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ON ALGEBRAS OF SECOND LOCAL TYPE, I

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Throughout this paper, A denotes a (left and right) artinian ring with identity 1, J its Jacobson radical and all modules are (unital and) finitely generated.

Let n be any natural number. Then we say that A is of *right n-th local type* in case for every indecomposable right A-module M, the n-th top $\mathrm{top}^n M:=M/MJ^n$ of M is indecomposable. (Note that if $\mathrm{top}^n M$ is indecomposable, then so is M since A is artinian and M is finitely generated.) Hence for such a ring A, the question of indecomposability of right A-modules can be reduced to the corresponding problem of right A/J^n-modules. In [11] H. Tachikawa has studied the case $n=1$ and obtained a necessary and sufficient condition for algebras (by algebra we always mean a finite dimensional algebra over a field k) to be of this type. Further the representation theory of algebras with square-zero radical is well known [5], [6], [7]. So in this paper, we examine the case $n=2$ and give some necessary conditions for rings with selfduality to be of this type. Further in particular for QF (=quasi-Frobenius) rings, we give necessary and sufficient conditions to be of this type. More precisely, we show the following two theorems:

Theorem 1. *Let A be a ring with selfduality which is of right 2nd local type and e any primitive idempotent in A. Then*

(1) *J^2e is a uniserial waist in Ae if $J^2e\neq 0$* (see section 2 for definition of a waist),

(2) *eJ^m is a direct sum of local modules for every $m\geqslant 2$,*

(3) *for each local direct summand L of eJ^2, LJ^2 is uniserial (thus eJ^4 is a direct sum of uniserial modules).*

Further if A is an algebra, we have

(4) *Ae is uniserial if $h(Ae)\geqslant 5$.*

In particular if the base field k is, in addition, an algebraically closed field, then

(5) *Ae is uniserial if $h(Ae)\geqslant 4$,*

and then

(6) *eJ^2 is a direct sum of uniserial modules.*

Theorem 2. *Let A be a QF ring. Then the following statements are equivalent*:

(1) *A is of right 2nd local type.*

(2) *A is of right 2nd colocal type* (see section 1 for definition).

(3) *For any primitive idempotent e in A, eA is uniserial if $h(eA)\geqslant 4$.*

(4) *A/J^t is QF for every $t\geqslant 3$.*

(5) *For each M_A indecomposable with $h(M)\geqslant 3$, there is a primitive idempotent e in A such that $M\cong eA/eJ^{h(M)}$.*

(6) *$A=A_1\times A_2$ for some QF rings A_1 and A_2 such that A_1 has cube-zero radical and A_2 is a serial ring.*

Furthermore, each of these conditions are equivalent to the corresponding left side version.

In the theorems above $h(M)$ denotes the height ($=$Loewy length) of M, namely $h(M):=\min\{n\in \boldsymbol{N}\cup\{0\}\mid MJ^n=0\}$. We remark that Theorem 1 (5) and (6) remain valid also in the case where k is a splitting field for A.

In section 1, we introduce the basic tools used in the following sections. Section 2 is devoted to the structure of an indecomposable projective left module and in section 3, we examine the structure of an indecomposable projective right module mainly using the technique of Sumioka [10]. In section 4, we give the proof of Theorem 2. Finally in section 5, we give some examples.

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1. Preliminaries

1.1. Throughout the paper, we write homomorphisms on the opposite side to scalar multiplications, and for homomorphisms p: $K\rightarrow L$ and q: $L\rightarrow M$ of left A-modules and for a decomposition D: $L=\bigoplus_{i=1}^{n}L_i$ of L, $(p, D)=(p_i)_{n=1}^{i}$ and $(D, q)=(q_i)^{T}{}_{i=1}^{n}$ are matrix expressions of p and q relative to D, respectively (for homomorphisms of right A-modules, we write as $(p, D)=(p_i)^{T}{}_{i=1}^{n}$ and $(D, q)=(q_i)_{i=1}^{n}$). In addition to the definition of right n-th local type for n any natural number, we define the dual notion: A is called to be of *left n-th colocal type* in case for every indecomposable left A-module M, the n-th socle $\mathrm{soc}^n M:=$ (the right annihilator of J^n in M) of M is indecomposable. It should be noted that if A has a selfduality, then A is of right n-th local type iff A is of left n-th colocal type. Further noting that the composition lengths of the projective covers (over A) of all indecomposable right A/J^n-modules have a bound if A/J^n is of finite representation type (i.e. it has only finitely many isomorphism classes of indecomposable right modules), we see easily that when A is of right n-th local type, A is of finite representation type iff so is A/J^n (See

Auslander [3]).

Since the property to be of n-th local (colocal) type is Morita invariant, we may assume that A is a basic ring. We put $\mathrm{pi}(A):=\{e_1, \cdots, e_p\}$ to be a basic set of primitive idempotents of A.

DEFINITION 1.2 ([2]). Let D: $L=\bigoplus_{i=1}^{n} L_i$ be a decomposition of a right A-module L and p: $K\to L$ be a homomorphism, and j in $\{1, \cdots, n\}$. Then the pair (p, D) (or simply p: $K\to\bigoplus_{i=1}^{n} L_i$) is called *j-fusible* in case there is a homomorphism q: $\bigoplus_{i\neq j} L_i\to L_j$ such that the diagram

$$\begin{array}{ccc} K & \xrightarrow{(p_i)_{i\neq j}} & \bigoplus_{i\neq j} L_i \\ \| & & \downarrow q \\ K & \xrightarrow{p_j} & L_j \end{array}$$

commutes where $(p, D)=(p_i)_{i=1}^{T\ n}$. The pair (p, D) is called *fusible* in case (p, D) is j-fusible for some $j=1, \cdots, n$. Finally (p, D) is called *infusible* in case (p, D) is not fusible.

Corollary 1.2.1 ([2, Corollary 1.4]). *Let $K_i\lneqq L_i$ for each $i=1, 2$ and h: $K_1\to K_2$ be an isomorphism. Define $p_1=k_1$, $p_2=k_2h$ where k_i: $K_i\to L_i$ is the inclusion map for each i. Then h or h^{-1} is extendable to a homomorphism $L_1\to L_2$ or $L_2\to L_1$, respectively iff p: $K_1\to L_1\oplus L_2$ is fusible.* //

Proposition 1.2.2 ([2, Proposition 1.1]). *Consider an exact sequence $K\xrightarrow{p}L\xrightarrow{q}M\to 0$ of right A-modules and let D: $L=\bigoplus_{i=1}^{n} L_i$ be a decomposition of L, $(p, D)=(p_i)_{i=1}^{T\ n}$, $(D, q)=(q_i)_{i=1}^{n}$ and j in $\{1, \cdots, n\}$. Then the following statements are equivalent*:

(1) *(p, D) is j-fusible.*

(2) *There is a homomorphism $r=(r_i)_{i=1}^{n}$: $\bigoplus_{i=1}^{n} L_i\to X$ such that $rp=0$ and r_j is an isomorphism.*

(3) *q_j is a split monomorphism.*

Proof. See [2]. //

REMARK. In [2] the fusible maps were defined by the condition (2) above.

Proposition 1.3 *Let $0\to K\xrightarrow{p}L\xrightarrow{q}M\to 0$ be a nonsplit exact sequence of right A-modules and D: $L=\bigoplus_{i=1}^{n} L_i$ be a decomposition of L $(n\geqslant 2)$. Then we have*

(1) *if M is indecomposable, then (p, D) is infusible,*

(2) *if K is simple, each L_i is local and (p, D) is infusible, then M is indecomposable.*

Proof. See [1] or [2]. //

1.4. Let I be a two-sided ideal of A and e and f in pi(A). Then we have the canonical isomorphisms $\mathrm{Hom}_A(fA, eA/eI) \xrightarrow{\sim} eAf/eIf \xrightarrow{\sim} \mathrm{Hom}_A(Ae, Af/If)$. We denote by p^* the image of every p in $\mathrm{Hom}_A(fA, eA/eI)$ or the inverse image of every p in $\mathrm{Hom}_A(Ae, Af/If)$ under the composition of these isomorphisms.

Proposition 1.4.1 *Let $e, f_1, \cdots, f_n$ be in* pi(A), $l>m$, j *in* $\{1, \cdots, n\}$ *and* $p=(p_i)_{i=1}^n: \bigoplus_{i=1}^n f_iA \to eJ^m/eJ^l$ *be a homomorphism. Then the following statements are equivalent*:

(1) $p(f_jA) \leqslant \sum_{i \neq j} p(f_iA)$.

(2) $p^*: Ae/J^{l-m}e \to \bigoplus_{i=1}^n Af_i/J^lf_i$ *is j-fusible, where p^* is the map induced by the homomorphism* $(p_i^*)_{i=1}^n$.

Proof. There is some u_i in eJ^mf_i such that each p_i^* is the left multiplication by u_i. Then p has the property stated in (1) iff $(u_jf_jA+eJ^l)/eJ^l \leqslant (\sum_{i \neq j} u_if_iA + eJ^l)/eJ^l$

iff $u_jA \leqslant \sum_{i \neq j} u_iA + eJ^l$

iff $u_j = \sum_{i \neq j} u_ia_i + b$, for some a_i in f_iA and b in eJ^l

iff $u_j = \sum_{i \neq j} u_ia_i + b$, for some a_i in f_iAf_j and b in eJ^lf_j

iff $u_j = \sum_{i \neq j} u_ia_i + b$, for some a_i in f_iAf_j and b in J^lf_j

iff p^* is j-fusible. //

In future p^* shall always mean the above induced homomorphism when the domain of p is of the form as above.

Corollary 1.4.2. *Under the same situation as above but $l=m+1$, the following are equivalent*:

(1) $\bar{p}: \bigoplus_{i=1}^n f_iA/f_iJ \to eJ^m/eJ^{m+1}$ *(the induced map) is a monomorphism.*

(2) $p^*: Ae/Je \to \bigoplus_{i=1}^n Af_i/J^{m+1}f_i$ *is infusible.*

In particular if $p: \bigoplus_{i=1}^n f_iA \to eJ^m$ is a projective cover of eJ^m, then $p^*: Ae/Je \to \bigoplus_{i=1}^n Af_i/J^{m+1}f_i$ is infusible. //

Corollary 1.4.3. *Let $p: \bigoplus_{i=1}^n f_iA \to eJ^m$ be a projective cover of eJ^m and* $0 \to$

$Ae/Je \overset{p^*}{\to} \bigoplus_{i=1}^{n} Af_i/J^{m+1}f_i \to M \to 0$ *be an exact sequence. Then M is indecomposable.*

Proof. Clear from (1.4.2) and (1.3). //

2. Structure of an indecomposable projective left module

For an A-module M, we put $|M| :=$ the composition length of M.

Proposition 2.1. *Let A be of right n-th local type, n any natural number and e in* pi(A). *Then $J^n e$ is uniserial.*

Proof. It is sufficient to prove that $|J^m e/J^{m+1}e| \leqslant 1$ for every $m \geqslant n$. Suppose $|J^m e/J^{m+1}e| \geqslant 2$ for some $m \geqslant n$. Then we have a homomorphism $p: Af_1 \oplus Af_2 \to J^m e/J^{m+1}e$; f_1, f_2 in pi(A) such that the induced map $\bar{p}: (Af_1/Jf_1) \oplus (Af_2/Jf_2) \to J^m e/J^{m+1}e$ is a monomorphism. Putting $L = (f_1 A/f_1 J^{m+1}) \oplus (f_2 A/f_2 J^{m+1})$, we have an exact sequence $0 \to eA/eJ \overset{p^*}{\to} L \to M \to 0$ where M is indecomposable by (1.4.2) and (1.3). But since $p^*(eA/eJ) \leqslant LJ^m \leqslant LJ^n$, $\mathrm{top}^n M \cong \mathrm{top}^n L$ is decomposable. This is a contradiction. //

DEFINITION 2.2 ([4]). Let ${}_A L \leqslant {}_A M$. Then L is called to be a *waist* in M in case $0 \neq L \neq M$ and for each ${}_A N \leqslant {}_A M$, it holds that $L \leqslant N$ or $N \leqslant L$.

Proposition 2.2.1. *Let A be a ring with selfduality which is of right 2nd local type and e in* pi(A). *Then $J^2 e$ is a waist in Ae if $J^2 e \neq 0$.*

Proof. Deduced from the following three lemmas for an artinian ring A:

Lemma 2.2.2 ([9, Lemma 1.2]). *Let ${}_A M$ be nonsimple indecomposable. Then* soc(JM)=soc M.

Proof. Let S be any simple submodule of M and X be any proper submodule of M. If $S+X=M$ then S is not contained in X. Thus $S \cap X = 0$. Hence $S=M$, a contradiction. Therefore S is small in M i.e. $S \leqslant JM$. Hence soc $M \leqslant JM$ and soc M=soc(JM). //

Lemma 2.2.3. *Let ${}_A M$ be local and* $\mathrm{soc}^2 M$ *indecomposable. Then* soc ($J^2 M$)=soc M *if $J^2 M \neq 0$.*

Proof. Clear from (2.2.2) nothing that JM is nonsimple indecomposable since $J^2 M \neq 0$ and $\mathrm{soc}^2 M \leqslant JM$. //

Lemma 2.2.4. *Let A be a ring of left 2nd colocal type, ${}_A M$ be local and $J^2 M$ be a nonzero uniserial module. Then $J^2 M$ is a waist in M.*

Proof. Suppose that $J^2 M$ is not a waist in M. Then for some $X \leqslant M$,

$J^2M \not\leqslant X$ and $X \not\leqslant J^2M$. And, $J^2M \cap X = J^tM$ for some $t \geqslant 3$. Hence $M/J^t \geqslant (J^2M/J^tM) \oplus (X/J^tM)$ where $J^2M/J^tM \neq 0$ and $X/J^tM \neq 0$. On the other hand since $\mathrm{soc}^2(M/J^tM)$ is indecomposable and $J^2(M/J^tM) \neq 0$, we have that $\mathrm{soc}(M/J^tM) = \mathrm{soc}(J^2M/J^tM)$ is simple by (2.2.3). This is a consradiction. //

We get Theorem 1 (1) from Propositions 2.1 and 2.2.1.

Corollary 2.2.5. *Let A be a ring with selfduality which is of right 2nd local type, e in* pi(A) *and $h = h(Ae)$. Then we have $\mathrm{soc}^{h-t}(Ae) = J^te$ for every $t = 0, \cdots, h$.*

Proof. It is clear from Theorem 1 (1) in case $t \geqslant 2$. The other cases ($t = 0, 1$) are trivial.

Lemma 2.3.1. *Let ${}_AL_1$ and ${}_AL_2$ be local of height $\geqslant 3$ such that for each $i = 1, 2$, $\mathrm{soc}^3 L_i$ is uniserial and J^2e_i is a uniserial waist in Ae_i where Ae_i is the projective cover of $\mathrm{soc}^3 L_i$. Suppose that ${}_AK$ is simple and there exists an isomorphism $p_i\colon K \to \mathrm{soc}\, L_i$ for each $i = 1, 2$. Consider an exact sequence:*

$$0 \to K \xrightarrow{p = (p_1, p_2)} L_1 \oplus L_2 \xrightarrow{q = \begin{bmatrix} q_1 \\ -q_2 \end{bmatrix}} M \to 0 .$$

Then $\mathrm{soc}^2 M$ is decomposable if $p\colon K \to \mathrm{soc}^2 L_1 \oplus \mathrm{soc}^2 L_2$ is fusible.

Proof. Assume that $p\colon K \to \mathrm{soc}^2 L_1 \oplus \mathrm{soc}^2 L_2$ is fusible, say 2-fusible. Then we have a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{p_1} & \mathrm{soc}^2 L_1 \\ \| & & \downarrow r \\ K & \xrightarrow{p_2} & \mathrm{soc}^2 L_2 . \end{array}$$

And, $M \geqslant (\mathrm{soc}^2 L_1)q_1 + L_2q_2 = U \oplus L_2q_2$ where $U = (\mathrm{soc}^2 L_1)\ (q_1 - rq_2) \neq 0$. Now for each x in $\mathrm{soc}^2 M$, $x = l_1q_1 + l_2q_2$ for some (l_1, l_2) in $L_1 \oplus L_2$. Since $ux = 0$ for each u in J^2, we have $ul_1q_1 = -ul_2q_2 \in L_1q_1 \cap L_2q_2 = Kp_1q_1$ $(=: S)$. Hence $J^2l_1q_1 = J^2l_2q_2 \leqslant S$ where S is simple. In particular, $\mathrm{soc}^2 M \leqslant \mathrm{soc}^3 L_1q_1 + \mathrm{soc}^3 L_2q_2$.

i) In case for each x in $\mathrm{soc}^2 M$, there are l_1, l_2 with $x = l_1q_1 + l_2q_2$ such that $J^2l_1q_1 = J^2l_2q_2 = 0$. Then we have $J^2l_1 = 0$ for q_1 is monic. Thus l_1 is in $\mathrm{soc}^2 L_1$ and x is in $U \oplus L_2q_2$. Therefore $\mathrm{soc}^2 M \leqslant U \oplus L_2q_2$. Hence $\mathrm{soc}^2 M$ is decomposable.

ii) In case for some x in $\mathrm{soc}^2 M$, there are l_1, l_2 with $x = l_1q_1 + l_2q_2$ such that $J^2l_1q_1 = J^2l_2q_2 = S$. We may assume that $x = ex$ for some e in pi(A). Since S is simple and q_i are monic, $J^3l_1 = J^3l_2 = 0$. Thus l_i is in $\mathrm{soc}^3 L_i \backslash \mathrm{soc}^2 L_i$ for each i. Also, we may assume that $l_i = el_i$ for each i since $x = ex$. Further we have $\mathrm{soc}^3 L_i = Ael_i$ for each $i = 1, 2$ since $\mathrm{soc}^3 L_i$ are uniserial. Hence we may assume that $e=e_1=e_2$. Define a homomorphism s: $\mathrm{soc}^3L_1 \to \mathrm{soc}^3L_2$ by $ael_1 \mapsto ael_2$ for each a in A. Then s is well-defined. In fact, if t is in Ae and $tl_1=0$, then t is in $\mathrm{Ann}_{Ae}(l_1)$, the annihilator of l_1 in Ae. On the other hand, by the fact that $J^2el_1 \neq 0$, we see $\mathrm{Ann}_{Ae}(l_1)$ does not contain J^2e which is a uniserial waist in Ae. Hence $\mathrm{Ann}_{Ae}(l_1)$ is contained in J^3e and t is in J^3e. Thus tl_2 is in $J^3l_2=0$.

Further the diagram

$$\begin{array}{ccc} K & \xrightarrow{p_1} & \mathrm{soc}^3L_1 \\ \| & & \downarrow s \\ K & \xrightarrow{p_2} & \mathrm{soc}^3L_2 \end{array}$$

is commutative. For, $J^2(l_1, l_2)$ $(\neq 0)$ is contained in the simple module $\mathrm{Im}\, p$ since $J^2(l_1, l_2)q=0$. Hence $J^2(l_1, l_2)=\mathrm{Im}\, p$. Let c be a nonzero element in K. Then $K=Ac$ and $cp=(ul_1, ul_2)$ for some u in J^2. Therefore $c(p_1s)=ul_1s=ul_2$ $=cp_2$. Thus $p_1s=p_2$.

Then putting $V:=(\mathrm{soc}^3L_1)(q_1-sq_2)$, the same argument as in i) shows that $\mathrm{soc}^2M \leqslant V \oplus L_2q_2$ and soc^2M is decomposable. //

Proposition 2.3.2. *Let A be a ring with selfduality which is of right 2nd local type and ${}_AL_1$, ${}_AL_2$ be local of height$\geqslant 3$ such that soc^3L_i are uniserial and $|L_1| \leqslant |L_2|$. Then for every isomorphism r: $\mathrm{soc}L_1 \to \mathrm{soc}L_2$, r is extendable to a monomorphism $L_1 \to L_2$ if r is extendable to a homomorphism $\mathrm{soc}^2L_1 \to \mathrm{soc}^2L_2$.*

Proof. Put $K=\mathrm{soc}\, L_1$, $p_1=$identity map of $\mathrm{soc}\, L_1$ and $p_2=r$. Consider an exact sequence $0 \to K \xrightarrow{p=(p_1, p_2)} L_1 \oplus L_2 \xrightarrow{q} M \to 0$. If r is extendable to a homomorphism $\mathrm{soc}^2L_1 \to \mathrm{soc}^2L_2$, then p: $K \to \mathrm{soc}^2L_1 \oplus \mathrm{soc}^2L_2$ is fusible. Hence by (2.3.1), soc^2M is decomposable, thus M is decomposable. Therefore p: $K \to L_1 \oplus L_2$ is fusible by (1.3). Hence by (1.2.1), r is extendable to a homomorphism q: $L_1 \to L_2$ since $|L_1| \leqslant |L_2|$ where q is monic since $\mathrm{soc}\, L_1$ is simple. //

2.4. Throughout the rest of this section, A is a ring with selfduality which is of right 2nd local type. Here, we examine indecomposable projective left A-modules of height$\geqslant 4$.

Proposition 2.4.1. *Let e and f be in $\mathrm{pi}(A)$ and $fJe/fJ^2e \neq 0$. Then Af is uniserial if $h(Ae) \geqslant 4$.*

Proof. Take some u in $fJe \backslash fJ^2e$ and define p: $Af \to Je$ by the right multiplication by u. Then $\mathrm{Ker}\, p \leqslant J^2f$ or $\mathrm{Ker}\, p \geqslant J^2f$ since J^2f is a waist in Af (if $J^2f \neq 0$). Assume that $\mathrm{Ker}\, p \geqslant J^2f$. Then $h(\mathrm{Im}\, p) \leqslant 2$ since $\mathrm{Im}\, p \cong Af/\mathrm{Ker}\, p$ is an epimorph of Af/J^2f. Hence $\mathrm{Im}\, p \leqslant \mathrm{soc}^2(Ae) \leqslant J^2e$ for $h\,(Ae) \geqslant 4$ and $\mathrm{soc}^2(Ae)=J^{h(Ae)-2}e$. But by the definition of p we have $\mathrm{Im}\, p \leqslant J^2e$, a contradiction.

Accordingly, $\mathrm{Ker}\, p \leqslant J^2 f$. Then $\mathrm{Ker}\, p = J^t f$ for some $t \geqslant 2$ and $Af/J^t f$ is embedded into Je. Therefore $|Jf/J^2 f| = 1$ since $Jf/J^t f$ is embedded into $J^2 e$ which is uniserial. Hence Af is uniserial. //

Proposition 2.4.2. *Assume that e is in* pi(A), $h(Ae) \geqslant 4$ *and Ae is not uniserial. Then*

(1) *all simple submodules of* $Je/J^2 e$ *are pairwise isomorphic, and*

(2) $J^2 e/J^3 e \cong J^3 e/J^4 e$.

Proof. Let $u\colon \bigoplus_{i=1}^{n} Af_i \to Je/J^4 e$ be a projective cover of $Je/J^4 e$. Then $n \geqslant 2$ since Ae is not uniserial. Putting $L_i := (Af_i)u$, we have $L_i \cap L_j = J^2 e/J^4 e$, $L_i \gneqq J^2 e/J^4 e$ for each $i \neq j$ in $\{1, \cdots, n\}$. By (2.4.1), each L_i is uniserial and $h(L_i) = 3$. Further $\mathrm{soc}\, L_i = J^3 e/J^4 e$ is simple and $\mathrm{soc}^2 L_i = J^2 e/J^4 e$ for each $i = 1, \cdots, n$.

(1) For any $i \neq j$ in $\{1, \cdots, n\}$, the identity map $p\colon \mathrm{soc}\, L_i \to \mathrm{soc}\, L_j$ is extendable to a homomorphism $\mathrm{soc}^2 L_i \to \mathrm{soc}^2 L_j$ since $L_i \cap L_j = J^2 e/J^4 e = \mathrm{soc}^2 L_i = \mathrm{soc}^2 L_j$. Hence by (2.3.2), p is extendable to an isomorphism $L_i \to L_j$. Thus all simple submodules of $Je/J^2 e$ are pairwise isomorphic.

(2) Putting $p_i\colon J^2 e/J^4 e \to L_i$ and $q_i\colon L_i \to L_1 + L_2$ to be inclusion maps for $i = 1, 2$, we have an exact sequence

$$0 \to J^2 e/J^4 e \xrightarrow{(p_1,\, p_2)} L_1 \oplus L_2 \xrightarrow{\begin{bmatrix} q_1 \\ -q_2 \end{bmatrix}} L_1 + L_2 \to 0$$

where $L_1 + L_2$ is colocal. Hence the identity map $r\colon \mathrm{soc}^2 L_1 \to \mathrm{soc}^2 L_2$ is not extendable to any isomorphism $L_1 \to L_2$. On the other hand, the identity map $p\colon \mathrm{soc}\, L_1 \to \mathrm{soc}\, L_2$ is extendable to an isomorphism $s\colon L_1 \to L_2$ since $r|(\mathrm{soc}\, L_1) = p$. As a consequence, $s|(\mathrm{soc}^2 L_1) \neq r$. But if $J^2 e/J^3 e \not\cong J^3 e/J^4 e$, then the restriction map

$$\mathrm{Hom}_A(\mathrm{soc}^2 L_1, \mathrm{soc}^2 L_2) \to \mathrm{Hom}_A(\mathrm{soc}\, L_1, \mathrm{soc}\, L_2)$$

is an injection. This implies that $s|(\mathrm{soc}^2 L_1) = r$ since both $s|(\mathrm{soc}^2 L_1)$ and r are extensions of p. This is a contradiction. //

Proposition 2.4.3. *Assume that e, f and g are in* pi(A), $h(Ae) \geqslant 5$, *Ae is not uniserial,* $fJe/fJ^2 e \neq 0$ *and* $J^2 e/J^3 e \cong Ag/Jg$. *Then* $fAf/fJf \cong gAg/gJg$ *as rings.*

Proof. There exists a submodule L of $Je/J^4 e$ such that L is uniserial of height 3 and $\mathrm{top}\, L \cong Af/Jf$, $\mathrm{top}\, JL \cong Ag/Jg$. We identify these isomorphic modules. Further Af and Ag are both uniserial by (2.4.1) and the fact that $h(Ae) \geqslant 5$ and also $h(Af) \geqslant 4$. Then we can define a homomorphism $t\colon \mathrm{End}_A(Af/Jf) \to \mathrm{End}_A(Ag/Jg)$ by $t(p) := \overline{(q|Jf/J^3 f)}$ for each p in $\mathrm{End}_A(Af/Jf)$ where p is induced by some q in $\mathrm{End}_A(Af/J^3 f)$ and $\bar{r}$ is the map in $\mathrm{End}_A(Jf/J^2 f)$ induced by r for every r in $\mathrm{End}_A(Jf/J^3 f)$. (We identified $\mathrm{End}_A(Jf/J^2 f) = \mathrm{End}_A(Ag/Jg)$.)

Then t is well-defined and injective since for each q in $\mathrm{End}_A(Af/J^3f)$, $(Af/J^3f)q \leqslant Jf/J^3f$ iff $(Jf/J^3f)q \leqslant J^2f/J^3f$ (See [10, section 3]). Further by (2.3.2), every automorphism p of soc L is extendable to an automorphism of L if p is extendable to an automorphism of $\mathrm{soc}^2 L$. Thus t is surjective. (Note that both Af/J^3f and Jf/J^3f are quasi-projective since we have $Jf/J^3f \simeq Ag/J^2g$ from the fact that Ag is uniserial.) Hence $fAf/fJf \simeq \mathrm{End}_A(Af/Jf) \simeq \mathrm{End}_A(Ag/Jg) \simeq gAg/gJg$ as rings.

REMARK. In the above, if A is a k-algebra, then the isomorphism defined as above is a k-algebra isomorphism.

2.4.4. Proof of Theorem 1 (4) and (5). Assume that A is an algebra and suppose that Ae is not uniserial, and $h(Ae) \geqslant 4$. Let $p\colon \bigoplus_{i=1}^{n} P_i \rightarrow Je/J^3e$ be a projective cover of Je/J^3e where each ${}_AP_i$ is indecomposable. Then $n \geqslant 2$. By (2.4.2), there is an f in pi(A) such that every P_i is isomorphic to Af. And, $J^2e/J^3e \simeq Ag/Jg$ for some g in pi(A). If we put $L_i := (P_i)p$ for $i=1, 2$, then $L_i \simeq Af/J^2f$, $J^2e/J^3e \lneqq L_i \leqslant Je/J^3e$, $L_1 \cap L_2 = J^2e/J^3e$ and top $L_i \simeq Af/Jf$ for each $i=1, 2$. Since we have an exact sequence

$$0 \rightarrow J^2e/J^3e \rightarrow L_1 \oplus L_2 \rightarrow L_1 + L_2 \rightarrow 0$$

where $J^2e/J^3e \simeq Ag/Jg$, $L_1 \oplus L_2 \simeq (Af/J^2)^{(2)}$ and $L_1 + L_2$ is colocal, there exists an infusible homomorphism $Ag/Jg \rightarrow (Af/J^2f)^{(2)}$ by (1.3; 1). Therefore $(fA/fJ)^{(2)}$ is isomorphic to a direct summand of gJ/gJ^2 by (1.4.2). Hence $\dim(gJf/gJ^2f)_{fAf/fJf} \geqslant 2$. If $h(Ae) \geqslant 5$ or k is algebraically closed, then by (2.4.3), $d := \dim_{gAg/gJg}(gJf/gJ^2f) = \dim(gJf/gJ^2f)_{fAf/fJf} \geqslant 2$. Hence $(Ag/Jg)^{(d)}$ is isomorphic to a direct summand of Jf/J^2f and $d \geqslant 2$. Thus $|Jf/J^2f| \geqslant 2$. This contradicts the uniseriality of Af. Hence Ae must be uniserial. //

3. Structure of an indecomposable projective right module

Lemma 3.1. *Let $0 \rightarrow K \xrightarrow{p} L \xrightarrow{q} M \rightarrow 0$ be an exact sequence of left A-modules such that K is simple, $D\colon L = \bigoplus_{i=1}^{n} L_i$ is a decomposition of L $(n \geqslant 2)$ and for each $i=1, \cdots, n$, $L_i = Ae_i/I_i$ for some e_i in* pi(A) *and $J^{m+1}e_i \leqslant I_i \lneqq J^m e_i$ $(m \geqslant 1)$. Then $JM = \mathrm{soc}^m M$ if (p, D) is infusible.*

Proof. Put $l_i := e_i + I_i$, $\bar{l}_i = l_i + JL$, $m_i := l_i q$, $\bar{m}_i := m_i + JM$ and $m_i' := m_i + \mathrm{soc}^m M$. Then we have $\bigoplus_{i=1}^{n} A\bar{l}_i = L/JL \simeq M/JM = \bigoplus_{i=1}^{n} A\bar{m}_i$ where each $A\bar{m}_i$ is simple. It follows from $h(M) \leqslant m+1$ that $JM \leqslant \mathrm{soc}^m M$. Assume that $JM \lneqq \mathrm{soc}^m M$. Then we show that (p, D) is fusible. (Clearly, we may assume that each $p_i \neq 0$ i.e. each p_i is a monomorphism where $(p, D) = (p_i)_{i=1}^{n}$.) By assumption the sum $M/\mathrm{soc}^m M=\sum_{i=1}^{n} Am'_i$ is redundant i.e. $Am'_j \leqslant \sum_{i\neq j} Am'_i$ for some j, say $j=1$. So $m'_1=\sum_{i\neq 1} -a_i m'_i$ for some a_i in A. By putting $a_1=1$, we have $\sum_{i=1}^{n} a_i m_i \in \mathrm{soc}^m M$ and $J^m(a_i l_i)_{i=1}^{n}\cdot q=0$. Thus $J^m(a_i l_i)_{i=1}^{n} \leqslant \mathrm{Im}\, p$. Further putting $e:=e_1$ we may assume that $a_i=ea_i$ for each $i\neq 1$. Put $l:=(a_i l_i)_{i\neq 1}$. Then we have $l_1\in L_1$, $l\in \bigoplus_{i\neq 1} L_i$, $l_1=el_1$, $l=el$ and $J^m(l_1, l)\leqslant \mathrm{Im}\, p$. On the other hand, it holds that $J^m(l_1, l)\neq 0$ since we have $J^m l_1\neq 0$ by the assumption $I_i \lneqq J^m e_i$. Accordingly, $J^m(l_1, l)=\mathrm{Im}\, p$ since $\mathrm{Im}\, p$ is simple. Define a map $r\colon L_1\to \bigoplus_{i\neq 1} L_i$ by $xl_1 \mapsto xl$ for each $xl_1\in L_1$. Then r is well-defined. In fact, if $xl_1=0$, then $xe\in I_1\leqslant J^m$ and then $xe(l_1, l)\in \mathrm{Im}\, p$. Thus $xe(l_1, l)=sp$ for some s in K. Therefore $sp_1=xel_1=xl_1=0$ and $s(p_i)_{i\neq 1}=xel$. But since p_1 is a monomorphism, we have $s=0$ and $xl=xel=0$. Further by the similar argument as in (2.3.1), $p_1 r=(p_i)_{i\neq 1}$ i.e. (p, D) is fusible. //

Proposition 3.2. *Let A be a ring with selfduality which is of right 2nd local type, $m\geqslant 2$, $e, f_1, \cdots, f_n (n\geqslant 2)$ in* pi(A) *and $p\colon \bigoplus_{i=1}^{n} f_i A\to eJ^m/eJ^{m+1}$ be a projective cover of eJ^m/eJ^{m+1}. Then $p^*\colon Ae/Je\to \bigoplus_{i=1}^{n} Jf_i/J^{m+1}f_i$ is infusible.*

Proof. Let $0\to Ae/Je \xrightarrow{p^*} \bigoplus_{i+1}^{n} Af_i/J^{m+1}f_i\to M\to 0$ be an exact sequence. Then M is indecomposable by (1.4.3). By (3.1), $JM=\mathrm{soc}^m M$. Accordingly, JM is indecomposable since $JM\geqslant \mathrm{soc}^2 M$ and $\mathrm{soc}^2 M$ is indecomposable. Then from the exact sequence $0\to Ae/Je \xrightarrow{p^*} \bigoplus_{i=1}^{n} Jf_i/J^{m+1}f_i\to JM\to 0$, we obtain that $p^*\colon Ae/Je \to \bigoplus_{i=1}^{n} Jf_i/J^{m+1}f_i$ is infusible by (1.3). //

3.3. Proof of Theorem 1 (2). Let $p\colon \bigoplus_{i=1}^{n} f_i A\to eJ^m$ be a projective cover of eJ^m and f_i in pi(A) for each $i=1, \cdots, n$. If $n=1$, then the assertion is trivial. So we may assume that $n\geqslant 2$. There is some u_i in $eJ^m f_i\backslash eJ^{m+1}f_i$ such that the i-th coordinate map of p is the right multiplication by u_i for each $i=1, \cdots, n$. Put $\bar{u}_i:=u_i+eJ^{m+1}$, $u'_i:=u_i+J^{m+1}f_i$ and $e':=e+Je$. Then $eJ^m=\sum_{i=1}^{n} u_i A$ where each $u_i A$ is local. Suppose that eJ^m is not a direct sum of local modules. Then $\sum_{i=1}^{n} u_i a_i=0$ for some a_i in A and $u_j a_j\neq 0$ for some $j=1, \cdots, n$. We may assume that there is some g in pi(A) such that $u_j a_j g\neq 0$ and $a_i=f_i a_i g$ for each $i=1, \cdots, n$. Then it holds that a_i is in $f_i Jg$ for each i. In fact, if $f_i\neq g$, then $a_i\in f_i Ag=f_i Jg$. And, in case $f_i=g$, we have $f_i Ag/f_i Jg=f_i Af_i/f_i Jf_i$ is a division ring. Furthermore, $\sum_{i=1}^{n} u_i a_i=0$ implies $\sum_{i=1}^{n} \bar{u}_i a_i=0$ and hence each $\bar{u}_i a_i=0$, since $\bar{u}_i A$ are independent. Then putting $\bar{a}_i:=a_i+f_i Jg$, we have that $\bar{u}_i\bar{a}_i$ is defined and is zero. Hence if a_i is not in $f_i Jg$, then $\bar{u}_i=(\bar{u}_i\bar{a}_i)\bar{a}_i^{-1}=0$, a contradiction. Further $Au_i=J^m f_i$ since $J^m f_i$ is uniserial for $m\geqslant 2$. Therefore we may assume that $Au_i a_i \leqslant Au_n a_n$ for each i and $Au_n a_n = J^s g$ for some $s\geqslant m+1\geqslant 3$. Define a homomorphism q_i: $Af_i/J^{m+1}f_i \to Ag/J^{s+1}g$ by $x\mapsto xa_i$ for each $i=1, \cdots, n$. Then q_n is a monomorphism since $\mathrm{soc}(Af_n/J^{m+1}f_n)=J^m f_n/J^{m+1}f_n$ is simple and is mapped by q_n onto the simple module $J^s g/J^{s+1}g$. Further putting $q_i' := q_i|(Jf_i/J^{m+1}f_i)$, we have $\mathrm{Im}\ q_i' \leqslant \mathrm{soc}^m(Jg/J^{s+1}g)=J^{s+1-m}g/J^{s+1}g=\mathrm{Im}\ q_n'$ for each $i=1, \cdots, n$. Hence if we put $q_i'' := q_i' : Jf_i/J^{m+1}f_i \to J^{s+1-m}g/J^{s+1}g$ and $q := (q_i'')^T{}_{i=1}^{n}$, then p^*: $Ae/Je \to \bigoplus_{i=1}^{n} Jf_i/J^{m+1}f_i$ is fusible since $e'p^*q=0$ and q_n'' is an isomorphism. This contradicts (3.2). Hence eJ^m must be a direct sum of local modules.

3.4. Proof of Theorem 1 (3) and (6). Suppose that $|LJ^s/LJ^{s+1}|\geqslant 2$ for some $s\geqslant 1$. LJ^s is a direct sum of local modules for LJ^s is a direct summand of eJ^{2+s}. Further $L=vA$ for some v in $eJ^2g\backslash eJ^3g$ and for some g in $\mathrm{pi}(A)$. Hence $LJ^s=vJ^s=u_1A\oplus u_2A\oplus \cdots$ for some u_i in $eJ^{2+s}f_i\backslash eJ^{3+s}f_i$ where f_i are in $\mathrm{pi}(A)$. Then for each $i=1, 2$, there is some a_i in $gJ^s f_i$ such that $u_i=va_i$. Define a map p_i: $Ag/J^3g \to Af_i/J^{s+3}f_i$ by $x\mapsto xa_i$ for each $i=1, 2$. Then p_1 and p_2 are both monomorphisms since putting $v' := v+J^3g$ and $u_i' := u_i+J^{s+3}f_i$, $\mathrm{soc}(Ag/J^3g)=J^2g/J^3g=Av'$ and $\mathrm{soc}(J^s f_i/J^{s+3}f_i)=J^{s+2}f_i/J^{s+3}f_i=Au_i'$ are simple modules and $(Av')p_i=Au_i'$ for each $i=1, 2$. In particular, Ag is uniserial by Theorem 1 (1).

i) In case $s\geqslant 2$. By the above,

$$Av' \xrightarrow{(p_1, p_2)} (J^s f_1/J^{s+3}f_1)\oplus(J^s f_2/J^{s+3}f_2)$$

is fusible. Also, $\mathrm{soc}^3(Af_i/J^{s+3}f_i)=J^s f_i/J^{s+3}f_i$ is uniserial. Hence

$$Av' \xrightarrow{(p_1, p_2)} (Af_1/J^{s+3}f_1)\oplus(Af_2/J^{s+3}f_2)$$

is fusible by (2.3.2), say 2-fusible. Then for some a in f_1Af_2, the diagram

$$\begin{array}{ccc} Av' & \xrightarrow{p_1} & Af_1/J^{s+3}f_1 \\ \downarrow & & \downarrow \text{right multiplication by } a \\ Av' & \xrightarrow{p_2} & Af_2/J^{s+3}f_2 \end{array}$$

is commutative. Therefore $u_2'=u_1'a$. Putting $\bar{u}_i := u_i+eJ^{s+3}$ for each $i=1, 2$, we have $\bar{u}_2=\bar{u}_1a$ since u_2 is in $u_1a+eJ^{s+3}f_2$. Thus $\bar{u}_2A\leqslant\bar{u}_1A$. This contradicts the linear independency of $\bar{u}_1A$ and $\bar{u}_2A$.

ii) In case the base field k is algebraically closed. It remains only the case $s=1$. Similarly, it holds that

$$Av' \xrightarrow{(p_1, p_2)} (Jf_1/J^4f_1)\oplus(Jf_2/J^4f_2)$$

is fusible. But since $0 \neq u_i \in eJ^3f_i \leqslant J^3f_i$ for each $i=1, 2$, $h(Af_i) \geqslant 4$ and then Af_i/J^4f_i is uniserial of length 4 and $Jf_i/J^4f_i = \mathrm{soc}^3(Af_i/J^4f_i)$ by Theorem 1 (5). Then

$$Av' \xrightarrow{(p_1, p_2)} (Af_1/J^4f_1) \oplus (Af_2/J^4f_2)$$

is fusible by (2.3.2). Hence by the same argument as in i) we have a contradiction. //

4. *QF* rings of right 2nd local type

Lemma 4.1. *Let A be a QF ring and e and f be in* pi(A) *such that $fJe/fJ^2e \neq 0$. Then*

(a) *If Je/J^2e is simple, then $h(Af) \geqslant h(Ae)$; and*

(b) *If fJ/fJ^2 is simple, then $h(eA) \geqslant h(fA)$.*

Proof. (a). It follows from the fact that Je/J^2e is simple and $fJe/fJ^2e \neq 0$ that there is an epimorphism $p\colon Af \to Je$. If p is a monomorphism, then Je is injective and is a direct summand of Ae. Thus $Je=0$ for Je is small in Ae. But this is impossible since Je/J^2e is simple. Therefore $\mathrm{Ker}\, p \geqslant \mathrm{soc}\, Af = J^{h(Af)-1}f$ since Af is colocal. Hence $h(Af) \geqslant h(Je)+1 = h(Ae)$.

(b) Similar. //

4.2. Proof of Theorem 2. Let $(x)'$ be the left side version of (x) for each $x=1, 3$. We show the following implications: (1)$\Rightarrow$(3)$'\Leftrightarrow$(3)$\Rightarrow$(6)$\Rightarrow$(4)$\Rightarrow$(5)$\Rightarrow$(1). Note that (2)$\Leftrightarrow$(1)$'$ is clear since A has a selfduality. Denote by D the selfduality $\mathrm{Hom}_A(?, A)$ of A.

(1)$\Rightarrow$(3)$'$. Let e be in pi(A) and $h := h(Ae) \geqslant 4$. Then J^2e is a uniserial waist in Ae. Hence $\mathrm{soc}^2 eA = D(Ae/J^2e)$ is a waist in $eA = D(Ae)$ and $\mathrm{soc}^2 eA = eJ^{h-2}$ is a direct sum of local modules for $h-2 \geqslant 2$. But since $eJ^{h-2} \leqslant eA$ and eA is colocal, eJ^{h-2} is local. Hence $|Je/J^2e| = |\mathrm{soc}^2(eA)/\mathrm{soc}(eA)| = 1$ and Ae is uniserial.

(3)$'\Leftrightarrow$(3). Clear from the fact that both height and uniseriality are preserved by D.

(3)$\Rightarrow$(6). By the equivalence (3)$\Leftrightarrow$(3)$'$ and left-right symmetry, it is sufficient to prove that under the assumption (3)$'$, if A is an indecomposable ring and $J^3 \neq 0$, then A is a left serial ring. Let Q be the left quiver of A, namely the oriented graph with vertex set $\{1, \cdots, p\}$ where pi$(A) = \{e_1, \cdots, e_p\}$ and with n_{ji} arrows $i \to j$ iff $\dim_{(e_jAe_j/e_jJe_j)}(e_jJe_i/e_jJ^2e_i) = n_{ji}$. Note that A is an indecomposable ring iff Q is connected. It follows from $J^3 \neq 0$ that $h(Ae_i) \geqslant 4$ for some $i=1, \cdots, p$ and then Ae_i is uniserial by (3)$'$. By 4.1 and the selfduality D, we have $h(Ae_j) \geqslant h(Ae_i)$ ($\geqslant 4$) if either

(a) there is an arrow $i \to j$; or

(b) there is an arrow $j \to i$.

Hence Ae_j is uniserial of height$\geqslant 4$ for any $j=1, \cdots, p$ by (4.1), (3)′ and the fact that Q is connected. Thus A is a left serial ring.

(6)$\Rightarrow$(4). Clear from the fact that for a serial ring A, A is QF iff the admissible sequence of A is constant.

(4)$\Rightarrow$(5). Let M_A be indecomposable of height $h \geqslant 3$. Then A/J^h is QF by (4). Let $0 \to K \hookrightarrow \bigoplus_{i=1}^{m} P_i \to M \to 0$ be a projective cover of M over A/J^h with each P_i indecomposable. Then $\operatorname{soc}(\bigoplus_{i=1}^{m} P_i) \nleqslant K$ implies that soc $P_i \nleqslant K$ for some $i=1, \cdots, m$ and then $P_i \cap K=0$ since P_i is colocal. Hence P_i is embedded into M. But since P_i is injective, P_i is isomorphic to a direct summand of M. Hence $P_i \simeq M$ for M is indecomposable. Further $P_i \simeq eA/eJ^h$ for some e in pi(A).

(5)$\Rightarrow$(1). Clear. //

5. Examples

In this section, we give some examples using bounden quiver algebras over an algebraically closed field k. (See Gabriel [8] for details concerning bounden quiver algebras.)

EXAMPLE 1. Let A be the algebra defined by the following bounden quiver:

$$\alpha \circlearrowright 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2; \quad \beta\alpha = \alpha\gamma = 0 ,\ \alpha^2 = \gamma\beta ,$$

namely, the algebra having $\{e_1, e_2, \alpha, \beta, \gamma, \gamma\beta, \beta\gamma\}$ as k-basis and with multiplication given by the following table:

left \ right	e_1	e_2	α	β	γ	$\gamma\beta$	$\beta\gamma$
e_1	e_1		α		γ	$\gamma\beta$	
e_2		e_2		β			$\beta\gamma$
α	α		$\gamma\beta$				
β	β				$\beta\gamma$		
γ		γ		$\gamma\beta$			
$\gamma\beta$	$\gamma\beta$						
$\beta\gamma$		$\beta\gamma$					

(each blank is zero).

Then A is weakly symmetric and hence QF. Further as easily seen, A has cube-zero radical. Therefore A is of right (and left) 2nd local type by Theorem 2. But since A is not a serial ring, A is neither of right (1st) local type nor of left (1st) local type.

Example 2. Let A be the algebra defined by the following quiver Q:

$$\begin{array}{ccccccc} & & 5 & & & & \\ & & \big\downarrow{\scriptstyle\delta} & & & & \\ 4 & \xrightarrow{\ \gamma\ } & 1 & \xleftarrow{\ \alpha\ } & 2 & \xleftarrow{\ \beta\ } & 3\,, \end{array}$$

namely, the algebra having $\{e_1, e_2, e_3, e_4, e_5, \alpha, \beta, \gamma, \delta, \alpha\beta\}$ as k-basis with multiplication given by the following table:

left \ right	e_1	e_2	e_3	e_4	e_5	α	β	γ	δ	$\alpha\beta$
e_1	e_1					α		γ	δ	$\alpha\beta$
e_2		e_2					β			
e_3			e_3							
e_4				e_4						
e_5					e_5					
α		α					$\alpha\beta$			
β			β							
γ				γ						
δ					δ					
$\alpha\beta$			$\alpha\beta$							

(each blank is zero).

Then as easily verified, A satisfies all the conditions stated in Theorem 1. But it is not of right 2nd local type. For instance, let M be the right A-module corresponding to the following k-representation of Q^{op} (the opposite quiver of Q, with all arrows reversed)

$$\begin{array}{ccccccc} & & k & & & & \\ & & \big\uparrow{\scriptstyle(1,0)} & & & & \\ k & \xleftarrow[(0,1)]{} & k\oplus k & \xrightarrow[\begin{pmatrix}1&0\\0&1\end{pmatrix}]{} & k\oplus k & \xrightarrow[(1,1)]{} & k \end{array}$$

namely, the module having $\{m_1, m_1', m_2, m_2', m_3, m_4, m_5\}$ as k-basis and with right A-action given by the following table:

	e_1	e_2	e_3	e_4	e_5	α	β	γ	δ	$\alpha\beta$
m_1	m_1					m_2			m_4	m_3
m_1'	m_1'					m_2'		m_4		m_3
m_2		m_2					m_3			
m_2'		m_2'					m_3			
m_3			m_3							
m_4				m_4						
m_5					m_5					

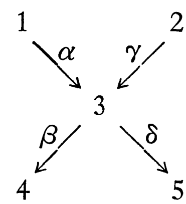
(each blank is zero).

Then M is indecomposable but $\mathrm{top}^2 M$ is decomposable:

$$\mathrm{top}^2 M = \left[\begin{array}{ccccccc} & & 0 & & & & \\ & & \uparrow & & & & \\ k & \underset{1}{\leftarrow} & k & \underset{1}{\rightarrow} & k & \rightarrow & 0 \end{array}\right] \oplus \left[\begin{array}{ccccccc} & & k & & & & \\ & & \uparrow 1 & & & & \\ 0 & \leftarrow & k & \underset{1}{\rightarrow} & k & \rightarrow & 0 \end{array}\right].$$

Hence the conditions stated in Theorem 1 are not sufficient for algebras (even if k is algebraically closed) to be of right 2nd local type.

EXAMPLE 3. Let A be the algebra defined by the following bounded quiver:



$$; \quad \beta\alpha = \delta\gamma = 0 .$$

Then we can see that A has just 13 indecomposable left modules (up to isomorphism), all of which have indecomposable second tops and second socles since the indecomposable left A-modules of height$\geqslant 3$ are both projective and injective. Hence A is of right and left 2nd local type.[1)] But it is neither of right (1st) local type nor of left (1st) local type. For instance, let M_1 and M_2 be the left A-modules corresponding to the following k-representations of the bounden quiver:

1) In Part II of this series of papers, we shall give some necessary and sufficient conditions for artinian rings to be of right and left n-th local type for any natural number n. Using this result, it is clear that the algebra defined in Example 3 is of right and left 2nd local type.

$$\begin{array}{ccccc} 0 & & & & 0\\ & \searrow & & \swarrow & \\ & & k & & \\ & {}^{1}\swarrow & & \searrow{}^{1} & \\ k & & & & k \end{array} \quad\text{and}\quad \begin{array}{ccccc} k & & & & k\\ & {}^{1}\searrow & & \swarrow{}^{1} & \\ & & k & & \\ & \swarrow & & \searrow & \\ 0 & & & & 0 \end{array}\ , \text{ respectively.}$$

Then M_1 and M_2 are indecomposable but M_1 is not colocal and M_2 is not local.

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