfrak{S}_n$ given by $\rho(\omega) = \omega E_n$ where E_n is the unit matrix. Then

$$\begin{aligned}
\psi &= \frac{1}{2}\left(\frac{1}{\omega}{}^t\theta\theta - n\log\omega\right),\\
\eta^i &= -\frac{\partial\psi}{\partial\theta_i} = -\frac{\theta_i}{\omega}, \quad \xi = -\frac{\partial\psi}{\partial\omega} = \frac{1}{2}\left(\frac{{}^t\theta\theta}{\omega^2} + \frac{n}{\omega}\right),\\
\theta &= -\frac{n}{2}\eta\left(\xi - \frac{1}{2}{}^t\eta\eta\right)^{-1}, \quad \omega = \frac{n}{2}\left(\xi - \frac{1}{2}{}^t\eta\eta\right)^{-1}.
\end{aligned}$$

The image of the gradient mapping is a domain lying above a paraboloid;

$$\left\{(\xi, \eta) \in \mathbf{R} \times \mathbf{R}^n \,\middle|\, \xi - \frac{1}{2}{}^t\eta\eta > 0\right\}.$$

The Legendre transform ψ_L of ψ is

$$\psi_L = -\frac{n}{2}\log\left(\xi - \frac{1}{2}{}^t\eta\eta\right) + n\log\frac{n}{2}.$$

By [12] the Hessian sectional curvature of the Hessian structure $(D', g = D'd\psi_L)$ is $2/n$. This implies that the space of probability distributions on $\mathbf{R}^n$ defined by

$$p(x;\mu,\omega) = \left(\frac{\omega}{2\pi}\right)^{n/2}\exp\left\{-\frac{\omega}{2}{}^t(x-\mu)(x-\mu)\right\}$$

where $\mu \in \mathbf{R}^n$, $\omega \in \mathbf{R}^+$ is the space of constant curvature $-1/(2n)$ with respect to the Fisher information metric.

In case of a Lie subgroup G of $GL(V^m)$ acting on Ω we assume further that G admits a matrix representation f such that

(A.2) $\rho(s\omega) = f(s)\rho(\omega)\,{}^tf(s)$ for $s \in G$, $\omega \in V^m$.

Then G acts on $\mathbf{R}^n \times \Omega$ by $s(\theta, \omega) = (f(s)\theta, s\omega)$. Since $\psi(f(s)\theta, s\omega) - \psi(\theta, \omega)$ is a constant, the Hesian metric $g = Dd\psi$ is G-invariant.

O.S. Rothaus studied the case of Ω being a homogeneous convex cone and ρ satisfying (A.1), (A.2). He showed that the set $(\xi, \theta, \omega) \in \mathbf{R} \times \mathbf{R}^n \times \Omega$ fulfilling $\xi - {}^t\theta\rho(\omega)^{-1}\theta > 0$ is a homogeneous convex cone, and that all homogeneous convex cones are obtained from lower dimensional ones in this manner [7].

EXAMPLE 3. Let $M(n, \mathbf{R})$ be the set of all $n \times n$ matrices, and let ρ be a linear mapping of $\mathfrak{S}_n$ into the space $\mathrm{End}(M(n, \mathbf{R}))$ of endomorphisms of $M(n, \mathbf{R})$ given by

$$\rho(\omega)x = \omega x + x\omega$$

for $\omega \in \mathfrak{S}_n$, $x \in M(n, \mathbf{R})$. Then $\rho(\omega)$ is symmetric with respect to the inner product $\langle x, y\rangle = \mathrm{Tr}\, {}^t xy$, and positive definite for positive definite matrix ω. Let f be a representation of $O(n)$ on $M(n, \mathbf{R})$ defined by

$$f(s)x = sx\, {}^t s.$$

Then ${}^t f(s)x = {}^t sxs$ and

$$\rho(f(s)\omega) = f(s)\rho(\omega)\, {}^t f(s).$$

Setting $\mathfrak{A}_n = \{x \in M(n, \mathbf{R}) \mid {}^t x = -x\}$, we have $M(n, \mathbf{R}) = \mathfrak{S}_n + \mathfrak{A}_n$, and

$$\rho(\omega)\mathfrak{S}_n \subset \mathfrak{S}_n, \quad \rho(\omega)\mathfrak{A}_n \subset \mathfrak{A}_n.$$

Hence ρ induces an equivariant linear mapping ρ^+ and ρ^- of $\mathfrak{S}_n$ into $\mathrm{End}(\mathfrak{S}_n)$ and $\mathrm{End}(\mathfrak{A}_n)$ respectively. The Hessian structure on $\mathfrak{A}_n \times PD\mathfrak{S}_n$ induced by ρ^- is related to the theory of stable state feedback systems [4][5].

4. Vector bundles over compact hyperbolic affine manifolds

A flat affine manifold M is said to be hyperbolic if the universal covering of M is affinely isomorphic to an open convex cone not containing full straight line [3]. Hence a compact hyperbolic affine manifold is expressed by $\Gamma\backslash\Omega$ where Ω is an open convex cone with vertex 0 in V^m not containing full straight line, and Γ is a discrete subgroup of $GL(V^m)$ acting properly discontinuously and freely on Ω. Suppose that a compact hyperbolic affine manifold $\Gamma\backslash\Omega$ admits a linear mapping ρ of V^m into $\mathfrak{S}_n$ satisfying the conditions (A.1) and (A.2). We denote by $\pi_E : E(\Gamma\backslash\Omega, \rho) \longrightarrow \Gamma\backslash\Omega$ the vector bundle over $\Gamma\backslash\Omega$ associated with the universal covering $\pi : \Omega \longrightarrow \Gamma\backslash\Omega$ and ρ. Since the Hessian structure $(D, g = Dd\psi)$ on $\mathbf{R}^n \times \Omega$ is Γ-invariant, it induces a Hessian structure on the vector bundle $E(\Gamma\backslash\Omega, \rho)$. The Hessian metric defines a fiber metric on each fiber $\pi_E^{-1}(\pi(\omega)) = \{\omega\theta \mid \theta \in \mathbf{R}^n\}$ by

$$(\omega\theta, \omega\theta') = {}^t\theta\rho(\omega)^{-1}\theta'.$$

Using this fiber metric we prove

Theorem 3. *The p-th cohomology group of the complex of $E(\Gamma\backslash\Omega, \rho)$-valued forms on $\Gamma\backslash\Omega$ vanishes for $p \geq 1$.*

Theorem 3 is generalized as the following Theorem 4.

Let $\pi_E : E \longrightarrow M$ be a locally constant vector bundle over a compact hyperbolic affine manifold M. Then there exists an open covering $\{U_\lambda\}$ of M admitting

(i) affine coordinate system $\{u_{\lambda 1}, \cdots, u_{\lambda m}\}$ on U_λ,

(ii) local frames $\{s_\lambda^1, \cdots, s_\lambda^n\}$ on U_λ whose transition functions are constants.

The universal covering of M being a convex cone, M admits a Hessian structure (D, h) and a vector field H such that

1) $D_X H = X$ for all vector field X on M,

2) $L_H h = 0$ where L_H is the Lie differentiation by H.

Let $\mathbf{A}^p(M, E)$ denote the space of E-valued p-forms on M. E being locally constant we can define the exterior differentiation $d : \mathbf{A}^p(M, E) \longrightarrow \mathbf{A}^{p+1}(M, E)$. Let $\mathbf{H}^p(M, E)$ be the p-th cohomology group of the complex $\{\mathbf{A}^p(M, E), d\}$. The following theorem is a generalization of Koszul's theorem [3].

Theorem 4. *Let $\pi_E : E \longrightarrow M$ be a locally constant vector bundle over a compact hyperbolic affine manifold M. Suppose that the vector bundle admits a fiber metric satisfying the following property;*

(C) *there exists a constant $c \neq 0$ such that*

$$H(s_\lambda^i, s_\lambda^j) = c(s_\lambda^i, s_\lambda^j).$$

Then we have

$$\mathbf{H}^p(M, E) = \{0\} \quad (p \geq 1).$$

Using the vector field H the proof is done under the same line as in [2], so it will be omitted.

Let $T_s^r(M)$ be the tensor bundle of type (r, s) over a compact hyperbolic affine manifold M. Then the fiber metric induced by the Hessian metric h satisfies the condition (C) in Theorem 4 where $c = 2(s - r)$. Hence we have

Corollary 5 ([3]). *If $r \neq s$, then we have*

$$\mathbf{H}^p(M, T_s^r(M)) = \{0\} \quad (p \geq 1).$$

Proof of Theorem 3. For each $(\theta, \omega) \in \mathbf{R}^n \times \Omega$ we denote by $\omega\theta$ the image of (θ, ω) by the projection $\mathbf{R}^n \times \Omega \longrightarrow E(\Gamma\backslash\Omega, \rho)$. Then each $\omega \in \Omega$ defines a linear isomorphism $\mathbf{R}^n \ni \theta \longrightarrow \omega\theta \in \pi_E^{-1}(\pi(\omega))$. Let $\{U_\lambda\}$ be an open covering of $\Gamma\backslash\Omega$ satisfying the local triviality on each U_λ, that is, there exists a diffeomorphism

$$\Phi_\lambda : \pi^{-1}(U_\lambda) \ni \omega \longrightarrow (\pi(\omega), \phi_\lambda(\omega)) \in U_\lambda \times \Gamma$$

where ϕ_λ is a mapping of $\pi^{-1}(U_\lambda)$ into Γ such that

$$\phi_\lambda(\gamma^{-1}\omega) = \phi_\lambda(\omega)\gamma$$

for $\omega \in \pi^{-1}(U_\lambda)$, $\gamma \in \Gamma$. Define

$$\Psi_\lambda : \pi_E^{-1}(U_\lambda) \ni \omega\theta \longrightarrow (\pi(\omega), f(\phi_\lambda(\omega))\theta) \in U_\lambda \times \mathbf{R}^n.$$

Then $\{\Psi_\lambda\}$ gives a local triviality for $E(\Gamma\backslash\Omega, \rho)$. By (5) the Hessian metric $g = Dd\psi$ defines a fiber metric on each fiber $\pi_E^{-1}(\pi(\omega)) = \{\omega\theta \mid \theta \in \mathbf{R}^n\}$ by

$$(\omega\theta, \omega\theta') = {}^t\theta\rho(\omega)^{-1}\theta'.$$

Let $s_\lambda^i : U_\lambda \longrightarrow \pi_E^{-1}(U_\lambda)$ be a section given by $s_\lambda^i(u) = \Psi_\lambda^{-1}(u, \mathbf{e}^i)$ where $\mathbf{e}^i$ is a vector in $\mathbf{R}^n$ whose j-th component is δ^{ij}. Then $\{s_\lambda^1, \cdots, s_\lambda^n\}$ is a local frame field of $E(\Gamma\backslash\Omega, \rho)$ over U_λ, and

$$(s_\lambda^i, s_\lambda^j) = {}^t\mathbf{e}^i(\rho \circ \sigma_\lambda)^{-1}\mathbf{e}^j,$$

where σ_λ is a section on U_λ given by $\sigma_\lambda(u) = \Phi_\lambda^{-1}(u, identity)$. Let $\{u_{\lambda 1}, \cdots, u_{\lambda m}\}$ be an affine local coordinate system on U_λ such that $u_{\lambda\alpha} \circ \pi = \omega_\alpha$. The vector field $\tilde{H} = \sum_\alpha \omega_\alpha \partial/\partial\omega_\alpha$ is π-projectable and $H = \pi_*(\tilde{H}) = \sum_\alpha u_{\lambda\alpha}\partial/(\partial u_{\lambda\alpha})$ on U_λ. Since $\sum_\alpha \omega_\alpha \partial\rho(\omega)^{-1}/\partial\omega_\alpha = -\rho(\omega)^{-1}$ we have

$$\begin{aligned}
H(s_\lambda^i, s_\lambda^j) &= {}^t\mathbf{e}^i \left\{ \sum_\alpha u_{\lambda\alpha} \frac{\partial}{\partial u_{\lambda\alpha}} (\rho \circ \sigma_\lambda)^{-1} \right\} \mathbf{e}^j \\
&= {}^t\mathbf{e}^i \left\{ \left(\sum_\alpha \omega_\alpha \frac{\partial}{\partial \omega_\alpha} \rho^{-1} \right) \circ \sigma_\lambda \right\} \mathbf{e}^j \\
&= -{}^t\mathbf{e}^i(\rho \circ \sigma_\lambda)^{-1}\mathbf{e}^j \\
&= -(s_\lambda^i, s_\lambda^j).
\end{aligned}$$

Thus the vector bundle $E(\Gamma\backslash\Omega, \rho)$ admits a fiber metric satisfying the condition (C) of Theorem 4, so the proof of Theorem 3 is completed. $\square$

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