Limiting behaviors of generalized elephant random walks

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We study the limiting behaviors of a generalized elephant random walk on the integer lattice. This random walk is defined by using two sequences of parameters expressing the memory at each step from the whole past and the drift of each step to the right, respectively. This model is also regarded as a dependent Bernoulli process. Our results reveal how the scaling factors are determined by the behaviors of the parameters. In particular, we allow the degeneracy of the parameters. We further present several examples in which the scaling factors are explicitly computed.

I. INTRODUCTION

We are concerned with the limiting behaviors of a generalized elephant random walk on the integer lattice. This random walk is defined by using the two sequences of parameters expressing the memory at each step from the whole past and the drift of each step to the right, respectively. Our purpose in this paper is to reveal how the behaviors of these two sequences would determine the scaling factors of the random walk under consideration.

Drezner and Farnum[6] introduced a generalized binomial distribution associated with the correlated Bernoulli sequence, and calculated the moments and variance. Heyde[10] (Theorem 1) utilized the martingale theory (see, e.g.,[8]) to show the existence of the phase transition on the Gaussian fluctuation around the mean in terms of the correlation parameter. The approach of Heyde[10] is further developed and applied to the proofs of several limit theorems for more general dependent Bernoulli sequences (see, e.g.,[11,14,17,18]). As we see from the previous works above, if the Bernoulli sequence does not degenerate, then the almost sure linear scaling limit exists for the associated process.

On the other hand, Schütz and Trimper[15] introduced the model of an elephant random walks in which the law of each step is given by the steps until just before with the independent random drift. They observed the existence of the phase transition on the diffusivity by revealing the asymptotic behavior of the second moment. Coletti, Gava and Schütz[5] and Bercu[1] also applied the martingale theory to establish the limit theorems as Heyde[10]. For the superdiffusive elephant random walk with bias, Kubota and Takei[13] showed that if the bias decays, then the L^2-scaling limit exhibits the phase transition, together with the explicit scaling factor. When the variance does not degenerate, they also revealed the Gaussian fluctuations around both the mean and the random drift. As pointed in[13], a class of elephant random walks corresponds to the dependent Bernoulli sequences by simple relations. For the recent development on elephant random walks, see, e.g.,[2,3,7,9].

Motivated by[13], we have two preliminary questions as follows:

(i) Almost sure scaling limit of the elephant random walk with decaying bias;

(ii) Gaussian fluctuations of the elephant random walk with degenerate variance.

In this paper, we address these two questions for generalized elephant random walks. More precisely, for (i), we establish the almost sure scaling limit in Corollary 4. As we see from examples in Section III, the scaling factor is not necessarily linear and explicitly affected by the convergence rates of the sequences of the parameters. We further see that the L^2-scaling limit of[13] (Theorem 4) remains true as the almost sure convergence (Remark 6). For (ii), we find suitable scaling factors for the Gaussian fluctuations by taking into consideration the degeneracy of the variance in Theorems 14 and 22. As its application, we have the scaling limit result interpolating Corollary 4 and Theorem 5 (Corollary 17).

Our approach here is based on the martingale theory as in the previous works ([1,5,10,11,13,14,17,18]). For (i), Corollary 4 follows from the law of the large numbers (Theorem 3). To prove Theorem 3, we calculate the quadratic variation of some martingale associated with the scaling factor. This approach is a simple modification of[5] (Theorem 1),[11] (Theorem 2.1),[13] (Theorem 1) and[17] (Theorem

2). Here we would like to mention that, as far as the author knows, no other results are available about the almost sure scaling limit except for the linear scaling. For (ii), we show the Gaussian fluctuations (Theorems 14 (1) and 22) by getting the asymptotic behavior of the quadratic variation of a square integrable martingale $\{M_n\}_{n=1}^{\infty}$ (Lemma 1 (3) and (19)). To do so, we obtain the exact decay rate for the linear scaling limit as an application of the law of the large numbers (Lemma 15 (2)).

Related to our preliminary questions (i) and (ii), we would like to mention that Konno[12] clarified a structural similarity between correlated random walks and quantum walks. Using this similarity, he further calculated the characteristic function and proved the convergence in distribution for the correlated random walk even if the correlation parameter degenerates.

The rest of this paper is organized as follows: In Section II, we first introduce a model of generalized elephant random walks. We then present the moment formula and elementary calculations which are necessary for the subsequent sections. In Section III, we first prove the law of the large numbers, almost sure and also L^2-scaling limits of the random walks. We then compute the scaling factors in examples. In Section IV, we establish the Gaussian fluctuations around the mean and the random drift by proving the central limit theorems and the laws of the iterated logarithm. In connection with Section III, we also discuss again the scaling limit problem. We then compute scaling factors related to the Gaussian fluctuations. Appendix A includes the calculations of series and the computations of the scaling factors in the examples of Sections III and IV.

II. PRELIMINARIES

We first introduce a model of generalized elephant random walks. Let $q \in [0,1]$, and let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\varepsilon_n\}_{n=1}^{\infty}$ be $[0,1]$-valued sequences. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of $\{-1,1\}$-valued random variables defined on a probability space $(\Omega, \mathscr{F}, P)$ such that

$$P(X_1 = 1) = q, \quad P(X_1 = -1) = 1-q$$

and

$$P(X_{n+1} = \pm 1 \mid X_1, \dots, X_n) = \alpha_n \frac{\sharp\{i = 1, \dots, n \mid X_i = \pm 1\}}{n} + (1-\alpha_n)\frac{1 \pm \varepsilon_n}{2} \quad (n \geq 1). \tag{1}$$

We then define $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ $(n \geq 1)$. We call $\{S_n\}_{n=0}^{\infty}$ a *generalized elephant random walk* with correlation parameters $\{\alpha_n\}_{n=1}^{\infty}$ and drift parameters $\{\varepsilon_n\}_{n=1}^{\infty}$. By (1), the $(n+1)$st step X_{n+1} of $\{S_n\}_{n=0}^{\infty}$ follows a uniformly chosen past step among $X_1, \dots, X_n$ with probability α_n, and the random walk with drift ε_n with probability $1-\alpha_n$. If $\alpha_n = \alpha$ $(n \geq 1)$ for some $\alpha \in [0,1]$, then $\{S_n\}_{n=0}^{\infty}$ is the so-called elephant random walk with bias ([1,5,13,15]).

Let $X'_n = (1+X_n)/2$. If we take $\varepsilon_n = 2q-1$ $(n \geq 1)$, then $\{X'_n\}_{n=1}^{\infty}$ is a sequence of the correlated Bernoulli random variables introduce by Drezner and Farnum[6] (Section 3). Under the current setting, $\{X'_n\}_{n=1}^{\infty}$ forms the dependent Bernoulli sequence studied by[11,17]. In particular, if we let $S'_0 = 0$ and $S'_n = X'_1 + \cdots + X'_n$ $(n \geq 1)$, then $\{S_n\}_{n=0}^{\infty}$ corresponds to the dependent Bernoulli process $\{S'_n\}_{n=0}^{\infty}$ by the relation $S_n = 2S'_n - n$.

Let $a_1 = 1$ and $b_1 = 1$, and let

$$a_n = \prod_{k=1}^{n-1}\left(1 + \frac{\alpha_k}{k}\right), \quad b_n = \prod_{k=1}^{n-1}\left(1 + \frac{2\alpha_k}{k}\right) \quad (n \geq 2).$$

For $n \geq 1$, we define $\mathscr{F}_n = \sigma(X_1, \dots, X_n)$ and $M_n = (S_n - E[S_n])/a_n$. The next lemma can be proved in a way similar to[13] (Section 3).

Lemma 1. (1) *For any* $n \geq 1$,

$$E[X_{n+1} \mid \mathscr{F}_n] = \frac{\alpha_n}{n} S_n + (1-\alpha_n)\varepsilon_n$$

and

$$E[S_{n+1} \mid \mathscr{F}_n] = \left(1+\frac{\alpha_n}{n}\right) S_n + (1-\alpha_n)\varepsilon_n.$$

Moreover, for any $n \geq 1$,

$$E[S_n] = a_n \left(2q-1+\sum_{k=1}^{n-1} \frac{(1-\alpha_k)\varepsilon_k}{a_k}\right).$$

(2) *For any* $n \geq 1$,

$$E[S_{n+1}^2 \mid \mathscr{F}_n] = \left(1+\frac{2\alpha_n}{n}\right) S_n^2 + 2(1-\alpha_n)\varepsilon_n S_n + 1$$

and

$$E[S_n^2] = b_n \left(\sum_{k=1}^{n} \frac{1}{b_k} + 2\sum_{k=1}^{n-1} \frac{(1-\alpha_k)\varepsilon_k}{b_{k+1}} E[S_k]\right).$$

(3) $\{M_n\}_{n=1}^{\infty}$ *is a square integrable martingale with respect to the filtration* $\{\mathscr{F}_n\}_{n\geq 1}$.

We next present asymptotic properties of series and products which will be used in the subsequent sections. Let $g_1 = 1$ and $l_1 = 1$, and let

$$g_n = \prod_{k=1}^{n-1}\left(1+\frac{\alpha}{k}\right), \quad l_n = \prod_{k=1}^{n-1}\left(1+\frac{\alpha_k-\alpha}{k+\alpha}\right) \quad (n \geq 2). \tag{2}$$

Then by definition, $a_n = g_n l_n$. Let $\rho_1 = 1$ and

$$\rho_n = \exp\left(\sum_{k=1}^{n-1} \frac{\alpha_k - \alpha}{k}\right) \quad (n \geq 2).$$

Lemma 2. (1) *For any* $\alpha \in [0,1]$,

$$n^{1-\alpha}\left(g_n - \frac{n^{\alpha}}{\Gamma(\alpha+1)}\right) \to -\frac{\alpha(1-\alpha)}{2\Gamma(\alpha+1)} \quad (n \to \infty). \tag{3}$$

(2) *There exists a positive constant* C_0 *such that* $b_n \sim C_0 a_n^2$.

(3) $a_n/n \to 0$ $(n \to \infty)$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{1-\alpha_n}{n} = \infty. \tag{4}$$

(4) *Assume that the sequence* $\{\alpha_n\}_{n=1}^{\infty}$ *is convergent to some* $\alpha \in [0,1]$. *Then the sequence* $\{l_n\}_{n=1}^{\infty}$ *is slowly varying, that is, for any* $\lambda > 0$,

$$\lim_{n\to\infty} \frac{l_{[\lambda n]}}{l_n} = 1.$$

Moreover, there exists a positive constant C_* *such that* $l_n \sim C_* \rho_n$. *In particular,*

$$a_n \sim \frac{C_*}{\Gamma(\alpha+1)} n^{\alpha} \rho_n$$

and $\{a_n\}_{n=1}^{\infty}$ *is a regularly varying sequence with index* α.

(5) *If* $\alpha = 1$, *then* (4) *holds if and only if*

$$\sum_{n=1}^{\infty} \frac{1-\alpha_n}{a_n} = \infty. \tag{5}$$

Moreover, if this condition is valid, then

$$\lim_{n\to\infty} \rho_n \sum_{k=1}^{n} \frac{1-\alpha_k}{a_k} = \frac{1}{C_*} \tag{6}$$

and

$$\lim_{n\to\infty} \frac{a_n}{n} \sum_{k=1}^{n} \frac{1-\alpha_k}{a_k} = 1. \tag{7}$$

Proof. (3) is proved in[17] (Lemma 2). We prove (1), (2), (4) and (5) in this order.

By the Stirling formula for the Gamma function (see, e.g.,[16] (Table 1)),

$$\Gamma(t) = \sqrt{\frac{2\pi}{t}} \left(\frac{t}{e}\right)^t \left(1 + \frac{1}{12t} + O\left(\frac{1}{t^2}\right)\right) \quad (t \to \infty).$$

Hence we have as $n \to \infty$,

$$g_n = \frac{\Gamma(n+\alpha)}{\Gamma(n)\Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha+1)e^{\alpha}} \sqrt{\frac{n}{n+\alpha}} \left(1+\frac{\alpha}{n}\right)^n (n+\alpha)^{\alpha} \left(1 + O\left(\frac{1}{n^2}\right)\right).$$

Then by elementary calculus, we obtain (1).

Since

$$\frac{b_n}{a_n} = \prod_{k=1}^{n-1} \frac{k+2\alpha_k}{k+\alpha_k} = \prod_{k=1}^{n-1} \left(1 + \frac{\alpha_k}{k+\alpha_k}\right) = a_n \prod_{k=1}^{n-1} \left(1 - \frac{\alpha_k^2}{k^2+2k\alpha_k+\alpha_k^2}\right)$$

and

$$0 \le \sum_{n=1}^{\infty} \frac{\alpha_n^2}{n^2+2n\alpha_n+\alpha_n^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

the infinite product

$$C_0 := \prod_{n=1}^{\infty} \left(1 - \frac{\alpha_n^2}{n^2+2n\alpha_n+\alpha_n^2}\right)$$

is convergent so that we have (2).

By the Taylor theorem, for any $x > -1$, there exists a constant $\theta \in (0,1)$ such that

$$\log(1+x) = x - \frac{(\theta x)^2}{2}.$$

Hence for all sufficiently large $k \ge 1$, there exist constants $\theta_k \in (0,1)$ such that

$$\log\left(1+\frac{\alpha_k-\alpha}{k+\alpha}\right) = \frac{\alpha_k-\alpha}{k+\alpha} - \frac{\theta_k^2}{2}\left(\frac{\alpha_k-\alpha}{k+\alpha}\right)^2 = \frac{\alpha_k-\alpha}{k} - \left(\frac{\alpha(\alpha_k-\alpha)}{k(k+\alpha)} + \frac{\theta_k^2}{2}\left(\frac{\alpha_k-\alpha}{k+\alpha}\right)^2\right).$$

Combining this relation with the expression

$$l_n = \exp\left\{\sum_{k=1}^{n-1} \log\left(1+\frac{\alpha_k-\alpha}{k+\alpha}\right)\right\},$$

we have $l_n/\rho_n \to C_*$ $(n \to \infty)$ with

$$C_* := \exp\left\{-\sum_{n=1}^{\infty}\left(\frac{\alpha(\alpha_n-\alpha)}{n(n+\alpha)}+\frac{\theta_n^2}{2}\left(\frac{\alpha_n-\alpha}{n+\alpha}\right)^2\right)\right\}.$$

Since the sequence $\{\rho_n\}_{n=1}^{\infty}$ is slowly varying, so is the sequence $\{l_n\}_{n=1}^{\infty}$. Then by (1), the proof of (4) is complete.

Suppose that $\alpha = 1$. If (4) holds, then (5) follows by (3). We now assume that (5) holds but (4) fails. Then by (4),

$$\sum_{k=1}^{n-1}\frac{1-\alpha_k}{a_k} \sim \frac{1}{C_*}\sum_{k=1}^{n}\frac{1-\alpha_k}{k\rho_{k+1}} = \frac{1}{C_*}\sum_{k=1}^{n}\frac{1-\alpha_k}{k}\exp\left(\sum_{l=1}^{k}\frac{1-\alpha_l}{l}\right). \tag{8}$$

Since the right hand side above is convergent as $n \to \infty$ by assumption, we have a contradiction so that (4) holds. Then (A2) and (8) yield (6). Combining this with (4), we get (7). □

III. LAW OF THE LARGE NUMBERS

In this section, we establish the law of the large numbers for $\{S_n\}_{n=0}^{\infty}$.

Theorem 3. *Let $\{r_n\}_{n=1}^{\infty}$ be a positive sequence such that $a_n/r_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} 1/r_n^2 < \infty$. Then*

$$\lim_{n\to\infty}\frac{S_n - E[S_n]}{r_n} = 0, \quad P\text{-a.s.}$$

If $r_n = n$ $(n \geq 1)$, then this theorem is proved by[5] (Theorem 1),[11] (Theorem 2.1),[13] (Theorem 1) and[17] (Theorem 1). As we will see from the proof of Theorem 3 below, their approach still works for more general sequences $\{r_n\}_{n=1}^{\infty}$ because the sequence $\{(X_n - E[X_n \mid \mathscr{F}_{n-1}])/r_n\}_{n=1}^{\infty}$ is a martingale difference.

Proof of Theorem 3. Let

$$d_j = \begin{cases} M_1 & (j=1), \\ M_j - M_{j-1} & (j \geq 2) \end{cases}$$

and $\gamma_j = 1 + \alpha_j/j$. Then by Lemma 1 (1),

$$\begin{aligned} d_j &= \frac{S_j - E[S_j] - \gamma_{j-1}(S_{j-1} - E[S_{j-1}])}{a_j} = \frac{S_j - \gamma_{j-1}S_{j-1} - (1-\alpha_{j-1})\varepsilon_{j-1}}{a_j} \\ &= \frac{S_j - E[S_j \mid \mathscr{F}_{j-1}]}{a_j} = \frac{X_j - E[X_j \mid \mathscr{F}_{j-1}]}{a_j}, \end{aligned} \tag{9}$$

which yields

$$\frac{X_j - E[X_j \mid \mathscr{F}_{j-1}]}{r_j} = \frac{d_j a_j}{r_j}. \tag{10}$$

By assumption,

$$\sum_{j=1}^{\infty} E\left[\left(\frac{X_j - E[X_j \mid \mathscr{F}_{j-1}]}{r_j}\right)^2 \mid \mathscr{F}_{j-1}\right] \leq \sum_{j=1}^{\infty}\frac{4}{r_j^2} < \infty.$$

Hence by[8] (Theorem 2.15) and (10), the series

$$\sum_{j=1}^{\infty}\frac{X_j - E[X_j \mid \mathscr{F}_{j-1}]}{r_j} = \sum_{j=1}^{\infty}\frac{d_j}{r_j/a_j}$$

converges almost surely. Since $a_n/r_n \to 0$ $(n \to \infty)$ by assumption, we obtain by Kronecker's lemma,

$$\frac{S_n - E[S_n]}{r_n} = \frac{a_n}{r_n} M_n = \frac{a_n}{r_n} \sum_{j=1}^{n} d_j \to 0, \quad P\text{-a.s.}$$

The proof is complete. □

By Theorem 3 with Lemma 1 (1), we have the growth exponent of $\{S_n\}_{n=0}^{\infty}$:

Corollary 4. *Let*

$$r_n = a_n \sum_{k=1}^{n} \frac{(1-\alpha_k)\varepsilon_k}{a_k}.$$

If $a_n/r_n \to 0$ $(n \to \infty)$ *and* $\sum_{n=1}^{\infty} 1/r_n^2 < \infty$, *then*

$$\lim_{n\to\infty} \frac{S_n}{r_n} = 1, \quad P\text{-a.s.} \tag{11}$$

We can also prove the L^2-convergence of S_n/r_n as in[13] (Theorem 4). Let $w_n = \sum_{k=1}^{n} 1/a_k^2$.

Theorem 5. *Let*

$$r_n = a_n \sum_{k=1}^{n} \frac{(1-\alpha_k)\varepsilon_k}{a_k}.$$

If $a_n/r_n \to 0$ $(n \to \infty)$, *then* $E[S_n] \sim r_n$. *Moreover, if* $a_n\sqrt{w_n}/r_n \to 0$ $(n \to \infty)$, *then* $E[S_n^2] \sim E[S_n]^2$ *and*

$$\lim_{n\to\infty} \frac{S_n}{r_n} = 1 \quad \text{in } L^2(P). \tag{12}$$

Proof. Assume that $a_n/r_n \to 0$ $(n \to \infty)$. Then by Lemma 1 (1), $E[S_n] \sim r_n$. Hence by Lemma 2 (2) and Lemma 25,

$$\begin{aligned} 2b_n \sum_{k=1}^{n-1} \frac{(1-\alpha_k)\varepsilon_k}{b_{k+1}} E[S_k] &\sim 2a_n^2 \sum_{k=1}^{n} \frac{(1-\alpha_k)\varepsilon_k}{a_k} \left(\sum_{l=1}^{k} \frac{(1-\alpha_l)\varepsilon_l}{a_l} \right) \\ &\sim a_n^2 \left(\sum_{k=1}^{n} \frac{(1-\alpha_k)\varepsilon_k}{a_k} \right)^2 \sim E[S_n]^2. \end{aligned}$$

Lemma 2 (2) also implies that for some positive constants c_1 and c_2, $c_1 \sum_{k=1}^{n} 1/b_k \le w_n \le c_2 \sum_{k=1}^{n} 1/b_k$ $(n \ge 1)$. Therefore, if $a_n\sqrt{w_n}/r_n \to 0$ $(n \to \infty)$, then Lemma 1 (2) yields $E[S_n^2] \sim E[S_n]^2$. Since

$$\lim_{n\to\infty} \frac{E[(S_n - E[S_n])^2]}{E[S_n]^2} = 0,$$

we obtain (12). □

Remark 6. Let $\alpha_n = \alpha \in [0,1)$ and $\varepsilon_n = n^{-\rho}$ for some $\rho > 0$. Then by Corollary 4 and Theorem 5, we have:

- If $0 \le \alpha < 1-\rho$ and $0 < \rho < 1/2$, then

$$\lim_{n\to\infty} \frac{S_n}{n^{1-\rho}} = \frac{1-\alpha}{1-(\alpha+\rho)}, \ P\text{-a.s. and in } L^2(P).$$

- If $\alpha = 1-\rho$ and $0 < \rho < 1/2$, then

$$\lim_{n\to\infty} \frac{S_n}{n^{\alpha}\log n} = 1-\alpha, \ P\text{-a.s. and in } L^2(P).$$

- If $\alpha = \rho = 1/2$, then

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}\log n} = \frac{1}{2}, \ P\text{-a.s. and in } L^2(P)$$

(see the subsequent examples for the validity of these calculations). The L^2-convergence results above are already proved in[13] (Theorem 4).

We now apply Theorem 3 to find the scaling factor of $\{S_n\}_{n=0}^{\infty}$. We first see that if $\varepsilon > 0$, then $\{S_n\}_{n=0}^{\infty}$ grows linearly under reasonable conditions.

Example 7. Let $0 \le \varepsilon \le 1$.
(i) Assume that $0 \le \alpha < 1$. Since $\{a_n\}_{n=1}^{\infty}$ is a regularly varying sequence with index α, we have

$$\lim_{n\to\infty} \frac{a_n}{n} \sum_{k=1}^{n} \frac{1}{a_k} = \frac{1}{1-\alpha}$$

and thus

$$\lim_{n\to\infty} \frac{r_n}{n} = \lim_{n\to\infty} \frac{a_n}{n} \sum_{k=1}^{n} \frac{(1-\alpha_k)\varepsilon_k}{a_k} = \varepsilon. \tag{13}$$

As $a_n/n \to 0$ $(n\to\infty)$, we see that if $\varepsilon > 0$, then by Corollary 4 and Theorem 5,

$$\lim_{n\to\infty} \frac{S_n}{n} = \varepsilon, \quad P\text{-a.s. and in } L^2(P). \tag{14}$$

This equality is valid also for $\varepsilon = 0$ by Theorem 3 and the calculation of $E[(S_n/n)^2]$. Moreover, since it follows by (1) that

$$P(X_n = \pm 1 \mid \mathscr{F}_{n-1}) = \frac{\alpha_n}{2}\left(1 \pm \frac{S_n}{n}\right) + (1-\alpha_n)\frac{1\pm\varepsilon_n}{2},$$

we have

$$P(X_n = \pm 1 \mid \mathscr{F}_{n-1}) \to \frac{1\pm\varepsilon}{2}, \quad P\text{-a.s. and in } L^2(P). \tag{15}$$

(ii) Assume that $\alpha = 1$ and $\sum_{n=1}^{\infty}(1-\alpha_n)/n = \infty$. Since (13) remains valid by (7), we get (14) and (15).

We next see that if $\varepsilon = 0$ and $0 \le \alpha < 1$, then the convergence rate of $\{\varepsilon_n\}_{n=1}^{\infty}$ affects the scaling factor of $\{S_n\}_{n=0}^{\infty}$.

Example 8. Let $\varepsilon = 0$ and $0 \le \alpha < 1$. Assume that for some $\rho \in [0, 1-\alpha)$, the sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ is regularly varying with index $-\rho$. Let r_n be as in Corollary 4. Since $\alpha + \rho < 1$, we have

$$\frac{r_n}{a_n} = \sum_{k=1}^{n} \frac{(1-\alpha_k)\varepsilon_k}{a_k} \sim \frac{1-\alpha}{1-(\alpha+\rho)} \frac{n\varepsilon_n}{a_n}.$$

Namely, $a_n/r_n \to 0$ and

$$r_n \sim \frac{1-\alpha}{1-(\alpha+\rho)} n\varepsilon_n.$$

We now impose the following condition on α, ρ and $\{\varepsilon_n\}_{n=1}^{\infty}$:

- $0 \le \rho < 1/2 \wedge (1-\alpha)$, or
- $\rho = 1/2$ and for some positive constants η and c_1,

$$\sqrt{n}\varepsilon_n \ge c_1(\log n)^{(1+\eta)/2} \ (n \ge 1). \tag{16}$$

Note that under this condition, $0 \le \alpha < 1/2$. Since $\sum_{n=1}^{\infty} 1/r_n^2 < \infty$, we have by Corollary 4,

$$\lim_{n\to\infty} \frac{S_n}{n\varepsilon_n} = \frac{1-\alpha}{1-(\alpha+\rho)}, \quad P\text{-a.s.} \tag{17}$$

If we replace the condition (16) with $\lim_{n\to\infty} \sqrt{n}\varepsilon_n = \infty$, then by Theorem 5, (17) holds also in $L^2(P)$.

Example 9. Let $\varepsilon = 0$ and $0 \le \alpha < 1$. Assume that the sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ is regularly varying with index $-(1-\alpha)$. Let r_n be as in Corollary 4. Then the scaling factor of $\{S_n\}_{n=0}^{\infty}$ depends also on the asymptotic behavior of $\{\alpha_n\}_{n=1}^{\infty}$. In what follows, we assume that for some constants η, κ and $\theta > 0$,

$$\varepsilon_n = \frac{(\log n)^{\eta-1}}{n^{1-\alpha}}, \quad \alpha_n = \alpha + \frac{\kappa}{(\log n)^{\theta}} \quad (n \ge 2).$$

(i) Let $0 < \theta < 1$. If $\kappa < 0$, then by Example 26 with Lemma 2 (4),

$$r_n \sim \frac{1-\alpha}{-\kappa} n^{\alpha} (\log n)^{\eta+\theta-1}$$

and $a_n/r_n \to 0$ $(n \to \infty)$. In particular, if

- $0 < \theta < 1$, $\alpha > 1/2$ and $\kappa < 0$, or
- $0 < \theta < 1$, $\alpha = 1/2$, $\kappa < 0$ and $\eta + \theta > 3/2$,

then $\sum_{n=1}^{\infty} 1/r_n^2 < \infty$ so that $S_n/r_n \to 1$ P-a.s. by Corollary 4. If we replace the condition $\eta + \theta > 3/2$ with $\eta + \theta/2 > 1$ and keep other conditions, then $a_n\sqrt{w_n}/r_n \to 0$ $(n \to \infty)$ by Examples 26 and 27. Hence $S_n/r_n \to 1$ in $L^2(P)$ by Theorem 5.
(ii) Let $\theta = 1$. If $\eta \ge \kappa$, then by Example 26 with Lemma 2 (4),

$$r_n \sim \begin{cases} \dfrac{1-\alpha}{\eta-\kappa} n^{\alpha} (\log n)^{\eta} & (\theta = 1, \eta > \kappa), \\ (1-\alpha) n^{\alpha} (\log n)^{\eta} \log\log n & (\theta = 1, \eta = \kappa) \end{cases}$$

and $a_n/r_n \to 0$ $(n \to \infty)$. In particular, if

- $\theta = 1$, $\alpha > 1/2$ and $\eta \ge \kappa$, or
- $\theta = 1$, $\alpha = 1/2$, and $\eta = \kappa \ge 1/2$ or $\eta > \kappa \vee (1/2)$,

then by Examples 26 and 27, $\sum_{n=1}^{\infty} 1/r_n^2 < \infty$ and $a_n\sqrt{w_n}/r_n \to 0$ $(n \to \infty)$ so that $S_n/r_n \to 1$ P-a.s. and in $L^2(P)$ by Corollary 4 and Theorem 5, respectively.
(iii) Let $\theta > 1$ or $\kappa = 0$. If $\eta \ge 0$, then all the calculations for $\theta = 1$ remain valid by taking $\kappa = 0$. Therefore, if

- $\theta > 1$, $\alpha > 1/2$ and $\eta \ge 0$, or
- $\theta > 1$, $\alpha = 1/2$ and $\eta > 1/2$,

then $S_n/r_n \to 1$ P-a.s. and in $L^2(P)$ by Corollary 4 and Theorem 5, respectively.

Example 10. Assume that for some positive constants η, κ and θ,

$$\varepsilon_n = \frac{1}{(\log n)^\eta}, \quad \alpha_n = 1 - \frac{\kappa}{(\log n)^\theta} \quad (n \geq 1).$$

By Example 26 with Lemma 2 (4), if $0 < \theta < 1$, or if $\theta = 1$ and $\kappa \geq \eta$, then

$$r_n \sim \begin{cases} \dfrac{n}{(\log n)^\eta} & (0 < \theta < 1), \\ \dfrac{\kappa}{\kappa - \eta} \dfrac{n}{(\log n)^\eta} & (\theta = 1, \kappa > \eta), \\ \dfrac{\kappa n \log \log n}{(\log n)^\kappa} & (\theta = 1, \kappa = \eta) \end{cases}$$

and $a_n/r_n \to 0$ $(n \to \infty)$. Since $\sum_{n=1}^\infty 1/r_n^2 < \infty$ and $\sum_{n=1}^\infty 1/a_n^2 < \infty$, we have $S_n/r_n \to 1$ P-a.s. and in $L^2(P)$ by Corollary 4 and Theorem 5, respectively.

IV. GAUSSIAN FLUCTUATIONS

In this section, we first reveal the Gaussian fluctuation of $\{S_n\}_{n=0}^\infty$ around the mean by proving the central limit theorem and law of the iterated logarithm. We next show that even if such a fluctuation is unavailable, the fluctuation around the random drift $a_n M_\infty$ is Gaussian. The former is proved in[5,11,13,17] for the dependent Bernoulli processes and superdiffusive elephant random walks with $\varepsilon < 1$; the latter is proved in[13] for the superdiffusive elephant random walks with $\varepsilon < 1$. Our results in this section extend the both results above to the random walks under consideration which can be degenerate in the sense that $\varepsilon = 1$.

A. Central limit theorem and law of the iterated logarithm

Let $w_n = \sum_{k=1}^n 1/a_k^2$ and $\phi(t) = \sqrt{2t \log \log t}$. We state the results separately in Theorems 11 and 14 below.

Theorem 11. (1) *Let* $\varepsilon \in [0,1)$. *If* $\sum_{n=1}^\infty 1/a_n^2 = \infty$, *then*

$$\sum_{k=1}^n E[d_k^2 \mid \mathscr{F}_{k-1}] \sim (1-\varepsilon^2) w_n.$$

Moreover,

$$\frac{S_n - E[S_n]}{a_n \sqrt{w_n}} \to N(0, 1-\varepsilon^2) \quad \text{in distribution}$$

and

$$\limsup_{n\to\infty} \pm \frac{S_n - E[S_n]}{a_n \phi(w_n)} = \sqrt{1-\varepsilon^2}, \quad P\text{-a.s.}$$

(2) *Let* $\varepsilon \in [0,1]$. *If* $\sum_{n=1}^\infty 1/a_n^2 < \infty$, *then*

$$\lim_{n\to\infty} \frac{S_n - E[S_n]}{a_n} = M_\infty, \quad P\text{-a.s. and in } L^2(P). \tag{18}$$

In particular, $E[M_\infty] = 0$ *and* $P(M_\infty \neq 0) > 0$.

We omit the proof of Theorem 11 because this theorem follows by[17] (Theorems 3 and 4).

Remark 12. Theorem 11 says that if $\varepsilon \in [0,1)$ and $\alpha \neq 1/2$, then the fluctuations of $\{\varepsilon_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ do not essentially affect that of $\{S_n\}_{n=0}^{\infty}$ around the mean. In fact, we have:

- If $0 \le \alpha < 1/2$, then $a_n\sqrt{w_n} \sim \sqrt{n/(1-2\alpha)}$ so that $\{S_n\}_{n=1}^{\infty}$ exhibits the diffusive behavior.
- If $1/2 < \alpha < 1$, then $\{S_n\}_{n=1}^{\infty}$ exhibits the superdiffusive behavior because $a_n \sim n^{\alpha} l_n/\Gamma(\alpha+1)$ with the slowly varying sequence $\{l_n\}_{n=1}^{\infty}$ in (2).

On the other hand, if $\alpha = 1/2$, then the behavior of $\{S_n\}_{n=1}^{\infty}$ depends on the convergence rate of $\{\alpha_n\}_{n=1}^{\infty}$ because this rate affects whether $\sum_{n=1}^{\infty} 1/a_n^2$ is convergent or not (see Example 23 below for details). This fact is already observed in[11] for the constant bias $\varepsilon_n = \varepsilon$ $(n \ge 1)$, and in[13] for the constant correlation $\alpha_n = \alpha$ $(n \ge 1)$.

We are now concerned with the fluctuation of $\{S_n\}_{n=1}^{\infty}$ around the mean for $\varepsilon = 1$ and $\sum_{n=1}^{\infty} 1/a_n^2 = \infty$, which is not covered by Theorem 11. Let us make the next assumptions on the sequences $\{\varepsilon_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$.

Assumption 13. The sequences $\{\varepsilon_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ satisfy the next conditions:

(i) $\varepsilon = 1$ and $\sum_{n=1}^{\infty} 1/a_n^2 = \infty$.

(ii) The sequence $\{1-\varepsilon_n\}_{n=1}^{\infty}$ is regularly varying with index $-\rho$ for some $\rho \in [0,1/2)$.

Let $v_n = \sum_{k=1}^{n}(1-\varepsilon_k^2)/a_k^2$ and

$$c_{\alpha,\rho} = \frac{(1-\alpha)(1-\rho)}{1-(\alpha+\rho)}.$$

Theorem 14. *If Assumption 13 is fulfilled, then the next assertions hold*:

(1) *If* $\sum_{n=1}^{\infty}(1-\varepsilon_n)/a_n^2 = \infty$, *then*

$$\sum_{k=1}^{n} E[d_k^2 \mid \mathscr{F}_{k-1}] \sim c_{\alpha,\rho} v_n. \tag{19}$$

Moreover,

$$\frac{S_n - E[S_n]}{a_n\sqrt{v_n}} \to N(0, c_{\alpha,\rho}) \quad \textit{in distribution}$$

and

$$\limsup_{n\to\infty} \pm \frac{S_n - E[S_n]}{a_n \phi(v_n)} = \sqrt{c_{\alpha,\rho}}, \quad P\text{-}a.s.$$

(2) *If* $\sum_{n=1}^{\infty}(1-\varepsilon_n)/a_n^2 < \infty$, *then the same assertion as in Theorem* 11 (2) *holds.*

Theorem 14 reveals how the convergence rate of the bias $\{\varepsilon_n\}_{n=1}^{\infty}$ affects the fluctuation of $\{S_n\}_{n=0}^{\infty}$ around the mean. In particular, we find that $\rho = 1-2\alpha$ is the borderline between (1) and (2).

We also note that if Assumption 13 is fulfilled, then (14) and (15) hold by Example 7. Therefore, if we assume in addition that $\sum_{n=1}^{\infty}(1-\varepsilon_n)/a_n^2 = \infty$, then

$$v_n \sim \sum_{k=1}^{n} \frac{4}{a_k^2} P(X_k = 1)(1 - P(X_k = 1))$$

and

$$\lim_{n\to\infty} \frac{1}{v_n} \sum_{k=1}^{n-1} \frac{1}{a_k^2} = \infty.$$

Namely, we can not apply[17] (Theorems 3 and 4) to the proof of Theorem 14.

To prove Theorem 14, we find the convergence rate for Example 7 with $\varepsilon = 1$.

Lemma 15. *Let Assumption* 13 *hold.*

(1) *There exists a positive constant c_0 such that for any $n \geq 2$,*

$$\left|(1-\alpha)\frac{a_{n-1}}{n}\sum_{k=1}^{n-1}\frac{1}{a_k}-1-\frac{a_{n-1}}{n}\sum_{k=1}^{n-1}\frac{\alpha_k-\alpha}{a_k}\right| \leq \frac{c_0 a_{n-1}}{n}.$$

(2) *The next assertion holds*:

$$\lim_{n\to\infty}\frac{1}{1-\varepsilon_n}\left(\frac{S_n}{n}-1\right) = -\frac{1-\alpha}{1-(\alpha+\rho)},\ P\text{-a.s.}$$

Proof. We first prove (1). Let

$$\begin{aligned}(1-\alpha)\frac{a_{n-1}}{n}\sum_{k=1}^{n-1}\frac{1}{a_k}-1-\frac{a_{n-1}}{n}\sum_{k=1}^{n-1}\frac{\alpha_k-\alpha}{a_k} &= (1-\alpha)\frac{a_{n-1}}{n}\left\{\sum_{k=1}^{n-1}\left(\frac{1}{g_k}-\frac{\Gamma(\alpha+1)}{k^\alpha}\right)\frac{1}{l_k}\right\}\\ &+\frac{a_{n-1}}{n}\left\{\Gamma(\alpha+1)\left((1-\alpha)\sum_{k=1}^{n-1}\frac{1}{k^\alpha l_k}-\frac{n^{1-\alpha}}{l_n}\right)-\sum_{k=1}^{n-1}\frac{\alpha_k-\alpha}{a_k}\right\}+\left(\Gamma(\alpha+1)\frac{a_{n-1}}{n^\alpha l_n}-1\right)\\ &= (\mathrm{I})+(\mathrm{II})+(\mathrm{III}).\end{aligned} \tag{20}$$

Since Lemma 2 (1) yields

$$\frac{1}{g_n}-\frac{\Gamma(\alpha+1)}{n^\alpha} = -\frac{\Gamma(\alpha+1)}{n^\alpha g_n}\left(g_n-\frac{n^\alpha}{\Gamma(\alpha+1)}\right)\sim\frac{c_1}{n^{1+\alpha}} \tag{21}$$

for some $c_1 > 0$, we have

$$\left|\sum_{k=1}^{n-1}\left(\frac{1}{g_k}-\frac{\Gamma(\alpha+1)}{k^\alpha}\right)\frac{1}{l_k}\right| \leq \sum_{k=1}^{n-1}\left|\frac{1}{g_k}-\frac{\Gamma(\alpha+1)}{k^\alpha}\right|\frac{1}{l_k} \leq c_2\sum_{k=1}^{\infty}\frac{1}{k^{1+\alpha}l_k} < \infty$$

so that

$$|(\mathrm{I})| \leq c_3\frac{a_n}{n}.$$

For $k \geq 2$, since

$$\frac{1-\alpha}{k^\alpha} \leq k^{1-\alpha}-(k-1)^{1-\alpha},$$

we get

$$\begin{aligned}\frac{1-\alpha}{k^\alpha l_k} &\leq \left(\frac{k^{1-\alpha}}{l_k}-\frac{(k-1)^{1-\alpha}}{l_{k-1}}\right)+(k-1)^{1-\alpha}\left(\frac{1}{l_{k-1}}-\frac{1}{l_k}\right)\\ &= \left(\frac{k^{1-\alpha}}{l_k}-\frac{(k-1)^{1-\alpha}}{l_{k-1}}\right)+\frac{(k-1)^{1-\alpha}(\alpha_k-\alpha)}{(k+\alpha)l_k}.\end{aligned}$$

Combining (21) with the relation

$$\frac{(k-1)^{1-\alpha}}{k+\alpha}-\frac{1}{k^\alpha}\sim-\frac{1}{k^{\alpha+1}},$$

we have for some $c_4 > 0$,

$$\Gamma(\alpha+1)\frac{(k-1)^{1-\alpha}}{k+\alpha}-\frac{1}{g_k}\sim-\frac{c_4}{k^{\alpha+1}}.$$

Therefore, we obtain for some $c_5 > 0$ and for any $n \geq 1$,

$$\Gamma(\alpha+1)(1-\alpha)\sum_{k=1}^{n}\frac{1}{k^{\alpha}l_k} \leq c_5 + \Gamma(\alpha+1)\frac{n^{1-\alpha}}{l_n} + \sum_{k=1}^{n}\frac{\alpha_k-\alpha}{a_k}.$$

In the same way, we have for some $c_6 > 0$ and for any $n \geq 1$,

$$\Gamma(\alpha+1)(1-\alpha)\sum_{k=1}^{n}\frac{1}{k^{\alpha}l_k} \geq -c_6 + \Gamma(\alpha+1)\frac{n^{1-\alpha}}{l_n} + \sum_{k=1}^{n}\frac{\alpha_k-\alpha}{a_k}.$$

Hence there exists a positive constant c_7 such that for any $n \geq 1$,

$$|(\mathrm{II})| \leq \frac{c_7 a_{n-1}}{n}.$$

By the triangle inequality and (3),

$$\begin{aligned}|(\mathrm{III})| &= \left|\frac{\Gamma(\alpha+1)g_{n-1}}{n^{\alpha}}\frac{l_{n-1}}{l_n} - 1\right| \\ &\leq \frac{\Gamma(\alpha+1)}{n^{\alpha}}\frac{l_{n-1}}{l_n}\left|g_{n-1} - \frac{(n-1)^{\alpha}}{\Gamma(\alpha+1)}\right| + \frac{l_{n-1}}{l_n}\left\{1-\left(1-\frac{1}{n}\right)^{\alpha}\right\} + \left|\frac{l_{n-1}}{l_n} - 1\right| \\ &\leq \frac{c_8}{n}.\end{aligned}$$

Combining the estimates of (I), (II), (III) above with (20), we have (1).

We next prove (2). By Lemma 1 (1),

$$\begin{aligned}\frac{E[S_n]}{n} - 1 &= \frac{a_{n-1}}{n}\left(2q-1+\sum_{k=1}^{n-1}\frac{(1-\alpha_k)\varepsilon_k}{a_k}\right) - 1 \\ &= \frac{a_{n-1}}{n}(2q-1) - \frac{a_{n-1}}{n}\sum_{k=1}^{n-1}\frac{(1-\alpha_k)(1-\varepsilon_k)}{a_k} + \left((1-\alpha)\frac{a_{n-1}}{n}\sum_{k=1}^{n-1}\frac{1}{a_k} - 1 - \frac{a_{n-1}}{n}\sum_{k=1}^{n-1}\frac{\alpha_k-\alpha}{a_k}\right) \\ &= A_n - B_n + C_n.\end{aligned} \tag{22}$$

Since $\alpha+\rho < 1$ by assumption, we have

$$\lim_{n\to\infty}\frac{A_n}{1-\varepsilon_n} = 0, \quad \lim_{n\to\infty}\frac{B_n}{1-\varepsilon_n} = \frac{1-\alpha}{1-(\alpha+\rho)}.$$

By (1), we also get

$$\lim_{n\to\infty}\frac{C_n}{1-\varepsilon_n} = 0$$

so that

$$\lim_{n\to\infty}\frac{1}{1-\varepsilon_n}\left(\frac{E[S_n]}{n} - 1\right) = -\frac{1-\alpha}{1-(\alpha+\rho)}.$$

On the other hand, since $\rho < 1/2$ by assumption, we can apply Theorem 3 with $r_n = n(1-\varepsilon_n)$ to obtain

$$\lim_{n\to\infty}\frac{1}{1-\varepsilon_n}\left(\frac{S_n - E[S_n]}{n}\right) = 0, \quad P\text{-a.s.} \tag{23}$$

These two expressions above imply the desired assertion. □

Proof of Theorem 14. Since $X_n^2 = 1$ for any $n \in \mathbb{N}$, we have by Lemma 1 (1),

$$\begin{aligned} E[(X_j - E[X_j \mid \mathscr{F}_{j-1}])^2 \mid \mathscr{F}_{j-1}] &= 1 - \left(\frac{\alpha_{j-1}}{j-1} S_{j-1} + (1-\alpha_{j-1})\varepsilon_{j-1} \right)^2 \\ &= \left\{ \alpha_{j-1} \left(1 - \frac{S_{j-1}}{j-1} \right) + (1-\alpha_{j-1})(1-\varepsilon_{j-1}) \right\} \left\{ 1 + \alpha_{j-1} \frac{S_{j-1}}{j-1} + (1-\alpha_{j-1})\varepsilon_{j-1} \right\}. \end{aligned} \tag{24}$$

Then by Lemma 15 (2) and Example 7 (i),

$$\begin{aligned} \alpha_{j-1} \left(1 - \frac{S_{j-1}}{j-1} \right) + (1-\alpha_{j-1})(1-\varepsilon_{j-1}) &\sim \frac{\alpha(1-\alpha)}{1-(\alpha+\rho)}(1-\varepsilon_j) + (1-\alpha)(1-\varepsilon_j) \\ &= c_{\alpha,\rho}(1-\varepsilon_j) \end{aligned}$$

and

$$1 + \alpha_{j-1} \frac{S_{j-1}}{j-1} + (1-\alpha_{j-1})\varepsilon_{j-1} \sim 2 \sim 1 + \varepsilon_j.$$

We thus have as $j \to \infty$,

$$E[(X_j - E[X_j \mid \mathscr{F}_{j-1}])^2 \mid \mathscr{F}_{j-1}] \sim c_{\alpha,\rho}(1-\varepsilon_j^2). \tag{25}$$

If $\sum_{n=1}^{\infty}(1-\varepsilon_n)/a_n^2 = \infty$, then (25) implies that as $n \to \infty$,

$$\sum_{j=1}^{n} E[d_j^2 \mid \mathscr{F}_{j-1}] = \sum_{j=1}^{n} \frac{1}{a_j^2} E[(X_j - E[X_j \mid \mathscr{F}_{j-1}])^2 \mid \mathscr{F}_{j-1}] \sim c_{\alpha,\rho} \sum_{j=1}^{n} \frac{1-\varepsilon_j^2}{a_j^2} = c_{\alpha,\rho} v_n,$$

that is, (19) holds. Hence we get (1) by[11] (Lemmas 3.4 and 3.5), which follow from[8] (Corollary 3.1, Theorems 4.7 and 4.8).

On the other hand, if $\sum_{n=1}^{\infty}(1-\varepsilon_n)/a_n^2 < \infty$, then by (24) and Lemma 15 (2), we have for some random positive constant C,

$$\sum_{n=1}^{\infty} E[d_n^2 \mid \mathscr{F}_{n-1}] \le C \sum_{n=1}^{\infty} \frac{1-\varepsilon_n}{a_n^2} < \infty.$$

Hence by[8] (Theorem 2.15), $M_\infty := \sum_{n=1}^{\infty}(M_n - M_{n-1})$ exists P-a.s. and $M_n \to M_\infty$ in $L^2(P)$. This yields

$$\lim_{n\to\infty} \frac{S_n - E[S_n]}{a_n} = \lim_{n\to\infty} M_n = M_\infty, \ P\text{-a.s. and in } L^2(P)$$

and

$$E[M_\infty] = \lim_{n\to\infty} E[M_n] = 0, \quad E[M_\infty^2] = \sum_{n=1}^{\infty} E[(M_n - M_{n-1})^2] \in (0,\infty).$$

The proof is complete. □

Remark 16. In the proof of Theorem 14, we used the condition $\rho < 1/2$ in Assumption 13 only for the validity of (23); the rest of the proof is still valid for $\rho < 1-\alpha$. At present, we do not know if the statement of Theorem 14 is true or not for $\rho < 1-\alpha$.

B. Growth exponent

In this subsection, we discuss again the growth exponent of $\{S_n\}_{n=0}^{\infty}$ to interpolate Corollary 4 and Theorem 5. Let $w_n = \sum_{k=1}^{n} 1/a_k^2$ and $v_n = \sum_{k=1}^{n}(1-\varepsilon_k^2)/a_k^2$. As a corollary of Theorems 11 and 14, we have

Corollary 17. (1) *Assume that $\varepsilon = 0$, $\sum_{n=1}^{\infty} 1/a_n^2 = \infty$ and the following limit exists as a finite value*:

$$\mu := \lim_{n\to\infty} \frac{1-\alpha}{\sqrt{w_n}} \sum_{k=1}^{n} \frac{\varepsilon_k}{a_k}. \tag{26}$$

Then

$$\frac{S_n}{a_n\sqrt{w_n}} \to N(\mu, 1) \quad \text{in distribution}. \tag{27}$$

(2) *If*

$$\sum_{n=1}^{\infty} \frac{(1-\alpha_n)\varepsilon_n}{a_n} < \infty, \tag{28}$$

then $E[S_n] \sim c_* a_n$ with

$$c_* = 2q - 1 + \sum_{n=1}^{\infty} \frac{(1-\alpha_n)\varepsilon_n}{a_n}.$$

Moreover, if $\sum_{n=1}^{\infty} 1/a_n^2 < \infty$, or if Assumption 13 is fulfilled and $\sum_{n=1}^{\infty}(1-\varepsilon_n)/a_n^2 < \infty$, then

$$\frac{S_n}{a_n} \to M_\infty + c_* \quad P\text{-a.s. and in } L^2(P). \tag{29}$$

Proof. Under the assumption of (1), Lemma 1 (1) yields

$$\lim_{n\to\infty} \frac{E[S_n]}{a_n\sqrt{w_n}} = \mu.$$

Then (1) follows by Theorem 11. If (28) is fulfilled, then by Lemma 1 (1), $E[S_n] \sim c_* a_n$. Therefore, the proof is complete by Theorems 11 (2) and 14. □

Note that if $0 < \varepsilon \le 1$ and $\sum_{n=1}^{\infty} 1/a_n^2 = \infty$, then $0 \le \alpha \le 1/2$ so that $\mu = \infty$ in (26).

Example 18. Let $0 < \varepsilon \le 1$. If $0 \le \alpha < 1$, or if $\alpha = 1$ and $\sum_{n=1}^{\infty}(1-\alpha_n)/n = \infty$, then by Example 7, $S_n/n \to \varepsilon$, P-a.s. and in $L^2(P)$. If $\alpha = 1$ and $\sum_{n=1}^{\infty}(1-\alpha_n)/n < \infty$, then $\sum_{n=1}^{\infty}(1-\alpha_n)/a_n < \infty$ by Lemma 2 (5). Hence (29) holds by Corollary 17 (2).

Example 19. Let $\varepsilon = 0$ and $0 \le \alpha < 1$. Assume that the sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ is regularly varying with index $-\rho$ for some $\rho \ge 0$.
(i) Let $\alpha + \rho < 1$. We first assume that

- $1/2 < \rho < 1 - \alpha$.

Since $\mu = 0$, we have by Corollary 17 (1),

$$\frac{S_n}{\sqrt{n}} \to N\left(0, \frac{1}{1-2\alpha}\right) \quad \text{in distribution.}$$

We next assume that

- $0 < \alpha < 1/2$ and $\rho = 1/2$, and the limit $\mu_0 := \lim_{n\to\infty} \sqrt{n}\varepsilon_n$ exists as a finite value.

Let μ be as in (26). Since

$$\mu = \frac{\mu_0\sqrt{1-2\alpha}(1-\alpha)}{1-(\alpha+\rho)},$$

Corollary 17 (1) yields

$$\frac{S_n}{\sqrt{n}} \to N\left(\frac{\mu_0(1-\alpha)}{1-(\alpha+\rho)}, \frac{1}{1-2\alpha}\right) \quad \text{in distribution.}$$

(ii) Let $\alpha+\rho>1$. Then (28) is valid. Moreover, if $\sum_{n=1}^{\infty} 1/a_n^2=\infty$, then $\mu=0$ so that by Corollary 17 (1),

$$\frac{S_n}{a_n\sqrt{w_n}} \to N(0,1) \quad \text{in distribution.}$$

On the other hand, if $\sum_{n=1}^{\infty} 1/a_n^2<\infty$, then Corollary 17 (2) yields (29).

Example 20. For some constants η, κ and $\theta>0$, let

$$\varepsilon_n=\frac{(\log n)^{\eta-1}}{n^{1-\alpha}}, \quad \alpha_n=\alpha+\frac{\kappa}{(\log n)^{\theta}} \ (n\geq 2).$$

If $0\leq\alpha<1/2$, then $\sum_{n=1}^{\infty} 1/a_n^2=\infty$ and $\mu=0$. Hence by Corollary 17 (1),

$$\frac{S_n}{\sqrt{n}} \to N\left(0, \frac{1}{1-2\alpha}\right) \quad \text{in distribution.}$$

In what follows, we assume that $1/2\leq\alpha<1$. See Examples 26 and 27 for the asymptotic behaviors of a_n and $a_n\sqrt{w_n}$.
(i) Let $0<\theta<1$ and $\kappa\neq 0$. Then Corollary 17 implies the following: if $\alpha=1/2$, $\kappa<0$, and $\eta+\theta/2<1$, then

$$\frac{S_n}{a_n\sqrt{w_n}} \to N(0,1) \quad \text{in distribution;}$$

if $\alpha=1/2$, $\kappa<0$ and $\eta+\theta/2=1$, then

$$\frac{S_n}{a_n\sqrt{w_n}} \to N\left(\frac{1}{\sqrt{-2\kappa}}, 1\right) \quad \text{in distribution.}$$

On the other hand, if $1/2\leq\alpha<1$ and $\kappa>0$, then by Corollary 17 (2), (29) holds.
(ii) Let $\theta=1$. Then Corollary 17 implies the following: if $\alpha=1/2$, $\kappa<1/2$, and $\eta<1/2$, then

$$\frac{S_n}{a_n\sqrt{w_n}} \to N(0,1) \quad \text{in distribution;}$$

if $\alpha=1/2$, $\kappa<1/2$ and $\eta=1/2$, then

$$\frac{S_n}{a_n\sqrt{w_n}} \to N\left(\frac{1}{\sqrt{1-2\kappa}}, 1\right) \quad \text{in distribution.}$$

On the other hand, if

- $\alpha=1/2$ and $\kappa>\eta\vee(1/2)$, or
- $1/2<\alpha<1$ and $\eta<0$,

then by Corollary 17 (2), (29) holds.
(iii) Let $\theta>1$ or $\kappa=0$. Then by Corollary 17, if $\alpha=1/2$ and $\eta<1/2$, then

$$\frac{S_n}{a_n\sqrt{w_n}} \to N(0,1) \quad \text{in distribution;}$$

if $\alpha = 1/2$ and $\eta = 1/2$, then

$$\frac{S_n}{a_n\sqrt{w_n}} \to N(1,1) \quad \text{in distribution.}$$

On the other hand, if $1/2 < \alpha < 1$ and $\eta < 0$, then (29) holds by Corollary 17 (2).

Combining (i)–(iii) above with Example 9, we can draw graphs describing the phase transition on the growth exponent of $\{S_n\}_{n=0}^{\infty}$ in terms of κ and η. We here present the graphs for $\alpha = 1/2$ in Figures 1, 2 and 3. The colors of the graphs correspond to the equation numbers respectively as we see in Figures 1. See Example 27 for the behavior of $a_n\sqrt{w_n}$ in (i)–(iii).

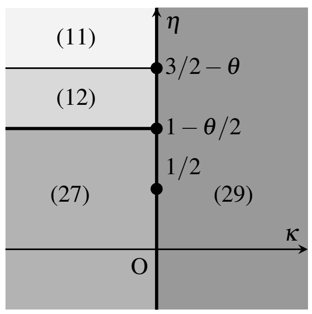

FIG. 1. $0 < \theta < 1$ and $\alpha = 1/2$

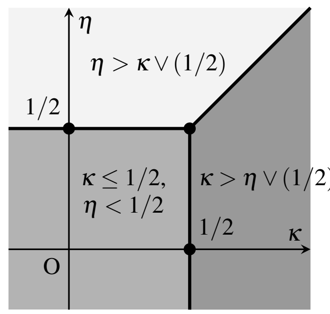

FIG. 2. $\theta = 1$ and $\alpha = 1/2$

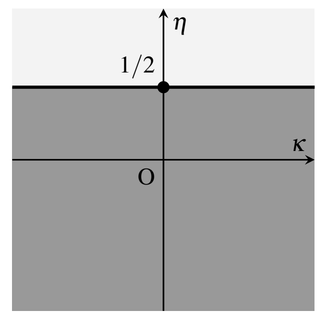

FIG. 3. $\theta > 1$ and $\alpha = 1/2$

Example 21. For some positive constants η, κ and θ, let

$$\varepsilon_n = \frac{1}{(\log n)^{\eta}}, \quad \alpha_n = 1 - \frac{\kappa}{(\log n)^{\theta}} \ (n \geq 2).$$

Let

$$r_n = a_n \sum_{k=1}^{n} \frac{(1-\alpha_k)\varepsilon_k}{a_k}.$$

We saw in Example 10 that if $0 < \theta < 1$, or if $\theta = 1$ and $\kappa \geq \eta$, then $S_n/r_n \to 1$, P-a.s. and in $L^2(P)$. On the other hand, if $\theta = 1$ and $\eta > \kappa$, or if $\theta > 1$, then $\sum_{n=1}^{\infty}(1-\alpha_n)\varepsilon_n/a_n < \infty$ by Example 26. Since $\sum_{n=1}^{\infty} 1/a_n^2 < \infty$, Corollary 17 (2) yields (29).

C. Gaussian fluctuation around the random drift

Let $z_n = \sum_{k=n}^{\infty} 1/a_k^2$ and $t_n = \sum_{k=n}^{\infty}(1-\varepsilon_k^2)/a_k^2$. Let $\psi(t) = \sqrt{2t\log|\log t|}$.

Theorem 22. (1) *Let $\varepsilon \in [0,1)$ and $\sum_{n=1}^{\infty} 1/a_n^2 < \infty$. If $\alpha = 1$, then assume also that $a_n/n \to 0$ as $n \to \infty$. Then*

$$\frac{S_n - E[S_n] - a_n M_{\infty}}{a_n\sqrt{z_n}} \to N(0, 1-\varepsilon^2) \quad \textit{in distribution}$$

and

$$\limsup_{n\to\infty} \pm \frac{S_n - E[S_n] - a_n M_{\infty}}{a_n \psi(z_n)} = \sqrt{1-\varepsilon^2}, \quad P\text{-a.s.}$$

(2) *Assume that $\varepsilon = 1$ and $\alpha < 1$, and*

(a) *$\sum_{n=1}^{\infty} 1/a_n^2 < \infty$ and the sequence $\{1-\varepsilon_n\}_{n=1}^{\infty}$ is regularly variyng with index $-\rho$ for some $\rho \in [0, 1-\alpha)$, or*

(b) *Assumption* 13 *is fulfilled and* $\sum_{n=1}^{\infty}(1-\varepsilon_n)/a_n^2 < \infty$.

Then

$$\frac{S_n - E[S_n] - a_n M_\infty}{a_n\sqrt{t_n}} \to N(0, c_{\alpha,\rho}) \quad \textit{in distribution} \tag{30}$$

and

$$\limsup_{n\to\infty} \pm \frac{S_n - E[S_n] - a_n M_\infty}{a_n \psi(t_n)} = \sqrt{c_{\alpha,\rho}}, \quad P\textit{-a.s.} \tag{31}$$

Proof. We show (2) under the condition (b) only because the rest of the assertions is proved in a similar way. We take the same approach as the proof of[13] (Theorem 3). Let Assumption 13 hold and $\sum_{n=1}^{\infty}(1-\varepsilon_n)/a_n^2 < \infty$. Define

$$V_n^2 = \sum_{k=n}^{\infty} E[d_k^2 \mid \mathscr{F}_{k-1}], \quad \sigma_n^2 = \sum_{k=n}^{\infty} E[d_k^2].$$

Then (25) yields $V_n^2 \sim c_{\alpha,\rho} t_n$, P-a.s. and $\sigma_n^2 \sim c_{\alpha,\rho} t_n$ so that $V_n^2/\sigma_n^2 \to 1$ as $n\to\infty$, P-a.s.

We first prove (30). By assumption, $\rho + 2\alpha \geq 1$ and the sequence $\{(1-\varepsilon_n)/a_n^2\}_{n=1}^{\infty}$ is regularly varying with index $-(\rho+2\alpha)$. Then by[4] (Lemma 1.5.9 b), we have for some positive constant c_1,

$$a_n^2 t_n = n(1-\varepsilon_n)\left(\frac{n(1-\varepsilon_n)}{a_n^2}\right)^{-1} \sum_{k=n}^{\infty} \frac{1-\varepsilon_k^2}{a_k^2} \geq c_1 n(1-\varepsilon_n). \tag{32}$$

Since $|d_n| \leq 2/a_n$ by (9), we obtain by (32) and Assumption 13 (ii),

$$\sum_{n=1}^{\infty} \frac{1}{\sigma_n^4} E[d_n^4 \mid \mathscr{F}_{n-1}] \leq c_2 \sum_{n=1}^{\infty} \frac{1}{(a_n^2 t_n)^2} \leq c_3 \sum_{n=1}^{\infty} \frac{1}{n^2(1-\varepsilon_n)^2} < \infty. \tag{33}$$

Let $K_0 = 0$ and

$$K_n = \sum_{k=1}^{n} \frac{1}{\sigma_k^2}(d_k^2 - E[d_k^2 \mid \mathscr{F}_{k-1}]) \quad (n \geq 1).$$

Then $\{K_n\}_{n=0}^{\infty}$ is a martingale. Let

$$\Delta K_n = K_n - K_{n-1} = \frac{d_n^2 - E[d_n^2 \mid \mathscr{F}_{n-1}]}{\sigma_n^2}.$$

Since

$$(\Delta K_n)^2 \leq \frac{c_4}{\sigma_n^4}(d_n^4 + E[d_n^4 \mid \mathscr{F}_{n-1}]),$$

we have by (33),

$$\sum_{n=1}^{\infty} (\Delta K_n)^2 \leq c_5 \sum_{n=1}^{\infty} \frac{1}{\sigma_n^4}(d_n^4 + E[d_n^4 \mid \mathscr{F}_{n-1}]) < \infty.$$

Hence by[8] (Theorem 2.15), the series

$$\sum_{k=1}^{\infty} \frac{1}{\sigma_k^2}(d_k^2 - E[d_k^2 \mid \mathscr{F}_{k-1}])$$

is almost surely convergent. Then by Kronecker's lemma,

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{k=n}^{\infty} (d_k^2 - E[d_k^2 \mid \mathscr{F}_{k-1}]) = 0, \quad P\text{-a.s.}$$

and thus

$$\frac{1}{\sigma_n^2}\sum_{k=n}^{\infty} d_k^2 = \frac{1}{\sigma_n^2}\sum_{k=n}^{\infty}(d_k^2 - E[d_k^2 \mid \mathscr{F}_{k-1}]) + \frac{1}{\sigma_n^2}\sum_{k=n}^{\infty} E[d_k^2 \mid \mathscr{F}_{k-1}] \to 1, \quad P\text{-a.s.} \tag{34}$$

Since the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing by definition and $|d_n| \le 2/a_n$, we have

$$\sup_{k\ge n}(d_k^2) \le \frac{4}{a_n^2} \to 0 \ (n\to\infty)$$

so that by (32),

$$\frac{E[\sup_{k\ge n}(d_k^2)]}{\sigma_n^2} \le \frac{4}{a_n^2\sigma_n^2} \le \frac{c_6}{a_n^2 t_n} \le \frac{c_7}{n(1-\varepsilon_n)} \to 0 \ (n\to\infty).$$

Combining this relation with (34), we obtain by[8] (Corollary 3.5),

$$\frac{M_\infty - M_n}{\sigma_{n+1}} \to N(0,1) \quad \text{in distribution.}$$

As $M_n = (S_n - E[S_n])/a_n$ and $\sigma_n \sim \sqrt{c_{\alpha,\rho}}\sqrt{t_n}$, we have (30).

We next prove (31). For any $\eta > 0$, we have as for (33),

$$\sum_{n=1}^{\infty}\frac{1}{\sigma_n}E[|d_n|;|d_n|>\eta\sigma_n] \le \frac{1}{\eta^3}\sum_{n=1}^{\infty}\frac{1}{\sigma_n^4}E[d_n^4] < \infty$$

and

$$\sum_{n=1}^{\infty}\frac{1}{\sigma_n^4}E[d_n^4;|d_n|\le\eta\sigma_n] \le \sum_{n=1}^{\infty}\frac{1}{\sigma_n^4}E[d_n^4] < \infty.$$

By these calculations with (34), we can apply[8] (Theorem 4.9) to show that

$$\limsup_{n\to\infty} \pm\frac{M_\infty - M_n}{\psi(\sigma_n^2)} = 1, \quad P\text{-a.s.}$$

This implies (31). □

D. Examples

Example 23. Assume that $\varepsilon \in [0,1)$. Then the Gaussian fluctuation results in[13] remain true.
(i) If $0 \le \alpha < 1/2$, then by Theorem 11,

$$\frac{S_n - E[S_n]}{\sqrt{n}} \to N\left(0, \frac{1-\varepsilon^2}{1-2\alpha}\right) \quad \text{in distribution}$$

and

$$\limsup_{n\to\infty} \pm\frac{S_n - E[S_n]}{\sqrt{2n\log\log n}} = \sqrt{\frac{1-\varepsilon^2}{1-2\alpha}}, \quad P\text{-a.s.}$$

(ii) If $1/2 < \alpha < 1$, or if $\alpha = 1$ and $a_n/n \to 0$ as $n\to\infty$, then by Theorem 22,

$$\frac{S_n - E[S_n] - a_n M_\infty}{\sqrt{n}} \to N\left(0, \frac{1-\varepsilon^2}{2\alpha-1}\right) \quad \text{in distribution}$$

and

$$\limsup_{n\to\infty} \pm \frac{S_n - E[S_n] - a_n M_\infty}{\sqrt{2n\log\log n}} = \sqrt{\frac{1-\varepsilon^2}{2\alpha-1}}, \quad P\text{-a.s.}$$

(iii) For some constants κ and $\theta > 0$, let

$$\alpha_n = \frac{1}{2} + \frac{\kappa}{(\log n)^\theta} \quad (n \geq 2).$$

Then by Example 27, if $0 < \theta < 1$ and $\kappa > 0$, or if $\theta = 1$ and $\kappa > 1/2$, then the assertions of Theorem 11 (2) and Theorem 22 (1) are valid; otherwise, that of Theorem 11 (1) holds. See Example 27 for the calculations of the scaling factors $a_n\sqrt{w_n}$ and $a_n\sqrt{z_n}$ in Theorem 11 (1) and Theorem 22 (1). Note that if $0 < \theta < 1$, then the asymptotic behaviors of $a_n\sqrt{w_n}$ and $a_n\sqrt{z_n}$ are dependent of the value of θ.

Example 24. Let $\varepsilon = 1$ and $0 \leq \alpha < 1$. We assume that the sequence $\{1-\varepsilon_n\}_{n=1}^\infty$ is regularly varying of index $-\rho$ for some $\rho \in [0, 1-\alpha)$.
(i) Let $1/2 < \alpha < 1$. Since $\sum_{n=1}^\infty 1/a_n^2 < \infty$, we have the assertion in Theorem 22 (2) with

$$t_n \sim \frac{1}{2\alpha+\rho-1}\frac{n(1-\varepsilon_n)}{a_n^2}, \quad a_n\sqrt{t_n} \sim \sqrt{\frac{n(1-\varepsilon_n)}{2\alpha+\rho-1}}. \tag{35}$$

(ii) Let $0 \leq \alpha < 1/2$. Then $\sum_{n=1}^\infty 1/a_n^2 = \infty$. If $0 < \rho < (1-2\alpha) \wedge (1/2)$, then the assertion of Theorem 14 (1) holds with

$$v_n \sim \frac{1}{1-(2\alpha+\rho)}\frac{n(1-\varepsilon_n)}{a_n^2}, \quad a_n\sqrt{v_n} \sim \sqrt{\frac{n(1-\varepsilon_n)}{1-(2\alpha+\rho)}}.$$

On the other hand, if $1-2\alpha < \rho < 1/2$, then the assertion of Theorem 22 (2) holds with (35).

We here assume that $\rho = 1-2\alpha$ and $0 < \rho < 1/2$, that is, $1/4 < \alpha < 1/2$. Then the scaling factors in the Gaussian fluctuations depend on the convergence rates of $\{\varepsilon_n\}_{n=1}^\infty$ and $\{\alpha_n\}_{n=1}^\infty$. For instance, we first take for some constants η, κ and $\theta > 0$,

$$\varepsilon_n = 1 - \frac{(\log n)^{\eta-1}}{n^{1-2\alpha}}, \quad \alpha_n = \alpha + \frac{\kappa}{(\log n)^\theta} \quad (n \geq 2).$$

We assume one of the following conditions:

- $0 < \theta < 1$, $\kappa < 0$,
- $\theta = 1$ and $\eta \geq 2\kappa$,
- $\theta > 1$ or $\kappa = 0$, and $\eta \geq 0$.

Then the assertion of Theorem 14 (1) holds; otherwise, that of Theorem 22 (2) holds. See Example 28 for the calculations of the scaling functions $a_n\sqrt{v_n}$ and $a_n\sqrt{t_n}$.
(iii) Let $\alpha = 1/2$. If $\rho \in (0, 1/2)$, then $\sum_{n=1}^\infty (1-\varepsilon_n)/a_n^2 < \infty$ so that the assertion of Theorem 22 (2) holds.

If $\rho = 0$, then the convergence rates of $\{\varepsilon_n\}_{n=1}^\infty$ and $\{\alpha_n\}_{n=1}^\infty$ affect the scaling factors in the Gaussian fluctuations. For instance, we first take for some constants $\eta > 0$, κ and $\theta > 0$,

$$\varepsilon_n = 1 - \frac{1}{(\log n)^\eta}, \quad \alpha_n = \frac{1}{2} + \frac{\kappa}{(\log n)^\theta} \quad (n \geq 2).$$

We assume one of the following conditions:

- $0 < \theta < 1$, $\kappa < 0$,

- $\theta = 1$ and $0 < \eta \le 1 - 2\kappa$,
- $\theta > 1$ or $\kappa = 0$, and $0 < \eta \le 1$.

Then the assertion of Theorem 14 (1) is valid; otherwise, that of Theorem 22 (2) is valid. See Example 29 also for the calculations of the scaling factors $a_n\sqrt{v_n}$ and $a_n\sqrt{t_n}$ in Theorems 14 (1) and 22 (2).

Appendix A: Appendix

1. Asymptotic properties of series

The next lemma is used for the proofs of Lemma 2 and Theorem 5.

Lemma 25. *Let $\{c_n\}_{n=1}^{\infty}$ be a nonnegative sequence such that $c_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} c_n = \infty$. Then*

$$\sum_{k=1}^{n} c_k \left(\sum_{l=1}^{k} c_l \right) = \sum_{k=1}^{n} c_k \left(\sum_{l=k}^{n} c_l \right) \sim \frac{1}{2} \left(\sum_{k=1}^{n} c_k \right)^2 \tag{A1}$$

and

$$\sum_{k=1}^{n} c_k \exp\left(\sum_{l=1}^{k} c_l \right) \sim \exp\left(\sum_{k=1}^{n} c_k \right). \tag{A2}$$

Proof. Since

$$\sum_{k=1}^{n} c_k \left(\sum_{l=k}^{n} c_l \right) = \sum_{l=1}^{n} c_l \left(\sum_{k=1}^{l} c_k \right) = \sum_{k=1}^{n} c_k \left(\sum_{l=1}^{k} c_l \right),$$

we have

$$\sum_{k=1}^{n} c_k \left(\sum_{l=k}^{n} c_l \right) = \frac{1}{2} \left(\sum_{k=1}^{n} c_k \right)^2 + \frac{1}{2} \sum_{k=1}^{n} c_k^2.$$

As $c_n \to 0$ $(n \to \infty)$, we obtain the first assertion.

Let $T_0 = 0$ and $T_n = \sum_{k=1}^{n} c_k$ $(n \ge 1)$. Then

$$\sum_{k=1}^{n} c_k \exp\left(\sum_{l=1}^{k} c_l \right) = \sum_{k=1}^{n} (T_k - T_{k-1}) e^{T_k}$$

and

$$\int_{T_{k-1}}^{T_k} e^x \, \mathrm{d}x \le (T_k - T_{k-1}) e^{T_k} \le e^{c_k} \int_{T_{k-1}}^{T_k} e^x \, \mathrm{d}x.$$

As $c_n \to 0$ $(n \to \infty)$, we get the second assertion by elementary calculus. □

2. Examples

We present the asymptotic behaviors of sequences related to the examples in Sections III and IV.

Example 26. For some constants $\alpha > 0$, κ and $\theta > 0$, let

$$\alpha_n = \alpha + \frac{\kappa}{(\log n)^\theta} \quad (n \geq 2).$$

Let C_* be the same constant as in Lemma 2 (4). If $0 < \theta < 1$, then the limit

$$\lim_{n\to\infty}\left(\sum_{k=2}^{n-1}\frac{1}{k(\log k)^\theta} - \int_1^n \frac{1}{x(\log x)^\theta}\,\mathrm{d}x\right) = \lim_{n\to\infty}\left(\sum_{k=2}^{n-1}\frac{1}{k(\log k)^\theta} - \frac{(\log n)^{1-\theta}}{1-\theta}\right) \tag{A3}$$

exists and takes a positive value so that

$$\rho_n \sim c_1 \exp\left(\frac{\kappa}{1-\theta}(\log n)^{1-\theta}\right), \quad a_n \sim \frac{c_1 C_*}{\Gamma(\alpha+1)} n^\alpha \exp\left(\frac{\kappa}{1-\theta}(\log n)^{1-\theta}\right).$$

For $\theta = 1$, the same argument above applies and thus

$$\rho_n \sim c_2 (\log n)^\kappa, \quad a_n \sim \frac{c_2 C_*}{\Gamma(\alpha+1)} n^\alpha (\log n)^\kappa.$$

For $\theta > 1$, the sequence $\{\rho_n\}_{n=1}^\infty$ is convergent and $a_n \sim (\rho_\infty C_*/\Gamma(\alpha+1))n^\alpha$ with $\rho_\infty = \lim_{n\to\infty}\rho_n$.

According to the calculations above, we see that if $\alpha = 1/2$, then

$$\sum_{n=1}^\infty \frac{1}{a_n^2} < \infty \iff 0 < \theta < 1 \text{ and } \kappa > 0, \text{ or } \theta = 1 \text{ and } \kappa > \frac{1}{2}.$$

On the other hand, since $\{a_n\}_{n=1}^\infty$ is a regularly varying sequence with index $-\alpha$, we have $\sum_{n=1}^\infty 1/a_n^2 = \infty$ for $\alpha \in (0, 1/2)$, and $\sum_{n=1}^\infty 1/a_n^2 < \infty$ for $\alpha > 1/2$.

Example 27. For some constants κ and $\theta > 0$, let

$$\alpha_n = \frac{1}{2} + \frac{\kappa}{(\log n)^\theta} \quad (n \geq 2).$$

Let $w_n = \sum_{k=1}^n 1/a_k^2$ and $z_n = \sum_{k=n}^\infty 1/a_k^2$. By the calculations in Example 26, we have the following: Assume first that $0 < \theta < 1$. If $\kappa < 0$, then

$$w_n \sim \frac{c_1}{-2\kappa}(\log n)^\theta \exp\left(\frac{-2\kappa}{1-\theta}(\log n)^{1-\theta}\right), \quad a_n\sqrt{w_n} \sim \frac{1}{\sqrt{-2\kappa}}\sqrt{n(\log n)^\theta}.$$

If $\kappa = 0$, then

$$w_n \sim c_2 \log n, \quad a_n\sqrt{w_n} \sim \sqrt{n\log n}.$$

If $\kappa > 0$, then

$$z_n \sim \frac{c_3}{2\kappa}(\log n)^\theta \exp\left(-\frac{2\kappa}{1-\theta}(\log n)^{1-\theta}\right), \quad a_n\sqrt{z_n} \sim \sqrt{\frac{n(\log n)^{1-\theta}}{2\kappa}}.$$

Assume next that $\theta = 1$. If $\kappa < 1/2$, then

$$w_n \sim \frac{c_4}{1-2\kappa}(\log n)^{1-2\kappa}, \quad a_n\sqrt{w_n} \sim \sqrt{\frac{n\log n}{1-2\kappa}}.$$

If $\kappa = 1/2$, then

$$w_n \sim c_5 \log\log n, \quad a_n\sqrt{w_n} \sim \sqrt{n\log n\log\log n}.$$

If $\kappa > 1/2$, then

$$z_n \sim \frac{c_6}{2\kappa-1}\frac{1}{(\log n)^{2\kappa-1}}, \quad a_n\sqrt{z_n} \sim \sqrt{\frac{n\log n}{2\kappa-1}}.$$

We finally assume that $\theta > 1$. Then

$$w_n \sim c_7 \log n, \quad a_n\sqrt{w_n} \sim \sqrt{n\log n}.$$

Example 28. Let $0 < \alpha < 1/2$. For some constants η, κ and $\theta > 0$, let

$$\varepsilon_n = 1 - \frac{(\log n)^{\eta-1}}{n^{1-2\alpha}}, \quad \alpha_n = \alpha + \frac{\kappa}{(\log n)^{\theta}}.$$

We first assume that $\theta = 1$ or $\kappa = 0$. If $\eta > 2\kappa$, then

$$v_n \sim \frac{c_1}{\eta - 2\kappa}(\log n)^{\eta-2\kappa}, \quad a_n\sqrt{v_n} \sim \sqrt{\frac{n^{2\alpha}(\log n)^{\eta}}{\eta - 2\kappa}}.$$

If $\eta = 2\kappa$, then

$$v_n \sim c_2 \log\log n, \quad a_n\sqrt{v_n} \sim \sqrt{n^{2\alpha}\log\log n}.$$

If $\eta < 2\kappa$, then

$$t_n \sim \frac{c_3}{(2\kappa - \eta)(\log n)^{2\kappa-\eta}}, \quad a_n\sqrt{v_n} \sim \sqrt{\frac{n^{2\alpha}(\log n)^{\eta}}{2\kappa - \eta}}.$$

We next assume that $\theta > 1$. Independently of the value of κ, the calculations above for $\theta = 1$ remain valid by taking formally $\kappa = 0$.

We finally assume that $0 < \theta < 1$. If $\kappa > 0$, then

$$t_n \sim \frac{c_4(\log n)^{\eta+\theta}}{2\kappa} \exp\left(-\frac{2\kappa(\log n)^{1-\theta}}{1-\theta}\right), \quad a_n\sqrt{t_n} \sim \sqrt{\frac{n^{2\alpha}(\log n)^{\eta+\theta}}{2\kappa}}.$$

If $\kappa < 0$, then

$$v_n \sim \frac{c_5(\log n)^{\eta+\theta}}{-2\kappa} \exp\left(\frac{-2\kappa(\log n)^{1-\theta}}{1-\theta}\right), \quad a_n\sqrt{v_n} \sim \sqrt{\frac{n^{2\alpha}(\log n)^{\eta+\theta}}{-2\kappa}}.$$

Example 29. For some constants $\eta > 0$, κ and $\theta > 0$, let

$$\varepsilon_n = 1 - \frac{1}{(\log n)^{\eta}}, \quad \alpha_n = \frac{1}{2} + \frac{\kappa}{(\log n)^{\theta}} \quad (n \geq 2).$$

We first assume that $\theta = 1$ or $\kappa = 0$. If $0 < \eta < 1 - 2\kappa$, then

$$v_n \sim \frac{2c_1(\log n)^{1-(\eta+2\kappa)}}{1-(\eta+2\kappa)}, \quad a_n\sqrt{v_n} \sim \sqrt{\frac{2n(\log n)^{1-\eta}}{1-(\eta+2\kappa)}}.$$

If $\eta = 1 - 2\kappa$, then

$$v_n \sim 2c_2 \log\log n, \quad a_n\sqrt{v_n} \sim \sqrt{2n(\log n)^{1-\eta}(\log\log n)}.$$

If $\eta > 1 - 2\kappa$, then

$$t_n \sim \frac{2c_3}{(\eta+2\kappa-1)(\log n)^{\eta+2\kappa-1}}, \quad a_n\sqrt{t_n} \sim \sqrt{\frac{2n}{(\eta+2\kappa-1)(\log n)^{\eta-1}}}.$$

We next assume that $\theta > 1$. Then independently of the value of κ, the calculations above for $\theta = 1$ remain valid by taking formally $\kappa = 0$.

We finally assume that $0 < \theta < 1$. If $\kappa > 0$, then

$$t_n \sim \frac{c_4}{2\kappa}(\log n)^{\theta-\eta} \exp\left(-\frac{2\kappa}{1-\theta}(\log n)^{1-\theta}\right), \quad a_n\sqrt{t_n} \sim \sqrt{\frac{n(\log n)^{\theta-\eta}}{2\kappa}}.$$

If $\kappa < 0$, then

$$v_n \sim \frac{c_5}{-2\kappa}(\log n)^{\theta-\eta} \exp\left(\frac{-2\kappa}{1-\theta}(\log n)^{1-\theta}\right), \quad a_n\sqrt{v_n} \sim \sqrt{\frac{n(\log n)^{\theta-\eta}}{-2\kappa}}.$$

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