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GEOMETRY ASSOCIATED WITH NORMAL DISTRIBUTIONS

Hirohiko Shima and Jian Hua Hao

(Received June 26, 1998)

1. Introduction

It is known that manifolds of smooth families of probability distributions admit dualistic structures. S. Amari proposed Information Geometry, whose keywords are dualistic connections [1]. Among them the case of dual connections being flat is interesting. Many important families of probability distributions, e.g. exponential families admit flat dual connections.

The notion of flat dual connections is the same with Hessian structures which have being developed from a different viewpoint [9]-[12].

In this paper, for a linear mapping \( p \) of a domain \( \Omega \) into the space of positive definite symmetric matrices we construct an exponential family of probability distributions \( \{ p(x; \theta, \omega) \} \) on \( \mathbb{R}^n \) parametrized by \( \theta \in \mathbb{R}^n \), \( \omega \in \Omega \), and study a Hessian structure on \( \mathbb{R}^n \times \Omega \) given by the exponential family. Such families contain \( n \)-dimensional normal distributions (Example 1) and a family of constant negative curvature (Example 2).

In case of a Lie group acting on \( \Omega \), \( p \) is assumed to be equivariant. O.S. Rothaus and I. Satake studied such a linear mapping \( p \) for homogeneous convex cones [7][8]. Using \( p \) we introduce a Hessian structure on a vector bundle over a compact hyperbolic affine manifold and prove a certain vanishing theorem (Theorem 2).

2. Hessian structures

We first review some fundamental facts on Hessian structures needed in this paper [1][11].

Let \( U \) be an \( n \)-dimensional real vector space with canonical flat connection \( D \). Let \( \Omega \) be a domain in \( U \) with a convex function \( \psi \), i.e. the Hessian \( Dd\psi \) is positive definite on \( \Omega \). Then the metric \( g = Dd\psi \) is called a Hessian metric and the pair \( (D, g) \) a Hessian structure on \( \Omega \). Let \( \{ u_1, \cdots, u_n \} \) and \( \{ u^*_1, \cdots, u^*_n \} \) be dual basis of \( U \) and \( U^* \) (the dual vector space of \( U \)) respectively. We denote by \( \{ x^1, \cdots, x^n \} \) (resp. \( \{ x^*_1, \cdots, x^*_n \} \)) the linear coordinate system with respect to \( \{ u_1, \cdots, u_n \} \) (resp. \( \{ u^*_1, \cdots, u^*_n \} \)). Let \( \iota : \Omega \longrightarrow U^* \) be a mapping given by

\[
x^*_i \circ \iota = \frac{\partial \psi}{\partial x^i},
\]
which will be called the \textit{gradient mapping}. We define a flat affine connection $D'$ on $\Omega$ by

$$\iota_*(D' X) = D_x^* \iota_*(Y),$$

where the right hand side is the covariant differentiation along $\iota$ induced by the flat affine connection $D^*$ on $U^*$. Then we have

$$D' = 2\nabla - D,$$

$$X_g(Y, Z) = g(D_X Y, Z) + g(Y, D_X Z),$$

where $\nabla$ is the Levi-Civita connection of $g$. Putting $x'_i = -\partial \psi / \partial x^i$ we have an affine coordinate system $\{x'_1, \ldots, x'_n\}$ with respect to $D'$. A function $\psi_L$ on $\Omega$ defined by

$$\psi_L = -\sum_i x'_i x^i - \psi$$

is called the \textit{Legendre transform} of $\psi$. Then we have

$$g = D'd\psi_L.$$

The Hessian structure $(D', g = D'd\psi_L)$ is said to be the \textit{dual Hessian structure} of $(D, g = Dd\psi)$. The \textit{divergence} $D$ for the Hessian structure $(D, g = Dd\psi)$ is defined by

$$D(p, q) = \psi(p) + \psi_L(q) - \sum_i x'_i(p)x'_i(q) \quad (p, q \in \Omega).$$

3. Probability distributions and Hessian structures induced by $\rho$

Let $\Omega$ be a domain in a real vector space $V^m$ of dimension $m$. Let $\rho$ be an injective linear mapping of $V^m$ into the space $\mathbb{S}_n$ of $n \times n$ symmetric matrices such that

(A.1) \hspace{1cm} \rho(\omega) \in PD\mathbb{S}_n \quad \text{for} \quad \omega \in \Omega,

where $PD\mathbb{S}_n$ is the set of positive definite matrices in $\mathbb{S}_n$. For column vectors $x, \mu \in \mathbb{R}^n$ and $\omega \in \Omega$ we define a density function of $x \in \mathbb{R}^n$ by

(1) \hspace{1cm} p(x; \mu, \omega) = (2\pi)^{-n/2}(\det \rho(\omega))^{1/2} \exp \left\{-\frac{1}{2}(x - \mu)\rho(\omega)(x - \mu)\right\}.

The family $\{p(x; \mu, \omega) | \mu \in \mathbb{R}^n, \omega \in \Omega\}$ parametrized by $\mu, \omega$ is called the probability distributions induced by $\rho$.

Proposition 1. The probability distributions induced by $\rho$ is an exponential family parametrized by $(\theta, \omega) \in \mathbb{R}^n \times \Omega$ where $\theta = \rho(\omega)\mu$. The Fisher information metric
coincides with the Hessian metric $g = Dd\psi$ where $D$ is the canonical flat connection on $\mathbb{R}^n \times V^m$ and

$$\psi(\theta, \omega) = \frac{1}{2} \left[ \theta \rho(\omega)^{-1} \theta - \log \det \rho(\omega) \right].$$

Proof. For $x = [x^i], \mu = [\mu^i] \in \mathbb{R}^n$ and $\omega = \sum \omega_a v^a \in \Omega$ where $\{v^1, \cdots, v^m\}$ is a basis of $V^m$, we set

$$F^a(x) = -\frac{1}{2} x^i \rho(v^a) x,$$

$$\theta = \rho(\omega) \mu.$$

Then we have

$$p(x; \mu, \omega) = p(x; \theta, \omega) = \exp \left\{ \sum_j \theta_j x^j + \sum_a \omega_a F^a(x) - \psi(\theta, \omega) - \frac{n}{2} \log 2\pi \right\}.$$

This implies that the family $\{p(x; \theta, \omega)\}$ of probability distributions parametrized by $(\theta, \omega) \in \mathbb{R}^n \times \Omega$ is an exponential family, and the Fisher information metric coincides with the Hessian metric $g = Dd\psi$.

A straightforward calculation shows

$$\frac{\partial \psi}{\partial \theta_i} = e^i \rho(\omega)^{-1} \theta,$$

$$\frac{\partial \psi}{\partial \omega_a} = -\frac{1}{2} \left[ \theta \rho(\omega)^{-1} \rho(v^a) \rho(\omega)^{-1} \theta + \text{Tr} \rho(\omega)^{-1} \rho(v^a) \right],$$

$$\left[ \frac{\partial^2 \psi}{\partial \theta_i \partial \theta_j} \right] = \rho(\omega)^{-1},$$

$$\frac{\partial^2 \psi}{\partial \theta_i \partial \omega_a} = -e^i \rho(\omega)^{-1} \rho(v^a) \rho(\omega)^{-1} \theta,$$

$$\frac{\partial^2 \psi}{\partial \omega_a \partial \omega_\beta} = i \theta \rho(\omega)^{-1} \rho(v^a) \rho(\omega)^{-1} \rho(v^\beta) \rho(\omega)^{-1} \theta$$

$$+ \frac{1}{2} \text{Tr} \rho(\omega)^{-1} \rho(v^a) \rho(\omega)^{-1} \rho(v^\beta),$$

where $e^i$ is the vector in $\mathbb{R}^n$ whose $j$-th component is $\delta^{ij}$ (Kronecker's delta). The Legendre transform $\psi_L$ of $\psi$ is

$$\psi_L = \frac{1}{2} \log \det \rho(\omega).$$
Proposition 2. The divergence $D$ of the probability distributions \( \{p(x; \mu, \omega) \mid \mu \in \mathbb{R}^n, \omega \in \Omega \} \) is given by

\[
D(p, q) = \frac{1}{2} \{ (\mu(p) - \mu(q))\rho(\omega(p))(\mu(p) - \mu(q)) + \text{Tr}(\rho(\omega(p))\rho(\omega(q))^{-1}) \\
- \log \det(\rho(\omega(p)))\rho(\omega(q))^{-1} \} - n.
\]

Proof. Using (3), (4) we have

\[
D(p, q) = \sum_{i=1}^{n} (\theta_i(p) - \theta_i(q)) \frac{\partial \psi}{\partial \theta_i}(q) + \sum_{a=1}^{m} (\omega_a(q) - \omega_a(p)) \frac{\partial \psi}{\partial \omega_a}(q) \\
- (\psi(q) - \psi(p)) \\
= \{ \theta(q)\rho(\omega(q))^{-1}\theta(q) - \theta(p)\rho(\omega(q))^{-1}\theta(q) \} \\
- \frac{1}{2} \{ \theta(q)\rho(\omega(q))^{-1}\theta(q) + \text{Tr}(\rho(\omega(q))^{-1}\rho(\omega(q))) \\
- \theta(q)\rho(\omega(q))^{-1}\rho(\omega(p))\rho(\omega(q))^{-1}\theta(q) - \text{Tr}(\rho(\omega(q))^{-1}\rho(\omega(p))) \\
- \frac{1}{2} \{ \theta(q)\rho(\omega(q))^{-1}\theta(q) - \log \det(\rho(\omega(q))) \\
- \theta(p)\rho(\omega(p))^{-1}\theta(p) + \log \det(\rho(\omega(p))) \} \\
= \frac{1}{2} \{ (\mu(p) - \mu(q))\rho(\omega(p))(\mu(p) - \mu(q)) + \text{Tr}(\rho(\omega(p))\rho(\omega(q))^{-1}) \\
- \log \det(\rho(\omega(p)))\rho(\omega(q))^{-1} \} - n.
\]

\[\square\]

Example 1. Let $\Omega$ be the set of positive definite matrices in $\mathfrak{S}_n$, and let $\rho : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ be the identity mapping. Then $\{p(x; \mu, \omega)\}$ is the family of $n$-dimensional normal distributions. Then we have

\[
\psi = \frac{1}{2} (\theta \omega^{-1} \theta - \log \det \omega),
\]

for $\theta = [\theta_i] \in \mathbb{R}^n$, $\omega = [\omega_{ij}] \in \Omega$. Setting

\[
\eta^i = -\frac{\partial \psi}{\partial \theta_i} = -\xi^i \omega^{-1} \theta, \quad \xi^{ij} = -\frac{\partial \psi}{\partial \omega_{ij}} = \omega^{ij} + \frac{1}{2} \eta^i \eta^j
\]

where $[\omega^{ij}] = [\omega_{ij}]^{-1}$, we obtain

\[
\theta = -\left[ \xi^{ij} - \frac{1}{2} \eta^i \eta^j \right]^{-1} \eta, \quad \omega = \left[ \xi^{ij} - \frac{1}{2} \eta^i \eta^j \right]^{-1}.
\]
The image of the gradient mapping is the set of \((\eta, \xi) \in \mathbb{R}^n \times \mathbb{G}_n\) satisfying

\[ \xi - \frac{1}{2} \eta' \eta > 0. \]

This domain is an affine analogy of Siegel domains. Such domains have been treated in [2][14] in the course of the realization of homogeneous convex domains. The Legendre transform \(\psi_L\) of \(\psi\) is

\[ \psi_L(\eta, \xi) = -\frac{1}{2} \log \det \left( \xi - \frac{1}{2} \eta' \eta \right). \]

**Example 2.** Let \(\mathbb{R}^+\) be the set of positive numbers and let \(\rho\) be a linear mapping of \(\mathbb{R}\) into \(\mathbb{G}_n\) given by \(\rho(\omega) = \omega E_n\) where \(E_n\) is the unit matrix. Then

\[ \psi = \frac{1}{2} \left( \frac{1}{\omega} \theta \theta' - n \log \omega \right), \]

\[ \eta' = -\frac{\partial \psi}{\partial \theta_i} = -\frac{\theta_i}{\omega}, \quad \xi = -\frac{\partial \psi}{\partial \omega} = \frac{1}{2} \left( \frac{\theta \theta'}{\omega^2} + \frac{n}{\omega} \right), \]

\[ \theta = -\frac{n}{2} \eta \left( \xi - \frac{1}{2} \eta' \eta \right)^{-1}, \quad \omega = \frac{n}{2} \left( \xi - \frac{1}{2} \eta' \eta \right)^{-1}. \]

The image of the gradient mapping is a domain lying above a paraboloid;

\[ \left\{ (\xi, \eta) \in \mathbb{R} \times \mathbb{R}^n \mid \xi - \frac{1}{2} \eta' \eta > 0 \right\}. \]

The Legendre transform \(\psi_L\) of \(\psi\) is

\[ \psi_L = -\frac{n}{2} \log \left( \xi - \frac{1}{2} \eta' \eta \right) + n \log \frac{n}{2}. \]

By [12] the Hessian sectional curvature of the Hessian structure \((D', g = D'd\psi_L)\) is \(2/n\). This implies that the space of probability distributions on \(\mathbb{R}^n\) defined by

\[ p(x; \mu, \omega) = \left( \frac{\omega}{2\pi} \right)^{n/2} \exp \left\{ -\frac{\omega}{2} (x - \mu)(x - \mu) \right\} \]

where \(\mu \in \mathbb{R}^n\), \(\omega \in \mathbb{R}^+\) is the space of constant curvature \(-1/(2n)\) with respect to the Fisher information metric.

In case of a Lie subgroup \(G\) of \(GL(V^m)\) acting on \(\Omega\) we assume further that \(G\) admits a matrix representation \(f\) such that

\[ \rho(s\omega) = f(s)\rho(\omega)'f(s) \quad \text{for } s \in G, \omega \in V^m. \]
Then $G$ acts on $\mathbb{R}^n \times \Omega$ by $s(\theta, \omega) = (f(s)\theta, s\omega)$. Since $\psi(f(s)\theta, s\omega) - \psi(\theta, \omega)$ is a constant, the Hessian metric $g = Dd\psi$ is $G$-invariant.

O.S. Rothaus studied the case of $\Omega$ being a homogeneous convex cone and $\rho$ satisfying (A.1), (A.2). He showed that the set $(\xi, \theta, \omega) \in \mathbb{R} \times \mathbb{R}^n \times \Omega$ fulfilling $\xi - \theta \rho(\omega)^{-1}\theta > 0$ is a homogeneous convex cone, and that all homogeneous convex cones are obtained from lower dimensional ones in this manner [7].

**Example 3.** Let $M(n, \mathbb{R})$ be the set of all $n \times n$ matrices, and let $\rho$ be a linear mapping of $\mathfrak{S}_n$ into the space $\text{End}(M(n, \mathbb{R}))$ of endomorphisms of $M(n, \mathbb{R})$ given by

$$\rho(\omega)x = \omega x + x \omega$$

for $\omega \in \mathfrak{S}_n$, $x \in M(n, \mathbb{R})$. Then $\rho(\omega)$ is symmetric with respect to the inner product $\langle x, y \rangle = \text{Tr} x^\dagger y$, and positive definite for positive definite matrix $\omega$. Let $f$ be a representation of $O(n)$ on $M(n, \mathbb{R})$ defined by

$$f(s)x = sx^\dagger s.$$ 

Then $f(s)x = f(s)x^\dagger s$ and

$$\rho(f(s)\omega) = f(s)\rho(\omega)^\dagger f(s).$$

Setting $\mathfrak{A}_n = \{x \in M(n, \mathbb{R}) \mid x^\dagger = -x\}$, we have $M(n, \mathbb{R}) = \mathfrak{S}_n + \mathfrak{A}_n$, and

$$\rho(\omega)\mathfrak{S}_n \subset \mathfrak{S}_n, \quad \rho(\omega)\mathfrak{A}_n \subset \mathfrak{A}_n.$$ 

Hence $\rho$ induces an equivariant linear mapping $\rho^+$ and $\rho^-$ of $\mathfrak{S}_n$ into $\text{End}(\mathfrak{S}_n)$ and $\text{End}(\mathfrak{A}_n)$ respectively. The Hessian structure on $\mathfrak{A}_n \times PD\mathfrak{S}_n$ induced by $\rho^-$ is related to the theory of stable state feedback systems [4][5].

4. **Vector bundles over compact hyperbolic affine manifolds**

A flat affine manifold $M$ is said to be hyperbolic if the universal covering of $M$ is affinely isomorphic to an open convex cone not containing full straight line [3]. Hence a compact hyperbolic affine manifold is expressed by $\Gamma \backslash \Omega$ where $\Omega$ is an open convex cone with vertex 0 in $V^m$ not containing full straight line, and $\Gamma$ is a discrete subgroup of $GL(V^m)$ acting properly discontinuously and freely on $\Omega$. Suppose that a compact hyperbolic affine manifold $\Gamma \backslash \Omega$ admits a linear mapping $\rho$ of $V^m$ into $\mathfrak{S}_n$ satisfying the conditions (A.1) and (A.2). We denote by $\pi_E : E(\Gamma \backslash \Omega, \rho) \rightarrow \Gamma \backslash \Omega$ the vector bundle over $\Gamma \backslash \Omega$ associated with the universal covering $\pi : \Omega \rightarrow \Gamma \backslash \Omega$ and $\rho$. Since the Hessian structure $(D, g = Dd\psi)$ on $\mathbb{R}^n \times \Omega$ is $\Gamma$-invariant, it induces a Hessian structure on the vector bundle $E(\Gamma \backslash \Omega, \rho)$. The Hessian metric defines a fiber metric on each fiber $\pi_E^{-1}(\pi(\omega)) = \{\omega \theta \mid \theta \in \mathbb{R}^n\}$ by

$$(\omega \theta, \omega \theta') = \theta \rho(\omega)^{-1}\theta'.$$
Using this fiber metric we prove

**Theorem 3.** The $p$-th cohomology group of the complex of $E(\Gamma\backslash\Omega, \rho)$-valued forms on $\Gamma\backslash\Omega$ vanishes for $p \geq 1$.

Theorem 3 is generalized as the following Theorem 4.

Let $\pi_E : E \rightarrow M$ be a locally constant vector bundle over a compact hyperbolic affine manifold $M$. Then there exists an open covering $\{U_\lambda\}$ of $M$ admitting

(i) affine coordinate system $\{u_{\lambda,1}, \ldots, u_{\lambda,m}\}$ on $U_\lambda$,

(ii) local frames $\{s_1^\lambda, \ldots, s_m^\lambda\}$ on $U_\lambda$ whose transition functions are constants.

The universal covering of $M$ being a convex cone, $M$ admits a Hessian structure $(D, h)$ and a vector field $H$ such that

1) $D_X H = X$ for all vector field $X$ on $M$,

2) $L_H h = 0$ where $L_H$ is the Lie differentiation by $H$.

Let $A^p(M, E)$ denote the space of $E$-valued $p$-forms on $M$. $E$ being locally constant we can define the exterior differentiation $d : A^p(M, E) \rightarrow A^{p+1}(M, E)$. Let $H^p(M, E)$ be the $p$-th cohomology group of the complex $\{A^p(M, E), d\}$. The following theorem is a generalization of Koszul's theorem [3].

**Theorem 4.** Let $\pi_E : E \rightarrow M$ be a locally constant vector bundle over a compact hyperbolic affine manifold $M$. Suppose that the vector bundle admits a fiber metric satisfying the following property:

(C) there exists a constant $c \neq 0$ such that

$$H(s_1^\lambda, s_2^\lambda) = c(s_1^\lambda, s_2^\lambda).$$

Then we have

$$H^p(M, E) = \{0\} \quad (p \geq 1).$$

Using the vector field $H$ the proof is done under the same line as in [2], so it will be omitted.

Let $T^r_s(M)$ be the tensor bundle of type $(r, s)$ over a compact hyperbolic affine manifold $M$. Then the fiber metric induced by the Hessian metric $h$ satisfies the condition (C) in Theorem 4 where $c = 2(s - r)$. Hence we have

**Corollary 5 ([3]).** If $r \neq s$, then we have

$$H^p(M, T^r_s(M)) = \{0\} \quad (p \geq 1).$$

Proof of Theorem 3. For each $(\theta, \omega) \in \mathbb{R}^n \times \Omega$ we denote by $\omega \theta$ the image of $(\theta, \omega)$ by the projection $\mathbb{R}^n \times \Omega \rightarrow E(\Gamma\backslash\Omega, \rho)$. Then each $\omega \in \Omega$ defines a linear
isomorphism $R^n \ni \theta \rightarrow \omega \theta \in \pi_E^{-1}(\pi(\omega))$. Let $\{U_\lambda\}$ be an open covering of $\Gamma \backslash \Omega$ satisfying the local triviality on each $U_\lambda$, that is, there exists a diffeomorphism

$$
\Phi_\lambda : \pi^{-1}(U_\lambda) \ni \omega \rightarrow (\pi(\omega), \phi_\lambda(\omega)) \in U_\lambda \times \Gamma
$$

where $\phi_\lambda$ is a mapping of $\pi^{-1}(U_\lambda)$ into $\Gamma$ such that

$$
\phi_\lambda(y^{-1}\omega) = \phi_\lambda(\omega) \gamma
$$

for $\omega \in \pi^{-1}(U_\lambda)$, $\gamma \in \Gamma$. Define

$$
\Psi_\lambda : \pi_E^{-1}(U_\lambda) \ni \omega \theta \rightarrow (\pi(\omega), f(\phi_\lambda(\omega))\theta) \in U_\lambda \times R^n.
$$

Then $\{\Psi_\lambda\}$ gives a local triviality for $E(\Gamma \backslash \Omega, \rho)$. By (5) the Hessian metric $g = Dd\psi$ defines a fiber metric on each fiber $\pi_E^{-1}(\pi(\omega)) = \{\omega \theta | \theta \in R^n\}$ by

$$(\omega \theta, \omega \theta') = \theta \rho(\omega)^{-1} \theta'.$$

Let $s^i_\lambda : U_\lambda \rightarrow \pi_E^{-1}(U_\lambda)$ be a section given by $s^i_\lambda(u) = \Psi_\lambda^{-1}(u, e^i)$ where $e^i$ is a vector in $R^n$ whose $j$-th component is $\delta^i_j$. Then $\{s^1_\lambda, \ldots, s^n_\lambda\}$ is a local frame field of $E(\Gamma \backslash \Omega, \rho)$ over $U_\lambda$, and

$$(s^i_\lambda, s^j_\lambda) = \epsilon^i (\rho \circ \sigma_\lambda)^{-1} e^j,$$

where $\sigma_\lambda$ is a section on $U_\lambda$ given by $\sigma_\lambda(u) = \Phi_\lambda^{-1}(u, identity)$. Let $\{u_{\lambda 1}, \ldots, u_{\lambda m}\}$ be an affine local coordinate system on $U_\lambda$ such that $u_{\lambda \alpha} \circ \pi = \omega_\alpha$. The vector field

$$
\tilde{H} = \sum \omega_\alpha \partial / \partial \omega_\alpha
$$

is $\pi$-projectable and $H = \pi_* (\tilde{H}) = \sum u_{\lambda \alpha} \partial / (\partial u_{\lambda \alpha})$ on $U_\lambda$. Since $\sum \omega_\alpha \partial \rho(\omega)^{-1} / \partial \omega_\alpha = -\rho(\omega)^{-1}$ we have

$$
H(s^i_\lambda, s^j_\lambda) = \epsilon^i \left\{ \sum_{\alpha} \left. u_{\lambda \alpha} \frac{\partial}{\partial u_{\lambda \alpha}} (\rho \circ \sigma_\lambda)^{-1} \right\} e^j
$$

$$
= \epsilon^i \left\{ \sum_{\alpha} \omega_\alpha \frac{\partial}{\partial \omega_\alpha} \rho^{-1} \circ \sigma_\lambda \right\} e^j
$$

$$
= -\epsilon^i (\rho \circ \sigma_\lambda)^{-1} e^j
$$

$$
= -(s^i_\lambda, s^j_\lambda).
$$

Thus the vector bundle $E(\Gamma \backslash \Omega, \rho)$ admits a fiber metric satisfying the condition (C) of Theorem 4, so the proof of Theorem 3 is completed. \hfill \Box
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