A FEW REMARKS ON CLASS NUMBERS OF IMAGINARY QUADRATIC NUMBER FIELDS

Taira HONDA

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1. Let K be an imaginary quadratic number field with discriminant $-d$. As is well known, the class number $h(d)$ of K is given by the formula

$$(1) \qquad h(d) = -\frac{1}{d} \sum_{n=1}^{d} \chi(n)n \,,$$

where χ is the Jacobi symbol modulo d.

Let us consider the case where d is a prime number such that $p \equiv 3 \pmod 4$. Then

$$(2) \qquad h(p) = -\frac{1}{p} \sum_{n=1}^{p-1} \chi(n)n \,,$$

where χ is the Legendre symbol. From (2) we get

$$h(p) = -\frac{1}{p} \sum_{n=1}^{(p-1)/2} \{\chi(n)n + \chi(p-n)(p-n)\}$$

$$= -\frac{1}{p} \sum_{n=1}^{(p-1)/2} \{2\chi(n)n - \chi(n)p\}$$

$$= -\frac{2}{p} \sum_{n=1}^{(p-1)/2} \chi(n)n + \sum_{n=1}^{(p-1)/2} \chi(n)$$

Here it is well-known that

$$\sum_{n=1}^{(p-1)/2} \chi(n) = \{2 - \chi(2)\}h(p) \,.$$

Summing up, we get

$$\{1 - \chi(2)\}h(p) = \frac{2}{p} \sum_{n=1}^{(p-1)/2} \chi(n)n \,.$$

So, if $p \equiv -1 \pmod 7$, it holds that

(3)
$$h(p) = \frac{1}{p} \sum_{n=1}^{(q-1/2} \chi(n)n \, .$$

Now denote by $\left\{\dfrac{n}{p}\right\}$ the frational part of n/p, i.e. $\left\{\dfrac{n}{p}\right\} = \dfrac{n}{p} - \left[\dfrac{n}{p}\right]$.
Then we get

$$p \sum_{k=1}^{p-1} \left\{ \frac{k^2}{p} \right\} = 2 \sum_{\substack{n=1 \\ \chi(n)=1}}^{p-1} n \, ,$$

which implies

$$h(p) = -\frac{1}{p} \sum_{n=1}^{p-1} \chi(n)n$$

$$= -\frac{1}{p} \Big\{ \sum_{\substack{n=1 \\ \chi(n)=1}}^{p-1} n - \sum_{\substack{n=1 \\ \chi(n)=-1}}^{p-1} n \Big\}$$

$$= -\frac{1}{p} \Big\{ \sum_{\substack{n=1 \\ \chi(n)=1}}^{p-1} n - \sum_{n=1}^{p-1} n \Big\}$$

$$= -\sum_{k=1}^{p-1} \left\{ \frac{k^2}{p} \right\} + \frac{p-1}{2} \, .$$

Thus we have

(4)
$$h(p) = \frac{p-1}{2} - \sum_{k=1}^{p-1} \left\{ \frac{k^2}{p} \right\} \, .$$

Consider the area $S; 0<x<p, 0 \leq y < x^2/p$. Then (4) implies that $h(p)$ is the error term in estimating the lattice points in S. Since the hyperbola $y^2 = x^2/p$ has no center, this implies the difficulty of estimation of $h(p)$ comparative with the circle problem and the divisor problem.

2. Let p be the prime such that $p \equiv 3 \pmod 4$ as before. Then we see easily $(p-1)/2! \equiv \pm 1 \pmod p$. Put $(p-1)/2! \equiv \varepsilon_p \pmod p$ with $\varepsilon_p = \pm 1$. Then $\varepsilon_p = +1$ iff the number of the set $\left\{ 1 \leq n \leq (p-1)/2; \left(\dfrac{n}{p}\right) = 1 \right\}$ is even. From

$$h(p) = -\frac{1}{p} \sum_{n=1}^{p-1} \left(\frac{n}{p}\right) n$$

we have

$$h(p) = \frac{1}{2-\chi(2)} \sum_{n=1}^{(p-1)/2} \chi(n) \, ,$$

as is well known. Therefore if $\varepsilon_p = +1$,

$$h(p) = \frac{1}{2-\chi(2)}\left(\frac{p-1}{2} - 2s\right),$$

where s is the number of quadratic non-residue in $[1, (p-1)/2]$. Now

$$\frac{1}{2-\chi(2)} \cdot \frac{p-1}{2} \equiv 3 \pmod 4$$

as is easily verified. Therefore we have

$$h(p) \equiv -1 \pmod 4$$

regarding s is odd.

If $\varepsilon_p = -1$, we get

$$h(p) \equiv +1 \pmod 4$$

in the same way.

Summing up, we have

$$h(p) \equiv -\varepsilon_p \pmod 4.$$

It seems that the number of p with $\varepsilon_p = +1$ is asymptocally the same as that of p with $\varepsilon_p = -1$.

OSAKA CITY UNIVERSITY

Added in proof; The congruence

$$h(p) = -\varepsilon_p \pmod 4$$

is already known as the Jacobi-Mordell formula.