An analysis of nonlinear partial differential equations: quasilinear elliptic problems and semilinear parabolic problems

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Abstract

This thesis is devoted to the study of quasilinear elliptic partial differential equations and the semilinear parabolic differential equations. Both elliptic and parabolic partial differential equations are known to have applications in various fields outside of mathematics. Hence, it is necessary to develop a deep understanding for the behavior of solutions to the aforementioned type of partial differential equations to develop a strong mathematical framework which can be useful for mathematical models used in science and engineering fields for practical purposes. We focus on quasilinear elliptic partial differential equations in Chapter 2 and proceed to focus on semilinear parabolic equations in Chapter 3 and Chapter 4.

The first study in Chapter 2 aims to establish the existence of functions which can be represented by its generalized mean value that is also known as p-mean value. Here, we use Perron's Method suitable for our problem to establish the existence of the aforementioned functions which are called (variationally) p-harmonious functions. The second aim of this study is to show the uniform convergence of p-harmonious functions to p-harmonic functions for game-theoretic p-Laplace equation. We obtain this result by using an appropriate approximation scheme. When $p = 2$, our results here revert to the asymptotic mean value property for harmonic functions.

The second study in Chapter 3 aims to obtain the blow-up rate of semilinear heat equations with subcritical nonlinear term under Ambrosetti-Rabinowitz condition in an unbounded domain. We use variable transformation and parabolic argument to obtain our main results here. Moreover, we also extend the results from single equation case to a system of equations.

The third study in Chapter 4 aims to establish the existence of time-global solutions to a system of semilinear heat equations with subcritical nonlinearity under Ambrosetti-Rabinowitz condition in a bounded smooth domain. We use the compactness of the orbit in scale-invariant Lebesgue space and blow-up argument to obtain our results. For single equation, this method is also applicable for the critical case in the sense of Sobolev embedding.

Finally, we give conclusions for each study and explain possible future research based on the results presented here in Chapter 5.

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Chapter 1

Introduction

1.1 Background and motivation

This thesis is devoted to the study of both elliptic and parabolic partial differential equations which are known to be useful to provide mathematical framework for modeling various diffusion and heat transfer phenomena in physics and engineering. In fact, both elliptic and parabolic partial differential equations have a wide range of applications in physics and engineering. For instance, diffusion phenomenon and heat transfer can be modeled by both elliptic and parabolic partial differential equations (see e.g. [24, Chapter 2, pp. 17]).

In the case of elliptic partial differential equations, Laplace equation is widely used in physics to model the density of some quantity in equilibrium which is usually independent of time. More generally speaking, Laplace equation and Poisson equation are used in electrostatics and Einstein's theory of relativity. For an example of the use of Laplace equation and Poisson equation in electrostatics, see [27, Chapter 12]. As for an application in Einstein's theory of relativity, Persides solves the Laplace equation in Schwarzschild's space-time by using separation of variables in [76].

From the point of view of mathematics, Laplace equation has several nice properties. Particularly, the solutions to Laplace equation (also known as harmonic functions) satisfy the so called mean value property. On the other hand, Laplace equation can be generalized further into p-Laplace equation by replacing the usual Laplace operator Δ with p-Laplace operator. When $p \in (1, \infty)$, we can have the (formal) formulation of p-Laplace operator Δ_p as follows:

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

Benedikt, Girg, Kotrla, and Takáč give a historical exposition for the origin of p-Laplace equation and its applications outside of pure mathematics in [10]. This essentially gives a motivation from the point of view of application to study p-Laplace equation (see [9, Section 5.2, 5.10, 5.11, and 9.4] and [23] for further readings).

Naturally, we are led to ask the question whether the solutions to p-Laplace equation satisfy mean value property at least asymptotically. This problem has garnered a lot of attentions from researchers recently starting from the work of Manfredi, Parviainen, and Rossi in [62]. However, they use a normalized version of p-Laplace operator which is also known as *game-theoretic* (or *homogeneous*) *p-Laplace operator* Δ_p^G. Formally, game-theoretic p-Laplace operator can be defined as follows:

$$\Delta_p^G u = \frac{\operatorname{div}(|\nabla u|^{p-2}\nabla u)}{|\nabla u|^{p-2}} \text{ for } p \in [1,\infty), \quad \Delta_\infty^G u = \frac{\langle \nabla^2 u \nabla u, \nabla u\rangle}{|\nabla u|^2}$$

where $\nabla^2 u$ is the Hessian matrix of u. In fact, game-theoretic p-Laplace operator has an application in game theory. For further development of asymptotic mean value property of game-theoretic p-Laplace equation, see [61, 63, 79]. As for the applications in game theory, see [25, 39].

For parabolic differential equations, it is well-known for its application in heat transfer especially for general heat conduction which is described by the so called *Fourier-Biot equation* (see e.g. [58, pp. 49–98]). Other applications for parabolic equations can also be found in diffusion. Particularly, the so called *heat equations* (also *known as diffusion equations*)

$$\partial_t u = \Delta u$$

also appear in the study of Brownian motion (see [24, Chapter 2 pp. 44]).

From the point of view of mathematics, however, linear heat equations are relatively well-understood compared to nonlinear heat equations. Moreover, some natural phenomena cannot be described well by only using linear mathematical model. For instance, Bozzini, Monti, and Sgura use the so called *semilinear heat equations* to model Turing type electrochemical phase formation dynamics in [12]. Particularly, we are interested to study semilinear heat equations in the following form:

$$\partial_t u = \Delta u + f(u). \tag{1.1}$$

The existence of time-local solutions is typically obtained from fixed point argument (see e.g. [15, Section 5.2 pp. 64]). When $f(u) = u|u|^{p-2}$ and $p \in (2, 2^*)$ where 2^* is the critical exponent in the sense of Sobolev embedding, there are various results related to the behavior of solutions of (1.1). For instance, it is well-known that blow-up alternative occurs (see e.g. [15, Theorem 4.3.4 (ii), pp. 58]), i.e., time local solutions blow up in finite time under appropriate norm.

The natural question from blow-up alternative is to determine the blow-up rate of such solutions. For example, let us consider the following Dirichlet boundary problem:

$$\partial_t u = \Delta u - u|u|^{p-2} \text{ in } \mathbb{R}^N \times (0, T_m) \text{ and } u \text{ remains bounded as } |x| \to \infty, \quad (1.2)$$

where T_m denotes the maximal time existence of the solution itself. Giga and Kohn obtain the blow-up rate for the solutions to problem (1.2) which blows up in finite time in L^∞ (see [36]). Their results for blow-up rate can be written as follows:

Theorem 1.1.1 (Blow-up rate, Theorem 3.1 in [36])
Assume $p \in (2, \frac{6N+4}{3N-4})$ *or* $N = 1$*, then we have*

$$\sup_{\mathbb{R}^N \times [0,T_m)} (T_m - t)^{\frac{1}{p-2}} |u(x,t)| < \infty.$$

The theorem above tells us that the blow-up rate of the solution is controlled by p in the nonlinear term of (1.2). In fact, Giga and Kohn show that their results cover a system of semilinear heat equations and can be extended to a more general nonlinear term with polynomial principal term, see [36, Section 6 pp.30] for details. Thus, it is interesting to see whether we can find a nonlinearity without polynomial principal term which is not covered by Giga and Kohn in [36]. Additionally, we are interested to extend this result into a system of partial differential equations. These problems will be addressed in Chapter 2.

Another natural question from blow-up alternative is under what condition the solutions of (1.1) does not blow up in finite time. This leads us to consider under what kind of conditions that the solutions to (1.1) become time-global. Although we have been discussing blow-up in L^∞ up to this point, we are going to discuss the blow-up in H^1 to approach the aforementioned problem from another point of view. First, we consider the following Dirichlet Boundary condition.

$$\begin{cases} \partial_t u = \Delta u + u|u|^{p-2} & \text{in } \Omega \times (0, T_m), \\ u = 0 & \text{on } \partial\Omega \times (0, T_m), \\ u(\,\cdot\,, 0) = u_0 & \text{in } \Omega, \end{cases} \quad (1.3)$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ and $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is the initial data which is chosen for simplicity. One of the methods to determine whether the solutions of (1.3) are time-local or time-global is the *potential well argument* which was introduced by Payne and Sattinger in [73]. First, we begin with some energy functionals associated with (1.3) as follows:

$$I[u] = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p \tag{1.4}$$

and

$$J[u] = \|\nabla u\|_2^2 - \|u\|_p^p. \tag{1.5}$$

for any $u \in H_0^1(\Omega)$. Next, we define the *Nehari manifold* and *potential depth* respectively as follows:

$$\mathscr{N} := \{u \in H_0^1(\Omega) \mid J[u] = 0,\ u \neq 0\}, \tag{1.6}$$

$$d := \inf_{u\in\mathscr{N}} I[u]. \tag{1.7}$$

It is not difficult to see that $\mathscr{N}$ is not empty and $d > 0$. For details, see [73, pp.281–284]. Following this, we define the stable set and unstable set respectively as follows:

$$W^* = \{u \in H_0^1(\Omega) \mid I[u] < d, J[u] > 0\} \cup \{0\}, \tag{1.8}$$

$$V^* = \{u \in H_0^1(\Omega) \mid I[u] < d, J[u] < 0\} \cup \{0\}. \tag{1.9}$$

The following theorems given by Ikehata and Suzuki in [44] will give us a classification of solutions to (1.3).

Theorem 1.1.2 (Theorem 3.1 in [44]) *Let u be a time-local solution to the problem (1.3) with initial data $u_0 \in H_0^1(\Omega)$. Then, there exists a time $t_0 \in [0, T_m)$ such that $u(t_0) \in W^*$ if and only if $T_m = \infty$ and $\lim_{t\to\infty} \|\nabla u(t)\|_2 = 0$.*

Theorem 1.1.3 (Theorem 3.2 in [44]) *Let u be a time-local solution to the problem (1.3) with initial data $u_0 \in H_0^1(\Omega)$. Suppose that either $u_0 \geq 0$ or Ω is a convex set. Then, there exists a time $t_0 \in [0, T_m)$ such that $u(t_0) \in W^*$ if and only if $T_m = \infty$ and $\lim_{t\to\infty} \|\nabla u(t)\|_2 = 0$.*

Both Theorem 1.1.2 and Theorem 1.1.3 imply that the solutions to (1.3) decay in H_0^1-norm as $t \to \infty$ for sufficiently small initial data in the sense of H_0^1-norm and blow-up in finite time in H_0^1-norm as $t \to T_m$ for sufficiently large initial data in the sense of H_0^1-norm. See Remark 3.3 in [44] for details.

On the other hand, there is a possibility that the solutions to (1.3) do not enter neither W^* nor V^*. At the very least, we know that when the initial data is positive or the domain is convex, any solution which does not enter neither W^* nor V^* is a time-global solution, see [44, Corollary 3.6, pp. 482]. However, we do not know the behavior of such a solution. The answer to this problem is known in one dimensional space at least when the spatial domain is $\mathbb{R}_+$ for some initial data. In order to have a better understanding of the situation in one dimensional space, we will recall the results of Fašangová and Feireisl in [26].

Let us consider

$$\begin{cases} \partial_t u = \partial_{xx} u + u|u|^{p-2} & \text{in } \mathbb{R}_+ \times (0,\infty), \\ u(\,\cdot\,,0) = u_0 & \text{in } \mathbb{R}_+ \end{cases} \tag{1.10}$$

where $u_0 \in H^1_0(\mathbb{R}_+)$ with either a homogeneous Dirichlet or Neumann boundary condition at $x = 0$ as follows:

$$u(0,\,\cdot\,) = 0 \text{ in } (0,\infty) \tag{1.11}$$

or

$$\partial_x u(0,\,\cdot\,) = 0 \text{ in } (0,\infty). \tag{1.12}$$

One of the reasons to only consider nonnegative solutions is due to the relevancy to applications such as models of heat propagation along a rod, population dynamics, and flame propagation in chemical reactor theory, see [3] for details.

Next, we consider a one dimensional semilinear elliptic equation as follows:

$$-w'' + w|w|^{p-2} = 0 \text{ in } \mathbb{R}, \quad w \in C_0(\mathbb{R}), \quad 0 \neq \max w = w(0). \tag{1.13}$$

The unique solution to the problem (1.13) w_g is also known as *the ground state* for (1.10).

Now, we are ready to consider the behavior of solutions to (1.10) which do not enter neither W^* nor V^*.

Theorem 1.1.4 *[See Theorem 1.1 and Remark in [26]] Suppose* $\bar{u} \in H^1_0(\mathbb{R}_+)$ *is nonnegative, nondecreasing in* $[0,\gamma_0]$*, nonincreasing in* $[\gamma_0,\infty)$ *for some positive number* γ_0*, and* $\bar{u} \neq 0$*. Then, there exists* λ_N*,* λ_D*, and* $\lambda_{D'}$ *where* $0 < \lambda_N < \lambda_D \leq \lambda_{D'} < \infty$ *such that if* u^D *(or* u^N *respectively) is a solution to* (1.10) *and* (1.11) *(or* (1.12) *respectively) with the initial condition*

$$u(0) = \lambda\bar{u},$$

then we have

(i) if $0 \leq \lambda < \lambda_D$ *(or* $0 \leq \lambda < \lambda_N$ *respectively), then*

$$\lim_{t\to\infty} u^D(t) = 0 \text{ (or } \lim_{t\to\infty} u^N(t) = 0 \text{ respectively) in } H^1_0(\mathbb{R}_+); \tag{1.14}$$

(ii) *if* $\lambda = \lambda_N$, *then*

$$\lim_{t\to\infty} u^N(t) = w_g \ \textit{in} \ H_0^1(\mathbb{R}_+); \tag{1.15}$$

(iii) *if* $\lambda \in [\lambda_D, \lambda_{D'}]$, *then there exists a function* $y(t)$, *independent of* λ, *satisfying* $\lim_{t\to\infty} y(t) = \infty$, *such that*

$$\lim_{t\to\infty} [u^D(\,\cdot\,, t) - w_g(\,\cdot\, - y(t))] = 0 \ \textit{in} \ H_0^1(\mathbb{R}_+), \tag{1.16}$$

where w_g *is the ground state;*

(iv) *if* $\lambda > \lambda_N$, *then*

$$\lim_{t\to\infty} u^N(t) = \infty \ \textit{in} \ H_0^1(\mathbb{R}_+); \tag{1.17}$$

(v) *if* $\lambda > \lambda_{D'}$, *then*

$$\lim_{t\to\infty} u^D(t) = \infty \ \textit{in} \ H_0^1(\mathbb{R}_+). \tag{1.18}$$

Roughly speaking, the solution with medium size initial data in the sense of H^1-norm converges to a traveling pseudo-wave of the ground state function as $t \to \infty$. This relation connects the notion of parabolic partial differential equations with elliptic partial differential equations from mathematical point of view. Certainly, intuitively speaking, elliptic differential equations, particularly Laplace equation and Poisson equation can be regarded as the steady-state of a certain semilinear heat equation. The work of Fašangová and Feireisl in [26] emphasizes the connection between parabolic and elliptic partial differential equations. Unfortunately, we still do not have a clear criterion for time-global solutions in higher spatial dimension up to this point. This leads us to the following problem.

Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain, $\lambda \in \mathbb{R}$, and $p \in (2, 2^*]$ where 2^* is the critical Sobolev exponent in the sense of Sobolev embedding. Consider the following semilinear heat equations:

$$\begin{cases} \partial_t u = \Delta u + \lambda u + u|u|^{p-2} & \text{in } \Omega \times (0, T_m), \\ u = 0 & \text{on } \partial\Omega \times \times (0, T_m), \\ u(\,\cdot\, 0) = u_0 & \text{in } \Omega \end{cases} \tag{1.19}$$

where the initial data $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ for simplicity and T_m is the maximal time existence of solution. We only consider time-global solution, i.e., $T_m = \infty$. Here, we say that the time-global solution to (1.19) has an L^∞-global bound if and only if

$$\sup_{t\in[0,\infty)} \|u(t)\|_\infty < \infty.$$

In one dimensional spatial space, the compact Sobolev embedding $H_0^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ allows us to obtain L^∞-global bounds from the boundedness of time-global solutions in H^1-norm. Unfortunately, the lack of compact embedding in higher dimensions becomes a major issue to establish the existence of L^∞-global bounds. For subcritical case, i.e., $p \in (2, 2^*)$, a lot of works have been put into the existence of L^∞ time-global bounds, see e.g. [16, 71, 37, 77, 34]. However, the method being used in the aforementioned results rely on the subcriticality of the nonlinear term. Hence, another framework which can unify and cover both subcritical and critical case is necessary.

Ishiwata suggests a different method in [47] which is based on the compactness of the solution orbit in the scale-invariant Lebesgue space and the blow-up argument. Particularly, he obtains the necessary and sufficient condition for the existence of L^∞-global bounds for time-global solutions to the problem (1.19). Before we cite his result concerning L^∞-global bounds, we will need some mathematical notions from variational analysis (see [85] and [89] for reference).

First, we define the energy functionals associated with (1.19) as follows:

$$\tilde{I}[u] = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\|u\|_2^2 - \frac{1}{p}\|u\|_p^p \tag{1.20}$$

for any $u \in H_0^1(\Omega)$. Next, we introduce the notion of *Palais-Smale sequence*, see [85, pp. 70] for reference. Let

$$\mathscr{S} := \{u(t) \,|\, t \in [0, \infty)\}$$

be the orbit of solutions to the problem (1.19) and a subset of $H_0^1(\Omega)$. A sequence $(u_n)_{n\in\mathbb{N}} \subset \mathscr{S}$ is said to be a *Palais-Smale sequence of $\tilde{I}$ at level L in $\mathscr{S}$* (denoted as *$(PS)_L$ sequence in $\mathscr{S}$*) if it satisfies

(PS1) $\tilde{I}[u_n] \to L$ as $n \to \infty$,

(PS2) $(d\tilde{I})_{u_n} \to 0$ as $n \to \infty$ in $(H_0^1(\Omega))^*$ where $(d\tilde{I})_{u_n}$ is the Fréchet derivative of $\tilde{I}$ at u_n in $H_0^1(\Omega)$ and $(H_0^1(\Omega))^*$ denotes the dual space of $H_0^1(\Omega)$.

Following this, we introduce the *Palais-Smale condition*. The energy functional $\tilde{I}$ is said to satisfy *Palais-Smale condition at level* L *in* $\mathscr{S}$ (denoted as $(PS)_L$ *condition in* $\mathscr{S}$) if and only if

every $(PS)_L$ sequence in $\mathscr{S}$ has a strong convergence subsequence in $H_0^1(\Omega)$.

Now, we will cite the main result in [47].

Theorem 1.1.5 (Theorem 2.1 in [47]) *Let* $p \in (2, 2^*]$. *Then, for every time-global solution to* (1.19)*, the following two statements are equivalent.*

(a) *The energy functional* $\tilde{I}$ *satisfies the* $(PS)_L$ *condition in* $\mathscr{S}$*.*

(b) *The time-global solution* u *to the problem* (1.19) *has an* L^∞*-global bounds, i.e.,* $\sup_{t\in[0,\infty)} \|u(t)\|_\infty < \infty$.

Particularly, when p is subcritical in the sense of Sobolev embedding, the time-global solution to (1.19) has an L^∞-global bounds by [47, Corollary 2.1, pp. 1025]. Moreover, when $p = 2^*$, an example of the situation in which the time-global solution has an an L^∞-global bounds is also given by Ishiwata in [47, Corollary 2.2 and Remark 2.1, pp. 1025]. It is also confirmed that there exists an unbounded time-global solution which means the solution does not have an L^∞-global bounds in [47, Corollary 2.4, pp. 1026].

As a natural progression, we are interested in obtaining a similar result for a system of semilinear heat equations with more general nonlinearity. This problem will be discussed in Chapter 4.

1.2 Thesis outline

This thesis aims to analyze and to study the behavior of solutions to quasilinear elliptic and semilinear parabolic partial differential equations. Particularly for the quasilinear elliptic partial differential equations, we want to investigate the relation between a function which satisfies a generalized mean value property and a solution to game-theoretic p-Laplace equation. As for the semilinear parabolic differential equations, we have two aims in this thesis. The first aim is to find the blow-up rate of a certain type of semilinear heat equation which can be extended into a system of equations with nonlinearity which is not covered by Giga and Kohn in [36] with $\mathbb{R}^N$ as the spatial domain. Finally, the last aim is to establish the existence of L^∞-global bounds for a system of semilinear heat equations with a more general nonlinear term based on the method proposed by Ishiwata in [47].

Chapter 1 serves as an introduction which explains our research background and motivation. Moreover, we also briefly mention several examples of applications of elliptic and parabolic partial differential equations especially in physics and engineering to emphasize the importance of our research. We also discuss several important results which are related to the aims of this thesis.

Chapter 2 is devoted to the study quasilinear elliptic differential equations in the form of game-theoretic p-Laplace equation. Particularly, we are interested to find a generalization of mean value property which holds for harmonic functions in the form of asymptotic mean value property. There are two important results in this chapter. First, we establish the existence of functions which satisfies generalized mean value property which is called as (variationally) p-harmonious functions. The second result is to establish the relation between a p-harmonious function and the solution to game-theoretic p-Laplace equation (also known as p-harmonic function) by using an approximation scheme developed by Barles and Souganidis in [8].

Chapter 3 is devoted to the study of semilinear heat equations in $\mathbb{R}^N$. To be more precise, we are interested in analyzing the blow-up rate of a semilinear heat equations with subcritical nonlinear term under the Ambrosetti-Rabinowitz condition in $\mathbb{R}^N$. We mainly use similar variable transformation and parabolic argument to obtain the main results in this chapter. Finally, we extend our results to a system of semilinear heat equations in the last section of this chapter.

Chapter 4 is devoted to the study of a system of semilinear heat equations in a bounded smooth domain. We use the method developed by Ishiwata in [47] to establish the existence of L^∞-global bounds for the time-global solutions to a system of semilinear heat equations with subcritical nonlinearity under the Ambrosetti-Rabinowitz condition. Our method is also applicable to the system with a critical nonlinearity which is a subject of the forthcoming paper.

Finally, we discuss the conclusion for each chapter and the possible future direction of the research contained in this thesis in Chapter 5.

Chapter 2

Asymptotic mean value property for p-Laplace equations

This chapter focuses on quasilinear elliptic partial differential equations in the form of p-Laplace equations. Furthermore, the content presented here has been published in [17].

2.1 Introduction

In this study, we introduce and study (variationally) p-harmonious functions in order to show that asymptotic mean value property holds for quasilinear elliptic partial differential equation in the form of p-Laplace equation. First, we define the *p-mean* of a function $u \in L^p(\Omega)$ as a real number $\nu_p[u]$ such that

$$\|u - \nu_p[u]\|_p = \min_{\nu \in \mathbb{R}} \|u - \nu\|_p \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ and $p \in [1, \infty]$. The existence and uniqueness of $\nu_p[u]$ are guaranteed for $1 < p < \infty$ because it is a projection of u on the subspace of constant functions. For $p = 1$ we need to assume the continuity of u on Ω to guarantee the uniqueness of $\nu_1[u]$. When $p \in [1, \infty)$, $\nu_p[u]$ can be characterized as the unique root of the equation

$$\int_\Omega |u - \nu|^{p-2}(u - \nu)dx = 0 \tag{2.2}$$

as shown by Ishiwata, Magnanini, and Wadade in [46]. In fact, Ishiwata, Magnanini, and Wadade have also obtained the explicit formulas of $\nu_p[u]$ when $p = 1, 2, \infty$ in their aforementioned paper as follows:

$$\nu_1[u] = \operatorname*{med}_{\Omega} u, \quad \nu_2[u] = \fint_{\Omega} u(x)dx, \quad \text{and} \quad \nu_\infty[u] = \frac{1}{2}\big(\min_{\Omega} u + \max_{\Omega} u\big).$$

Let $\varepsilon > 0$. We proceed to introduce the operator μ_p^ε which acts on a continuous function $u \in C(\overline{\Omega})$ by the rule

$$\mu_p^\varepsilon[u](x) = \nu_p^{r_\varepsilon(x)}[u](x) \text{ for any } x \in \overline{\Omega}, \tag{2.3}$$

where $\nu_p^r[u]$ is the p-mean of u on a ball $B(x,r)$ with radius r centered at $x \in \mathbb{R}^N$ and $r_\varepsilon(x) = \min\{x, \operatorname{dist}(x, \partial\Omega)\}$ for any $x \in \overline{\Omega}$. Next, we consider the following Dirichlet problem:

$$\begin{cases} u = \mu_p^\varepsilon[u] & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where the boundary condition $g \in C(\partial\Omega)$.

When $p = 2$, the Dirichlet problem (2.4) becomes mean value property for harmonic functions. In the context of heat distribution, we can interpret the mean value property for harmonic functions in the following statement. Given any point $x \in \Omega$, the temperature $u(x)$ at x can be represented by the average temperature of a ball centered at x for any radius r sufficiently small to be contained inside Ω (see e.g. Evans [24] and Strauss [84]). A harmonic function which satisfies mean value property is called as *harmonious function*. In this study, we say that a function u is *p-harmonious* if and only if

$$u = \mu_p^\varepsilon[u] \text{ in } \Omega.$$

Thus, it is natural to ask whether mean value property (asymptotically) holds for our p-mean or not and whether p-harmonious functions have any relation to p-harmonic functions or not when $p \neq 2$.

We will clarify the precise meaning of p-harmonicity. We begin by introducing the *game-theoretic (homogeneous) p-Laplacian* denoted by Δ_p^G which is (formally) defined as follows:

$$\Delta_p^G u := \frac{1}{p}\Delta u + \frac{(p-2)}{p}\frac{\langle \nabla^2 u \nabla u, \nabla u\rangle}{|\nabla u|^2} \tag{2.5}$$

where $\nabla^2 u$ is the Hessian matrix of u. Particularly, when $p \in [1, \infty)$, we have

$$\Delta_p^G u = \frac{1}{p}\frac{\operatorname{div}(|\nabla u|^{p-2}\nabla u)}{|\nabla u|^{p-2}}, \tag{2.6}$$

whereas

$$\Delta_\infty^G u = \frac{\langle \nabla^2 u \nabla u, \nabla u \rangle}{|\nabla u|^2} \tag{2.7}$$

by taking the limit $p \to \infty$ in (2.5). We observe that Δ_p^G is uniformly elliptic but it has discontinuous coefficients and is not variational. Thus, we say a function u is *p-harmonic* (in the viscosity sense) if and only if

$$\Delta_p^G u = 0 \text{ in } \Omega.$$

Following this, we can consider the following Dirichlet problem:

$$\begin{cases} \Delta_p^G u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

Here, we understand the solution to Dirichlet problem (2.8) as a viscosity solution. Notice that the viscosity solution of (2.8) coincides with the weak solution of p-Laplace equation below:

$$\Delta_p u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega.$$

See [49] for further details.

There are a lot of reseachers who are interested in the study of p-mean values. For example, Hartenstine and Rudd in [41] also propose a p-mean value which is based on the median of a function. However, their p-mean value only produces asymptotic mean value property when $N = 2$. Kawohl, Manfredi, and Parviainen in [50] also develops their own version of p-mean value which produces asymptotic mean value property for $N \geq 2$. Notice that our p-mean value defined in (2.3) does not coincide with the other two versions of p-mean value mentioned above. In fact, our p-mean value has several nice properties such as continuity, monotonicity, additivity with constants, and homogeneity (see Theorem 2.4, Theorem 2.5, and Proposition 2.7 (i) and (ii) respectively in [46]).

We summarize the content of this chapter as follows. First, we introduce important definitions and some basic tools which are useful in Section 2.1. Next, we will show the existence and uniqueness for p-harmonious functions which satisfy Dirichlet problem (2.4) in Section 2.2 by using a modified Perron's method. Finally, we will show that our p-harmonious function converges to p-harmonic function in Section 2.3 by using a convergence scheme developed by Barles and Souganidis (see [8]).

2.1.1 Preliminaries

We begin this subsection by showing the continuity of ν_p. For similar result, see (Theorem 2.4, [46]).

Lemma 2.1.1 (Continuity of ν_p) *Let $p \in (1, \infty)$, $u \in L^p(\Omega)$, and $(u_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions which converge to u a.e. in Ω. Suppose that either one of the following statements hold:*

(i) *The sequence $(u_n)_{n\in\mathbb{N}}$ is increasing and nonnegative in Ω.*

(ii) *There exists a function $v \in L^p(\Omega)$ such that $|u_n| \leq v$ a.e. in Ω for any $n \in \mathbb{N}$.*

Then $\nu_p[u_n] \to \nu_p[u]$ as $n \to \infty$.

Proof of Lemma 2.1.1. (i) Set $\nu_n = \nu_p[u_n]$ and $\nu = \nu_p[u]$. Then $(\nu_n)_{n\in\mathbb{N}}$ either converges or diverges to $+\infty$ since $(u_n)_{n\in\mathbb{N}}$ is an increasing sequence and ν_p is monotone. For simplicity we let $\bar{\nu}$ be the limit of $(\nu_n)_{n\in\mathbb{N}}$. By the monotonicity of ν_p and the fact that $u_n \leq u$ (for any $n \in \mathbb{N}$), we have $\bar{\nu} \leq \nu$. It remains to show that $\bar{\nu} \geq \nu$ holds. In order to see this, observe that

$$0 = \int_\Omega |u_n - \nu_n|^{p-2}(u_n - \nu_n)dx \geq \int_\Omega |u_n - \bar{\nu}|^{p-2}(u_n - \bar{\nu})dx,$$

since the mapping $s \mapsto |t - s|^{p-2}(t - s)$ is decreasing and $\nu_n \leq \bar{\nu}$. In addition, the integrand in the right-hand side of the last inequality is bounded from below by the number $-|\bar{\nu}|^{p-2}\bar{\nu}$. Hence, we can apply Monotone Convergence Theorem to pass the limit inside the integral to obtain

$$\int_\Omega |u - \nu|^{p-2}(u - \nu)dx = 0 \geq \int_\Omega |u - \bar{\nu}|^{p-2}(u - \bar{\nu})dx.$$

The inequality above together with the fact that the mapping $s \mapsto |t - s|^{p-2}(t - s)$ is decreasing imply $\bar{\nu} \geq \nu$. Hence, the proof is complete.

(ii) The assumptions in (ii) allow us to use Dominated Convergence Theorem so that we obtain $u_n \to u$ in $L^p(\Omega)$ as $n \to \infty$. Then, the conclusion follows from Theorem 2.4 in [46]. ∎

Remark 2.1.1 *The conclusion of Lemma 2.1.1 remains valid for $p = 1$ when we assume further that $(u_n)_{n\in\mathbb{N}} \subset L^1(\Omega) \cap C(\Omega)$ and $u \in L^1(\Omega) \cap C(\Omega)$.*

When $p = \infty$, if assumption (i) holds, $(u_n)_{n\in\mathbb{N}} \subset C(\Omega)$, and $u \in C(\Omega)$, then the conclusion of Lemma 2.1.1 holds. Moreover, if $u_n \to u$ uniformly in Ω as $n \to \infty$, the conclusion of Lemma 2.1.1 also holds.

For the monotonicity of ν_p, the result has been obtained by Ishiwata, Magnanini, and Wadade in [46, Theorem 2.5]. The following lemma improves their aforementioned result.

Lemma 2.1.2 (Strict monotonicity of ν_p) *Let u_1, $u_2 \in L^p(\Omega)$ and $\omega \subset \Omega$ where $|\omega| > 0$. Assume that*

$$u_1 \leq u_2 \text{ a.e. in } \Omega \text{ and } u_1 < u_2 \text{ a.e. in } \omega.$$

Then, $\nu_p[u_1] < \nu_p[u_2]$. Particularly, if $u_1 \leq u_2$ a.e. in Ω and $\nu_p[u_1] = \nu_p[u_2]$, then we have $u_1 = u_2$ a.e. in Ω.

Proof of Lemma 2.1.2. When $p = \infty$, the result follows from an inspection. On the other hand, when $p \in [1, \infty)$, the assumptions on u_1 and u_2, and the fact that the mapping $t \mapsto |t - \nu|^{p-2}(t - \nu)$ is strictly increasing in $t \in \mathbb{R}$ give that

$$\int_\omega |u_1 - \nu|^{p-2}(u_1 - \nu)dx < \int_\omega |u_2 - \nu|^{p-2}(u_2 - \nu)dx$$

for any ν. The same result as the inequality above holds when ω is replaced with Ω. Thus, the conclusion of this lemma follows from the characterization of ν_p in (2.2). ∎

Corollary 2.1.1 (Monotonicity of μ_p^ε) *Assume u_1, $u_2 \in L^p(\Omega)$ when $p > 1$, and u_1, $u_2 \in L^p(\Omega) \cap C(\Omega)$ when $p = 1$. If $u_1 \leq u_2$ a.e. in Ω, then $\mu_p^\varepsilon[u_1] \leq \mu_p^\varepsilon[u_2]$ in $\overline{\Omega}$. Additionally, we have $\mu_p^\varepsilon[u_1] < \mu_p^\varepsilon[u_2]$ whenever the set*

$$\{x \in B(x, r(x)) \,|\, u_1(x) < u_2(x)\}$$

has a positive measure.

Next, we show that μ_p^ε acts naturally on $USC(\overline{\Omega})$ and $LSC(\overline{\Omega})$.

Proposition 2.1.1 (Invariance semicontinuity property) *Let $u \in LSC(\overline{\Omega})$ (or $USC(\overline{\Omega})$ respectively). Then, we have $\mu_p^\varepsilon[u] \in LSC(\overline{\Omega}) \cap C(\Omega)$ (or $USC(\overline{\Omega}) \cap C(\Omega)$ respectively).*

Proof of Proposition 2.1.1. For the sake of brevity, we will only prove for the lower semicontinuous case. We begin by showing that $\mu_p^\varepsilon[u] \in C(\Omega)$. First, we fix $x \in \Omega$ and $(x_n)_{n\in\mathbb{N}} \subset \Omega$ such that $x_n \to x$ as $n \to \infty$. Let us define $v_x : B(0,1) \to \mathbb{R}$ as follows:

$$v_x(z) := u(x + r_\varepsilon(x)z) \text{ for any } z \in B(0,1). \tag{2.9}$$

Since $u \in L^p_{loc}(\Omega)$ and r_ε is continuous, we deduce that $v_{x_n} \to v_x$ in $L^p(B(0,1))$ as $n \to \infty$. Therefore, [46, Theorem 2.4] gives that $\nu_p[v_{x_n}] \to \nu_p[v_x]$ as $n \to \infty$ which means $\mu_p^\varepsilon[u](x_n) \to \mu_p^\varepsilon[u](x)$ as $n \to \infty$.

It remains to prove the desired semicontinuity on $\partial\Omega$. We start the part of this proof by taking any sequence $(x_n)_{n\in\mathbb{N}} \subset \overline{\Omega}$ and $x \in \partial\Omega$ with $x_n \to x$ as $n \to \infty$. Since $\overline{\Omega}$ is a compact set, u attains its minimum $m > -\infty$. Next, we proceed as before and consider the function v_x in (2.9). v_{x_n} is well-defined when $x_n \in \Omega$. When $x_n \in \partial\Omega$, we let $v_{x_n} = u(x_n)$. By additivity with constants, we have

$$\nu_p[v_{x_n}] = \nu_p[v_{x_n} - m] + m.$$

By using the relation above together with the fact that $v_{x_n} \geq m$, we may assume $v_{x_n} \geq 0$ for simplicity. Following this, we use the semicontinuity of u and the continuity of r_ε to infer

$$\liminf_{n\to\infty} v_{x_n}(z) \geq v_x(z) \text{ for any } z \in B(0,1).$$

Now, we also have

$$\inf_{k\geq n} v_{x_k} \to \liminf_{n\to\infty} v_{x_n} \text{ in } B(0,1) \text{ as } n \to \infty,$$

and the convergence is motone increasing. Hence, we may apply Lemma 2.1.1 (i) to obtain

$$\lim_{n\to\infty} \nu_p\left[\inf_{k\geq n} v_{x_k}\right] = \nu_p\left[\liminf_{n\to\infty} v_{x_n}\right].$$

Note that the relation above also holds even if there are infinitely many terms of $(x_n)_{n\in\mathbb{N}}$ on $\partial\Omega$.

Thus, we conclude

$$\begin{aligned}\liminf_{n\to\infty} \mu_p^\varepsilon[u](x_n) &= \liminf_{n\to\infty} \nu_p[v_{x_n}] \geq \liminf_{n\to\infty} \nu_p\left[\inf_{k\geq n} v_{x_k}\right] \\ &= \nu_p\left[\liminf_{n\to\infty} v_{x_n}\right] \geq \nu_p[v_x] = \mu_p^\varepsilon[u](x).\end{aligned}$$

Therefore, $\mu_p^\varepsilon[u] \in LSC(\overline{\Omega})$. The proof for the upper semicontinuous case follows similarly. ∎

Proposition 2.1.2 *Let $\phi \in C^2(\Omega)$. Then, for any bounded open set $\omega \subset \Omega$ with $\overline{\omega} \subset \Omega$ and such that $\nabla\phi \neq 0$ in $\overline{\omega}$, it holds that*

$$\lim_{\varepsilon\to 0} \frac{\mu_p^\varepsilon[\phi] - \phi}{r_\varepsilon^2} = \frac{p}{2(N+p)} \Delta_p^G \phi \text{ uniformly in } \overline{\omega}. \tag{2.10}$$

Proof of L.et x_ε be a point in $\overline{\omega}$ maximizing the difference

$$\left|\frac{\mu_p^\varepsilon[\phi]-\phi}{r_\varepsilon^2}-\frac{p}{2(N+p)}\Delta_p^G\phi\right|$$

in $\overline{\omega}$. Then, we have $x_\varepsilon \to x$ up to a subsequence for some $x \in \overline{\omega}$, and hence $r_\varepsilon(x_\varepsilon) \to 0$ as $\varepsilon \to 0$. The conclusion then follows from Theorem 3.2 in [46]. ∎

Remark 2.1.2 (Examples of p-harmonious functions)

(i) It is obvious that constant functions are p-harmonious.
(ii) Let $\xi \in \mathbb{R}^N$, $x_1 \in \Omega$, $a(x) = \langle \xi, x\rangle + c$ be an affine function in Ω, and $a_{x_1}(x) = \langle \xi, x - x_1\rangle$. Then, we see that

$$a(x) = a_{x_1}(y) + \langle \xi, x_1\rangle + c \text{ for any } x \in \Omega.$$

Let $r > 0$. It is easy to see that $\nu_p[a_{x_1}](x_1) = 0$ over any ball $B(x_1, r)$ by the central symmetry of a_{x_1} with respect to x_1 and (2.2). Therefore, we conclude that

$$\nu_p[a](x_1) = \nu_p[a_{x_1}](x_1) + \nu_p[\langle \xi, x_1\rangle + c](x_1) = \langle \xi, x_1\rangle + c = a(x_1)$$

over any ball $B(x_1, r)$. Thus, it is clear that $\mu_p^\varepsilon[a] = a$, which means a is p-harmonious.

2.2 Existence and uniqueness of p-harmonious functions

In this section, we will prove the existence and uniqueness of p-harmonious functions by using a modified Perron's Method. First, we will introduce the notion of *(variationally) p-subharmonious* and *p-superharmonious* functions.

Definition 2.2.1 (p-subharmonious and p-superharmonious functions)
A function $u \in C(\Omega)$ is said to be (variationally) p-subharmonious (or p-superharmonious respectively) whenever

$$u \le \mu_p^\varepsilon[u] \quad (\text{ or } u \ge \mu_p^\varepsilon[u] \text{ respectively}) \text{ in } \Omega.$$

Remark 2.2.1 (i) By observing Remark 2.1.2 (ii), we can also deduce that convex (or concave respectively) functions are p-subharmonious (or p-superharmonious respectively).

(ii) Fix $\alpha > 0$ and set $\gamma_\alpha(x) = |x|^{-\alpha}$ for $x \neq 0$. Then, $\gamma_\alpha \in L^p_{loc}(\mathbb{R}^N)$ when $\alpha p < N$ and $\gamma_\alpha \in L^p_{loc}(\mathbb{R}^N \setminus \{0\})$ for any $p \in [1, \infty]$. Next, let $\Delta_p^G \phi$ as the *game-theoretic p-Laplacian* of a smooth function away from its critical points. We easily compute

$$\Delta_p^G \gamma_\alpha(x) = \frac{\alpha[\alpha(p-1)+p-N]}{p}|x|^{-\alpha+2} \text{ for any } x \neq 0.$$

By applying formula (3.6) of Theorem 3.2 in [46], we obtain

$$\frac{\gamma_\alpha(x) - \nu_p^r[\gamma_\alpha](x)}{r^2} = -\frac{\alpha[\alpha(p-1)+p-N]}{2(N+p)}|x|^{-(\alpha+2)} + o(1)$$

as $r \to 0$. Now, we let Ω be bounded open set such that $\overline{\Omega}$ does not contain the origin. Depending on the sign of $\alpha(p-1)+p-N$, there exists $r_\Omega > 0$ such that the inequality

$$\gamma_\alpha \geq \nu_p^r[\gamma_\alpha] \text{ or } \gamma_\alpha \leq \nu_p^r[\gamma_\alpha]$$

is satisfied uniformly in $\overline{\Omega}$. Therefore, for $r \in (0, r_\Omega)$, γ_α is p-subharmonious in Ω if

$$p > \frac{\alpha+N}{\alpha+1},$$

and p-superharmonious in Ω, if

$$p < \frac{\alpha+N}{\alpha+1}$$

Particularly, when $p = \frac{\alpha+N}{\alpha+1}$, we get the fundamental solution to p-Laplacian.

Proposition 2.2.1 (Weak comparison principle)
Let u, $v \in C(\overline{\Omega})$ be p-subharmonious and p-superharmonious functions in Ω respectively. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$.

Proof of Proposition 2.2.1. We argue by contradiction. Assume the conclusion does not hold. We have $u - v \in C(\overline{\Omega})$ and thus it attains its maximum $M > 0$ in $\overline{\Omega}$. Since $u \leq v$ on $\partial\Omega$, the following set

$$A = \big\{x \in \Omega \,|\, u(x) - v(x) = M\big\}$$

has at least one point x_0. Since $u - v \in C(\overline{\Omega})$ and $\{M\}$ is a closed set, then A must be a closed set.

Next, we wil show that A is an open set. To see this, we take any $x \in A$ and observe that

$$\mu_p^\varepsilon[u - M](x) \leq \mu_p^\varepsilon[v](x) \leq v(x)$$

since $u - M \leq v$ in Ω by the definition of M and v is p-superharmonious. Thus, we deduce that

$$v(x) \geq \mu_p^\varepsilon[u - M](x) = \mu_p^\varepsilon[u](x) - M \geq u(x) - M = v(x)$$

since u is p-subharmonious in Ω. Particularly,

$$\mu_p^\varepsilon[v](x) = \mu_p^\varepsilon[u - M](x)$$

which means

$$v = u - M \text{ in } B(x, r_\varepsilon(x))$$

in view of Lemma 2.1.2. In other words, $B(x, r_\varepsilon(x)) \subset A$ for any $x \in A$ and hence A is an open set.

Finally, we recall that Ω is a bounded domain in $\mathbb{R}^N$ which means it is a connected set. Thus, $A = \Omega$ and $u - v \equiv M$ in Ω. By continuity, we deduce that $M \equiv u - v$ on $\partial\Omega$. Thus, this leads to a contradiction since $0 < M \leq 0$. ∎

By following similar argument as above, we can obtain the following strong comparison principle.

Corollary 2.2.1 (Strong comparison principle)
Let u, $v \in C(\overline{\Omega})$ be a p-subharmonious and p-superharmonious functions in Ω respectively. If $u \leq v$ in Ω, then either $u < v$ in Ω or $u \equiv v$ in Ω.

Proof of Corollary 2.2.1. Assume $u - v = 0$ at some point in Ω. Then, we repeat the argument in the proof of Proposition 2.2.1 with

$$A = \big\{x \in \Omega \,|\, u(x) - v(x) = 0\big\} \text{ and } M = 0.$$

Then, we proceed to deduce that A is non-empty, closed, and open. Since Ω is connected, we deduce that $A = \Omega$. Therefore, $u \equiv v$ in Ω. ∎

Now, we fix a function $g \in C(\partial\Omega)$ to define two classes of continuous functions as follows:

$$S_g := \big\{v \in C(\overline{\Omega}) \,|\, v \text{ is } p\text{-subharmonious in } \Omega \text{ and } v \leq g \text{ on } \partial\Omega\big\} \tag{2.11}$$

and

$$S^g := \{v \in C(\overline{\Omega}) \,|\, v \text{ is } p\text{-superharmonious in } \Omega \text{ and } v \geq g \text{ on } \partial\Omega\}. \tag{2.12}$$

Any function $v \in S_g$ (or $v \in S^g$ respectively) is called as the (variational) *p-subsolution* (or *p-supersolution* respectively) of the Dirichlet problem (2.4). By (2.1), we see that

$$\mu_p^\varepsilon[c] = c \text{ in } \Omega$$

for any constant $c \in \mathbb{R}$. Therefore, it is easy to deduce that both S_g and S^g are nonempty sets since the constant functions $v_1 = \min_{\partial\Omega} g \in S_g$ and $v_2 = \max_{\partial\Omega} g \in S^g$. The desired solution of (2.4) u^ε can be obtained by checking the functions defined by

$$\underline{\mathrm{u}}^\varepsilon = \sup_{v \in S_g} v \text{ and } \bar{u}^\varepsilon = \inf_{v \in S^g}$$

coincide in $\overline{\Omega}$ provided $\partial\Omega$ satisfies some sufficient regularity assumptions. Then, we can define $u^\varepsilon := \underline{\mathrm{u}}^\varepsilon = \bar{u}^\varepsilon$ in $\overline{\Omega}$ as the solution of (2.4).

Lemma 2.2.1 *Let Ω be a bounded domain in $\mathbb{R}^N$ and $g \in C(\partial\Omega)$. Then, the following statements hold true:*

(i) *It holds that $\min_{\partial\Omega} g \leq \underline{u}^\varepsilon \leq \bar{u}^\varepsilon \leq \max_{\partial\Omega} g$.*

(ii) *$u^\varepsilon \in C(\overline{\Omega})$ is a solution of (2.4) if and only if $u \in S_g \cap S^g$.*

(iii) *If $u^\varepsilon \in C(\overline{\Omega})$ is a solution of (2.4), then $u^\varepsilon = \underline{u}^\varepsilon = \bar{u}^\varepsilon$ in $\overline{\Omega}$.*

Proof of Lemma 2.2.1. (i) By the weak comparison principle (Proposition 2.2.1), we see that $u \leq v$ in $\overline{\Omega}$ for every $u \in S_g$ and $v \in S^g$ since $u \leq g \leq v$ on $\partial\Omega$. Therefore, by using the properties of supremum and infimum, we infer that

$$\underline{\mathrm{u}}^\varepsilon \leq \bar{u}^\varepsilon.$$

The two other relevant inequalities easily follow from the fact that the two relevant constant functions belong to S_g and S^g.

(ii) The statement is trivial by the definitions of S_g and S^g.

(iii) If $u \in C(\overline{\Omega})$ is a solution of (2.4), then (ii) implies $u \in S_g \cap S^g$. Therefore, we have

$$\bar{u}^\varepsilon \leq u \leq \underline{\mathrm{u}}^\varepsilon \leq \bar{u}^\varepsilon \text{ in } \overline{\Omega},$$

which completes the proof for (iii). ∎

Next, similar to the classical Perron's Method (see e.g. Gilbarg and Trudinger [38]), we introduce the following barrier function for our Dirichlet Problem (2.4).

Definition 2.2.2 (Barrier functions) *Let $x_0 \in \partial\Omega$. A function $w = w_{x_0} \in C(\overline{\Omega})$ is said to be a barrier function at x_0 if and only if it is p-superharmonious in Ω, positive in $\overline{\Omega} \setminus \{x_0\}$, and $w(x_0) = 0$.*

In addition, we say that $x_0 \in \partial\Omega$ is a *regular point* for the Dirichlet problem in (2.4) if and only if there exists a barrier function at x_0.

Proposition 2.2.2 *If $x_0 \in \partial\Omega$ is a regular point for* (2.4)*, then*

$$\underline{u}^\varepsilon(x_0) = \bar{u}^\varepsilon(x_0) = g(x_0).$$

Proof of Proposition 2.2.2. The proof is similar to the classical case, see [38, Gilbarg and Trudinger Section 2.8, pp. 23]. Let $x_0 \in \partial\Omega$ and $\eta > 0$. Since $g \in C(\partial\Omega)$, we can find $\delta > 0$ such that $|g - g(x_0)| < \eta$. Let $w_{x_0} = w$ be the barrier function at x_0 and define

$$M_\delta := \max_{\partial\Omega \setminus B(x_0,\delta)} \frac{|g - g(x_0)|}{w}.$$

Hence, we obtain

$$|g - g(x_0)| \leq \eta + M_\delta w \text{ on } \partial\Omega,$$

which means

$$(g(x_0) + \eta + M_\delta w) \in S^g \text{ and } (g(x_0) - \eta - M_\delta w) \in S_g.$$

Thus, by the definition of S^g and S_g, we observe that

$$g(x_0) - \eta - M_\delta w \leq \underline{u}^\varepsilon \leq \bar{u}^\varepsilon \leq g(x_0) + \eta + M_\delta w \text{ in } \overline{\Omega},$$

whence it leads to

$$g(x_0) - \eta \leq \underline{u}^\varepsilon(x_0) \leq \bar{u}^\varepsilon(x_0) \leq g(x_0) + \eta,$$

since $w(x_0) = 0$. The conclusion of this proposition follows since $\eta > 0$ is arbitrary. ∎

Finally, we have our main theorem in this section as written below.

Theorem 2.2.2 (Existence and uniqueness of p-harmonious functions) *Let $\varepsilon_0 > 0$ such that Ω contains at least a ball of radius ε_0. Then, for any $\varepsilon \in (0, \varepsilon_0]$, the Dirichlet problem* (2.4) *admits a unique solution $u^\varepsilon \in C(\overline{\Omega})$ for any $g \in C(\partial\Omega)$ if and only if all boundary points of Ω are regular for* (2.4).

Proof of Theorem 2.2.2. We will divide the proof of this theorem in two steps.

Step 1. Assume that every point on $\partial\Omega$ is regular for (2.4). In view of Lemma 2.2.1 (i) and the definition of $\underline{u}^\varepsilon$, $\underline{u}^\varepsilon$ is bounded and lower semicontinuous in $\overline{\Omega}$, that is $\underline{u}^\varepsilon \in L^\infty(\overline{\Omega}) \cap LSC(\overline{\Omega})$. In fact, $\underline{u}^\varepsilon = g$ on $\partial\Omega$ in view of Proposition 2.2.2.

Next, we define a sequence of functions by following the iteration scheme below:

$$u_1 = \underline{u}^\varepsilon, \quad u_{j+1} = \mu_p^\varepsilon[u_j] \text{ in } \overline{\Omega} \text{ for any } j \in \mathbb{N}.$$

Proposition 2.1.1 ensures that $(u_j)_{j\in\mathbb{N}} \subset L^\infty(\overline{\Omega}) \cap LSC(\overline{\Omega})$. In fact, $u_j = g$ on $\partial\Omega$ for every $j \in \mathbb{N}$.

Since $u \le \underline{u}^\varepsilon$ in $\overline{\Omega}$ for any $u \in S_g$, we have

$$u \le \mu_p^\varepsilon[u] \le \mu_p^\varepsilon[\underline{u}^\varepsilon] \text{ in } \overline{\Omega}$$

for any $u \in S_g$. Thus, we can deduce $u_1 \le \mu_p^\varepsilon[u_1] = u_2$ in $\overline{\Omega}$ which in view of the monotonicity of μ_p^ε, we also have $u_2 \le \mu_p^\varepsilon[u_2] = u_3$ in $\overline{\Omega}$. By iterating this process, we see that

$$u_j \le u_{j+1} \text{ in } \overline{\Omega} \text{ for any } j \in \mathbb{N}.$$

Therefore, $(u_j)_{j\in\mathbb{N}}$ is increasing in $\overline{\Omega}$. By Lemma 2.2.1 (i) and weak comparison principle, we have

$$\min_{\partial\Omega} g \le u_1 = \underline{u}^\varepsilon \le \bar{u}^\varepsilon \le \max_{\partial\Omega} g \text{ in } \overline{\Omega},$$

whence by iteration process, it follows that

$$\min_{\partial\Omega} g \le u_j \le \max_{\partial\Omega} g \text{ in } \overline{\Omega} \text{ for any } j \in \mathbb{N}.$$

This means that $(u_j)_{j\in\mathbb{N}}$ is a bounded and increasing sequence and hence it converges to a function u_1^ε pointwise in $\overline{\Omega}$. Thus, Lemma 2.1.1 ensures that

$$\mu_p^\varepsilon[u_j] \to \mu_p^\varepsilon[u_1^\varepsilon] \text{ as } j \to \infty.$$

Similarly, by using the fact that $S^g = -S_{-g}$, we can obtain $(U_j)_{j\in\mathbb{N}} \subset L^\infty(\overline{\Omega}) \cap USC(\overline{\Omega})$. Clearly, such a sequence is obtained by choosing $U_1 = \bar{u}^\varepsilon$ and it is also decreasing in $\overline{\Omega}$. Hence, $(U_j)_{j\in\mathbb{N}}$ must converge to a function U_1^ε pointwise in $\overline{\Omega}$ and

$$\mu_p^\varepsilon[U_j] \to \mu_p^\varepsilon[U_1^\varepsilon] \text{ as } j \to \infty.$$

It only remains to prove that $u_1^\varepsilon = U_1^\varepsilon$ in $\overline{\Omega}$. First, observe that $U_1^\varepsilon - u_1^\varepsilon$ is nonnegative in $\overline{\Omega}$ and belongs to $USC(\overline{\Omega})$. Thus, it attains its maximum value $M \geq 0$ at some point $x_0 \in \overline{\Omega}$. Our goal is to show that $M = 0$.

For the sake of contradiction, assume $M > 0$ and consider the following set:

$$A = \left\{ x \in \overline{\Omega} \,|\, (U_1^\varepsilon - u_1^\varepsilon)(x) = M \right\}.$$

It is clear that A is nonempty since $x_0 \in A$. Furthermore, $x_0 \in \Omega$ and $A \subset \Omega$ since $U_1^\varepsilon - u_1^\varepsilon = 0$ on $\partial\Omega$. By Proposition 2.1.1, $U_1^\varepsilon - u_1^\varepsilon$ belongs to $C(\Omega)$ whence it follows that A is a closed set.

Next, we observe that $U_1^\varepsilon - U_1^\varepsilon(x_0) \leq u_1^\varepsilon - u_1^\varepsilon(x_0)$ in Ω since $U_1^\varepsilon - u_1^\varepsilon \leq U_1^\varepsilon(x_0) - u_1^\varepsilon(x_0) = M$ in Ω. By taking $x \in \Omega$, we deduce that

$$\mu_p^\varepsilon[U_1^\varepsilon](x) - U_1^\varepsilon(x_0) = \mu_p^\varepsilon[U_1^\varepsilon - U_1^\varepsilon(x_0)](x) \leq \mu_p^\varepsilon[u_1^\varepsilon - u_1^\varepsilon(x_0)](x) = \mu_p^\varepsilon[u_1^\varepsilon](x) - u_1^\varepsilon(x_0)$$

in view of the monotonicity of μ_p^ε. Therefore, we infer that

$$U_1^\varepsilon(x) - U_1^\varepsilon(x_0) = \mu_p^\varepsilon[U_1^\varepsilon - U_1^\varepsilon(x_0)] \leq \mu_p^\varepsilon[u_1^\varepsilon - u_1^\varepsilon(x_0)](x) = u_1^\varepsilon(x) - u_1^\varepsilon(x_0)$$

since both of U_1^ε and u_1^ε are p-harmonious in Ω. Next, we take any $x \in A \subset \Omega$. The definition of μ_p^ε and Lemma 2.1.2 imply $U_1^\varepsilon - U_1^\varepsilon(x_0) \equiv u_1^\varepsilon - u_1^\varepsilon(x_0)$ in $B(x, r_\varepsilon(x))$ and thus $U_1^\varepsilon - u_1^\varepsilon \equiv M$ in $B(x, r_\varepsilon(x))$. This means that $B(x, r_\varepsilon(x)) \subset A$ for any $x \in A$ and therefore A is an open set.

Following the previous argument, observe that A is nonempty, open, and closed. Additionally, Ω is connected. Hence, we infer that $A = \Omega$. Now, we take $(x_n)_{n\in\mathbb{N}} \subset \Omega$ and $x_* \in \partial\Omega$ so that $x_n \to x_*$ as $n \to \infty$. Since $U_1^\varepsilon - u_1^\varepsilon$ belongs to $USC(\overline{\Omega})$, we see that

$$\begin{aligned} 0 < M = U_1^\varepsilon(x_0) - u_1^\varepsilon(x_0) &= \limsup_{n\to\infty} (U_1^\varepsilon - u_1^\varepsilon)(x_n) \\ &\leq U_1^\varepsilon(x_*) - u_1^\varepsilon(x_*) = g(x_*) - g(x_*) = 0 \end{aligned}$$

which is a contradiction. Hence, we have shown the existence of solution to Dirichlet Problem (2.4).

It is not difficult to show the uniqueness of solution. To see this, we let u_1^ε and u_2^ε be the solution of (2.4). Observe that

$$|u_1^\varepsilon - u_2^\varepsilon| = |\mu_p^\varepsilon[u_1^\varepsilon] - \mu_p^\varepsilon[u_2^\varepsilon]| \text{ in } \Omega$$

which means

$$\mu_p^\varepsilon[|u_1^\varepsilon - u_2^\varepsilon|] = |u_1^\varepsilon - u_2^\varepsilon| \text{ in } \Omega \text{ and } |u_1^\varepsilon - u_2^\varepsilon| = 0 \text{ on } \partial\Omega.$$

By Lemma 2.2.1 (i) and (iii), it follows that $|u_1^\varepsilon - u_2^\varepsilon| = 0$ in $\overline{\Omega}$ which means our solution is unique.

Step 2. Now, we assume that the Dirichlet problem (2.4) admits a unique solution for any boundary data $g \in C(\partial\Omega)$. We will show that any point $x_0 \in \partial\Omega$ has a barrier function. To see this, we fix $x_0 \in \partial\Omega$ and choose boundary data $g_0 = |x - x_0|^2$. It is clear that g_0 is a convex function and therefore it is a p-subharmonious function in view of Remark 2.2.1 (i). Moreover, $g_0 \in S_{g_0}$ and the solution to Dirichlet problem (2.4) denoted as $u_0 \in S^{g_0}$ is p-superharmonious in view of Lemma 2.2.1 (ii). It is easy to see that u_0 is positive in $\overline{\Omega} \setminus \{x_0\}$ and $u_0(x_0) = g_0(x_0) = 0$ since $u_0 = g_0$ on $\partial\Omega$ and weak maximum principle holds. Therefore, we can choose u_0 as the barrier function at x_0 which completes the proof of this theorem. ∎

Corollary 2.2.2 *Assume Ω also satisfies the uniform exterior sphere condition on $\partial\Omega$. Then, there exists $\varepsilon_{\partial\Omega} \in (0, \varepsilon_0]$ such that for any $\varepsilon \in (0, \varepsilon_{\partial\Omega})$, all points on $\partial\Omega$ are regular.*

Particularly, the Dirichlet problem (2.4) admits a unique solution $u^\varepsilon \in C(\overline{\Omega})$.

Proof of Corollary 2.2.2. By the uniform exterior sphere condition, we have $R > 0$ for any $x_0 \in \partial\Omega$ so that we can obtain a ball $B(y_0, R)$ such that $\overline{B(y_0, R)} \cap \overline{\Omega} = \{x_0\}$. We proceed to define the following function:

$$w(x) := \frac{1}{R} - \frac{1}{|x - y_0|^\alpha} \text{ for any } x \in \overline{\Omega} \tag{2.13}$$

where $\alpha > 0$. It is clear that $w \in C(\overline{\Omega})$, w is positive in $\overline{\Omega} \setminus \{x_0\}$, and $w(x_0) = 0$. Finally, we choose $\alpha = (N + 1)(p - 1)$ as in Remark 2.2.1 (ii) to see that w is p-superharmonious in Ω for $\varepsilon \in (0, r_\Omega)$ in which we choose $\varepsilon_{\partial\Omega} = r_\Omega$ here.

Thus, w is a barrier function at x_0 which completes the proof since x_0 can be choosen arbitrarily from $\partial\Omega$. ∎

2.3 Convergence of p-harmonious functions

In this section, we will show that the solution to Dirichlet problem (2.4) converges uniformly to the (viscosity) solution of (2.8). First, we start with the classical definition of a viscosity solution of an elliptic degenerate equation. Consider a continuous mapping

$$F : \overline{\Omega} \times \mathbb{R} \times (\mathbb{R}^N \setminus \{0\}) \times S^N \to \mathbb{R}$$

where S^N is the set of $N \times N$ symmetric matrices.

Definition 2.3.1 (The upper and lower semicontinuous envelopes)
The upper semicontinuous envelopes F^* of F are the functions defined by

$$F^*(x,s,\xi,X) = \limsup_{(y,t,\eta,Y)\to(x,s,\xi,X)} F(y,t,\eta,Y)$$

for any $(x,s,\xi,X) \in \overline{\Omega} \times \mathbb{R} \times (\mathbb{R}^N \setminus \{0\}) \times S^N$.

The lower semicontinuous envelopes F_* of F are the functions which can be defined by

$$F_* = -(-F)^*.$$

We recall from [21] and [51] the following notions of *viscosity subsolutions* and *viscosity supersolutions*.

Definition 2.3.2 (Viscosity subsolutions and supersolutions)
A function u is a viscosity subsolution (or supersolution respectively) of $F = 0$ in $\overline{\Omega}$ if and only if for any $(x,\phi) \in \overline{\Omega} \times C^2(\overline{\Omega})$ with $\nabla\phi(x) \neq 0$ and such that $u - \phi$ has a local maximum (or local minimum respectively) at x with $\phi(x) = u(x)$, it holds that

$$F_*(x,\phi(x),\nabla\phi(x),\nabla^2\phi(x)) \leq 0 \quad \textit{(or } F^*(x,\phi(x),\nabla\phi(x),\nabla^2\phi(x)) \geq 0 \textit{ respectively).}$$

Hence, $u \in C(\overline{\Omega})$ is a *viscosity solution* of $F = 0$ in $\overline{\Omega}$ if and only if it is both a viscosity subsolution and supersolution.

Next, we set F as follows:

$$F(x,s,\xi,X) = -\frac{\mathrm{tr}(X)}{p} - \frac{(p-2)}{p}\frac{\langle X\xi, xi\rangle}{|\xi|^2}$$

for any $(x,s,\xi,X) \in \overline{\Omega} \times \mathbb{R} \times (\mathbb{R}^N \setminus \{0\}) \times S^N$ where $\mathrm{tr}(X)$ is the trace of matrix X.

In order to obtain the convergence of p-harmonious function, we will follow the arguments given by Barles and Souganidis in [8]. To this aim, we need to setup further notation.

Proposition 2.3.1 *Let $G : \overline{\Omega} \times \mathbb{R} \times (\mathbb{R}^N \setminus \{0\}) \times S^N \to \mathbb{R}$ be a mapping defined by*

$$G(x,s,\xi,X) := \begin{cases} -\frac{tr(X)}{p} - \frac{(p-2)}{p}\frac{\langle X\xi,\xi\rangle}{|\xi|^2} & \textit{if } x \in \Omega, \\ s - g(x) & \textit{if } x \in \partial\Omega, \end{cases} \tag{2.14}$$

for any $(x,s,\xi,X) \in \overline{\Omega} \times \mathbb{R} \times (\mathbb{R}^N \setminus \{0\}) \times S^N$. Then, we have

$$G^*(x,s,\xi,X) = G_*(x,s,\xi,X) = -\frac{tr(X)}{p} - \frac{(p-2)}{p}\frac{\langle X\xi,\xi\rangle}{|\xi|^2} \textit{ for any } x \in \Omega \tag{2.15}$$

and

$$G^*(x,s,\xi,X) = \max\left\{ -\frac{tr(X)}{p} - \frac{(p-2)}{p}\frac{\langle X\xi,\xi\rangle}{|\xi|^2}, s-g(x)\right\}, \tag{2.16}$$

$$G_*(x,s,\xi,X) = \min\left\{ -\frac{tr(X)}{p} - \frac{(p-2)}{p}\frac{\langle X\xi,\xi\rangle}{|\xi|^2}, s-g(x)\right\} \tag{2.17}$$

for any $x \in \partial\Omega$.

Proof of Proposition 2.3.1. The first formula in (2.15) follows from the continuity of G at interior points of Ω. When $x \in \partial\Omega$, we observe that for sufficiently small $\delta > 0$, the supremum of G in $(B(x,\delta)\overline{\Omega}) \times B(s,\delta) \times B(\xi,\delta) \times B(X,\delta)$ (where the ball must be intended in the relevant Euclidean spaces) is equal to

$$\max\left\{ -\frac{\mathrm{tr}(X)}{p} - \frac{(p-2)}{p}\frac{\langle X\xi,\xi\rangle}{|\xi|^2}, s-g(x)\right\}.$$

The formula for G^* in (2.16) follows from above. The formula for G_* easily follows from $G_* = -(-G)^*$. ∎

Remark 2.3.1 *Let $x \in \overline{\Omega}$ and $\phi \in C^2(\overline{\Omega})$ with $\nabla\phi(x) \neq 0$. Then, we have*

$$G(x,\phi(x),\nabla\phi(x),\nabla^2\phi(x)) = \begin{cases} -\Delta_p^G\phi(x) & \text{if } x \in \Omega, \\ \phi(x) - g(x) & \text{if } x \in \partial\Omega, \end{cases} \tag{2.18}$$

and thus

$$G^*(x,\phi(x),\nabla\phi(x),\nabla^2\phi(x)) = \max\{-\Delta_p^G\phi(x), \phi(x)\}, \tag{2.19}$$

$$G_*(x,\phi(x),\nabla\phi(x),\nabla^2\phi(x)) = \min\{-\Delta_p^G\phi(x), \phi(x)\}. \tag{2.20}$$

By Remark 2.3.1, Theorem 3.3 in [22], and the classical weak comparison principle for the viscosity solution of (2.8) (see [49, 55]), we immediately obtain the following weak comparison principle for $G = 0$ in $\overline{\Omega}$, see [8] (and also [6, 7, 45] for further references).

Proposition 2.3.2 (Comparison principle) *Let $u \in USC(\overline{\Omega}) \cap L^\infty(\Omega)$ and $v \in LSC(\overline{\Omega}) \cap L^\infty(\Omega)$. If u and v are viscosity subsolution and supersolution of $G = 0$ in $\overline{\Omega}$ respectively, then we have that $u \leq v$ in $\overline{\Omega}$.*

Following the previous proposition, we proceed to introduce an approximation scheme by p-harmonious functions. To this aim, we let $\varepsilon_0 > 0$ be sufficiently small so that there exists a ball with radius ε_0 which is contained in Ω and $\mathscr{A}_\varepsilon : \mathbb{R}\times\overline{\Omega}\times C(\overline{\Omega}) \to \mathbb{R}$ for $\varepsilon \in (0, \varepsilon_0)$ as follows:

$$\mathscr{A}_\varepsilon(s, x, u) = \begin{cases} \frac{2(N+p)\varepsilon}{p}\frac{s-\mu_p^\varepsilon[u](x)}{r_\varepsilon(x)^2} & \text{if } x \in \Omega, \\ \varepsilon(s - g(x)) & \text{if } x \in \partial\Omega \end{cases} \tag{2.21}$$

for $(s, x, u) \in \mathbb{R} \times \overline{\Omega} \times C(\overline{\Omega})$.

Lemma 2.3.1 *If $x \in \overline{\Omega}$ and $\phi \in C^2(\overline{\Omega})$ with $\nabla\phi(x) \neq 0$, then we have*

$$\limsup_{(\varepsilon,y,\delta)\to(0^+,x,0)} \frac{\mathscr{A}_\varepsilon(\phi(y) + \delta, y, \phi + \delta)}{\varepsilon} = G^*(x, \phi(x), \nabla\phi(x), \nabla^2\phi(x)), \tag{2.22}$$

$$\liminf_{(\varepsilon,y,\delta)\to(0^+,x,0)} \frac{\mathscr{A}_\varepsilon(\phi(y) + \delta, y, \phi + \delta)}{\varepsilon} = G_*(x, \phi(x), \nabla\phi(x), \nabla^2\phi(x)). \tag{2.23}$$

Proof of Lemma 2.3.1. By using the additivity with constants and homogeneity of p-means, for every $\delta \in \mathbb{R}$, we have

$$\frac{\mathscr{A}_\varepsilon(\phi(y) + \delta, y, \phi + \delta)}{\varepsilon} = \frac{2(N + p)}{p}\frac{\phi(y) - \mu_p^\varepsilon[\phi](y)}{r_\varepsilon(y)^2} \text{ when } y \in \Omega,$$

and

$$\frac{\mathscr{A}_\varepsilon(\phi(y) + \delta, y, \phi + \delta)}{\varepsilon} = \phi(y) + \delta - g(y) \text{ when } y \in \partial\Omega.$$

Then by uniform convergence in Proposition 2.1.2, continuity of ϕ and g, we deduce

$$\limsup_{(\varepsilon,y,\delta)\to(0^+,x,0)} \frac{\mathscr{A}_\varepsilon(\phi(y) + \delta, y, \phi + \delta)}{\varepsilon} = \begin{cases} -\Delta_p^G\phi(x) & \text{if } x \in \Omega, \\ \max\{-\Delta_p^G\phi(x), \phi(x) - g(x)\} & \text{if } x \in \partial\Omega. \end{cases}$$

Thus, (2.22) holds. The proof for (2.23) follows similarly. ∎

We are now ready to introduce our main theorem in this section.

Theorem 2.3.2 (Convergence of p-harmonious functions)
Assume Ω be a bounded C^2-smooth domain containing a ball of radius ε_0. Suppose there exists $\varepsilon_{\partial\Omega} \in (0, \varepsilon_0]$ such that for any $\varepsilon \in (0, \varepsilon_{\partial\Omega})$, every point of $\partial\Omega$ is a regular point for the Dirichlet problem (2.4).

Let u^ε be the unique solution in $C(\overline{\Omega})$ of (2.4) *for $\varepsilon \in (0, \varepsilon_{\partial\Omega})$. Then, for every $g \in C(\partial\Omega)$, there exists $u \in C(\overline{\Omega})$ such that $u^\varepsilon \to u$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$ and u is the unique viscosity solution of* (2.8).

Proof of Theorem 2.3.2. Up to re-defining ε_0, for every $\varepsilon \in (0, \varepsilon_0)$, there exists a unique function $u^\varepsilon \in C(\overline{\Omega})$ which solves

$$\mathscr{A}_\varepsilon(u^\varepsilon(x), x, u^\varepsilon) = 0 \text{ for any } x \in \overline{\Omega} \tag{2.24}$$

in view of Theorem 2.2.2. Moreover, u^ε is bounded in $\overline{\Omega}$ from below and above by the minimum and the maximum of g on $\partial\Omega$ by Lemma 2.2.1 (i). Thus, u^ε is uniformly bounded with respect to ε in $\overline{\Omega}$. Hence, by the aforementioned boundedness of u^ε, we can define

$$u_*(x) = \liminf_{(\varepsilon,y)\to(0^+,x)} u^\varepsilon(y) \text{ and } u^*(x) = \limsup_{(\varepsilon,y)\to(0^+,x)} u^\varepsilon(y)$$

which are bounded in $\overline{\Omega}$. In fact, u_* and u^* are lower semicontinuous and upper semicontinuous in $\overline{\Omega}$ respectively since $u^\varepsilon \in C(\overline{\Omega})$. We will show that $u_* \equiv u^*$. Since we know $u_* \leq u^*$, it is enough for us to show $u_* \geq u^*$. To this aim, we only need to prove that u_* and u^* are a viscosity supersolution and subsolution of $G = 0$ in $\overline{\Omega}$.

Following the previous argument, we recall Theorem 2.5 in [46] to see that the mapping $\mathscr{A}_\varepsilon$ is decreasing in the third variable, in the sense that for any $(s, x) \in \mathbb{R}\times\overline{\Omega}$ and u, $v \in C(\overline{\Omega})$, it holds that

$$\text{if } u \leq v \text{ in } \overline{\Omega}, \text{ then } \mathscr{A}_\varepsilon(s, x, u) \geq \mathscr{A}_\varepsilon(s, x, v).$$

Now, let $(x, \phi) \in \overline{\Omega} \times C(\overline{\Omega})$ with $\nabla\phi(x) \neq 0$ and $u^* - \phi$ has a local maximum at x with $u^*(x) = \phi(x)$. Without loss of generality, we can assume that the maximum is global and strict, that is

$$u^* - \phi < u^*(x) - \phi(x) = 0 \text{ in } \overline{\Omega} \setminus \{x\}.$$

By a standard argument in the theory of viscosity solutions (see [51]), we know that there exists a sequence of elements $((\varepsilon_j, x_j))_{j\in\mathbb{N}} \subset (0, \varepsilon_0, \overline{\Omega})$ such that for any fixed $j \in \mathbb{N}$, x_j is a global maximum point for $u^{\varepsilon_j} - \phi$ and

$$(\varepsilon_j, x_j, u^{\varepsilon_j}(x_j)) \to (0, x, u^*(x)) \text{ as } j \to \infty.$$

If we set $\delta_j = u^{\varepsilon_j}(x_j) - \phi(x_j)$, then $\delta_j \to 0$ as $j \to \infty$ and $u^{\varepsilon_j} - \phi \leq \delta_j$ in $\overline{\Omega}$.

Next, we use (2.24) at x_j for $\varepsilon = \varepsilon_j$ and $u^\varepsilon = u^{\varepsilon_j}$ to deduce that

$$\begin{aligned} 0 &= \mathscr{A}_{\varepsilon_j}(u^{\varepsilon_j}(x_j), x_j, u^{\varepsilon_j}) \\ &= \mathscr{A}_{\varepsilon_j}(\phi(x_j) + \delta_j, x_j, u^{\varepsilon_j}) \geq \mathscr{A}_{\varepsilon_j}(\phi(x_j) + \delta_j, x_j, \phi + \delta_j) \end{aligned}$$

in which the last inequality follows from the aforementioned monotonicity of $\mathscr{A}_{\varepsilon_j}$. Then, since $\nabla\phi(x) \neq 0$, we can apply Lemma 2.3.1 to see that

$$\begin{aligned} 0 &\geq \liminf_{j\to\infty} \frac{\mathscr{A}_{\varepsilon_j}(\phi(x_j)+\delta_j, x_j, \phi+\delta_j)}{\varepsilon_j} \\ &\geq \liminf_{(\varepsilon,y,\delta)\to(0^+,x,0)} \frac{\mathscr{A}_{\varepsilon_j}(\phi(y)+\delta, y, \phi+\delta)}{\varepsilon} \\ &= F_*(x, \phi(x), \nabla\phi(x), \Delta^2\phi(x)). \end{aligned}$$

Therefore, u^* is a viscosity subsolution of $G = 0$ in $\overline{\Omega}$. By following similar arguments, u_* is a viscosity supersolution of $G = 0$ in $\overline{\Omega}$. Thus, applying Proposition 2.3.2, we deduce that $u^* \leq u_*$ in $\overline{\Omega}$. Hence $u_* = u^* = u \in C(\overline{\Omega})$ is a viscosity solution of (2.8).

It remains to show that the convergence is uniform. First, we will prove that u^ε converges pointwise to a function $u = u_* = u^*$ in $\overline{\Omega}$ as $\varepsilon \to 0$ by the way it is constructed. To see this, we observe that

$$u(x) = u^*(x) \limsup_{(\varepsilon,y)\to(0^+,x)} u^\varepsilon(y) \geq \liminf_{(\varepsilon,y)\to(0^+,x)} u^\varepsilon(y) \tag{2.25}$$

and

$$\limsup_{(\varepsilon,y)\to(0^+,x)} u^\varepsilon(y) = u^*(x) \leq u_*(x) = \liminf_{(\varepsilon,y)\to(0^+,x)} u^\varepsilon(y) \tag{2.26}$$

for any $x \in \overline{\Omega}$ since u^* and u_* are a viscosity subsolution and supersolution respectively in view of Proposition 2.3.2. Therefore, we have

$$u(x) = \limsup_{(\varepsilon,y)\to(0^+,x)} u^\varepsilon(y) = \liminf_{(\varepsilon,y)\to(0^+,x)} u^\varepsilon(y)$$

for any $x \in \overline{\Omega}$ which means $u^\varepsilon \to u$ pointwise in $\overline{\Omega}$ as $\varepsilon \to 0$. Next, we fix $x \in \overline{\Omega}$. Then, by using (2.25) and (2.26), for any $\eta > 0$, there exists $\delta_{\eta,x} > 0$ such that

$$u(x) - \frac{\eta}{2} < \inf_{\max\{\varepsilon,|y-x|\}<\delta_{\eta,x}} u^\varepsilon(y) \leq \sup_{\max\{\varepsilon,|y-x|\}<\delta_{\eta,x}} u^\varepsilon(y) < u(x) + \frac{\eta}{2}$$

which leads to

$$|u^\varepsilon - u(x)| < \frac{\eta}{2} \text{ in } B(x, \delta_{\eta,x}) \cap \overline{\Omega} \tag{2.27}$$

for $\varepsilon \in (0, \delta_{\eta,x})$. Let $\Lambda = \bigcup_{x\in\overline{\Omega}} B(x, \delta_{\eta,x})$ so that $\overline{\Omega} \subset \Lambda$. By the compactness of $\overline{\Omega}$, there exists $n_0 \in \mathbb{N}$ and $(x_i)_{i=1}^{n_0} \subset \overline{\Omega}$ such that $\overline{\Omega} \subset \bigcup_{i=1}^{n_0} B(x_i, \delta_{\eta,x_i})$. Combining this fact and (2.27), we infer

$$|u^\varepsilon - u(x_i)| < \frac{\eta}{2} \text{ in } B(x_i, \delta_{\eta,x_i}) \cap \overline{\Omega} \tag{2.28}$$

for any $i \in \{1, 2, ..., n_0\}$ and $\varepsilon \in (0, \delta_{\eta,x_i})$. By setting

$$\delta_\eta := \min_{1 \le i \le n_0} \delta_{\eta,x_i},$$

we see that

$$|u^\varepsilon - u| < \eta \text{ in } \overline{\Omega}$$

for any $\varepsilon \in (0, \delta_\eta)$. Since $\eta > 0$ is arbitrary and δ_η does not depend on x, we conclude that

$$u^\varepsilon \to u \text{ as } \varepsilon \to 0 \text{ uniformly in } \overline{\Omega}$$

which completes the proof of this theorem. ∎

Several similar results to this theorem have been obtained in [22, 61, 63, 64] based on a different notion of p-mean value. Particularly, this theorem was expected to hold since our p-mean ν_p has all structural assumptions similar to the p-mean proposed in [22], for the convergence of the underlying dynamic programming principle. Those assumptions are additivity with constants, 1-homogeneity, and monotonicity for essentially bounded functions. However, Theorem 2.3.2 cannot be proven as a direct application of the general result in [22] since the definitions of the two p-mean values do not necessarily coincide.

Chapter 3

Blow-up rate of time-local solutions to a semilinear heat equation

In this chapter we consider a semilinear heat equations with subcritical Ambrosetti-Rabinowitz nonlinear term in the sense of Sobolev embedding. Particularly, we are interested to study the blow-up rate of time-local solutions which blow up in finite time. The methods used to analyze the blow-up rate are variable transformation and parabolic argument. In fact, our method is also applicable to a system of semilinear heat equations which will be covered in the last section of this chapter. Finally, we put a remark here that the results in this study is a natural progression of the results which have been obtained in [19].

3.1 Introduction and main results

Let $N \in \mathbb{N}$. The critical Sobolev exponent is denoted by 2^* which is defined as $2^* := \frac{2N}{N-2}$ when $N \geq 3$ and $2^* := \infty$ when $N = 1, 2$. We consider the following semilinear heat equation:

$$\begin{cases} \partial_t u = \Delta u - u + f(u) & \text{in } \mathbb{R}^N \times (0, T_m), \\ u(\,\cdot\,, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{P}$$

where $u_0 \in L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ for simplicity and T_m denotes the maximal existence time of a classical solution to (P). Our assumptions on the nonlinearity f are as follows:

(N1) The function $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz in $\mathbb{R}$, $f(0) = 0$, and $\lim_{|u|\to 0} \left|\frac{f(u)}{u}\right| = 0$.

(N2) There exists $p \in (2, 2^*)$, $\kappa > 0$, and $C_{N2} > 0$ such that $|f(u)| \leq C_{N2}|u|^{p-1}$ for any $u \in \mathbb{R}$ with $|u| \geq \kappa$.

(AR) Let us define $F(u) := \int_0^u f(\tau)d\tau$ for any $u \in \mathbb{R}$. Then, F satisfies the following two conditions:

  (AR1) F is positive in $\mathbb{R} \setminus \{0\}$.

  (AR2) There exists $\mu > 2$ such that $\mu F(u) \leq uf(u)$ for any $u \in \mathbb{R}$.

Namely, we assume that our nonlinearity has subcritical growth in the Sobolev sense (N2), and its primitive function F satisfies the positivity (AR1), and the so-called Ambrosetti-Rabinowitz condition (AR2).

Definition 3.1.1 *The time-local classical solution u to* (P) *is said to blow up in finite time if*

$$\|u(t)\|_\infty \to \infty \quad (t \uparrow T_m).$$

We assume $T_m < \infty$, i.e., we only consider time-local classical solutions to (P) which blows up in finite time throughout this chapter. The existence of time-local solution is well-known, see e.g. [15], [77], [88], and references therein. Furthermore, the fact that time-local solution blows up also follows from a similar argument as above. We would also like to clarify that our time-local classical solution means $u, \nabla u, \nabla^2 u$, and $\partial_t u$ are bounded and continuous in $\mathbb{R}^N \times (0, T_m)$.

Our main results read as follows:

Theorem 3.1.1 (Lower estimate of blow-up rate) *Under* (N1), (N2), *and* (AR), *every time-local solution u to* (P) *which blows up in finite time satisfies the following lower blow-up estimate:*

$$\text{there exists } C_L > 0 \ s.t. \ \frac{C_L}{(T_m - t)^{\frac{1}{p-2}}} \leq \|u(t)\|_\infty \qquad \text{(BRL)}$$

for any t sufficiently close to T_m.

Theorem 3.1.2 (Upper estimate of blow-up rate) *Assume* $p \in (2, \frac{6N+4}{3N-4})$ *and* $\mu = p$. *Then, under* (N1), (N2), *and* (AR), *every time-local solution* u *to* (P) *which blows up in finite time satisfies the following upper blow-up estimate:*

$$\text{there exists } C_U > 0 \; s.t. \; \|u(t)\|_\infty \leq \frac{C_U}{(T_m - t)^{\frac{1}{\mu-2}}} \tag{BRU}$$

for any $t \in [0, T_m)$.

Roughly speaking, the lower blow-up rate is controlled by exponent p in (N2) whereas the upper blow-up rate is controlled by μ in (AR). Notice that from Theorem 3.1.1 and Theorem 3.1.2 we can observe that $p \leq \mu$. However, the authors suspect that the assumption $\mu = p$ might not be an essential condition and therefore it is still an open problem to show that the upper blow-up rate in Theorem 3.1.2 holds when $p < \mu$. For ordinary differential equation case, the analog result of Theorem 3.1.2 does not require $\mu = p$, see [19].

Study of semilinear heat equations has been of great interests for many researchers (see e.g. [11], [13], [15], [31], [44], [57], [54], [52], etc). Let Ω be a bounded smooth domain (for simplicity). Consider the following semilinear heat equation:

$$\begin{cases} \partial_t u = \Delta u + \tilde{f}(u) & \text{in } \Omega \times (0, T_m), \\ u = 0 & \text{on } \partial\Omega \times (0, T_m), \\ u(\,\cdot\,, 0) = \tilde{u}_0 & \text{in } \Omega, \end{cases} \tag{P0}$$

where $\tilde{u}_0$ is the initial data. It is also well-known that some solutions to (P0) could blow up in finite time depending on $\tilde{f}$ and the size of initial data under some appropriate norms. For instance, the solution to (P0) blows up in finite time for some initial data in L^2 when the nonlinearity is locally Lipschitz in $\mathbb{R}$ and satisfies the compatibility condition (see e.g. [15]). In [44], Ikehata and Suzuki obtain equivalent condition for blow-up solutions to (P0) when $\tilde{f}(u) = u|u|^{p-2}$ with $p > 2$ in a convex bounded smooth domain by using stable and unstable sets from the argument of dynamical system (see e.g. [42]). When the domain is $\mathbb{R}^N$ and the solution is a radially symmetric nonnegative function, Mizoguchi shows that solutions to the semilinear heat equations with supercritical nonlinearity in the sense of Sobolev embedding blow up in finite time for sufficiently large initial data in [65]. As for the subcritical case, Fujita (see [30]) shows that every nontrivial solution blows up in finite time when $p \in (2, 2 + \frac{2}{N}]$ while there exists a nontrivial time-global classical solutions when $p > 2 + \frac{2}{N}$. For more results concerning the blow-up of solutions, see e.g. [54], [73], [29], and references therein.

We introduce the definition of Type I blow-up as follows:

Definition 3.1.2 (Type I blow-up) *The solution to semilinear heat equations which blows up in finite time denoted by u is said to have Type I blow-up if*

$$\text{there exits } C > 0 \text{ and } q > 2 \text{ such that } \|u(t)\|_\infty \leq C(T_m - t)^{\frac{-1}{q-2}}$$

for any $t \in [0, T_m)$.

Otherwise, the solution which blows up in finite time is said to have Type II blow-up. It is known that if the solution to the semilinear heat equation with polynomial nonlinearity is increasing in time and Ω is bounded, then Type I blow-up occurs. For subcritical polynomial nonlinearity in the sense of Sobolev embedding, Giga, Matsui, and Sasayama show that Type I blow-up occurs when the domain is $\mathbb{R}^N$ in [34]. Their results complement the results of Giga and Kohn in [36] which only cover $p \in (2, \frac{6N+4}{3N-4})$ when $N > 1$ for sign-changing solutions. For more results related to Type I blow-up, see also the monograph [78, pp.212].

In this chapter, we will study the lower and upper estimates of blow-up solutions to (P). Namely, we show that Type I blow-up occurs in view of Theorem 3.1.2. We also extend our results to a system of semilinear heat equations. In order to obtain Theorem 3.1.2, we mainly use similarity variable transformation and parabolic argument similar to the argument given by Giga and Kohn in [36]. However, Giga and Kohn's results mainly cover polynomial nonlinearity. Let p and q be subcritical in the sense of Sobolev embedding. Note that our results include nonlinear terms which are not covered in [36] such as

$$\tilde{f}(u) = \begin{cases} -u + u|u|^{q-2} & \text{if } u \in [-1, 1], \\ -u + u|u|^{p-2} & \text{if } u < -1 \text{ or } u > 1, \end{cases} \tag{3.1}$$

where $p < q$ and

$$\tilde{f}(u) = \begin{cases} -u + \left(1 + \frac{1}{2\pi L}\sin(\pi L u)\right)u|u|^6 & \text{if } |u| \leq 1, \\ -u + u|u|^2 & \text{if } u < -1 \text{ or } u > 1, \end{cases} \tag{3.2}$$

where we choose $L \in \mathbb{N}$, $p = \mu = 4$, $\kappa = 1$, and $N = 3$.

The structure of this chapter is as follows. In Section 2, we introduce some preliminary facts associated with (P) and some necessary tools whereas the proof of our main results will be given in Section 3. In Section 4, we extend our main results to a system of semilinear heat equations. The appendix deals with technical backgrounds.

3.2 Preliminaries

In this section, we introduce some basic tools concerning $L^p - L^q$ estimate, similarity variable transformation, and parabolic argument which would be applied to the time-local solution for (P).

We begin by considering the contraction semigroup $(S(t))_{t\geq 0}$ generated by Laplacian in $L^2(\mathbb{R}^N)$. We recall that our time-local solutions also satisfy the following integral equation:

$$u(t) = S(t)u_0 + \int_0^t S(t-\sigma)\big(-u(\sigma) + f(u(\sigma))\big)d\sigma \tag{3.3}$$

for any $t \in [0, T_m)$. See e.g. [15, pp.62, Proposition 5.1.1] for reference.

We also have the following ($L^p - L^q$ estimate) smoothing effect.

Lemma 3.2.1 (Smoothing effect) *Let $1 \leq \gamma_1 \leq \gamma_2 \leq \infty$. Then, for any $\varphi \in L^2(\mathbb{R}^N) \cap L^{\gamma_1}(\mathbb{R}^N)$ and for any $t > 0$, it holds that*

$$\|S(t)\varphi\|_{\gamma_2} \leq t^{-\frac{N}{2}\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)} \|\varphi\|_{\gamma_1}.$$

The proof of this lemma for bounded smooth domain can be found in [15] (see Proposition 3.5.7 pp.44). However, the argument for the proof still holds even when the domain is $\mathbb{R}^N$.

The use of similarity variable transformation is essential to change our time-local solution into a time-global solution. By changing our time-local solution into a time-global solution, we can find an energy functional with a good structure that satisfies non-increasing property and non-negativity (see e.g. [54]).

Now, we introduce the following similarity variable transformation. First, we fix an arbitrary point $a \in \mathbb{R}^N$. For convenience, we set

$$\beta := \frac{1}{\mu - 2}.$$

We let

$$y := (T_m - t)^{-\frac{1}{2}}(x - a), \text{ and } s := -\log(T_m - t), \tag{3.4}$$

to define

$$v_a(y, s) := (T_m - t)^{\beta} u(x, t). \tag{3.5}$$

Then, we apply (3.4) to (P) to obtain a new semilinear parabolic equation as follows:

$$\begin{cases} \partial s v_a = \Delta v_a - \frac{1}{2} y \cdot \nabla v_a - (\beta + e^{-s}) v_a + e^{-(\beta+1)s} f(e^{\beta s} v_a) & \text{in } \mathbb{R}^N \times (s_0, \infty), \\ v_a(\,\cdot\,, s_0) = e^{-\beta s_0} u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{Ps}$$

where $s_0 = -\log T_m$.

Next, we introduce a weight function as follows:

$$\rho(y) := e^{-\frac{|y|^2}{4}}$$

for any $y \in \mathbb{R}^N$. We use a weight function ρ here because we work in an unbounded domain and the first two terms of the right hand side of the first equation in (Ps) satisfies the following identity:

$$\nabla \cdot (\nabla v_a \rho) = \Delta v_a \rho - \frac{1}{2} y \cdot \nabla v_a \rho. \tag{3.6}$$

Following the definition of the weight function above, we introduce the following energy functionals associated with (Ps):

$$\begin{aligned} E[v_a(s)] := &\frac{1}{2} \|\nabla v_a(s)\|_{2,\rho}^2 + \frac{\beta + e^{-s}}{2} \|v_a(s)\|_{2,\rho}^2 \\ &+ \frac{1}{2} \int_{s_0}^{s} e^{-\sigma} \|v_a(\sigma)\|_{2,\rho}^2 d\sigma - e^{-(2\beta+1)s} \int_{\mathbb{R}^N} F(e^{\beta s} v_a(s)) \rho dy \\ &+ \beta \int_{s_0}^{s} e^{-(2\beta+1)\sigma} \int_{\mathbb{R}^N} \left[e^{\beta\sigma} v_a(\sigma) f(e^{\beta\sigma} v_a(\sigma)) - \mu F(e^{\beta\sigma} v_a(\sigma)) \right] \rho dy d\sigma \end{aligned} \tag{E01}$$

and

$$\begin{aligned} K[v_a(s)] := &\|\nabla v_a(s)\|_{2,\rho}^2 + (\beta + e^{-s}) \|v_a(s)\|_{2,\rho}^2 \\ &- e^{-(2\beta+1)s} \int_{\mathbb{R}^N} e^{\beta s} v_a(s) f(e^{\beta s} v_a(s)) \rho dy \end{aligned} \tag{K01}$$

for any $s \in [s_0, \infty)$. Our energy functionals satisfy the following proposition:

Proposition 3.2.1 (Energy equalities) *Let v_a be a solution of* (Ps). *Then, we have the following energy equalities:*

$$\frac{d}{ds} E[v_a(s)] = -\|\partial_s v_a(s)\|_{2,\rho}^2 \tag{E02}$$

and

$$\frac{1}{2}\frac{d}{ds}\|v_a(s)\|_{2,\rho}^2 = -K[v_a(s)] \tag{K02}$$

for any $s \in (s_0, \infty)$.

Proof of Proposition 3.2.1. We will divide the proof into two parts. The first part will show (E02) holds whereas the second part will show (K02).

Part 1. We multiply both sides of (Ps) by $\partial_s v_a$ together with ρ and then use (3.6) to see that

$$|\partial_s v_a(s)|^2\rho = \nabla\cdot(\nabla v_a \rho)\partial_s v_a - (\beta + e^{-s})v_a\partial_s v_a\rho + e^{-(\beta+1)s} f(e^{\beta s}v_a(s)).$$

Following the relation above, we integrate over spatial domain $\mathbb{R}^N$ and then we fix $s \in (s_0, \infty)$ to integrate over time domain $[s_0, s]$ to obtain the relation below:

$$\begin{aligned}\int_{s_0}^{s}\|\partial_\sigma v_a(\sigma)\|_{2,\rho}^2 d\sigma = &-\int_{s_0}^{s}\int_{\mathbb{R}^N}\frac{1}{2}\big(\partial_\sigma|\nabla v_a(\sigma)|^2 + (\beta + e^{-\sigma})\partial_\sigma|v_a(\sigma)|^2\big)\rho dy d\sigma \\ &+ \int_{s_0}^{s}\int_{\mathbb{R}^N} e^{-(\beta+1)\sigma} f(e^{\beta\sigma}v_a(\sigma))\partial_\sigma v_a \rho dy d\sigma.\end{aligned} \tag{3.7}$$

On the other hand, notice that

$$\begin{aligned}e^{-(\beta+1)\sigma} f(e^{\beta\sigma}v_a)\partial_\sigma v_a = &\ \partial_\sigma[e^{-(2\beta+1)\sigma}F(e^{\beta\sigma}v_a)] \\ &+ \beta e^{-(2\beta+1)\sigma}\Big[\mu F(e^{\beta\sigma}v_a) - e^{\beta\sigma}v_a f(e^{\beta\sigma}v_a)\Big]\end{aligned} \tag{3.8}$$

since F is the primitive of f.

By changing the order of integration for the first integral in the right hand side of (3.7) and substituting (3.8) into the second integral in the right hand side of (3.7), we obtain

$$\begin{aligned}\int_{s_0}^{s}\|\partial_\sigma v_a(\sigma)\|_{2,\rho}^2 d\sigma = &-E[v_a(s)] + \frac{1}{2}\|\nabla v_a(s_0)\|_{2,\rho}^2 + \frac{\beta + s_{s_0}}{2}\|v_a(s_0)\|_{2,\rho}^2 \\ &- e^{-(2\beta+1)s_0}\int_{\mathbb{R}^N} F(e^{\beta s_0}v_a(s_0))\rho dy.\end{aligned}$$

Therefore, we conclude that (E02) holds true.

Part 2. We multiply (Ps) by v_a together with ρ and then use (3.6) to see that

$$\frac{1}{2}\partial_s|v_a(s)|^2\rho = \nabla\cdot(\nabla v_a(s))v_a(s) - (\beta + e^{-s})|v(s)|^2\rho + e^{-(\beta+1)s}v_a(s)f(e^{\beta s}v_a(s))\rho.$$

Again, we integrate over spatial domain $\mathbb{R}^N$ and then we fix $s \in (s_0, \infty)$ to integrate over time domain $[s_0, s]$ as in **Part 1** to obtain the following relation:

$$\begin{aligned}\int_{s_0}^{s}\int_{\mathbb{R}^N}\frac{1}{2}\partial_\sigma|v_a(\sigma)|^2\rho dy = &-\int_{s_0}^{s}\int_{\mathbb{R}^N}|\nabla v_a(\sigma)|^2\rho + (\beta + e^{-\sigma})|v_a(\sigma)|^2\rho dy d\sigma\\ &+\int_{s_0}^{s}\int_{\mathbb{R}^N}e^{-(\beta+1)\sigma}v_a(\sigma)f(e^{\sigma v_a(\sigma)})\rho dy d\sigma.\end{aligned}$$

By changing the order of integration on the left hand side of the equation, we see that

$$\begin{aligned}\|v_a(s)\|_{2,\rho}^2 - \|v_a(s_0)\|_{2,\rho}^2 &= -\int_{s_0}^{s}\|\nabla v_a(\sigma)\|_{2,\rho}^2 + (\beta + e^{-\sigma})\|v_a(\sigma)|_{2,\rho}^2\\ &\quad - e^{-(2\beta+1)\sigma}\int_{\mathbb{R}^N}e^{\beta\sigma}v_a(\sigma)f(e^{\beta\sigma}v_a(\sigma))\rho dy d\sigma\\ &= -\int_{s_0}^{s}K[v_a(\sigma)]d\sigma\end{aligned}$$

which implies (K02) and thus we complete the proof for Proposition 3.2.1. ∎

We will show that the following concavity argument holds for our energy functional E.

Proposition 3.2.2 (Concavity argument, see e.g. [54]) *If v_a is a time-global solution to* (Ps)*, then*

$$E[v_a(s)] \geq 0$$

for any $s \in [s_0, \infty)$.

Proof of Proposition 3.2.2. We will prove by using contradiction argument. Assume on the contrary

$$\text{there exists } s_1 \in [s_0, \infty) \text{ s.t. } E[v_a(s_1)] < 0. \tag{3.9}$$

By combining (K02) together with (E01) and then using (AR2), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|v_a(s)\|_{2,\rho}^2 = -2E[v_a(s)] &+ \int_{s_0}^{s} e^{-\sigma}\|v_a(\sigma)\|_{2,\rho}^2 d\sigma \\ &+ e^{-(2\beta+1)s}\int_{\mathbb{R}^N} \underbrace{\Big[e^{\beta s}v_a(s)f(e^{\beta s}v_a(s)) - 2F(e^{\beta s}v_a(s))\Big]}_{\geq 0 \text{ from (AR2)}} \rho dy \\ &+ 2\beta\int_{s_0}^{s} e^{-(2\beta+1)\sigma}\int_{\mathbb{R}^N} \underbrace{\Big[e^{\beta\sigma}v_a(\sigma)f(e^{\beta\sigma}v_a(\sigma)) - \mu F(e^{\beta\sigma}v_a(\sigma))\Big]}_{\geq 0 \text{ from (AR2)}} \rho dy d\sigma\end{aligned}$$

which leads to the following inequality:

$$\frac{1}{2}\frac{d}{ds}\|v_a(s)\|_{2,\rho}^2 \geq -2E[v_a(s)] \geq -2E[v_a(s_0)] > 0 \tag{3.10}$$

for any $s \geq s_1$ since E is non-increasing and (3.9) holds.

Next, we set $Y(s) := \frac{1}{2}\int_{s_1}^{s}\|v_a(\sigma)\|_{2,\rho}^2 d\sigma$. Then, we use (3.10) to see that

$$Y''(s) \geq -2E[v(s_1)] > 0$$

for any $s \geq s_1$ which means

$$\lim_{s\to\infty} Y'(s) = \infty \text{ and } \lim_{s\to\infty} Y(s) = \infty. \tag{3.11}$$

We multiply Y by Y'' and use (3.10) to observe

$$\begin{aligned} Y(s)h''(s) &= \int_{s_1}^{2}\|v_a(\sigma)\|_{2,\rho}^2 d\sigma Y''(s) \\ &\geq \int_{s_1}^{2}\|v_a(\sigma)\|_{2,\rho}^2 d\sigma(-2E[v_a(s)]) \\ &\geq \frac{\mu}{2}\int_{s_1}^{2}\|v_a(\sigma)\|_{2,\rho}^2 d\sigma \int_{s_1}^{s}\|\partial_\sigma v_a(\sigma)\|_{2,\rho}^2 d\sigma \\ &\geq \frac{\mu}{2}\Big[\int_{s_1}^{s}\|v_a(\sigma)\|_{2,\rho}\|\partial_\sigma v_a(\sigma)\|_{2,\rho} d\sigma\Big]^2 \\ &\geq \frac{\mu}{2}\Big[\int_{s_1}^{s}\int_{\mathbb{R}^N}|v_a(\sigma)\partial_\sigma v_a(\sigma)|\rho dy d\sigma\Big]^2 \\ &\geq \frac{\mu}{2}[Y'(s) - Y'(s_1)]^2. \end{aligned}$$

This relation together with (3.11) allows us to choose $\theta \in (\frac{2}{\mu}, 1)$ and $s_\theta > s_1$ large enough so that

$$Y(s)Y''(s) \geq \frac{\mu}{2}\theta(Y'(s))^2 \text{ for any } s \in (s_\theta, \infty).$$

But according to Lemma .0.2, Y cannot exist time-globally which is impossible since v_a exists as the time-global solution to (Ps). Hence, (3.9) is false and the conclusion of this proposition must be true. ∎

We introduce the following general parabolic equation in the form of

$$\partial_s v - \nabla \cdot (A\nabla v) + B \cdot \nabla v + gv = 0 \tag{Pg}$$

with continuously varying coefficients. In this chapter, we only need the following two lemmas related to (Pg) to be applied to our transformed problem (Ps).

Lemma 3.2.2 *Let $2 \leq \mu' < \infty$ and $1 \leq r < \frac{2(\mu'+1)}{3}$. If $v \in L^{2\mu'}((0,1); B_R(0))$ and $\partial_s v \in L^2((0,1); B_R(0))$, then*

$$\sup_{0<\tau<1} \|v(\tau)\|_{L^r\left(B_R(0)\right)} \leq C\Bigg(\int_0^1 \|\partial_\tau v(\tau)\|^2_{L^2\left(B_R(0)\right)} + \|v(\tau)\|^2_{L^2\left(B_R(0)\right)} d\tau \Bigg)^{\frac{(1-\nu)}{2}} \times \Bigg(\int_0^1 \|v(\tau)\|^{2\mu'}_{L^{\mu'}\left(B_R(0)\right)} d\tau \Bigg)^{\frac{\nu}{2\mu'}}$$

for some constant $C > 0$ where $\nu = \frac{\mu'}{\mu'+1}$.

For details of the proof of Lemma 3.2.2, see [35] and the appendix in [16].

Lemma 3.2.3 *Let $v(y,\tau)$ solve* (Pg) *in $B_R(0)\times(0,1) \subset \mathbb{R}^N \times \mathbb{R}$, and assume further:*

(A1) *There exists $\lambda_0 > 0$ and $\lambda_1 > 0$ such that $|B(y,\tau)| \leq \lambda_1$ and $\lambda_0|\xi|^2 \leq (A(y,\tau)\xi, \xi) \leq \lambda_0^{-1}|\xi|^2$.*
(A2) *There exists $\lambda 2 > 0$ such that $\int_0^1 \int_{B_R(0)} |v|^2 dy d\tau \leq \lambda_2$.*
(A3) *There exists $\lambda_3 > 0$, $r > 0$, and $q \geq 1$ such that $\frac{1}{r} + \frac{N}{2q} < 1$ and $\int_0^1 \Bigg(\int_{B_R(0)} |g|^q dy \Bigg)^{\frac{r}{q}} d\tau \leq \lambda_3$.*

Then, there is a constant M (depending only on λ_0, λ_1, λ_2, λ_3, r, N, q, and R) such that

$$|v| \leq M \text{ in } B_{\frac{R}{2}}(0) \times \left(\frac{1}{2}, 1\right). \tag{3.12}$$

The proof of Lemma 3.2.3 can be found in [52] (See Theorem 7.1 pp. 181 and Theorem 8.1 pp. 192).

3.3 Proof of the main results

3.3.1 Proof of Theorem 3.1.1

We begin the proof of our first main theorem by choosing $T_0 \in [0, T_m)$ so that for any $t \in [T_0, T_m)$, we have

$$\|u(t)\|_\infty \geq \kappa,$$

which can be done since our solution blows up in finite time. We also fix $t \in [T_0, T_m)$ and $L_1 > 0$ small enough so that

$$L_1 \leq \min\left\{\frac{1}{4}, T_m - t, t\right\}. \tag{3.13}$$

Next, we fix τ and T_1 in (t, T_m) so that $t < \tau \leq t + L_1 < T_1 < T_m$ and

$$L_2(T_1) := \max_{\sigma \in [0,T_1]} \|u(\sigma)\|_\infty = 2\|u(t)\|_\infty. \tag{3.14}$$

By using (3.3) and taking $\gamma_1 = \gamma_2 = \infty$ together with $\varphi = -u + f(u)$ in Lemma 3.2.1, we see that

$$\begin{aligned}\|u(\tau)\|_\infty &\leq \|u(t)\|_\infty + \int_t^\tau \| - u(\sigma) + f(u(\sigma))\|_\infty d\sigma \\ &\leq \|u(t)\|_\infty + L_2(T_1)(\tau - t) + \int_t^\tau \|f(u(\sigma))\|_\infty d\sigma.\end{aligned}$$

Then, by using (N2) and (3.13) for the inequality above, we obtain

$$\|u(\tau)\|_\infty \leq \|u(t)\|_\infty + L_2(T_1)(\tau - t) + C_{N2} L_2(T_1)^{p-1}(T_m - t)$$

whence follows

$$L_2(T_1) \leq \|u(t)\|_\infty + \frac{1}{4} L_2(T_1) + C_{N2} L_2(T_1)^{p-1}(T_m - t)$$

in view of (3.14). We subtract $\frac{1}{4}L_2(T_1)$ from both sides of the inequality above and use (3.14) again to see that

$$\begin{aligned}\frac{3}{4} L_2(T_1) &\leq \|u(t)\|_\infty + C_{N2} L_2(T_1)^{p-1}(T_m - t) \\ \frac{1}{2}\|u(t)\|_\infty &\leq 2^{p-1} C_{N2} \|u(t)\|_\infty^{p-1}(T_m - t) \\ 1 &\leq 2^p C_{N2} \|u(t)\|_\infty^{p-2}(T_m - t)\end{aligned}$$

which implies

$$\frac{1}{2^p C_{N2}(T_m - t)^{\frac{1}{p-2}}} \leq \|u(t)\|_\infty.$$

By taking $C_L = \frac{1}{2^p C_{N2}}$, we complete the proof of Theorem 3.1.1. ∎

3.3.2 Proof of Theorem 3.1.2

Before we proceed to the details of the proof, we would like to discuss the idea of the proof. We use the parabolic estimates in Section 3.2 (Lemma 3.2.2 and Lemma 3.2.3) to bound v_a uniformly in a inside a ball. Then, we recall (3.5) to see that

$$(T_m - t)^\beta |u(a,t)| = |v_a(0,s)| \leq \sup_{s \in [s_0, \infty))} \|v_a(s)\|_{L^\infty\left(B_{\frac{R}{2}}(0)\right)}$$

for any $t \in [0, T_m)$ and for any $a \in \mathbb{R}^N$. Therefore, if we can show the following proposition, then the conclusion of Theorem 3.1.2 is immediate.

Proposition 3.3.1 *There exists $M > 0$ independent of $a \in \mathbb{R}^N$ such that*

$$\|v_a(s)\|_{L^\infty(B_{\frac{R}{2}}(0))} \leq M$$

for any $s \in [s_0, \infty)$.

Before we prove Proposition 3.3.1, we need the following proposition first.

Proposition 3.3.2 (Boundedness of energy functional) *Let $a \in \mathbb{R}^N$ and v_a be the solution of* (Ps)*. Then, the following inequalities hold true:*

$$\int_{s_0}^{\infty} \|\partial_\sigma v_a(\sigma)\|_{2,\rho}^2 d\sigma \leq E[v_a(s_0)] \tag{3.15}$$

$$\|v_a(s)\|_{2,\rho}^2 \leq M_1 \text{ for any } s \geq s_0 \tag{3.16}$$

$$\int_{s}^{s+1} \|v_a(\sigma)\|_{\mu,\rho}^{2\mu} d\sigma \leq M_1 \text{ for any } s \geq s_0 \tag{3.17}$$

The constant M_1 depends only on $E[v_a(s_0)]$, N, and μ.

Proof of Proposition 3.3.2. Although the proof is essentially similar to the one given in [36], we will give the sketch here for convenience.

First, we see that (3.15) follows immediately from Proposition 3.2.2 and (E02). Next, we set $g_1 := \|v_a\|_{2,\rho}$ and follow similar calculation from (3.10) to see

$$\frac{1}{2}\frac{d}{ds}g_1(s)^2 \geq -2E[v_a(s)] + (\mu-2)e^{-(2\beta+1)s}\int_{\mathbb{R}^N} F(e^{\beta s}v_a(s))\rho dy. \tag{3.18}$$

Then, we fix $\delta > 0$ and $s \in [s_0, \infty)$ to set $\Omega_{\delta,s} := \{y \in \mathbb{R}^N \mid |e^{\beta s}v_a(s)| \geq \delta\}$. By using (AR) and Lemma .0.1 we see that

$$\text{if } |e^{\beta s}v_a(s)| > \delta \text{ then } C_{\mu,\delta}\|v_a(s)\|^{\mu}_{L^{\mu}(\Omega_\delta),\rho} \leq e^{-(2\beta+1)s}\int_{\Omega_\delta} F(e^{\beta s}v_a(s))\rho dy \tag{3.19}$$

and

$$\text{if } |e^{\beta s}v_a(s)| \leq \delta \text{ then } C_{\mu,\delta}\|v_a(s)\|^{\mu}_{L^{\mu}(\mathbb{R}^N\setminus\Omega_\delta),\rho} \leq C_{\mu,\delta}\delta^{\mu}\int_{\mathbb{R}^N} \rho dy =: K(\delta). \tag{3.20}$$

From (3.18) and the positivity of F together with the fact that $\Omega_{\delta,s} \subset \mathbb{R}^N$, we obtain

$$\frac{1}{2}\frac{d}{ds}g_1(s)^2 \geq -2E[v_a(s)] + (\mu-2)e^{-(2\beta+1)s}\int_{\Omega\delta,s} F(e^{\beta s}v_a(s))\rho dy.$$

By adding $\mu K(\delta)$ on both sides of the inequality above and using (AR1), we have

$$\frac{1}{2}\frac{d}{ds}g_1(s)^2 + \mu K(\delta) \geq -2E[v_a(s)] + (\mu-2)e^{-(2\beta+1)s}\int_{\Omega_{\delta,s}} F(e^{\beta s}v_a(s))\rho dy + (\mu-2)K(\delta)$$

whence it follows that

$$\frac{1}{2}\frac{d}{ds}g_1(s)^2 + \mu K(\delta) \geq -2E[v_a(s_0)] + C_{\mu,\delta}\|v_a(s)\|^{\mu}_{\mu,\rho} \tag{3.21}$$

for any $s \in [s_0, \infty)$ in view of (3.19) and (3.20). By applying $L^p - L^q$ inequality on the right hand side of the inequality of (3.21), we arrive at

$$C_{\mu,\delta}g_1(s)^{\mu} \leq 2E[v_a(s_0)] + \mu K(\delta) + g_1(s)\frac{dg_1(s)}{ds} \tag{3.22}$$

for any $s \in [s_0, \infty)$. Then, we deduce either

$$g_1(s) \leq 1 \text{ or } C_{\mu,\delta}g(s)^{\mu-1} \leq \frac{dg_1(s)}{ds} + 2E[v_a(s_0)] + \mu K(\delta) \tag{3.23}$$

for any $s \in [s_0, \infty)$. If $g_1(s) \leq 1$, then we are done but if it is not then we need to bound $\frac{dg_1}{ds}$ by the following inequality

$$\begin{aligned}\left|\frac{dg_1}{ds}\right|^2 &= \left|\frac{1}{g_1}\int_{\mathbb{R}^N} v_a \partial_s v_a \rho dy\right|^2 \\ &\leq \frac{1}{g_1^2}\|v_a\|_{2,\rho}^2 \|\partial_s v_a\|_{2,\rho}^2 \\ &= \|\partial_s v_a\|_{2,\rho}^2\end{aligned}$$

which means

$$\begin{aligned}\int_s^{s+1} \left|\frac{dg_1(\sigma)}{d\sigma}\right|^2 d\sigma &\leq \int_s^{s+1} \|\partial_\sigma v_a(\sigma)\|_{2,\rho}^2 d\sigma \\ &\leq E[v_a(s_0)] \end{aligned} \tag{3.24}$$

for any $s \in [s_0, \infty)$. In view of (3.23), we see that

$$\int_s^{s+1} g_1(\sigma)^{2(\mu-1)} d\sigma \leq \frac{2}{C_{\mu,\delta}^2}\bigg(\int_s^{s+1} \left|\frac{dg_1(\sigma)}{d\sigma}\right|^2 d\sigma + (2E[v_a(s_0)] + \mu K(\delta))^2\bigg). \tag{3.25}$$

Since $\mu > 2$ and $g_1(s) > 1$, we also have

$$\int_s^{s+1} g_1(\sigma)^2 d\sigma \leq \frac{2}{C_{\mu,\delta}^2}\bigg(\int_s^{s+1} \left|\frac{dg_1(\sigma)}{d\sigma}\right|^2 d\sigma + (2E[v_a(s_0)] + \mu K(\delta))^2\bigg). \tag{3.26}$$

By using (3.24), (3.25), (3.26), and the following Sobolev inequality

$$\|g_1\|_{L^\infty(s,s+1)} \leq C\big(\|g_1'\|_{L^2(s,s+1)} + \|g_1\|_{L^2(s,s+1)}\big)^{\frac{1}{\mu}} \|g_1\|_{L^{2(\mu-1)}(s,s+1)}^{1-\frac{1}{\mu}}, \tag{3.27}$$

we deduce that (3.16) holds true.

In order to show (3.17), we only need to consider (3.21) and follow the calculation below

$$\begin{aligned}\|v_a(s)\|_{\mu,\rho}^{2\mu} &\leq \frac{2}{C_{\mu,\delta}^2}(E[v_a(s_0)] + \mu K(\delta))^2 + \frac{2}{C_{\mu,\delta}^2} g_1(s)^2 \left|\frac{dg_1(s)}{ds}\right|^2 \\ &\leq \frac{2}{C_{\mu,\delta}^2}\bigg((E[v_a(s_0)] + \mu K(\delta))^2 + \|g_1\|_\infty^2 \left|\frac{dg_1(s)}{ds}\right|^2\bigg).\end{aligned}$$

By fixing $s \in [s_0, \infty)$ and then integrating both sides of the inequality above in $(s, s+1)$, and observing that our bound does not depend on s in view of (3.24) together with (3.27), we immediately obtain (3.17) and complete the proof. ∎

Although our bound in the proof of Proposition 3.3.2 requires δ, it is essentially a constant which can be fixed arbitrarily and does not affect our results. For instance, we can simply choose $\delta = 1$ for convenience. However, we have not shown that our bound does not depend on $a \in \mathbb{R}^N$. In order to circumvent this problem, we also need the following lemma.

Lemma 3.3.1 *$E[v_a(s_0)]$ varies continuously with $a \in \mathbb{R}^N$ and stays uniformly bounded.*

Proof of Lemma 3.3.1. Fix $a \in \mathbb{R}^N$ and recall (E01) to see that

$$\begin{aligned}
E[v_a(s_0)] &= \frac{1}{2}\|\nabla v_a(s_0)\|_{2,\rho}^2 + \frac{\beta + e^{-s_0}}{2}\|v_a(s_0)\|_{2,\rho}^2 \\
&\quad - e^{-(2\beta+1)s_0}\int_{\mathbb{R}^N} F(e^{\beta s_0} v_a(s_0))\rho dy \\
&= T_m^{2\beta+1-\frac{N}{2}}\Bigg(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_0|^2 \rho(\frac{x-a}{\sqrt{T_m}})dx + \frac{\beta+1}{2}\int_{\mathbb{R}^N} |u_0|^2 \rho(\frac{x-a}{\sqrt{T_m}})dx \\
&\quad - \int_{\mathbb{R}^N} F(u_0)\rho(\frac{x-a}{\sqrt{T_m}})dx\Bigg) \qquad (3.28)
\end{aligned}$$

Since $|\rho| \leq 1$ and $u_0 \in L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, we see that

$$\begin{aligned}
E[v_a(s_0)] &\leq T_m^{2\beta+1-\frac{N}{2}}\Bigg(C_\beta \|u_0\|_{H^1}^2 + \max\Bigg\{\mu \int_{\mathbb{R}^N} F(u_0)dx, \sup_{|u|<\kappa}|F(u)| \int_{\mathbb{R}^{\mathbb{N}}} \rho(\frac{x}{\sqrt{T_m}})dx\Bigg\}\Bigg) \\
&\leq T_m^{2\beta+1-\frac{N}{2}}\Bigg(C_\beta \|u_0\|_{H^1}^2 + \max\Bigg\{C_{N2}\|u_0\|_p^p, C_{T_M}\sup_{|u|<\kappa}|F(u)|\Bigg\}\Bigg) \\
&\leq T_m^{2\beta+1-\frac{N}{2}}\Bigg(C_\beta \|u_0\|_{H^1}^2 + \max\Bigg\{C_{N2}\|u_0\|_\infty^{p-2}\|u_0\|_2^2, C_{T_M}\sup_{|u|<\kappa}|F(u)|\Bigg\}\Bigg) \\
&< \infty
\end{aligned}$$

which means $E[v_a(s_0)]$ is uniformly bounded with respect to $a \in \mathbb{R}^N$.

Continuity follows from standard argument by applying Lebesgue dominated convergence theorem to (3.28) for any sequence $(a_n)_{n\in\mathbb{N}}$ which converges to a as $n \to \infty$. Hence, our proof is complete. ∎

Now, we will prove Proposition 3.3.1. The idea here is to apply Lemma 3.2.3 and obtain an estimate inside an arbitrary ball that does not depend on s.

Proof of Proposition 3.3.1. First, we set $A^{ij} = \delta^{ij}$, $B^i = \frac{1}{2}y^i$, and

$$g = \beta + e^{-s} - \frac{e^{-(\beta+1)s} f(e^{\beta s} v_a)}{v_a}$$

for any $i, j \in \{1, 2, ..., N\}$ in (Pg). Then, we fix $R > 0$ and choose $\lambda_0 = 1$ and $\lambda_1 = R$ to see that (A1) is satisfied. (A2) is immediate from (3.16). It only remains to show that (A3) holds true.

In order to show (A3) holds true, we begin by setting $\frac{N}{2}(\mu - 2) \leq r < \frac{2(\mu+1)}{3}$ and $q = \frac{r}{\mu-2}$. Following this, we may use condition (N1) to see that

$$\left| \frac{e^{-(\beta+1)s} f(e^{\beta s} v_a)}{v_a} \right| \leq \max\left\{ e^{\frac{p-\mu}{\mu-2}s} |v_a(s)|, e^{-s} C_\kappa \right\} \leq e^{\frac{p-\mu}{\mu-2}s} |v_a(s)| + C_\kappa$$

for any $s \in [s_0, \infty)$. Then, we proceed with the following estimate

$$\begin{aligned}
\int_s^{s+1} \left(\int_{B_R(0)} |g|^q dy \right)^{\frac{r}{q}} d\tau &\leq \int_s^{s+1} \Bigg((\beta + e^{-\tau}) \text{measure}(B_R(0))^{\frac{1}{q}} \\
&\quad + \left\| e^{-(\beta+1)\tau} \frac{f(e^{\beta\tau} v_a(\tau))}{v_a(\tau)} \right\|_{L^q(B_R(0))} \Bigg)^r d\tau \\
&\leq \int_s^{s+1} \Bigg(C_{1,\beta,q,R} + C_\kappa \text{measure}(B_R(0))^{\frac{1}{q}} + \\
&\quad e^{\frac{p-\mu}{\mu-2}\tau} C_{N2} \Big(\int_{B_R(0)} |v_a(\tau)|^{(p-2)q} dy \Big)^{\frac{1}{q}} \Bigg)^r d\tau \\
&= \int_s^{s+1} \Bigg(C_{1,\beta,q,R,\kappa} + \\
&\quad e^{\frac{p-\mu}{\mu-2}\tau} C_{N2} \Big(\int_{B_R(0)} |v_a(\tau)|^{(p-2)q} dy \Big)^{\frac{1}{q}} \Bigg)^r d\tau
\end{aligned}$$

.

Since we assume $\mu = p$, we see that

$$\int_s^{s+1} \left(\int_{B_R(0)} |g|^q dy \right)^{\frac{r}{q}} d\tau \leq \int_s^{s+1} \left(C_{1,\beta,q,R,\kappa} + C_{N2} \Big(\int_{B_R(0)} |v_a(\tau)|^r dy \Big)^{\frac{1}{q}} \right)^r d\tau. \tag{3.29}$$

It remains to bound $\|v(\tau)\|^r_{L^r(B_R(0))}$ by using Lemma 3.2.2. Here, we take $\mu' = \mu = p \in (2, \frac{6N+4}{3N-4})$ and recall Proposition 3.3.2 with Lemma 3.3.1 to deduce

$$\sup_{\tau\in(s,s+1)} \int_{B_R(0)} |v_a(\tau)|^r d\tau \le C_{2,\mu,N}$$

for any $s \in [s_0, \infty)$ which means the right hand side of (3.29) is bounded by a constant whence it follows that

$$\int_s^{s+1} \left(\int_{B_R(0)} |g|^q dy \right)^{\frac{r}{q}} d\tau \le C_{3,N,\mu,R,\kappa}$$

in view of (3.29). By taking $\lambda_3 = C_{3,N,\mu,R,\kappa}$, we infer (A3) holds true.
Therefore, applying Lemma 3.2.3, we conclude that

$$\text{there exists } M_2 > 0 \text{ such that } |v_a| \le M_2 \text{ on } B_{\frac{R}{2}}(0) \times \left(s + \frac{1}{2}, s+1 \right)$$

for any $s \in [s_0, \infty)$. Here, M_2 only depends on N, μ, R, and κ. Therefore, we can deduce that

$$\sup_{s\in(s_0+\frac{1}{2},\infty)} \|v_a(s)\|_{L^\infty(B_{\frac{R}{2}}(0))} < M_2.$$

It remains to show the estimate for $[s_0, s_0 + \frac{1}{2}]$ which is immediate since our solution is a classical one. Hence, we conclude this proposition is true. ∎

Now, we are ready to prove Theorem 3.1.2 by observing that

$$(T_m - t)^\beta |u(a,t)| \le M$$

for any $a \in \mathbb{R}^N$ and for any $t \in [0, T_m)$. Note that M does not depend on a which means we can obtain

$$(T_m - t)^\beta \sup_{a\in\mathbb{R}} |u(a,t)| \le M$$

for any $t \in [0, T_m)$ which completes the proof of Theorem 3.1.2. ∎

3.4 Extension to a system of equations

In this section, we consider the following system of semilinear heat equations

$$\begin{cases} \partial_t u = \Delta u - u + f(u,v) & \text{in } \mathbb{R}^N \times (0,T_m) \\ \partial_t v = \Delta v - v + g(u,v) & \text{in } \mathbb{R}^N \times (0,T_m) \\ u(\,\cdot\,,0) = u_0 & \text{in } \mathbb{R}^N \\ v(\,\cdot\,,0) = v_0 & \text{in } \mathbb{R}^N \end{cases} \tag{P1}$$

where $u_0,\, v_0 \in L^\infty(\mathbb{R}^N) \cap H_0^1(\mathbb{R}^N)$. We impose the following conditions for f and g:

(SN1) Function $f,\, g : \mathbb{R}^2 \to \mathbb{R}$ are locally Lipschitz in $\mathbb{R}^2$, $f(0,0) = g(0,0) = 0$, and $\lim_{|(u,v)|\to 0} \frac{|f(u,v)|}{|(u,v)|} = \lim_{|(u,v)|\to 0} \frac{|g(u,v)|}{|(u,v)|} = 0$.

(SN2) There exists $\alpha' > 1$, $\beta' > 1$ with $\alpha' + \beta' = p \in (2, 2^*)$, $\kappa_S > 0$, and $C_{SN2} > 0$ such that $|f(u,v)| \le C_{SN2}|u|^{\alpha'-1}|v|^{\beta'}$ and $|g(u,v)| \le C_{SN2}|u|^{\alpha'}|v|^{\beta'-1}$ for any $(u,v) \in \mathbb{R}^2$ with $|(u,v)| \ge \kappa_S$.

(SAR) There exists $F : \mathbb{R}^2 \to \mathbb{R}$ which satifies the following conditions:

- (SAR0) $\nabla F(u,v) = (f(u,v), g(u,v))^\mathsf{T}$ For any $(u,v) \in \mathbb{R}^2$.
- (SAR1) F is positive in $\mathbb{R}^2 \setminus \{(0,0)\}$.
- (SAR2) There exists $\mu > 2$ such that $\mu F(u,v) \le uf(u,v) + vg(u,v)$ for any $(u,v) \in \mathbb{R}^2$.

Condition (SN1), (SN2), and (SAR) correspond to (N1), (N2), and (AR) respectively for the single equation case.

We only consider the time-local classical solution which blows up in finite time with respect to (P1) throughout this section which occurs as follows:

$$\|u(t)\|_\infty + \|v(t)\|_\infty \to \infty \text{ as } t \uparrow T_m \tag{3.30}$$

The corresponding main results for the system case read as follows:

Theorem 3.4.1 (Lower estimate of blow-up rate) *Under* (SN1), (SN2), *and* (SAR), *every time-local solution* (u,v) *to* (P1) *which blows up in finite time satisfies the following lower blow-up estimate:*

$$\text{there exists } C_L > 0 \text{ s.t. } \frac{C_L}{(T_m - t)^{\frac{1}{p-2}}} \le \|u(t)\|_\infty + \|v(t)\|_\infty \tag{SBRL}$$

for any t *sufficiently close to* T_m.

Theorem 3.4.1 corresponds to Theorem 3.1.1 in the single equation case. The proof of Theorem 3.4.1 follows the argument from the proof of Theorem 3.1.1 in the single equation case. Therefore, the details of the proof can be omitted for the sake of brevity.

Theorem 3.4.2 (Upper estimate of blow-up rate) *Assume $p \in (2, \frac{6N+4}{3N-4})$ and $\mu = p$. Then, under* (SN1), *(SN2), and* (SAR), *every time-local solution u to* (P) *which blows up in finite time satisfies the following upper blow-up estimate:*

$$\text{there exists } C_U > 0 \text{ s.t. } \|u(t)\|_\infty + \|v(t)\|_\infty \leq \frac{C_U}{(T_m - t)^{\frac{1}{\mu-2}}} \tag{SBRU}$$

for any $t \in [0, T_m)$.

Theorem 3.4.2 corresponds to Theorem 3.1.2 in the single equation case.

Now, we introduce similar variable transformation as in (3.4) to define:

$$\begin{cases} \tilde{u}_a(y,s) & := (T_m - t)^\beta u(x,t), \\ \tilde{v}_a(y,s) & := (T_m - t)^\beta v(x,t), \end{cases} \tag{3.31}$$

which corresponds to (3.5) in the single equation case. Following the similar variable transformation from above, (P1) is transformed into a new parabolic equation as follows

$$\begin{cases} \partial s\tilde{u}_a = \Delta\tilde{u}_a - \frac{1}{2}y \cdot \nabla\tilde{u}_a - (\beta + e^{-s})\tilde{u}_a + e^{-(\beta+1)s} f(e^{\beta s}\tilde{u}_a, e^{\beta s}\tilde{v}_a) & \text{in } \mathbb{R}^N \times (s_0 \times \infty), \\ \partial s\tilde{v}_a = \Delta\tilde{v}_a - \frac{1}{2}y \cdot \nabla\tilde{v}_a - (\beta + e^{-s})\tilde{v}_a + e^{-(\beta+1)s} g(e^{\beta s}\tilde{u}_a, e^{\beta s}\tilde{v}_a) & \text{in } \mathbb{R}^N \times (s_0 \times \infty), \\ \tilde{u}_a(\,\cdot\,, s_0) = e^{-\beta s_0} u_0 & \text{in } \mathbb{R}^N, \\ \tilde{v}_a(\,\cdot\,, s_0) = e^{-\beta s_0} v_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{P1s}$$

where $s_0 := -\log T_m$.

Next, we can define the following energy functionals associated with (P1s) as follows:

$$
\begin{aligned}
\tilde{E}[\tilde{u}_a, \tilde{v}_a(s)] :=& \frac{1}{2}(\|\nabla \tilde{u}_a(s)\|_{2,\rho}^2 + \|\nabla \tilde{v}_a(s)\|_{2,\rho}^2) + \frac{\beta + e^{-s}}{2}(\|\tilde{u}_a(s)\|_{2,\rho}^2 + \|\tilde{v}_a(s)\|_{2,\rho}^2) \\
&+ \frac{1}{2}\int_{s_0}^{s} e^{-\sigma}\|\tilde{u}_a(\sigma)\|_{2,\rho}^2 + \|\tilde{v}_a(\sigma)\|_{2,\rho}^2 d\sigma \\
&- e^{-(2\beta+1)s}\int_{\mathbb{R}^N} F(e^{\beta s}\tilde{u}_a(s), e^{\beta s}, \tilde{v}_a(s))\rho dy \\
&+ \beta\int_{s_0}^{s} e^{-(2\beta+1)\sigma}\int_{\mathbb{R}^N} \big[e^{\beta\sigma}\tilde{u}_a(\sigma) f(e^{\beta\sigma}\tilde{u}_a(\sigma), e^{\beta\sigma}\tilde{v}_a(\sigma)) \\
&+ e^{\beta\sigma}\tilde{v}_a(\sigma) g(e^{\beta\sigma}\tilde{u}_a(\sigma), e^{\beta\sigma}\tilde{v}_a(\sigma)) - \mu F(e^{\beta\sigma}\tilde{u}_a(\sigma), e^{\beta\sigma}\tilde{v}_a(\sigma))\big]\rho dy d\sigma
\end{aligned} \tag{sE01}
$$

and

$$
\begin{aligned}
\tilde{K}[\tilde{u}_a, \tilde{v}_a(s)] :=& \|\nabla \tilde{u}_a(s)\|_{2,\rho}^2 + \|\nabla \tilde{v}_a(s)\|_{2,\rho}^2 + (\beta + e^{-s})(\|\tilde{u}_a(s)\|_{2,\rho}^2 + \|\tilde{v}_a(s)\|_{2,\rho}^2) \\
&- e^{-(2\beta+1)s}\int_{\mathbb{R}^N} [e^{\beta s}\tilde{u}_a(s) f(e^{\beta s}\tilde{u}_a(s), e^{\beta s}\tilde{v}_a(s)) \\
&+ e^{\beta s}\tilde{v}_a(s) g(e^{\beta s}\tilde{u}_a(s), e^{\beta s}\tilde{v}_a(s))]\rho dy.
\end{aligned} \tag{sK01}
$$

Here, $\tilde{E}$ and $\tilde{K}$ correspond to E and K respectively in the single equation case. Similar to the single equation case, we also have the following propositions and lemma for the system of equations.

Proposition 3.4.1 (Energy equalities for system of equations)
Let $(\tilde{u}_a, \tilde{v}_a)$ *be a solution of* (P1s). *We have the following energy identities:*

$$
\frac{d}{ds}\tilde{E}[\tilde{u}_a(s), \tilde{v}_a(s)] = -(\|\partial_s \tilde{u}_a(s)\|_{2,\rho}^2 + \|\partial_s \tilde{v}_a(s)\|_{2,\rho}^2) \tag{sE02}
$$

and

$$
\frac{1}{2}\frac{d}{ds}(\|\tilde{u}_a(s)\|_{2,\rho}^2 + \|\tilde{v}_a(s)\|_{2,\rho}^2) = -\tilde{K}[\tilde{u}_a(s), \tilde{v}_a(s)]. \tag{sK02}
$$

Proposition 3.4.2 (Concavity argument for system of equations) *If* $(\tilde{u}_a, \tilde{v}_a)$ *is a time-global solution of* (P1s), *then*

$$
\tilde{E}[\tilde{u}_a(s), \tilde{v}_a(s)] \geq 0 \tag{3.32}
$$

for anys $s \in [s_0, \infty)$.

Lemma 3.4.1 *$\tilde{E}[\tilde{u}_a(s_0), \tilde{v}_a(s_0)]$ varies continuously with $a \in \mathbb{R}^N$ and stays uniformly bounded.*

Proposition 3.4.1, Proposition 3.4.2, and Lemma 3.4.1 correspond to Proposition 3.2.1, Proposition 3.2.2, and Lemma 3.3.1 respectively in the single equation case. Since the proof follows the argument as in the single equation case, it will be omitted for the sake of brevity.

We will prove Theorem 3.4.2 now. The proof of Theorem 3.4.2 is essentially the same as the single equation case. Therefore, we only need to prove the following proposition which corresponds to Proposition 3.3.1 in the single equation case.

Proposition 3.4.3 *There exists $M > 0$ independent of $a \in \mathbb{R}^N$ such that*

$$\|\tilde{u}_a(s)\|_{L^\infty\left(B_{\frac{R}{2}}(0)\right)} + \|\tilde{v}_a(s)\|_{L^\infty\left(B_{\frac{R}{2}}(0)\right)} \leq M$$

for any $s \in [s_0, \infty)$.

The proof of Proposition 3.4.3 is essentially the same with the proof of Proposition 3.3.1. However, we need to replace Proposition (3.3.2) with the following proposition:

Proposition 3.4.4 *Let $a \in \mathbb{R}^N$ and $(\tilde{u}_a, \tilde{v}_a)$ be the solution of* (P1s). *Then, the following relations hold true:*

$$\int_{s_0}^{\infty} \|\partial_\sigma \tilde{u}_a(\sigma)\|_{2,\rho}^2 + \|\partial_\sigma \tilde{v}_a(\sigma)\|_{2,\rho}^2 d\sigma = \tilde{E}[\tilde{u}_a(s_0), \tilde{v}_a(s_0)] \tag{5.4a}$$

$$\|\tilde{u}_a(s)\|_{2,\rho}^2 + \|\tilde{v}_a(s)\|_{2,\rho}^2 \leq M_1 \text{ for all } s \geq s_0 \tag{5.4b}$$

$$\int_s^{s+1} \int_{\mathbb{R}^N} (|\tilde{u}_a(s)|^2 + |\tilde{v}_a(s)|^2)^{\frac{\mu}{2}} \rho dy d\sigma \leq M_1 \text{ for all } s \geq s_0 \tag{5.4c}$$

The constant M_1 depends only on $\tilde{E}[\tilde{u}_a(s_0), \tilde{v}_a(s_0)]$, N, and μ.

The proof of Proposition 3.4.4 is essentially similar to the proof of Proposition 3.3.2 in the single equation case.

We follow the proof of Proposition 3.3.1 to show that Proposition 3.4.3 holds true by replacing Proposition 3.3.2 with Proposition 3.4.4. the argument to both $\tilde{u}_a$ and $\tilde{v}_a$. Hence, we can finally conclude that Theorem 3.4.2 is true and complete the proof our main results for the system of equations.

Chapter 4

L^∞ Bounds for time-global solutions to a system of semilinear heat equations

In this chapter, we consider a system of semilinear heat equations with a subcritical Ambrosetti-Rabinowitz nonlinear term in the sense of Sobolev embedding. Our main goal in this study is to establish the existence of L^∞ global bounds for the time-global solutions of such a system of semilinear heat equations. In order to obtain our main goal, we mainly use scaling argument to our time global solution to find nontrivial blow-up profile and proceed to show that it is actually trivial by using the compactness of the orbit of solutions in L^q space for $q \in [2, 2^*)$ which results in a contradiction. The method above is also applicable to show the existence of L^∞-global bounds for semilinear heat equations with critical polynomial nonlinearity.

4.1 Introduction and main result

Let $N \in \mathbb{N}$, A is a real symmetric 2×2 matrix, and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. The critical Sobolev exponent is denoted by 2^* which is defined as $2^* := \frac{2N}{N-2}$ when $N \geq 3$ and $2^* := \infty$ when $N = 1, 2$. We also let $\vec{U} := (u_1, u_2)$ and $\partial_t \vec{U} := (\partial_t u_1, \partial_t u_2)$. Consider the following system of semilinear heat equations:

$$\begin{cases} \partial_t \vec{U} = \Delta \vec{U} + A\vec{U}^{\mathsf{T}} + \nabla F(\vec{U}) & \text{in } \Omega \times (0, T_m), \\ \vec{U} = \vec{0} & \text{on } \partial\Omega \times (0, T_m), \\ \vec{U}(\,\cdot\,, 0) = \vec{U}_0 & \text{in } \Omega, \end{cases} \tag{P}$$

where $\vec{U}_0 \in \big(L^\infty(\Omega) \times L^\infty(\Omega)\big) \cap \big(H_0^1(\Omega) \times H_0^1(\Omega)\big)$ for simplicity and T_m denotes the maximal existence time of a classical solution to (P). Our assumptions on the nonlinearity ∇F are the following:

(N1) Let $\nabla F = (f_1, f_2)$. The functions $f_1,\ f_2 : \mathbb{R}^2 \to \mathbb{R}$ are locally Lipschitz in $\mathbb{R}^2$ and satisfy $f_1(\vec{0}) = f_2(\vec{0}) = 0$.

(N2) There exists $\alpha > 1$, $\beta > 1$ satisfying $\alpha + \beta = p \in [2 + \frac{2}{N}, 2^*)$, and $C_{N2} > 0$ satisfying $|f_1(\vec{U})| \le C_{N2}|u_1|^{\alpha-1}|u_2|^{\beta}$ and $|f_2(\vec{U})| \le C_{N2}|u_1|^{\alpha}|u_2|^{\beta-1}$ for any $\vec{U} \in \mathbb{R}^2$.

(AR) There exists $F : \mathbb{R}^2 \to \mathbb{R}$ which satisfies the following conditions:

(AR0) The function F is the antiderivative of ∇F.

(AR1) F is positive in $\mathbb{R}^2 \setminus \{\vec{0}\}$.

(AR2) There exists $\mu > 2$ such that $\mu F(\vec{U}) \le \vec{U} \cdot \nabla F(\vec{U})$ for any $\vec{U} \in \mathbb{R}^2$.

These conditions imply the nonlinearity has a variational form (AR0) with a subcritical growth in the Sobolev sense of (N2), and a positive primitive function F which satisfies the Ambrosetti-Rabinowitz condition (AR2).

In the proof of our main result, we are working in the space $L^{p_0}(\Omega)$ with $p_0 := \frac{N}{2}(p-2)$ which is invariant under the scaling (4.20) in Section 2. We see that

$$1 \le p_0 < 2^* \tag{4.1}$$

is assured by the assumption

$$p \ge 2 + \frac{2}{N} \text{ and } p < 2^*$$

respectively, i.e., the condition (N2) assures (4.1), the well-definedness of $L^{p_0}(\Omega)$ and the subcriticality of p_0 in the Sobolev sense.

An example which satisfies the assumptions above is

$$F(\vec{U}) = u_1^2 \log(1 + |u_1|^2) u_2^2 \tag{4.2}$$

which gives

$$\begin{cases} f_1(\vec{U}) & = 2u_1 \log(1+|u_1|^2) + \frac{2u_1^3 u_2^2}{1+|u_1|^2}, \\ f_2(\vec{U}) & = 2u_1^2 \log(1+|u_1|^2) u_2. \end{cases} \tag{4.3}$$

Indeed, it is easy to see that F, f_1 and f_2 satisfy (N1), (N2), and (AR) by taking $\alpha = 2$, $\beta = 2$ and $\mu = 2.1$.

Let A be a 2×2 symmetric real matrix in (P) where $\underline{\lambda}$ and $\overline{\lambda}$ are the eigenvalues of A satisfying $\underline{\lambda} \le \overline{\lambda}$. We assume

$$\text{(AN)} \quad \overline{\lambda} < \lambda_1,$$

where λ_1 denotes the first eigenvalue of $-\Delta$ in Ω with the homogeneous Dirichlet boundary condition. By recalling the Poincaré inequality

$$\|\phi\|_2^2 \le \frac{1}{\lambda_1} \|\nabla \phi\|_2^2 \quad (\phi \in H_0^1(\Omega)), \tag{4.4}$$

we see that

$$\int_\Omega (\phi_1, \phi_2) A (\phi_1, \phi_2)^\mathsf{T} dx \le \bar{\lambda}\big(\|\phi_1\|_2^2 + \|\phi_2\|_2^2\big) \le \frac{\bar{\lambda}}{\lambda_1}\big(\|\nabla\phi_1\|_2^2 + \|\nabla\phi_2\|_2^2\big)$$

which means

$$\|\nabla\phi_1\|_2^2 + \|\nabla\phi_2\|_2^2 - \int_\Omega (\phi_1, \phi_2) A (\phi_1, \phi_2)^\mathsf{T} dx \ge \left(1 - \frac{\bar{\lambda}}{\lambda_1}\right)\big(\|\nabla\phi_1\|_2^2 + \|\nabla\phi_2\|_2^2\big) \ge 0 \tag{4.5}$$

for $(\phi_1, \phi_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$ in view of (AN). This condition is used to assure the conclusion of Proposition 4.2.1 which claims the nonnegativity of the energy functional along the orbit. It also assures the solution stays bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$ along any time sequence under some appropriate conditions (see Lemma 4.2.3). The boundedness of time-global solutions in $H_0^1(\Omega) \times H_0^1(\Omega)$ in Lemma 4.3.1 is also assured by this condition.

We assume $T_m = \infty$, i.e., we only consider time-global classical solutions to (P) throughout this chapter. For the single equation case, the existence of a time-global solution is well-known, see e.g. [78], [87], and the references therein. The existence of a time-global solution for (P) also follows from a similar argument as above.

Let us define the following notation:

$$\|\vec{U}\|_\infty := \|u_1\|_\infty + \|u_2\|_\infty.$$

In this chapter, we are concerned with the existence of L^∞-global bounds for time-global solutions, which is defined as:

Definition 4.1.1 *A time-global classical solution $\vec{U}$ to* (P) *is said to have an L^∞-global bound if*

$$\sup_{t\in[0,\infty)} \|\vec{U}(t)\|_\infty < \infty.$$

Our main result reads as follows:

Theorem 4.1.1 (Main Theorem) *Under* (N1)*,* (N2)*,* (AR)*, and* (AN)*, every time-global classical solution to* (P) *has an L^∞-global bound.*

There have been a lot of studies on semilinear parabolic partial differential equations (see e.g. [11], [13], [15], [32], [44], [57], [54], [52], etc). However, most of the results are related to single semilinear parabolic equations with polynomial nonlinearity. As an example, let $N \in \mathbb{N}$, $a \in \mathbb{R}$, $\theta \in (2, 2^*)$, $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain, and consider

$$\begin{cases} \partial_t u = \Delta u + au + u|u|^{\theta-2} & \text{in } \Omega \times (0,\infty), \\ u(\,\cdot\,, t) = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\,\cdot\,, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{P0}$$

where $u_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$ (for simplicity). Generally, a solution to (P0) either blows up in finite time or exists globally in time (see e.g. [78, pp.188]). As for the analysis of the asymptotics of time-global solutions, the first step is to obtain several time-global bounds on the norm of solutions.

The first attempt in this direction was done by Ôtani in [69] and [70], in which time-global bounds for the Sobolev norm of solutions of (P0) with $\theta \in (2, 2^*)$ is obtained in the framework of the theory of abstract evolution equations governed by the difference of subdifferentials. This result also covers the case where the principal part is a nonlinear operator (say, p-Laplacian).

As for the concrete problem (P0), Ni, Sacks, and Tavantzis obtained the following results in [66] when Ω is a convex domain and $u_0 \geq 0$:

(i) If $\theta \in (2, 2+\frac{2}{N})$, then every time-global solution is uniformly bounded.
(ii) If $\theta \geq 2^*$ and $N \geq 3$, then there exists a time-global solution that is not uniformly bounded.

Following these results, Cazenave and Lions showed in [16] that every time-global solution to (P0) has a time-global bound in the sense of L^∞ if $\theta \in (2, 2^*)$ (subcritical case). The bounds they obtained only depend on the H_0^1-norm of u_0 if $2 < p$ for $N = 1$ and $2 < p < 2 + \frac{12}{3N-4}$ for $N \geq 2$. Also, Ôtani obtained in [71] the corresponding results for the case of Neumann boundary condition. For the homogeneous Dirichlet boundary condition case, Giga removed in [37] the restriction above on p for nonnegative time-global solutions. Finally, in [77], Quittner improved the results in [16, 37] by removing the nonnegativity assumption.

The known major method used in the works above is two-folds. One is based on the combined use of the energy estimates and parabolic estimates (see e.g. [16] or [71]) in which the subcriticality assumption is crucial to assure the validity of various inequalities used in the argument. The other one is based on the blow-up (or scaling) argument together with the Liouville property for the stationary problem in $\mathbb{R}^N$, i.e., the nonexistence of nontrivial nonnegative stationary solution (see e.g. [37]). Note that this method also essentially requires the subcriticality (and nonnegativity) assumption since the Liouville property in $\mathbb{R}^N$ only holds for the subcritical and nonnegative cases.

Ishiwata proposed the third method in [47] which is based on the compactness of the solution orbit in the scale-invariant Lebesgue space together with the blow-up argument. More precisely, it is proved in [47] that the compactness of the solution orbit in the scale-invariant Lebesgue space is equivalent to the existence of an L^∞-global bound. Since the Palais-Smale condition along the orbit assures the compactness of the orbit, the result above gives a unified approach including both the subcritical and the critical case.

Note that, in [47], we only consider single equations with polynomial nonlinearities. In this chapter, we will study the existence of time-global bounds in the spirit of [47] for solutions to a system of semilinear parabolic equations with general nonlinearities which is not covered in [47]. The discussion of the existence of an L^∞-global bound for a parabolic system with critical Sobolev nonlinearity in the same spirit will be given in the forthcoming paper [18].

The structure of this chapter is as follows. In section 2, we introduce some preliminary facts associated with (P) whereas the proof of our main result will be given in section 3. The appendix deals with technical backgrounds.

Let $\vec{U} := (u_1, u_2)$ and $H_0^1(\Omega) \times H_0^1(\Omega)$ as H throughout this chapter and define its norm as follows:

$$\|\vec{U}\|_H^2 := \|u_1\|_{H^1}^2 + \|u_2\|_{H^1}^2,$$
$$\|\nabla\vec{U}\|_2{}^2 := \|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2$$

for any $\vec{U} \in H$. Assume $q \in [1, 2^*)$. Similarly as above, we will also define the following notations:

$$\|\vec{U}\|_q^q := \|u_1\|_q^q + \|u_2\|_q^q,$$
$$|\vec{U}|_\infty := |u_1| + |u_2|. \tag{4.6}$$

For convenience, let X be an abstract space of functions and $\vec{V} = (v_1, v_2)$. To avoid confusion, we clarify the use of the following notations:

$$(X)^2 := X \times X,$$
$$\nabla\vec{U} \cdot \nabla\vec{V} := \nabla u_1 \cdot \nabla v_1 + \nabla u_2 \cdot \nabla v_2,$$
$$\vec{U} \circ \vec{V} := (u_1 v_1, u_2 v_2).$$

For $(t_n)_{n\in\mathbb{N}} \subset [0, \infty)$, we denote

$$\vec{U}(t_n) \text{ as } \vec{U}_n$$

throughout this chapter unless it is stated otherwise.

4.2 Preliminaries

In this section, we introduce basic tools concerning the energy structure and the scaling structure associated with (P). First, we define the energy and the Nehari functional associated with (P) as follows:

$$I[\vec{U}] := \frac{1}{2}\|\nabla\vec{U}\|_2^2 - \frac{1}{2}\int_\Omega \vec{U}A\vec{U}^\intercal dx - \int_\Omega F(\vec{U})dx, \tag{I1}$$
$$J[\vec{U}] := \|\nabla\vec{U}\|_2^2 - \int_\Omega \vec{U}A\vec{U}^\intercal dx - \int_\Omega \vec{U} \cdot \nabla F(\vec{U})dx, \tag{J1}$$

where $\vec{U} \in H$. Then we have:

Lemma 4.2.1 (Energy equalities) *Every solution* $\vec{U}$ *of* (P) *satisfies the following energy equalities:*

$$\frac{d}{dt}I[\vec{U}(t)] = -\|\partial_t \vec{U}(t)\|_2^2, \tag{I2}$$

$$\frac{1}{2}\frac{d}{dt}\|\vec{U}(t)\|_2^2 = -J[\vec{U}(t)], \tag{J2}$$

for $t \in (0, T_m)$.

The relation (I2) (respectively, (J2)) is easily obtained by multiplying $\partial_t \vec{U}$ (respectively, $\vec{U}$) to (P) and integrating over Ω.

Note that (I2) implies the energy functional I is non-increasing in time along the solution orbit. Now, we will show that the energy functional I is nonnegative along the solution orbit for the time-global solution which follows from the concavity argument (see e.g. [54, pp.373, Theorem I]).

Proposition 4.2.1 (Nonnegativity of energy functional)
Let $\vec{U}$ *be a global classical solution of* (P). *Then, we have*

$$I[\vec{U}(t)] \geq 0 \text{ for any } t \in [0, \infty). \tag{4.7}$$

The proof will be given in the appendix. By using (I2) and Proposition 4.2.1, we have the existence of $L \geq 0$ such that

$$I[\vec{U}(t)] \to L \text{ as } t \to \infty. \tag{4.8}$$

Associated with this limit, we introduce a condition on $(t_n)_{n\in\mathbb{N}} \subset [0, \infty]$ with $t_n \to \infty$ as $n \to \infty$:

$$\text{(PS)} \quad I[\vec{U}(t_n)] \to L \text{ and } (dI)_{\vec{U}(t_n)} \to 0 \text{ as } n \to \infty \text{ in } H^*,$$

where $(dI)_{\vec{U}(t_n)}$ is the Fréchet derivative of I at $\vec{U}(t_n)$ in H and H^* is the dual space of H. Note that for $(t_n)_{n\in\mathbb{N}}$ with (PS), the sequence $(\vec{U}(t_n))_{n\in\mathbb{N}}$ is a Palais-Smale sequence, see e.g. [85, pp.70] and [89, pp.13, Definition 1.16]. Now we show

Lemma 4.2.2 *Let* $(t_n)_{n\in\mathbb{N}}$ *be a sequence with* $t_n \to \infty$ *and* $\|\partial_t \vec{U}(t_n)\|_2^2 \to 0$ *as* $n \to \infty$. *Then,* $(t_n)_{n\in\mathbb{N}}$ *satisfies* (PS).

Proof of Lemma 4.2.2. For simplicity, we denote $\vec{U}(t_n)$ as $\vec{U}_n$ and $\partial_t\vec{U}(t_n)$ as $\partial_t\vec{U}_n$. For any fixed $\vec{\phi} \in H$ with $\|\phi\|_H = 1$, we see that

$$\begin{aligned}|(dI)_{\vec{U}_n}[\vec{\phi}]| &= \left| \int_\Omega \big(\nabla\vec{U}_n \cdot \nabla\vec{\phi} - \vec{U}_n A \vec{\phi}^\intercal - \vec{\phi}\cdot\nabla F(\vec{U}_n)\big) dx \right| \\ &= \left| \int_\Omega \partial_t\vec{U}_n \cdot \vec{\phi} dx \right| \\ &\leq \|\vec{\phi}\|_2 \|\partial_t\vec{U}_n\|_2 \leq \|\vec{\phi}\|_H \|\partial_t\vec{U}_n\|_2 = \|\partial_t\vec{U}_n\|_2,\end{aligned}$$

which together with the assumption and (4.8) yield the conclusion. ∎

Lemma 4.2.3 *Suppose* $(t_n)_{n\in\mathbb{N}} \subset [0,\infty)$ *satisfies* $t_n \to \infty$ *as* $n \to \infty$ *and* (PS). *Then* $(\vec{U}_n)_{n\in\mathbb{N}}$ *is bounded in* H *and there holds*

$$J[\vec{U}_n] \to 0, \quad \|\nabla\vec{U}_n\|_2^2 - \int_\Omega \vec{U}_n A \vec{U}_n^\intercal dx \leq \frac{L}{\frac{1}{2} - \frac{1}{\mu}} + o(1) \tag{4.9}$$

as $n \to \infty$, *where* $\vec{U}_n := \vec{U}(t_n)$.

Proof of Lemma 4.2.3. We observe that, by (PS),

$$\frac{\big|J[\vec{U}_n]\big|}{\|\vec{U}_n\|_H} = \frac{\big|(dI)_{\vec{U}_n}[\vec{U}_n]\big|}{\|\vec{U}_n\|_H} \leq \|(dI)_{\vec{U}_n}\|_{H^*} \to 0 \tag{4.10}$$

as $n \to \infty$. In view of (4.8) and (AR2), we also see that

$$\begin{aligned}L + o(1) &= I[\vec{U}_n] \\ &\geq \frac{1}{2}\left(\|\nabla\vec{U}_n\|_2^2 - \int_\Omega \vec{U}_n A \vec{U}_n^\intercal dx\right) - \frac{1}{\mu}\int_\Omega \vec{U}_n \cdot \nabla F(\vec{U}_n) dx \quad (n \to \infty),\end{aligned}$$

which together with (J1), yields

$$L + o(1) \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\left(\|\nabla\vec{U}_n\|_2^2 - \int_\Omega \vec{U}_n A \vec{U}_n^\intercal dx\right) + \frac{1}{\mu} J[\vec{U}_n] \tag{4.11}$$

as $n \to \infty$. By taking $(\phi_1, \phi_2) = \vec{U}_n$ in (4.5), we see that the quadratic form in the right hand side of (4.11) is nonnegative in view of (AN). From this fact, we will prove the boundedness of $(\vec{U}_n)_{n\in\mathbb{N}}$ in H by using (4.10) and (4.11). Indeed, if not, there exists a subsequence such that $\|\nabla\vec{U}_n\|_H \to \infty$ as $n \to \infty$ in view of the Poincaré inequality (4.4). Then we have, from (4.10),

$$\frac{\left|J[\vec{U}_n]\right|}{\|\nabla\vec{U}_n\|_2} \leq \left(1+\frac{1}{\lambda_1}\right)^{\frac{1}{2}} \frac{\left|J[\vec{U}_n]\right|}{\|\vec{U}_n\|_H} \to 0 \tag{4.12}$$

as $n \to \infty$.

By taking $(\phi_1, \phi_2) = \vec{U}_n$ in (4.5), substituting it into (4.11), and then dividing the result by $\|\nabla\vec{U}_n\|_2$, we have

$$o(1) = \left(\frac{1}{2}-\frac{1}{\mu}\right)\left(1-\frac{\bar{\lambda}}{\lambda_1}\right)\|\nabla\vec{U}_n\|_2$$

in view of (4.12) as $n \to \infty$, which is absurd since $\|\nabla\vec{U}_n\|_2 \to \infty$ as $n \to \infty$ and

$$1-\frac{\bar{\lambda}}{\lambda_1} > 0 \tag{4.13}$$

in view of (AN). Hence, $(\vec{U}_n)_{n\in\mathbb{N}}$ is uniformly bounded in H.

This boundedness together with (4.10) yields

$$J[\vec{U}_n] \to 0 \quad (n \to \infty),$$

the first relation in (4.9). This relation and (4.11) together with the fact that $\mu > 2$ imply the second relation in (4.9) which completes the proof. ∎

Now we show the compactness of the solution orbit in H along a time sequence satisfying (PS):

Proposition 4.2.2 *Assume $\vec{U}$ is a time-global solution to* (P)*. Then, for any time sequence $(t_n)_{n\in\mathbb{N}} \subset [0,\infty)$ with $t_n \to \infty$ as $n \to \infty$ satisfying* (PS)*, $(\vec{U}_n)_{n\in\mathbb{N}}$ is compact in H where $\vec{U}_n = \vec{U}(t_n)$.*

Proof of Proposition 4.2.2. Let $\vec{U}$ and $(t_n)_{n\in\mathbb{N}}$ be as in the assumption. By Lemma 4.2.3, $(\vec{U}_n)_{n\in\mathbb{N}}$ is weakly convergent in H along a subsequence. Since p in (N2) is subcritical, we can use the compact Sobolev embedding $H \hookrightarrow (L^p(\Omega))^2$ to deduce that, by taking a subsequence if necessary,

$$\vec{U}_n \to \vec{U}_* \text{ in } (L^p(\Omega))^2 \text{ and in } H \quad (n \to \infty) \tag{4.14}$$

where $\vec{U}_*$ is the weak limit of $(\vec{U}_n)_{n\in\mathbb{N}}$ in H.

Note that this convergence and (N2) assure $\nabla F(\vec{U}_n) \to \nabla F(\vec{U}_*)$ in $(L^{\frac{p}{p-1}}(\Omega))^2$ as $n \to \infty$, hence we obtain

$$\int_\Omega \vec{U}_* \cdot \nabla F(\vec{U}_n) dx \to \int_\Omega \vec{U}_* \cdot \nabla F(\vec{U}_*) dx \quad (n \to \infty). \tag{4.15}$$

Since $(t_n)_{n\in\mathbb{N}}$ satisfies (PS) and $U_* \in H$, we observe

$$0 = (dI)_{\vec{U}_n}[\vec{U}_*] + o(1) \quad (n \to \infty).$$

Then, by using the weak convergence of $(\vec{U}_n)_{n\in\mathbb{N}}$ in H and (4.15), we have

$$0 = (dI)_{\vec{U}_n}[\vec{U}_*] + o(1) = \|\nabla \vec{U}_*\|_2^2 - \int_\Omega \vec{U}_* A \vec{U}_*^{\mathsf{T}} dx - \int_\Omega \vec{U}_* \cdot \nabla F(\vec{U}_*) dx + o(1)$$

as $n \to \infty$, thus

$$\|\nabla \vec{U}_*\|_2^2 - \int_\Omega \vec{U}_* A \vec{U}_*^{\mathsf{T}} dx = \int_\Omega \vec{U}_* \cdot \nabla F(\vec{U}_*) dx. \tag{4.16}$$

On the other hand, by (4.14), we see that

$$\int_\Omega \vec{U}_n A \vec{U}_n^{\mathsf{T}} dx \to \int_\Omega \vec{U}_* A \vec{U}_*^{\mathsf{T}} dx \quad (n \to \infty). \tag{4.17}$$

Again by (4.14) and (4.15), we obtain

$$\int_\Omega \vec{U}_n \cdot \nabla F(\vec{U}_n) dx \to \int_\Omega \vec{U}_* \cdot \nabla F(\vec{U}_*) dx \tag{4.18}$$

as $n \to \infty$ up to a subsequence. By recalling $J[\vec{U}_n] \to 0$ as $n \to \infty$ in Lemma 4.2.3 and using (4.17) together with (4.18), we obtain

$$\begin{aligned} o(1) = J[\vec{U}_n] &= \|\nabla \vec{U}_n\|_2^2 - \int_\Omega \vec{U}_n A \vec{U}_n^{\mathsf{T}} dx - \int_\Omega \vec{U}_n \cdot \nabla F(\vec{U}_n) dx \\ &= \|\nabla \vec{U}_n\|_2^2 - \int_\Omega \vec{U}_* A \vec{U}_*^{\mathsf{T}} dx - \int_\Omega \vec{U}_* \cdot \nabla F(\vec{U}_*) dx + o(1) \end{aligned} \tag{4.19}$$

as $n \to \infty$. Now we conclude that

$$o(1) = \|\nabla \vec{U}_n\|_2^2 - \|\nabla \vec{U}_*\|_2^2 \quad (n \to \infty)$$

in view of (4.16) and (4.19), which together with (4.14) shows $(\vec{U}_n)_{n\mathbb{N}}$ is compact in H.
∎

Next, we introduce the scale transformation which will be used in the proof of our main result. Let $\lambda > 0$, $x_* \in \mathbb{R}^N$, $t_* \in [0, \infty)$, and we introduce the following variable transformations:

$$\begin{cases} y := \lambda(x - x_*),\ s := \lambda^2(t - t_*), \\ \tilde{U}_\lambda(y, s) := \lambda^{\frac{-2}{p-2}} \vec{U}(x, t). \end{cases} \tag{4.20}$$

Then, by substituting (4.20) into (P), we obtain

$$\partial_s \tilde{U}_\lambda = \Delta \tilde{U}_\lambda + \lambda^{-2} A \tilde{U}_\lambda^{\mathsf{T}} + \lambda^{\frac{-2}{p-2}} \nabla F(\lambda^{\frac{2}{p-2}} \tilde{U}_\lambda). \tag{P1}$$

We set

$$p_0 := \frac{N}{2}(p - 2). \tag{4.21}$$

Note that $p_0 \geq 1$ under the assumption $p \geq 2 + \frac{2}{N}$ in (N2) and we obtain the scale-invariance of L^{p_0}-norm as follows:

Lemma 4.2.4 *Let $\omega \subset \mathbb{R}^N$, $I = (a, b)$, $\omega_\lambda := \lambda(\omega - x_*)$, and $I_\lambda := (a_\lambda, b_\lambda)$, where $a_\lambda := \lambda^2(a - t_*)$ and $b_\lambda := \lambda^2(b - t_*)$. Then, the following statements hold true:*

(i) *The $(L^{p_0})^2$-norm is invariant under the variable transformation in* (4.20)*, i.e.,*

$$\|\vec{U}\|_{p_0,\omega} = \|\tilde{U}_\lambda\|_{p_0,\omega_\lambda}.$$

(ii) *Moreover, we also have*

$$\|\partial_s \tilde{U}_\lambda\|^2_{L^2(I_\lambda; L^2(\omega_\lambda))} = \lambda^{-(\frac{4}{p-2}+2-N)} \|\partial_t \vec{U}\|^2_{L^2(I; L^2(\omega))}.$$

Proof of Lemma 4.2.4.
(i) The conclusion for u easily follows from the relation

$$\|\tilde{U}_\lambda\|^{p_0}_{p_0,\Omega_\lambda} = \lambda^{\frac{-2p_0}{p-2}+N} \|\vec{U}\|^{p_0}_{p_0}$$

since $\frac{-2p_0}{p-2} + N = 0$.
(ii) The conclusion follows from straightforward calculation.
∎

Finally, we introduce the following result which is needed in the proof of our main result:

Lemma 4.2.5 (Convergence in Hölder space)
Assume $(w_n)_{n\in\mathbb{N}}$ *is uniformly bounded in* $C^{0,\beta';0,\beta'/2}_{loc}(\mathbb{R}^N \times [-1,0]))$ *for some* $\beta' \in (0,1)$. *Then, we have*

$$w_n \to w \text{ in } C^{0,\alpha';0,\alpha'/2}_{loc}(\mathbb{R}^N \times [-1,0]) \text{ as } n \to \infty \tag{4.22}$$

up to a subsequence for any $\alpha' \in [0,\beta')$.

See the appendix for the proof of this lemma.

4.3 Proof of the main result

4.3.1 A compactness of the orbit in $(L^{p_0}(\Omega))^2$

In this subsection, we show the compactness of the solution orbit in $(L^{p_0}(\Omega))^2$.

Lemma 4.3.1 (Boundedness in H) *Every time-global solution* $\vec{U}$ *to Problem* (P) *satisfies*

$$\sup_{t\in[0,\infty)} \|\vec{U}(t)\|_H^2 < \infty. \tag{4.23}$$

Proof of Lemma 4.3.1. In view of (4.5), we see that

$$\|\nabla\vec{U}\|_2^2 - \int_\Omega \vec{U}A\vec{U}^\mathsf{T} dx \geq 0$$

which means it is enough to show that

$$\limsup_{t\to\infty}\left(\|\nabla\vec{U}(t)\|_2^2 - \int_\Omega \vec{U}(t)A\vec{U}(t)^\mathsf{T} dx\right) \leq \frac{L}{\frac{1}{2}-\frac{1}{\mu}}. \tag{4.24}$$

We argue by contradiction. Suppose (4.24) does not hold, then there exist $\varepsilon > 0$ and $(t'_n)_{n\in\mathbb{N}}$ with $t'_n \to \infty$ as $n \to \infty$ which satisfy

$$\|\nabla\vec{U}(t'_n)\|_2^2 - \int_\Omega \vec{U}(t'_n)A\vec{U}(t'_n)^\mathsf{T} dx > \frac{L}{\frac{1}{2}-\frac{1}{\mu}} + \varepsilon + o(1) \tag{4.25}$$

as $n \to \infty$. On the other hand, notice that by (4.8), $I[\vec{U}(t)] \to L$ as $t \to \infty$. Then, we apply mean value theorem to find $t''_n \in (n, n+1)$ for any $n \in \mathbb{N}$ to obtain

$$\frac{d}{dt}I[\vec{U}(t''_n)] = I[\vec{U}(n+1)] - I[\vec{U}(n)] = o(1) \quad (n \to \infty),$$

which together with (I2) implies

$$-\|\partial_t \vec{U}(t''_n)\|_2^2 = \frac{d}{dt} I[\vec{U}(t''_n)] \to 0 \quad (n \to \infty).$$

As a consequence, by Lemma 4.2.2, we deduce that $(t''_n)_{n\in\mathbb{N}}$ satisfies (PS). Thus (4.9) for $(t''_n)_{n\in\mathbb{N}}$ follows from Lemma 4.2.3. Without loss of generality, we may assume that $t''_n < t'_n$ (up to a subsequence). By the intermediate value theorem for the mapping $t \mapsto \|\vec{U}(t)\|_2^2 - \int_\Omega \vec{U}(t) A \vec{U}(t)^\mathsf{T} dx$, (4.25) and the second relation in (4.9) for $(t''_n)_{n\in\mathbb{N}}$ yield the existence of $t_n \in (t''_n, t'_n)$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\|\nabla \vec{U}(t_n)\|_2^2 - \int_\Omega \vec{U}(t_n) A \vec{U}(t_n)^\mathsf{T} dx = \frac{L}{\frac{1}{2} - \frac{1}{\mu}} + \frac{\varepsilon}{2} + o(1) \tag{4.26}$$

as $n \to \infty$. Notice (4.26) implies $(\vec{U}(t_n))_{n\in\mathbb{N}}$ is uniformly bounded in H. Therefore, we can find $\vec{U}_1 \in H$ such that

$$\begin{cases} \vec{U}(t_n) \rightharpoonup \vec{U}_1 \text{ weakly in } H, \\ \vec{U}(t_n) \to \vec{U}_1 \text{ strongly in } (L^2(\Omega))^2 \text{ and } (L^p(\Omega))^2, \end{cases} \tag{4.27}$$

as $n \to \infty$ up to a subsequence. Let $\vec{U}_n(s) = \vec{U}(t_n + s)$ for $s \in [-1, 1]$. By using (I2) and (4.8), we see that

$$\int_{-1}^{1} \|\partial_s \vec{U} n(s)\|_2^2 ds = I[\vec{U}_n(-1)] - I[\vec{U}_n(1)] = o(1) \quad (n \to \infty). \tag{4.28}$$

This means $\|\partial_s \vec{U}_n(s_0)\|_2^2 \to 0$ for some $s_0 \in [-1, 1]$ as $n \to \infty$, which together with Lemma 4.2.2 yields $(t_n + s_0)_{n\in\mathbb{N}}$ satisfies (PS). Hence Proposition 4.2.2 guarantees the existence of $\vec{V}_1 \in H$ satisfying

$$\vec{U}_n(s_0) \to \vec{V}_1 \text{ strongly in } H \quad (n \to \infty) \tag{4.29}$$

up to a subsequence. Here, we see that

$$\|\vec{U}_1 - \vec{V}_1\|_2 \le \|\vec{U}_1 - \vec{U}_n(0)\|_2 + \|\vec{U}_n(0) - \vec{U}_n(s_0)\|_2 + \|\vec{U}_n(s_0) - \vec{V}_1\|_2. \tag{4.30}$$

The relation above together with (4.27), (4.28), and (4.29) yields

$$\begin{aligned} \|\vec{U}_1 - \vec{V}_1\|_2 &\le o(1) + \|\vec{U}_n(0) - \vec{U}_n(s_0)\|_2 = o(1) + \left\| \int_0^{s_0} \partial_s \vec{U}_n(s) ds \right\|_2 \\ &\le o(1) + \sqrt{s_0} \left(\int_{-1}^{1} \|\partial_s \vec{U}_n(s)\|_2^2 ds \right)^{\frac{1}{2}} \to 0 \end{aligned}$$

as $n \to \infty$. Hence there holds

$$\vec{U}_1 = \vec{V}_1 \text{ in } (L^2(\Omega))^2. \tag{4.31}$$

Furthermore, by using the argument similar to the derivation of (4.18), we deduce that

$$\int_\Omega \vec{U}_n \cdot \nabla F(\vec{U}_n) dx \to \int_\Omega \vec{U}_1 \cdot \nabla F(\vec{U}_1) dx \tag{4.32}$$

as $n \to \infty$ since $\vec{U} \mapsto \nabla F(\vec{U})$ is continuous in $(L^{\frac{p}{p-1}}(\Omega))^2$, and (4.27) holds. Moreover, by using (4.31), we see that

$$\int_\Omega \vec{U}_1 \cdot \nabla F(\vec{U}_1) dx = \int_\Omega \vec{V}_1 \cdot \nabla F(\vec{V}_1) dx. \tag{4.33}$$

Then, we recall Lemma 4.2.3 to conclude

$$J[\vec{U}_n(s_0)] = o(1) \quad (n \to \infty),$$

whence it follows (by similar derivation of (4.16))

$$\int_\Omega \vec{V}_1 \cdot \nabla F(\vec{V}_1) dx = \|\nabla \vec{V}_1\|_2^2 - \int_\Omega \vec{V}_1 A \vec{V}_1^\mathsf{T} dx \tag{4.34}$$

By applying Lemma 4.2.2 to $(t_n + s_0)_{n\in\mathbb{N}}$ and then using (4.33) together with (4.34), we conclude that

$$\int_\Omega \vec{U}_1 \cdot \nabla F(\vec{U}_1) dx \leq \frac{L}{\frac{1}{2} - \frac{1}{\mu}}. \tag{4.35}$$

Next, by using (4.26), we see that

$$\begin{aligned} L &= \frac{1}{2}\left(\|\nabla \vec{U}_n\|_2^2 - \int_\Omega \vec{U}_n) A \vec{U}_n^\mathsf{T} dx \right) - \int_\Omega F(\vec{U}_n) dx + o(1) \\ &= \frac{1}{2}\left(\frac{L}{\frac{1}{2} - \frac{1}{\mu}} + \frac{\varepsilon}{2} \right) - \int_\Omega F(\vec{U}_n) dx + o(1) \quad (n \to \infty). \end{aligned}$$

Then, we substitute (AR2) into the right hand side of the equation above and then use (4.32) to obtain the following inequality:

$$\int_\Omega \vec{U}_1 \cdot \nabla F(\vec{U}_1) dx \geq \frac{L}{\frac{1}{2} - \frac{1}{\mu}} + \frac{\mu\varepsilon}{4}. \tag{4.36}$$

Clearly, (4.35) and (4.36) lead to a contradiction since $\varepsilon > 0$. Therefore, (4.24) must be true which means our proof is complete. ∎

Now, by (4.1), we see that L^{p_0} is well-defined and p_0 is subcritical in the sense of the Sobolev embedding. Then the compactness of the Sobolev embedding $H \hookrightarrow (L^{p_0}(\Omega))^2$ together with Lemma 4.3.1 assures the following:

Proposition 4.3.1 (Compactness of orbit in $(L^{p_0}(\Omega))^2$)
For any time sequence $(t_n)_{n\in\mathbb{N}} \subset [0,\infty)$ with $t_n \to \infty$ as $n \to \infty$, there exists a subsequence (still denoted by the same notation) and $\vec{U}_* \in H$ such that

$$\vec{U}(t_n) \to \vec{U}_* \text{ in } (L^{p_0}(\Omega))^2 \text{ as } n \to \infty.$$

4.3.2 Proof of Theorem 4.1.1

We prove our main result by using contradiction following the argument in [47, pp.1030, Proof of Proposition 4.2] for the case of a single equation with polynomial nonlinearity.

Assume the conclusion of Theorem 4.1.1 does not hold. Then we have a time sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to \infty$ and $\|\vec{U}(t_n)\|_\infty \to \infty$ as $n \to \infty$. By taking a further subsequence if needed, we obtain the existence of $(t_n)_{n\in\mathbb{N}}$ satisfying

$$\sup_{t\in[0,t_n]} \|\vec{U}(t)\|_\infty = \|\vec{U}(t_n)\|_\infty \text{ for any } n \in \mathbb{N}, \tag{4.37}$$

together with the existence of $(x_n)_{n\in\mathbb{N}}$ and $x_0 \in \overline{\Omega}$ satisfying

$$\begin{cases} x_n \to x_0 \text{ as } n \to \infty, \\ \frac{1}{2}\|\vec{U}(t_n)\|_\infty \leq |\vec{U}(x_n, t_n)|_\infty \text{ for any } n \in \mathbb{N}. \end{cases} \tag{4.38}$$

by recalling the notation given in (4.6).

Let us take λ, x_* and t_* in (4.20) by λ_n, x_n and t_n respectively, where

$$\lambda_n^{\frac{2}{p-2}} := \|\vec{U}(t_n)\|_\infty \to \infty \tag{4.39}$$

as $n \to \infty$ by the assumption. In view of (P1), our original problem (P) is rewritten in the following form:

$$\partial_s \tilde{U}_n = \Delta \tilde{U}_n + \lambda_n^{-2} A \tilde{U}_n^{\mathsf{T}} + \lambda_n^{\frac{-2}{p-2}} \nabla F(\lambda_n^{\frac{2}{p-2}} \tilde{U}_n), \tag{Pn}$$

with a homogeneous boundary condition on $\partial(\lambda_n(\Omega - x_n))$, where a new variable (y,s) is in $\lambda_n(\Omega - x_n) \times (-\lambda_n^2 t_n, \infty)$.

Notice that by using (4.37) and (4.38), we have

$$\frac{1}{2} \leq |\tilde{U}_n(0_y, 0_s)|_\infty \text{ for any } n \in \mathbb{N} \tag{4.40}$$

and

$$\|\tilde{U}_n\|_{L^\infty(-2,0;L^\infty)} \leq 1 \text{ for any } n \in \mathbb{N} \text{ sufficiently large.} \tag{4.41}$$

From now on, we will divide our argument into two cases:

Case 1. $\limsup_{n\to\infty} \lambda_n \mathrm{dist}(x_n, \partial\Omega) = \infty$.
Case 2. $\limsup_{n\to\infty} \lambda_n \mathrm{dist}(x_n, \partial\Omega) < \infty$.

We denote $\gamma = \limsup_{n\to\infty} \lambda_n \mathrm{dist}(x_n, \partial\Omega)$ for the sake of simplicity.

Case 1. We assume $\gamma = \infty$. It is known that

$$\lambda_n(\Omega - x_n) \to \mathbb{R}^N \text{ as } n \to \infty.$$

See e.g. [78, pp. 190 Case (i) and pp. 343 *Case 1*]) and [37, pp. 417 *Case 1*] for references.

Step 1. We first show that $\tilde{U}_n$ has a nontrivial limit profile $\tilde{U}$. To this end, by applying L^p-estimate for parabolic operators (see e.g. [57, pp. 172, Theorem 7.13]) together with (4.41), we have the boundedness of $(\tilde{U}_n)_{n\in\mathbb{N}}$ in $\left(W^{2,1}_{q,loc}(\mathbb{R}^N \times (-1,0))\right)^2$ for sufficiently large $q > 1$. Thus, $(\tilde{U}_n)_{n\in\mathbb{N}}$ is also a bounded sequence in $\left(C^{0,\alpha';0,\alpha'/2}_{loc}(\mathbb{R}^N \times [-1,0])\right)^2$ by recalling the Sobolev embedding $W^{2,1}_{q,loc} \hookrightarrow C^{0,\alpha';0,\alpha'/2}_{loc}$ (see e.g. [52, pp. 80, Lemma 3.3]) where $\alpha' \in (0,1)$. By taking the components of $(\tilde{U}_n)_{n\in\mathbb{N}}$ as $(w_n)_{n\in\mathbb{N}}$ in Lemma 4.2.5, we see that (by taking a subsequence if necessary)

$$\tilde{U}_n \to \tilde{U} \text{ in } \left(C^{0,\alpha';0,\alpha'/2}_{loc}(\mathbb{R}^N \times [-1,0])\right)^2 \text{ as } n \to \infty. \tag{4.42}$$

Next, we will show that the following lemma holds:

Lemma 4.3.2 *Let* $(\tilde{U}_n)_{n\in\mathbb{N}}$ *be a solution to* (Pn)*. Then, we have*

$$\|\partial_s \tilde{U}_n\|^2_{L^2(-1,0;L^2)} \to 0 \text{ as } n \to \infty. \tag{4.43}$$

Proof of Lemma 4.3.2. First, we apply Lemma 4.2.4 (ii) and then use (I2) to obtain

$$\begin{aligned}\|\partial_s\tilde{U}_n\|^2_{L^2(-1,0;L^2)} &= \lambda_n^{-(\frac{4}{p-2}+2-N)}\|\partial_t\vec{U}\|^2_{L^2((t_n-\frac{1}{\lambda_n^2},t_n);L^2)}\\ &\le \lambda_n^{-(\frac{4}{p-2}+2-N)} I\left[\vec{U}\left(t_n-\frac{1}{\lambda_n^2}\right)\right].\end{aligned}$$

Following this, we recall that I is non-increasing and $p<2^*$ so that $-(\frac{4}{p-2}+2-N)<0$. Therefore, we can deduce that

$$\|\partial_s\tilde{U}_n\|^2_{L^2(-1,0;L^2)} \le \lambda_n^{-(\frac{4}{p-2}+2-N)} I[\vec{U}_0]\to 0 \quad (n\to\infty)$$

since $\lambda_n\to\infty$ as $n\to\infty$ which completes the proof of this lemma. ∎

Lemma 4.3.2 yields the following corollary which claims that $\tilde{U}$ is s-independent:

Corollary 4.3.1 *Assume that* (4.43) *holds. Then, we have*

$$\int_{-1}^0\int_{\mathbb{R}^N}\tilde{U}\circ\partial_s\vec{\varphi}dyds=\vec{0} \tag{4.44}$$

for any test function $\vec{\varphi}\in(C_0^\infty(\mathbb{R}^N\times[-1,0]))^2$.

Proof of Corollary 4.3.1. Fix $\vec{\varphi}\in(C_0^\infty(\mathbb{R}^N\times[-1,0]))^2$ and let K be the compact support of $\vec{\varphi}$. In view of (4.42), we see that

$$\begin{aligned}\int_{-1}^0\int_{\mathbb{R}^N}\tilde{U}\circ\partial_s\vec{\varphi}dyds &= \int_0^{-1}\int_K\tilde{U}_n\circ\partial_s\vec{\varphi}dyds+o(1)\\ &= -\int_0^{-1}\int_K\partial_s\tilde{U}_n\circ\vec{\varphi}dyds+o(1)\quad(n\to\infty).\end{aligned} \tag{4.45}$$

Using (4.45), the equivalence of the Euclidean norm and the taxicab norm, and Cauchy-Schwartz inequality, we see that

$$\left|\int_{-1}^0\int_{\mathbb{R}^N}\tilde{U}\circ\partial_s\vec{\varphi}dyds\right|\le C\|\vec{\varphi}\|_{L^2(-1,0;L^2)}\|\partial_s\tilde{U}_n\|^2_{L^2(-1,0;L^2)}+o(1)\to 0$$

for some $C>0$ as $n\to\infty$ in view of (4.43). Thus, the proof is complete. ∎

Next, by using (4.42) together with (4.40) and the continuity of $\tilde{U}$, there exists $r>0$ small enough so that

$$\|\tilde{U}\|_{L^{p_0}(B_y(0,r))}>0, \tag{4.46}$$

where p_0 is the exponent of the scale-invariant Lebesgue space introduced in (4.21).

Step 2. On the contrary to (4.46), we will show that the limit profile $\tilde{U}$ is trivial by using the compactness of the orbit in the scale-invariant Lebesgue space. By taking r as in (4.46), it is easily seen that

$$B_x(x_0;\varepsilon) \subset \Omega \text{ and } B_x(x_n;\frac{r}{\lambda_n}) \subset B_x(x_0;\varepsilon) \tag{4.47}$$

for ε small enough and sufficiently large n. Thus, by using Lemma 4.2.5 (uniform convergence) and Lemma 4.2.4 (i) (scale-invariance), we see that

$$\|\tilde{U}\|_{L^{p_0}(B_y(0;r))} = \|\tilde{U}_n(0_s)\|_{L^{p_0}(B_y(0;r))} + o(1) = \|\vec{U}_n\|_{L^{p_0}(B_x(x_n;\frac{r}{\lambda_n}))} + o(1)$$

as $n \to \infty$. Moreover, (4.47) and Proposition 4.3.1 yield

$$\|\vec{U}_n\|_{L^{p_0}(B_x(x_n;\frac{r}{\lambda_n}))} \le \|\vec{U}_n\|_{L^{p_0}(B_x(x_0;\varepsilon))} = \|\vec{U}_*\|_{L^{p_0}(B_x(x_0;\varepsilon))} + o(1) \quad (n \to \infty).$$

By combining these two relations, we have

$$\|\tilde{U}\|_{L^{p_0}(B_y(0;r))} \le \|\vec{U}_*\|_{L^{p_0}(B_x(x_0;\varepsilon))}$$

and, by letting $\varepsilon \downarrow 0$, we see that

$$\|\tilde{U}\|_{L^{p_0}(B_y(0;r))} = 0. \tag{4.48}$$

Clearly (4.48) contradicts (4.46) and thus we have the desired conclusion.

Case 2. We assume $\gamma < \infty$. Note that, since $\lambda_n \to \infty$ as $n \to \infty$ (see (4.39)), we observe

$$\lambda_n(\Omega - x_n) \to \mathbb{R}^N_+ \text{ as } n \to \infty \text{ and } x_0 \in \partial\Omega$$

up to spatial rotation and translation (see e.g. [78, Equation (22.6) pp. 190 and pp. 344 *Case 2*] and [37, pp. 418, *Case 2*]), where

$$\mathbb{R}^N_+ := \{y \in \mathbb{R}^N \mid y^N > -\gamma\}.$$

Now we repeat the procedure as in **Case 1** by replacing the interior estimate with the boundary estimate. Following this, we see that $(\tilde{U}_n)_{n\in\mathbb{N}}$ is uniformly bounded in $\big(W^{2,1}_{q,q;loc}(\mathbb{R}^N_+ \times (-1,0))\big)^2$ for any $q > 1$ sufficiently large, hence the boundedness in $\big(C^{0,\alpha';0,\alpha'/2}_{loc}(\mathbb{R}^N_+ \times [-1,0])\big)^2$.

Recall that (Pn) has a homogeneous boundary condition on $\partial(\lambda_n(\Omega - x_n))$ and $|\tilde{U}(0_y)| \ge \frac{1}{2}$. Thus, $\gamma \ne 0$. Then, by replacing $B_y(0,r)$ with $B_y(0,r) \cap \mathbb{R}^N_+$, we have the same contradiction as in **Case 1.** This completes the proof.

Chapter 5

Conclusions and future problems

5.1 Conclusions

In chapter 2, we introduce the p-mean value in the sense of [46] and define the notion of (variationally) p-harmonious functions, i.e., the functions which can be represented by the p-mean value. The existence of p-harmonious functions is confirmed in Theorem 2.2.2. The idea of the proof is by applying Perron's Method which is modified to suit our problem. Next, we also establish the relation between p-harmonious functions and p-harmonic functions for game-theoretic p-Laplace equation with the same Dirichlet boundary condition. In this case, we confirm that a p-harmonious function converges to a p-harmonic function uniformly in a sufficiently smooth bounded domain in $\mathbb{R}^N$ in view of Theorem 2.3.2. Particularly, we use the convergence scheme which is developed by Barles and Souganidis in [8] to prove Theorem 2.3.2. Our results here generalizes the notion of asymptotic mean value property and harmonic functions. In particular, when $p = 2$, we obtain the usual asymptotic mean value property for harmonic functions.

In chapter 3, we study the blow-up rate of semilinear heat equation with subcritical nonlinear term under Ambrosetti-Rabinowitz condition in $\mathbb{R}^N$. We obtain the blow-up rate of the aforementioned problem in the form of the lower estimate of blow-up rate and the upper estimate of blow-up rate as written in Theorem 3.1.1 and Theorem 3.1.2 respectively. In fact, our main results cover some functions which are not included by Giga and Kohn in [36] such as (3.1) and (3.2). We mainly use similar variable transformation and parabolic argument to prove our main results in this chapter. Finally, we also extend Theorem 3.1.1 and Theorem 3.1.2 to Theorem 3.4.1 and Theorem 3.4.2 respectively for a system of semilinear heat equations with subcritical nonlinear term under Ambrosetti-Rabinowitz condition.

In chapter 4, we study the time-global solutions to a system of semilinear heat quations with subcritical nonlinearity under Ambrosetti-Rabinowitz under Ambrosetti-Rabinowitz condition in a bounded smooth domain. Particularly, we establish the existence of L^∞-global bounds for the aforementioned time-global solutions as stated by Theorem 4.1.1. Here, we extend the method developed by Ishiwata in [47] which is based on the compactness of the orbit in scale-invariant Lebesgue space and the blow-up argument from a single equation case to a system of equations. This method particularly works for critical case in the sense of Sobolev embedding for single equation. The critical case for a system of equations will be discussed in our forthcoming paper [18].

5.2 Future problems

In this section, we will discuss several possible directions for future research based on the results we have presented especially in Chapter 3 and Chapter 4.

5.2.1 Blow-up rate of semilinear heat equations and Ambrosetti-Rabinowitz condition

First, let us introduce the following semilinear heat equations given by Giga, Matsui, and Sasayama in [34]. Let $N \geq 3$, $p \in (2, 2^*)$, and $u_0 \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for simplicity and consider

$$\begin{cases} \partial_t u = \Delta u + u|u|^{p-2} & \text{in } \mathbb{R}^N \times (0, T_m), \\ u(\,\cdot\,, 0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

In [36], Giga and Kohn needs to assume nonnegativity of u_0 to assure to obtain the blow-up rate for $p \in (2, 2^*)$. Otherwise, their results are only valid for $p \in (2, \frac{6N+4}{3N-4})$. Hence, the result of Giga, Matsui, and Sasayama in [34] improves the results in [36] by including the case $p \in [\frac{6N+4}{3N-4}, 2^*)$ without assuming the nonnegativity of the solution. Observe that we also need to assume $p \in (2, \frac{6N+4}{3N-4})$ in Theorem 3.1.2.

On the other hand, we also need the assumption of $\mu = p$ in Theorem (3.1.2) to obtain the upper estimate of the blow-up rate for our problem in Chapter 3. However, such an assumption is not necessary in the case of ordinary differential equations as obtained in [19]. By removing the assumption $\mu = p$, we can obtain the blow-up rate for a larger class of functions of the nonlinear term in our problem. For example, we can obtain the blow-up rate for

$$f(u) = u|u|^2 \log(1 + |u|^2) \tag{5.1}$$

by taking $N = 3$, $p = 4$, and $\mu = 3$ if we can omit the assumption $\mu = p$ in Theorem 3.1.2.

By the exposition above, we consider the following things for future research:

1. Improve the assumption on p from $p \in (2, \frac{6N+4}{3N-4})$ to $p \in (2, 2^*)$ for Theorem 3.1.2.

2. Remove the assumption $\mu = p$ in Theorem 3.1.2.

5.2.2 The existence of L^∞-global bounds and the lack of compactness

In Chapter 1, we cite the result of Ishiwata in [47] about the sufficient and necessary condition for the existence of L^∞-global bounds of the solutions to a semilinear heat equation. The method being used in the aforementioned paper also works for the critical case in the sense of Sobolev embedding. It is also well-known that the embedding $H_0^1(\Omega) \hookrightarrow L^{2*}(\Omega)$ is not compact. Thus, the lack of compactness coming from the nonlinear term raises a problem which might be essential in nature instead of a technical one. The extension of our result in Theorem 4.1.1 for the critical case will be discussed further in our forthcoming paper [18].

On the other hand, the lack of compactness can also arise from an unbounded domain. For instance, let $N \geq 3$ and consider the following system of semilinear heat equations:

$$\begin{cases} \partial_t \vec{U} = \Delta \vec{U} + A\vec{U} + \nabla F(\vec{U}) & \text{in } \mathbb{R}^N \times (0, T_m), \\ \vec{U}(\,\cdot\,, 0) = \vec{U}_0 & \text{in } \mathbb{R}^N. \end{cases}$$

where ∇F is given as in the problem (P) in Chapter 4, A is a real symmetric 2×2 matrix, and $\vec{U}_0 \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for simplicity. In this case, we do not have a compact Sobolev embedding even when $p \in (2, 2^*)$. Furthermore, there is no known result for a single equation case related to the aforementioned problem to the extent of our knowledge at the time of writing this thesis.

Based on the explanation above, we consider the following things for possible future research:

1. Verify the existence of L^∞-global bounds for time-global solutions to a system of semilinear heat equations in $\mathbb{R}^N$ for subcritical $p \in (2, 2^*)$.

2. Verify the existence of L^∞-global bounds for time-global solutions to a system of semilinear heat equations in $\mathbb{R}^N$ for $p = 2^*$, i.e., p is a critical Sobolev exponent in the sense of Sobolev embedding.

5.2.3 Behavior of solutions to semilinear heat equations and solutions to quasilinear heat equations

Let $N \geq 3$ and consider the following semilinear heat equations:

$$\begin{cases} \partial_t u = \Delta u - u + f(u) & \text{in } \mathbb{R}^N \times (0, T_m), \\ u(\,\cdot\,, 0) = u_0 & \text{in } \mathbb{R}^N \end{cases}$$

where f is given as in the Problem (P) in Chapter 3 and $u_0 \in L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ for simplicity. When the spatial domain is bounded and $f(u) = u + u|u|^{p-2}$ with $p > 1$, Ikehata and Suzuki have classified the solutions into stable and unstable sets. Particularly, in this case, the solution decays as $t \to \infty$ if the initial data is sufficiently small and the solution blows up in finite time in the sense of H_0^1-norm if the solution is sufficiently large. Moreover, when $N = 1$, the solution is also bounded in L^∞-norm in view of the compact embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. As a result, if we take the spatial domain of such a problem from $\mathbb{R}$ to $\mathbb{R}_+$, the solution converges to a travelling pseudo-wave of a ground state solution for some nonnegative initial data as given by Fašangová and Feireisl in [26]. This exposition ties up the connection between elliptic partial differential equations (Laplace or Poisson equation) and parabolic partial differential equations (semilinear heat equation).

In order to obtain the connection above between elliptic partial differential equations and parabolic partial differential equations for $N > 1$, we need to know the behavior of solutions of semilinear heat equations in $\mathbb{R}^N$. As the first step, establishing the existence of L^∞-global bounds for time-global solution is essential to extend the results of Fašangová and Feireisl in [26] from one dimensional spatial domain to higher dimensional spatial domain.

Another consideration for future research is also related to game-theoretic p-Laplace operator as heavily discussed in Chapter 2. Particularly, Ishiwata, Magnanini, and Wadade have also introduced the p-mean value for parabolic equations involving game-theoretic p-Laplace operator in [46]. Next, let us consider the following quasilinear parabolic equation:

$$\begin{cases} \partial_t u = \Delta_p u - u|u|^{p-2} & \text{in } \Omega \times (0, T_m), \\ u = 0 & \text{on } \partial\Omega \times (0, T_m), \\ u(\,\cdot\,, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{5.2}$$

where Ω is a bounded smooth domain and $u_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$ for simplicity. The existence of solution to this problem has been obtained by Otani in [72, Theorem 6.1, pp. 600] for any initial data in $L^2(\Omega)$. However, at the time of writing this thesis, we still do not fully understand the behavior of solutions to (5.2). For instance, we do not know whether time-global solution exists or not and if it exists, we also do not know whether such a solution is bounded under a suitable norm or not.

In view of the exposition so far in this subsection, we have the following things to consider for possible future research:

1. Verify the existence of L^∞-global bounds for semilinear heat equations with subcritical nonlinearity under Ambrosetti-Rabinowitz in $\mathbb{R}^N$.

2. Analyze the stable and unstable sets of the solutions to the semilinear heat equations with subcritical nonlinearity under Ambrosetti-Rabinowitz in $\mathbb{R}^N$.

3. Verify whether the results given by Fašangová and Feireisl in [26] can be extended to a higher dimension ($N \geq 3$).

4. Analyze the behavior of solutions to quasilinear parabolic equation in the form of(5.2) and its connection with the corresponding quasilinear elliptic equation in the form of p-Laplace equation.

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Appendix

Proof of Proposition 4.2.1

Lemma .0.1 *Assume condition* (AR) *in Chapter 3 holds. Then, for any* $\delta > 0$*, there exists* $C_{\mu,\delta} > 0$ *such that*

$$\textit{if } |u| \geq \delta \textit{ then } C_{\mu,\delta}|u|^{\mu} \leq F(u)$$

for any $u \in \mathbb{R}$*.*

Following Lemma .0.1, we have similar property for condition (SAR) in chapter 3.

Remark .0.1 *Assume* (SAR) *in Chapter 3 holds. Then, for any* $\delta > 0$*, there exists* $C_{\mu,\delta} > 0$ *such that*

$$\textit{if } |(u,v)| \geq \delta \textit{ then } C_{\mu,\delta}|(u,v)|^{\mu} \leq F(u,v)$$

for any $(u,v) \in \mathbb{R}^2$*.*

For the proof of both Lemma .0.1 and Remark .0.1, see [19].

We recall the following lemma.

Lemma .0.2 (See e.g. [54]) *There exists no nonnegative and increasing function* $Y \in C^2(\bar{t},\infty)$ *with* $\bar{t} \in \mathbb{R}$ *such that for some* $\beta > 0$*,*

$$YY'' \geq (\beta+1)[Y']^2 \textit{ on } (\bar{t},\infty) \textit{ and } \lim_{t\to\infty} Y(t) = \infty.$$

Now we are ready to prove Proposition 4.2.1. Assume on the contrary,

$$\text{there exists } t_0 \in [0,\infty) \text{ such that } I[\vec{U}(t_0)] < 0. \tag{3}$$

The condition (AR2) and the definition of I yield

$$\begin{aligned}
J[\vec{U}] &= \|\nabla \vec{U}\|_2^2 - \int_\Omega \vec{U} A \vec{U}^{\mathsf{T}} dx - \int_\Omega (u_1 f(\vec{U}) + u_2 g(\vec{U}))dx \\
&\leq \|\nabla \vec{U}\|_2^2 - \int_\Omega \vec{U} A \vec{U}^{\mathsf{T}} dx - \mu \int_\Omega F(\vec{U})dx \\
&= -\left(\frac{\mu}{2} - 1\right)\left(\|\nabla \vec{U}\|_2^2 - \int_\Omega \vec{U} A \vec{U}^{\mathsf{T}} dx\right) + \mu I[\vec{U}]. \qquad (4)
\end{aligned}$$

By taking $(\phi_1, \phi_2) = \vec{U}$ in (4.5), the quadratic form in (4) is nonnegative in view of (AN). However, since $\mu > 2$, the first term in the right hand side of (4) is not positive. Thus, we have

$$J[\vec{U}] \leq \mu I[\vec{U}], \qquad (5)$$

which together with (J2) yields

$$\frac{1}{2}\frac{d}{dt}\|\vec{U}(t)\|_2^2 = -J[\vec{U}(t)] \geq -\mu I[\vec{U}(t)].$$

Combining this relation with the fact that our energy functional I is strictly negative when $t \geq t_0$ since it is non-increasing, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\vec{U}(t)\|_2^2 \geq -\mu I[\vec{U}(t)] \geq -\mu I[\vec{U}(t_0)] > 0 \qquad (6)$$

for any $t \geq t_0$. Therefore, for $Y(t) := \frac{1}{2}\int_{t_0}^t \|\vec{U}(s)\|_2^2 ds$, we see that

$$Y''(t) \geq -\mu I[\vec{U}(t_0)] > 0 \text{ for any } t \geq t_0, \qquad (7)$$

which means

$$\lim_{t\to\infty} Y'(t) = \infty \text{ and } \lim_{t\to\infty} Y(t) = \infty. \qquad (8)$$

Next, observe that for any $t \geq t_0$, we have

$$\begin{aligned}
\left|Y'(t) - Y'(t_0)\right| &= \left|\int_{t_0}^t Y''(s)ds\right| = \left|\int_{t_0}^t \int_\Omega \vec{U}(s) \cdot \partial_s \vec{U}(s) dx ds\right| \\
&\leq \int_{t_0}^t \|\vec{U}(s)\|_2 \|\partial_s \vec{U}(s)\|_2 ds \\
&\leq \left(\int_{t_0}^t \|\vec{U}(s)\|_2^2 ds\right)^{\frac{1}{2}} \left(\int_{t_0}^t \|\partial_s \vec{U}(s)\|_2 ds\right)^{\frac{1}{2}},
\end{aligned}$$

which yields

$$Y(t)Y''(t) \geq \frac{\mu}{2}\big(Y'(t) - Y'(t_0)\big)^2 \quad (t \geq t_0) \tag{9}$$

in view of (I2) and (6). The relation above together with (8) allows us to choose $\theta \in \left(\frac{2}{\mu}, 1\right)$ and $t_\theta > t_0$ large enough so that

$$Y(t)Y''(t) \geq \frac{\mu}{2}\theta(Y'(t))^2 \text{ for any } t > t_\theta.$$

This relation and (8) assure the assumption in Proposition .0.2 whence Y cannot exist time-globally, which contradicts the time-global existence of $\vec{U}$. Hence (3) is false and we have the conclusion. ∎

Proof of Lemma 4.2.5 First, we fix $\alpha' \in [0, \beta')$ and a compact set $K \subset \mathbb{R}^N \times [-1, 0]$. Let $C_{\alpha'}(K) := C^{0,\alpha';0,\alpha'/2}(K)$ for convenience and define

$$\begin{aligned}\|\varphi\|_{C_{\alpha'}(K)} := \|\varphi\|_{C(K)} + \sup_{s\in[-1,0]} \sup_{y_1\neq y_2} \frac{|\varphi(y_1, s) - \varphi(y_2, s)|}{|y_1 - y_2|^{\alpha'}} \\ + \sup_{y\in\overline{\Omega}} \sup_{s_1\neq s_2} \frac{|\varphi(y, s_1) - \varphi(y, s_2)|}{|s_1 - s_2|^{\frac{\alpha'}{2}}}\end{aligned}$$

for any $\varphi \in C_{\alpha'}(K)$. Observe that $(w_n)_{\in\mathbb{N}}$ is uniformly bounded in $C_{\alpha'}(K)$ and equicontinuous in K since w is uniformly bounded in $C_{\beta'}(K)$ and $\alpha' < \beta'$. Therefore, we can apply Ascoli-Arzela Theorem to see that

$$w_n \to w \text{ in } C_{loc}(\mathbb{R}^N \times [-1, 0]) \quad (n \to \infty) \tag{10}$$

up to a subsequence. If $\alpha = 0$, then our proof is complete and thus we assume $\alpha \neq 0$ throughout the remaining part of the proof. It remains to show that

$$\sup_{s\in[-1,0]} \sup_{y_1\neq y_2} \frac{|w_n(y_1, s) - w(y_1, s) - (w_n(y_2, s) - w(y_2, s))|}{|y_1 - y_2|^{\alpha'}} \to 0 \quad (n \to \infty) \tag{11}$$

and

$$\sup_{y\in\overline{\Omega}} \sup_{s_1\neq s_2} \frac{|w_n(y, s_1) - w(y, s_1) - (w_n(y, s_2) - w(y, s_2))|}{|s_1 - s_2|^{\frac{\alpha'}{2}}} \to 0 \quad (n \to \infty). \tag{12}$$

Here, we set

$$h_n := w_n - w$$

for convenience and show (11) holds. Next, by using (10), we observe that

$$|w(y_1,s)-w(y_2,s)| = |w_n(y_1,s)-w_n(y_2,s)| + o(1) \le C|y_1-y_2|^{\alpha'} + o(1)$$

and

$$|w(y,s_1)-w(y,s_2)| = |w_n(y,s_1)-w_n(y,s_2)| + o(1) \le C|s_1-s_2|^{\frac{\alpha'}{2}} + o(1)$$

as $n\to\infty$ for any $y_1,y_2\in K$ and for any $s_1,s_2\in[-1,0]$. Therefore, we see that $w\in C_{\beta'}(K)\subset C_{\alpha'}(K)$ since $\alpha'<\beta'$. In fact, $(h_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C_{\alpha'}(K)$ and $C_{\beta'}(K)$. By using (10), we see that

$$h_n \to 0 \text{ in } C(K) \text{ as } n\to\infty.$$

Then, we define

$$A_{1,n}(s) := \sup\left\{ \frac{|h_n(y_1,s)-h_n(y_2,s)|}{|y_1-y_2|^{\alpha'}} \,\middle|\, y_1\ne y_2 \text{ and } |y_1-y_2|\le\delta \right\}$$

$$A_{2,n}(s) := \sup\left\{ \frac{|h_n(y_1,s)-h_n(y_2,s)|}{|y_1-y_2|^{\alpha'}} \,\middle|\, y_1\ne y_2 \text{ and } |y_1-y_2|>\delta \right\}$$

so that we can deduce

$$A_{1,n}(s) \le \delta^{\beta'-\alpha'}\|h_n\|_{C_{\beta'}(K)} \le C\delta^{\beta'-\alpha'}. \tag{13}$$

since $(h_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C_{\beta'}(K)$. Again, by using the uniform boundedness of $(h_n)_{n\in\mathbb{N}}$ in $C_{\beta'}(K)$ and the definition of $A_{2,n}(s)$, we infer

$$A_{2,n}(s) \le 2\delta^{-\alpha'}\|h_n\|_{C(K)} \to 0 \text{ as } n\to\infty. \tag{14}$$

Combining (13) and (14), we see that

$$\begin{aligned}\lim_{n\to\infty}\sup_{s\in[-1,0]}\sup_{y_1\ne y_2}\frac{|h_n(y_1,s)-h_n(y_2,s)|}{|y_1-y_2|^{\alpha'}} &\le \lim_{n\to\infty}\sup_{s\in[-1,0]}\big(A_{1,n}(s)+A_{2,n}(s)\big)\\ &\le C\delta^{\beta'-\alpha'} + \lim_{n\to\infty}\sup_{s\in[-1,0]}A_{2,n}(s)\\ &\le C\delta^{\beta'-\alpha'} \to 0 \text{ as } \delta\downarrow 0\end{aligned}$$

which means that (11) holds. Similarly, we repeat the argument for (11) to show that (12) holds. However, we replace $A_{1,n}$ and $A_{2,n}$ with

$$B_{1,n}(y) := \sup\left\{ \frac{|h_n(y,s_1)-h_n(y,s_2)|}{|s_1-s_2|^{\frac{\alpha'}{2}}} \,\middle|\, s_1\ne s_2 \text{ and } |s_1-s_2|\le\delta \right\}$$

$$B_{2,n}(y) := \sup\left\{ \frac{|h_n(y,s_1)-h_n(y,s_2)|}{|s_1-s_2|^{\frac{\alpha'}{2}}} \,\middle|\, s_1\ne s_2 \text{ and } |s_1-s_2|>\delta \right\}$$

respectively and proceed as in the proof of (11) to complete our proof. ∎

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List of Works in Progress

1. Chandra, E.W., Ishiwata, M. "Type I blow-up rate analysis for Ambrosetti-Rabinowitz nonlinearity in semilinear heat equations", work-in-progress.

2. Chandra, E.W., Ishiwata, M. "On bounds for time-global solutions to a system of semilinear heat equations with subcritical Ambrosetti-Rabinowitz nonlinearity", work-in-progress.

3. Chandra, E.W., Ishiwata, M. "On bounds for global solutions of semilinear parabolic equations system with critical nonlinearity", work-in-progress.

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