Osaka University Knowledge Archive

| | |
|---|---|
| Title | On algebras of second local type. I |
| Author(s) | Asashiba, Hideto |
| Citation | Osaka Journal of Mathematics. 1984, 21(2), p. 327-342 |
| Version Type | VoR |
| URL | https://doi.org/10.18910/8987 |
| rights | |
| Note | |

*Osaka University Knowledge Archive : OUKA*

https://ir.library.osaka-u.ac.jp/

Osaka University

Asashiba, H.
Osaka J. Math.
21 (1984), 327-342

# ON ALGEBRAS OF SECOND LOCAL TYPE, I

HIDETO ASASHIBA

(Received December 20, 1982)

Throughout this paper, $A$ denotes a (left and right) artinian ring with identity 1, $J$ its Jacobson radical and all modules are (unital and) finitely generated.

Let $n$ be any natural number. Then we say that $A$ is of *right n-th local type* in case for every indecomposable right $A$-module $M$, the $n$-th top $\mathrm{top}^n M := M/MJ^n$ of $M$ is indecomposable. (Note that if $\mathrm{top}^n M$ is indecomposable, then so is $M$ since $A$ is artinian and $M$ is finitely generated.) Hence for such a ring $A$, the question of indecomposability of right $A$-modules can be reduced to the corresponding problem of right $A/J^n$-modules. In [11] H. Tachikawa has studied the case $n=1$ and obtained a necessary and sufficient condition for algebras (by algebra we always mean a finite dimensional algebra over a field $k$) to be of this type. Further the representation theory of algebras with square-zero radical is well known [5], [6], [7]. So in this paper, we examine the case $n=2$ and give some necessary conditions for rings with selfduality to be of this type. Further in particular for $QF$ (=quasi-Frobenius) rings, we give necessary and sufficient conditions to be of this type. More precisely, we show the following two theorems:

**Theorem 1.** *Let $A$ be a ring with selfduality which is of right 2nd local type and $e$ any primitive idempotent in $A$. Then*

(1) *$J^2e$ is a uniserial waist in $Ae$ if $J^2e \neq 0$* (see section 2 for definition of a waist),

(2) *$eJ^m$ is a direct sum of local modules for every $m \geqslant 2$,*

(3) *for each local direct summand $L$ of $eJ^2$, $LJ^2$ is uniserial (thus $eJ^4$ is a direct sum of uniserial modules).*

*Further if $A$ is an algebra, we have*

(4) *$Ae$ is uniserial if $h(Ae) \geqslant 5$.*

*In particular if the base field $k$ is, in addition, an algebraically closed field, then*

(5) *$Ae$ is uniserial if $h(Ae) \geqslant 4$,*

*and then*

(6) *$eJ^2$ is a direct sum of uniserial modules.*

**Theorem 2.** *Let $A$ be a QF ring. Then the following statements are equivalent:*

(1) *$A$ is of right 2nd local type.*

(2) *$A$ is of right 2nd colocal type* (see section 1 for definition).

(3) *For any primitive idempotent $e$ in $A$, $eA$ is uniserial if $h(eA)\geqslant 4$.*

(4) *$A/J^t$ is QF for every $t\geqslant 3$.*

(5) *For each $M_A$ indecomposable with $h(M)\geqslant 3$, there is a primitive idempotent $e$ in $A$ such that $M\cong eA/eJ^{h(M)}$.*

(6) *$A=A_1\times A_2$ for some QF rings $A_1$ and $A_2$ such that $A_1$ has cube-zero radical and $A_2$ is a serial ring.*

*Furthermore, each of these conditions are equivalent to the corresponding left side version.*

In the theorems above $h(M)$ denotes the height (=Loewy length) of $M$, namely $h(M):=\min\{n\in N\cup\{0\}\mid MJ^n=0\}$. We remark that Theorem 1 (5) and (6) remain valid also in the case where $k$ is a splitting field for $A$.

In section 1, we introduce the basic tools used in the following sections. Section 2 is devoted to the structure of an indecomposable projective left module and in section 3, we examine the structure of an indecomposable projective right module mainly using the technique of Sumioka [10]. In section 4, we give the proof of Theorem 2. Finally in section 5, we give some examples.

The author would like to thank Professor T. Sumioka and Dr. T. Okuyama for fruitful conversations.

## 1. Preliminaries

**1.1.** Throughout the paper, we write homomorphisms on the opposite side to scalar multiplications, and for homomorphisms $p$: $K\to L$ and $q$: $L\to M$ of left $A$-modules and for a decomposition $D$: $L=\bigoplus_{i=1}^{n}L_i$ of $L$, $(p, D)=(p_i)_{n=1}^{i}$ and $(D, q)=(q_i)_{i=1}^{T\,n}$ are matrix expressions of $p$ and $q$ relative to $D$, respectively (for homomorphisms of right $A$-modules, we write as $(p, D)=(p_i)_{i=1}^{T\,n}$ and $(D, q)=(q_i)_{i=1}^{n}$). In addition to the definition of right $n$-th local type for $n$ any natural number, we define the dual notion: $A$ is called to be of *left $n$-th colocal type* in case for every indecomposable left $A$-module $M$, the $n$-th socle $\mathrm{soc}^n M:=$ (the right annihilator of $J^n$ in $M$) of $M$ is indecomposable. It should be noted that if $A$ has a selfduality, then $A$ is of right $n$-th local type iff $A$ is of left $n$-th colocal type. Further noting that the composition lengths of the projective covers (over $A$) of all indecomposable right $A/J^n$-modules have a bound if $A/J^n$ is of finite representation type (i.e. it has only finitely many isomorphism classes of indecomposable right modules), we see easily that when $A$ is of right $n$-th local type, $A$ is of finite representation type iff so is $A/J^n$ (See

Auslander [3]).

Since the property to be of $n$-th local (colocal) type is Morita invariant, we may assume that $A$ is a basic ring. We put $\mathrm{pi}(A):=\{e_1, \cdots, e_p\}$ to be a basic set of primitive idempotents of $A$.

DEFINITION 1.2 ([2]). Let $D$: $L=\bigoplus_{i=1}^{n} L_i$ be a decomposition of a right $A$-module $L$ and $p$: $K\to L$ be a homomorphism, and $j$ in $\{1, \cdots, n\}$. Then the pair $(p, D)$ (or simply $p$: $K\to\bigoplus_{i=1}^{n} L_i$) is called *$j$-fusible* in case there is a homomorphism $q$: $\bigoplus_{i\neq j} L_i\to L_j$ such that the diagram

$$\begin{tikzcd} K \arrow[r, "(p_i)_{i\neq j}"] \arrow[d, equal] & \bigoplus_{i\neq j} L_i \arrow[d, "q"] \\ K \arrow[r, "p_j"] & L_j \end{tikzcd}$$

commutes where $(p, D)=(p_i)_{i=1}^{T,n}$. The pair $(p, D)$ is called *fusible* in case $(p, D)$ is $j$-fusible for some $j=1, \cdots, n$. Finally $(p, D)$ is called *infusible* in case $(p, D)$ is not fusible.

**Corollary 1.2.1** ([2, Corollary 1.4]). *Let $K_i \lneqq L_i$ for each $i=1, 2$ and $h$: $K_1\to K_2$ be an isomorphism. Define $p_1=k_1$, $p_2=k_2h$ where $k_i$: $K_i\to L_i$ is the inclusion map for each $i$. Then $h$ or $h^{-1}$ is extendable to a homomorphism $L_1\to L_2$ or $L_2\to L_1$, respectively iff $p$: $K_1\to L_1\oplus L_2$ is fusible.* //

**Proposition 1.2.2** ([2, Proposition 1.1]). *Consider an exact sequence $K\xrightarrow{p} L \xrightarrow{q} M\to 0$ of right $A$-modules and let $D$: $L=\bigoplus_{i=1}^{n} L_i$ be a decomposition of $L$, $(p, D)=(p_i)_{i=1}^{T,n}$, $(D, q)=(q_i)_{i=1}^{n}$ and $j$ in $\{1, \cdots, n\}$. Then the following statements are equivalent*:

(1) *$(p, D)$ is $j$-fusible.*

(2) *There is a homomorphism $r=(r_i)_{i=1}^{n}$: $\bigoplus_{i=1}^{n} L_i\to X$ such that $rp=0$ and $r_j$ is an isomorphism.*

(3) *$q_j$ is a split monomorphism.*

Proof. See [2]. //

REMARK. In [2] the fusible maps were defined by the condition (2) above.

**Proposition 1.3** *Let $0\to K\xrightarrow{p} L\xrightarrow{q} M\to 0$ be a nonsplit exact sequence of right $A$-modules and $D$: $L=\bigoplus_{i=1}^{n} L_i$ be a decomposition of $L$ $(n\geqslant 2)$. Then we have*

(1) *if $M$ is indecomposable, then $(p, D)$ is infusible,*

(2) *if $K$ is simple, each $L_i$ is local and $(p, D)$ is infusible, then $M$ is indecomposable.*

Proof. See [1] or [2]. //

**1.4.** Let $I$ be a two-sided ideal of $A$ and $e$ and $f$ in $\mathrm{pi}(A)$. Then we have the canonical isomorphisms $\mathrm{Hom}_A(fA, eA/eI) \xrightarrow{\sim} eAf/eIf \xrightarrow{\sim} \mathrm{Hom}_A(Ae, Af/If)$. We denote by $p^*$ the image of every $p$ in $\mathrm{Hom}_A(fA, eA/eI)$ or the inverse image of every $p$ in $\mathrm{Hom}_A(Ae, Af/If)$ under the composition of these isomorphisms.

**Proposition 1.4.1** *Let $e, f_1, \cdots, f_n$ be in $\mathrm{pi}(A)$, $l>m$, $j$ in $\{1, \cdots, n\}$ and $p=(p_i)_{i=1}^n: \bigoplus_{i=1}^n f_iA \to eJ^m/eJ^l$ be a homomorphism. Then the following statements are equivalent:*

(1) $p(f_jA) \leqslant \sum_{i \neq j} p(f_iA)$.

(2) $p^*: Ae/J^{l-m}e \to \bigoplus_{i=1}^n Af_i/J^l f_i$ *is $j$-fusible, where $p^*$ is the map induced by the homomorphism $(p_i^*)_{i=1}^n$.*

Proof. There is some $u_i$ in $eJ^m f_i$ such that each $p_i^*$ is the left multiplication by $u_i$. Then $p$ has the property stated in (1) iff $(u_jf_jA+eJ^l)/eJ^l \leqslant (\sum_{i \neq j} u_if_iA + eJ^l)/eJ^l$

iff $u_jA \leqslant \sum_{i \neq j} u_iA + eJ^l$

iff $u_j = \sum_{i \neq j} u_ia_i + b$, for some $a_i$ in $f_iA$ and $b$ in $eJ^l$

iff $u_j = \sum_{i \neq j} u_ia_i + b$, for some $a_i$ in $f_iAf_j$ and $b$ in $eJ^lf_j$

iff $u_j = \sum_{i \neq j} u_ia_i + b$, for some $a_i$ in $f_iAf_j$ and $b$ in $J^lf_j$

iff $p^*$ is $j$-fusible. //

In future $p^*$ shall always mean the above induced homomorphism when the domain of $p$ is of the form as above.

**Corollary 1.4.2.** *Under the same situation as above but $l=m+1$, the following are equivalent:*

(1) $\bar{p}: \bigoplus_{i=1}^n f_iA/f_iJ \to eJ^m/eJ^{m+1}$ *(the induced map) is a monomorphism.*

(2) $p^*: Ae/Je \to \bigoplus_{i=1}^n Af_i/J^{m+1}f_i$ *is infusible.*

*In particular if $p: \bigoplus_{i=1}^n f_iA \to eJ^m$ is a projective cover of $eJ^m$, then $p^*: Ae/Je \to \bigoplus_{i=1}^n Af_i/J^{m+1}f_i$ is infusible.* //

**Corollary 1.4.3.** *Let $p: \bigoplus_{i=1}^n f_iA \to eJ^m$ be a projective cover of $eJ^m$ and $0 \to$*

$Ae/Je \xrightarrow{p^*} \bigoplus_{i=1}^{n} Af_i/J^{m+1}f_i \to M \to 0$ *be an exact sequence. Then $M$ is indecomposable.*

Proof. Clear from (1.4.2) and (1.3). //

## 2. Structure of an indecomposable projective left module

For an $A$-module $M$, we put $|M| :=$ the composition length of $M$.

**Proposition 2.1.** *Let $A$ be of right $n$-th local type, $n$ any natural number and $e$ in* $\mathrm{pi}(A)$. *Then $J^n e$ is uniserial.*

Proof. It is sufficient to prove that $|J^m e/J^{m+1}e| \leqslant 1$ for every $m \geqslant n$. Suppose $|J^m e/J^{m+1}e| \geqslant 2$ for some $m \geqslant n$. Then we have a homomorphism $p: Af_1 \oplus Af_2 \to J^m e/J^{m+1}e$; $f_1, f_2$ in $\mathrm{pi}(A)$ such that the induced map $\bar{p}: (Af_1/Jf_1) \oplus (Af_2/Jf_2) \to J^m e/J^{m+1}e$ is a monomorphism. Putting $L = (f_1A/f_1J^{m+1}) \oplus (f_2A/f_2J^{m+1})$, we have an exact sequence $0 \to eA/eJ \xrightarrow{p^*} L \to M \to 0$ where $M$ is indecomposable by (1.4.2) and (1.3). But since $p^*(eA/eJ) \leqslant LJ^m \leqslant LJ^n$, $\mathrm{top}^n M \cong \mathrm{top}^n L$ is decomposable. This is a contradiction. //

DEFINITION 2.2 ([4]). Let ${}_AL \leqslant {}_AM$. Then $L$ is called to be a *waist* in $M$ in case $0 \neq L \neq M$ and for each ${}_AN \leqslant {}_AM$, it holds that $L \leqslant N$ or $N \leqslant L$.

**Proposition 2.2.1.** *Let $A$ be a ring with selfduality which is of right 2nd local type and $e$ in* $\mathrm{pi}(A)$. *Then $J^2 e$ is a waist in $Ae$ if $J^2 e \neq 0$.*

Proof. Deduced from the following three lemmas for an artinian ring $A$:

**Lemma 2.2.2** ([9, Lemma 1.2]). *Let ${}_AM$ be nonsimple indecomposable. Then* $\mathrm{soc}(JM) = \mathrm{soc}\, M$.

Proof. Let $S$ be any simple submodule of $M$ and $X$ be any proper submodule of $M$. If $S + X = M$ then $S$ is not contained in $X$. Thus $S \cap X = 0$. Hence $S = M$, a contradiction. Therefore $S$ is small in $M$ i.e. $S \leqslant JM$. Hence $\mathrm{soc}\, M \leqslant JM$ and $\mathrm{soc}\, M = \mathrm{soc}(JM)$. //

**Lemma 2.2.3.** *Let ${}_AM$ be local and* $\mathrm{soc}^2 M$ *indecomposable. Then* $\mathrm{soc}(J^2M) = \mathrm{soc}\, M$ *if $J^2M \neq 0$.*

Proof. Clear from (2.2.2) nothing that $JM$ is nonsimple indecomposable since $J^2M \neq 0$ and $\mathrm{soc}^2 M \leqslant JM$. //

**Lemma 2.2.4.** *Let $A$ be a ring of left 2nd colocal type, ${}_AM$ be local and $J^2M$ be a nonzero uniserial module. Then $J^2M$ is a waist in $M$.*

Proof. Suppose that $J^2M$ is not a waist in $M$. Then for some $X \leqslant M$,

$J^2M \nleqslant X$ and $X \nleqslant J^2M$. And, $J^2M \cap X = J^tM$ for some $t \geqslant 3$. Hence $M/J^t \geqslant (J^2M/J^tM) \oplus (X/J^tM)$ where $J^2M/J^tM \neq 0$ and $X/J^tM \neq 0$. On the other hand since $\mathrm{soc}^2(M/J^tM)$ is indecomposable and $J^2(M/J^tM) \neq 0$, we have that $\mathrm{soc}(M/J^tM) = \mathrm{soc}(J^2M/J^tM)$ is simple by (2.2.3). This is a consradiction. //

We get Theorem 1 (1) from Propositions 2.1 and 2.2.1.

**Corollary 2.2.5.** *Let $A$ be a ring with selfduality which is of right 2nd local type, $e$ in* $\mathrm{pi}(A)$ *and $h = h(Ae)$. Then we have* $\mathrm{soc}^{h-t}(Ae) = J^te$ *for every* $t = 0, \cdots, h$.

Proof. It is clear from Theorem 1 (1) in case $t \geqslant 2$. The other cases ($t = 0, 1$) are trivial.

**Lemma 2.3.1.** *Let ${}_AL_1$ and ${}_AL_2$ be local of height $\geqslant 3$ such that for each $i = 1, 2$, $\mathrm{soc}^3L_i$ is uniserial and $J^2e_i$ is a uniserial waist in $Ae_i$ where $Ae_i$ is the projective cover of* $\mathrm{soc}^3L_i$. *Suppose that ${}_AK$ is simple and there exists an isomorphism $p_i$: $K \to \mathrm{soc}\, L_i$ for each $i = 1, 2$. Consider an exact sequence*:

$$0 \to K \xrightarrow{p=(p_1, p_2)} L_1 \oplus L_2 \xrightarrow{q=\begin{bmatrix} q_1 \\ -q_2 \end{bmatrix}} M \to 0 .$$

*Then* $\mathrm{soc}^2M$ *is decomposable if $p$: $K \to \mathrm{soc}^2L_1 \oplus \mathrm{soc}^2L_2$ is fusible.*

Proof. Assume that $p$: $K \to \mathrm{soc}^2L_1 \oplus \mathrm{soc}^2L_2$ is fusible, say 2-fusible. Then we have a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{p_1} & \mathrm{soc}^2L_1 \\ \| & & \downarrow r \\ K & \xrightarrow{p_2} & \mathrm{soc}^2L_2 . \end{array}$$

And, $M \geqslant (\mathrm{soc}^2L_1)q_1 + L_2q_2 = U \oplus L_2q_2$ where $U = (\mathrm{soc}^2L_1)\ (q_1 - rq_2) \neq 0$. Now for each $x$ in $\mathrm{soc}^2M$, $x = l_1q_1 + l_2q_2$ for some $(l_1, l_2)$ in $L_1 \oplus L_2$. Since $ux = 0$ for each $u$ in $J^2$, we have $ul_1q_1 = -ul_2q_2 \in L_1q_1 \cap L_2q_2 = Kp_1q_1$ $(=: S)$. Hence $J^2l_1q_1 = J^2l_2q_2 \leqslant S$ where $S$ is simple. In particular, $\mathrm{soc}^2M \leqslant \mathrm{soc}^3L_1q_1 + \mathrm{soc}^3L_2q_2$.

i) In case for each $x$ in $\mathrm{soc}^2M$, there are $l_1$, $l_2$ with $x = l_1q_1 + l_2q_2$ such that $J^2l_1q_1 = J^2l_2q_2 = 0$. Then we have $J^2l_1 = 0$ for $q_1$ is monic. Thus $l_1$ is in $\mathrm{soc}^2L_1$ and $x$ is in $U \oplus L_2q_2$. Therefore $\mathrm{soc}^2M \leqslant U \oplus L_2q_2$. Hence $\mathrm{soc}^2M$ is decomposable.

ii) In case for some $x$ in $\mathrm{soc}^2M$, there are $l_1$, $l_2$ with $x = l_1q_1 + l_2q_2$ such that $J^2l_1q_1 = J^2l_2q_2 = S$. We may assume that $x = ex$ for some $e$ in $\mathrm{pi}(A)$. Since $S$ is simple and $q_i$ are monic, $J^3l_1 = J^3l_2 = 0$. Thus $l_i$ is in $\mathrm{soc}^3L_i \backslash \mathrm{soc}^2L_i$ for each $i$. Also, we may assume that $l_i = el_i$ for each $i$ since $x = ex$. Further we have $\mathrm{soc}^3L_i = Ael_i$ for each $i = 1, 2$ since $\mathrm{soc}^3L_i$ are uniserial. Hence we

may assume that $e=e_1=e_2$. Define a homomorphism $s$: $\mathrm{soc}^3L_1\to\mathrm{soc}^3L_2$ by $ael_1\mapsto ael_2$ for each $a$ in $A$. Then $s$ is well-defined. In fact, if $t$ is in $Ae$ and $tl_1=0$, then $t$ is in $\mathrm{Ann}_{Ae}(l_1)$, the annihilator of $l_1$ in $Ae$. On the other hand, by the fact that $J^2el_1\neq 0$, we see $\mathrm{Ann}_{Ae}(l_1)$ does not contain $J^2e$ which is a uniserial waist in $Ae$. Hence $\mathrm{Ann}_{Ae}(l_1)$ is contained in $J^3e$ and $t$ is in $J^3e$. Thus $tl_2$ is in $J^3l_2=0$.

Further the diagram

$$\begin{tikzcd} K \arrow[r, "p_1"] \arrow[d, equal] & \mathrm{soc}^3L_1 \arrow[d, "s"] \\ K \arrow[r, "p_2"] & \mathrm{soc}^3L_2 \end{tikzcd}$$

is commutative. For, $J^2(l_1, l_2)$ $(\neq 0)$ is contained in the simple module $\mathrm{Im}\,p$ since $J^2(l_1, l_2)q=0$. Hence $J^2(l_1, l_2)=\mathrm{Im}\,p$. Let $c$ be a nonzero element in $K$. Then $K=Ac$ and $cp=(ul_1, ul_2)$ for some $u$ in $J^2$. Therefore $c(p_1s)=ul_1s=ul_2=cp_2$. Thus $p_1s=p_2$.

Then putting $V:=(\mathrm{soc}^3L_1)\,(q_1-sq_2)$, the same argument as in i) shows that $\mathrm{soc}^2M\leqslant V\oplus L_2q_2$ and $\mathrm{soc}^2M$ is decomposable. //

**Proposition 2.3.2.** *Let $A$ be a ring with selfduality which is of right 2nd local type and ${}_AL_1$, ${}_AL_2$ be local of height $\geqslant 3$ such that $\mathrm{soc}^3L_i$ are uniserial and $|L_1|\leqslant|L_2|$. Then for every isomorphism $r$: $\mathrm{soc}L_1\to\mathrm{soc}L_2$, $r$ is extendable to a monomorphism $L_1\to L_2$ if $r$ is extendable to a homomorphism $\mathrm{soc}^2L_1\to\mathrm{soc}^2L_2$.*

Proof. Put $K=\mathrm{soc}\,L_1$, $p_1=$identity map of $\mathrm{soc}\,L_1$ and $p_2=r$. Consider an exact sequence $0\to K\xrightarrow{p=(p_1,\,p_2)}L_1\oplus L_2\xrightarrow{q}M\to 0$. If $r$ is extendable to a homomorphism $\mathrm{soc}^2L_1\to\mathrm{soc}^2L_2$, then $p$: $K\to\mathrm{soc}^2L_1\oplus\mathrm{soc}^2L_2$ is fusible. Hence by (2.3.1), $\mathrm{soc}^2M$ is decomposable, thus $M$ is decomposable. Therefore $p$: $K\to L_1\oplus L_2$ is fusible by (1.3). Hence by (1.2.1), $r$ is extendable to a homomorphism $q$: $L_1\to L_2$ since $|L_1|\leqslant|L_2|$ where $q$ is monic since $\mathrm{soc}\,L_1$ is simple. //

**2.4.** Throughout the rest of this section, $A$ is a ring with selfduality which is of right 2nd local type. Here, we examine indecomposable projective left $A$-modules of height $\geqslant 4$.

**Proposition 2.4.1.** *Let $e$ and $f$ be in $\mathrm{pi}(A)$ and $fJe/fJ^2e\neq 0$. Then $Af$ is uniserial if $h(Ae)\geqslant 4$.*

Proof. Take some $u$ in $fJe\backslash fJ^2e$ and define $p$: $Af\to Je$ by the right multiplication by $u$. Then $\mathrm{Ker}\,p\leqslant J^2f$ or $\mathrm{Ker}\,p\geqslant J^2f$ since $J^2f$ is a waist in $Af$ (if $J^2f\neq 0$). Assume that $\mathrm{Ker}\,p\geqslant J^2f$. Then $h(\mathrm{Im}\,p)\leqslant 2$ since $\mathrm{Im}\,p\cong Af/\mathrm{Ker}\,p$ is an epimorph of $Af/J^2f$. Hence $\mathrm{Im}\,p\leqslant\mathrm{soc}^2(Ae)\leqslant J^2e$ for $h\,(Ae)\geqslant 4$ and $\mathrm{soc}^2(Ae)=J^{h(Ae)-2}e$. But by the definition of $p$ we have $\mathrm{Im}\,p\leqslant J^2e$, a contradiction.

Accordingly, $\operatorname{Ker} p \leqslant J^2 f$. Then $\operatorname{Ker} p = J^t f$ for some $t \geqslant 2$ and $Af/J^t f$ is embedded into $Je$. Therefore $|Jf/J^2 f| = 1$ since $Jf/J^t f$ is embedded into $J^2 e$ which is uniserial. Hence $Af$ is uniserial. //

**Proposition 2.4.2.** *Assume that $e$ is in* $\mathrm{pi}(A)$, $h(Ae) \geqslant 4$ *and $Ae$ is not uniserial. Then*

(1) *all simple submodules of $Je/J^2 e$ are pairwise isomorphic, and*

(2) $J^2 e/J^3 e \cong J^3 e/J^4 e$.

Proof. Let $u: \bigoplus_{i=1}^{n} Af_i \to Je/J^4 e$ be a projective cover of $Je/J^4 e$. Then $n \geqslant 2$ since $Ae$ is not uniserial. Putting $L_i := (Af_i)u$, we have $L_i \cap L_j = J^2 e/J^4 e$, $L_i \gneqq J^2 e/J^4 e$ for each $i \neq j$ in $\{1, \cdots, n\}$. By (2.4.1), each $L_i$ is uniserial and $h(L_i) = 3$. Further $\operatorname{soc} L_i = J^3 e/J^4 e$ is simple and $\operatorname{soc}^2 L_i = J^2 e/J^4 e$ for each $i = 1, \cdots, n$.

(1) For any $i \neq j$ in $\{1, \cdots, n\}$, the identity map $p$: $\operatorname{soc} L_i \to \operatorname{soc} L_j$ is extendable to a homomorphism $\operatorname{soc}^2 L_i \to \operatorname{soc}^2 L_j$ since $L_i \cap L_j = J^2 e/J^4 e = \operatorname{soc}^2 L_i = \operatorname{soc}^2 L_j$. Hence by (2.3.2), $p$ is extendable to an isomorphism $L_i \to L_j$. Thus all simple submodules of $Je/J^2 e$ are pairwise isomorphic.

(2) Putting $p_i$: $J^2 e/J^4 e \to L_i$ and $q_i$: $L_i \to L_1 + L_2$ to be inclusion maps for $i = 1, 2$, we have an exact sequence

$$0 \to J^2 e/J^4 e \xrightarrow{(p_1, p_2)} L_1 \oplus L_2 \xrightarrow{\begin{bmatrix} q_1 \\ -q_2 \end{bmatrix}} L_1 + L_2 \to 0$$

where $L_1 + L_2$ is colocal. Hence the identity map $r$: $\operatorname{soc}^2 L_1 \to \operatorname{soc}^2 L_2$ is not extendable to any isomorphism $L_1 \to L_2$. On the other hand, the identity map $p$: $\operatorname{soc} L_1 \to \operatorname{soc} L_2$ is extendable to an isomorphism $s$: $L_1 \to L_2$ since $r|(\operatorname{soc} L_1) = p$. As a consequence, $s|(\operatorname{soc}^2 L_1) \neq r$. But if $J^2 e/J^3 e \ncong J^3 e/J^4 e$, then the restriction map

$$\operatorname{Hom}_A(\operatorname{soc}^2 L_1, \operatorname{soc}^2 L_2) \to \operatorname{Hom}_A(\operatorname{soc} L_1, \operatorname{soc} L_2)$$

is an injection. This implies that $s|(\operatorname{soc}^2 L_1) = r$ since both $s|(\operatorname{soc}^2 L_1)$ and $r$ are extensions of $p$. This is a contradiction. //

**Proposition 2.4.3.** *Assume that $e$, $f$ and $g$ are in* $\mathrm{pi}(A)$, $h(Ae) \geqslant 5$, *$Ae$ is not uniserial, $fJe/fJ^2 e \neq 0$ and $J^2 e/J^3 e \cong Ag/Jg$. Then $fAf/fJf \cong gAg/gJg$ as rings.*

Proof. There exists a submodule $L$ of $Je/J^4 e$ such that $L$ is uniserial of height 3 and top $L \cong Af/Jf$, top $JL \cong Ag/Jg$. We identify these isomorphic modules. Further $Af$ and $Ag$ are both uniserial by (2.4.1) and the fact that $h(Ae) \geqslant 5$ and also $h(Af) \geqslant 4$. Then we can define a homomorphism $t$: $\operatorname{End}_A(Af/Jf) \to \operatorname{End}_A(Ag/Jg)$ by $t(p) := \overline{(q|Jf/J^3 f)}$ for each $p$ in $\operatorname{End}_A(Af/Jf)$ where $p$ is induced by some $q$ in $\operatorname{End}_A(Af/J^3 f)$ and $\bar{r}$ is the map in $\operatorname{End}_A(Jf/J^2 f)$ induced by $r$ for every $r$ in $\operatorname{End}_A(Jf/J^3 f)$. (We identified $\operatorname{End}_A(Jf/J^2 f) = \operatorname{End}_A(Ag/Jg)$.)

Then $t$ is well-defined and injective since for each $q$ in $\mathrm{End}_A(Af/J^3f)$, $(Af/J^3f)q \leqslant Jf/J^3f$ iff $(Jf/J^3f)q \leqslant J^2f/J^3f$ (See [10, section 3]). Further by (2.3.2), every automorphism $p$ of soc $L$ is extendable to an automorphism of $L$ if $p$ is extendable to an automorphism of $\mathrm{soc}^2L$. Thus $t$ is surjective. (Note that both $Af/J^3f$ and $Jf/J^3f$ are quasi-projective since we have $Jf/J^3f \cong Ag/J^2g$ from the fact that $Ag$ is uniserial.) Hence $fAf/fJf \cong \mathrm{End}_A(Af/Jf) \cong \mathrm{End}_A(Ag/Jg) \cong gAg/gJg$ as rings.

REMARK. In the above, if $A$ is a $k$-algebra, then the isomorphism defined as above is a $k$-algebra isomorphism.

**2.4.4.** Proof of Theorem 1 (4) and (5). Assume that $A$ is an algebra and suppose that $Ae$ is not uniserial, and $h(Ae) \geqslant 4$. Let $p$: $\bigoplus_{i=1}^{n} P_i \to Je/J^3e$ be a projective cover of $Je/J^3e$ where each ${}_AP_i$ is indecomposable. Then $n \geqslant 2$. By (2.4.2), there is an $f$ in pi$(A)$ such that every $P_i$ is isomorphic to $Af$. And, $J^2e/J^3e \cong Ag/Jg$ for some $g$ in pi$(A)$. If we put $L_i := (P_i)p$ for $i=1, 2$, then $L_i \cong Af/J^2f$, $J^2e/J^3e \lneqq L_i \leqslant Je/J^3e$, $L_1 \cap L_2 = J^2e/J^3e$ and top $L_i \cong Af/Jf$ for each $i=1, 2$. Since we have an exact sequence

$$0 \to J^2e/J^3e \to L_1 \oplus L_2 \to L_1 + L_2 \to 0$$

where $J^2e/J^3e \cong Ag/Jg$, $L_1 \oplus L_2 \cong (Af/J^2)^{(2)}$ and $L_1 + L_2$ is colocal, there exists an infusible homomorphism $Ag/Jg \to (Af/J^2f)^{(2)}$ by (1.3; 1). Therefore $(fA/fJ)^{(2)}$ is isomorphic to a direct summand of $gJ/gJ^2$ by (1.4.2). Hence $\dim (gJf/gJ^2f)_{fAf/fJf} \geqslant 2$. If $h(Ae) \geqslant 5$ or $k$ is algebraically closed, then by (2.4.3), $d := \dim_{gAg/gJg}(gJf/gJ^2f) = \dim(gJf/gJ^2f)_{fAf/fJf} \geqslant 2$. Hence $(Ag/Jg)^{(d)}$ is isomorphic to a direct summand of $Jf/J^2f$ and $d \geqslant 2$. Thus $|Jf/J^2f| \geqslant 2$. This contradicts the uniseriality of $Af$. Hence $Ae$ must be uniserial. //

## 3. Structure of an indecomposable projective right module

**Lemma 3.1.** *Let $0 \to K \xrightarrow{p} L \xrightarrow{q} M \to 0$ be an exact sequence of left $A$-modules such that $K$ is simple, $D$: $L = \bigoplus_{i=1}^{n} L_i$ is a decomposition of $L$ $(n \geqslant 2)$ and for each $i=1, \cdots, n$, $L_i = Ae_i/I_i$ for some $e_i$ in* pi$(A)$ *and $J^{m+1}e_i \leqslant I_i \lneqq J^m e_i$ $(m \geqslant 1)$. Then $JM = \mathrm{soc}^m M$ if $(p, D)$ is infusible.*

Proof. Put $l_i := e_i + I_i$, $\bar{l}_i = l_i + JL$, $m_i := l_i q$, $\bar{m}_i := m_i + JM$ and $m_i' := m_i + \mathrm{soc}^m M$. Then we have $\bigoplus_{i=1}^{n} A\bar{l}_i = L/JL \cong M/JM = \bigoplus_{i=1}^{n} A\bar{m}_i$ where each $A\bar{m}_i$ is simple. It follows from $h(M) \leqslant m+1$ that $JM \leqslant \mathrm{soc}^m M$. Assume that $JM \lneqq \mathrm{soc}^m M$. Then we show that $(p, D)$ is fusible. (Clearly, we may assume that each $p_i \neq 0$ i.e. each $p_i$ is a monomorphism where $(p, D) = (p_i)_{i=1}^{n}$.) By

assumption the sum $M/\mathrm{soc}^m M=\sum_{i=1}^{n} Am'_i$ is redundant i.e. $Am'_j \leqslant \sum_{i\neq j} Am'_i$ for some $j$, say $j=1$. So $m'_1=\sum_{i\neq 1} -a_i m'_i$ for some $a_i$ in $A$. By putting $a_1=1$, we have $\sum_{i=1}^{n} a_i m_i \in \mathrm{soc}^m M$ and $J^m(a_i l_i)_{i=1}^{n}\cdot q=0$. Thus $J^m(a_i l_i)_{i=1}^{n} \leqslant \mathrm{Im}\, p$. Further putting $e:=e_1$ we may assume that $a_i=ea_i$ for each $i\neq 1$. Put $l:=(a_i l_i)_{i\neq 1}$. Then we have $l_1\in L_1$, $l\in \bigoplus_{i\neq 1} L_i$, $l_1=el_1$, $l=el$ and $J^m(l_1, l)\leqslant \mathrm{Im}\, p$. On the other hand, it holds that $J^m(l_1, l)\neq 0$ since we have $J^m l_1\neq 0$ by the assumption $I_i \lneqq J^m e_i$. Accordingly, $J^m(l_1, l)=\mathrm{Im}\, p$ since $\mathrm{Im}\, p$ is simple. Define a map $r$: $L_1\to \bigoplus_{i\neq 1} L_i$ by $xl_1\mapsto xl$ for each $xl_1\in L_1$. Then $r$ is well-defined. In fact, if $xl_1=0$, then $xe\in I_1\leqslant J^m$ and then $xe(l_1, l)\in \mathrm{Im}\, p$. Thus $xe(l_1, l)=sp$ for some $s$ in $K$. Therefore $sp_1=xel_1=xl_1=0$ and $s(p_i)_{i\neq 1}=xel$. But since $p_1$ is a monomorphism, we have $s=0$ and $xl=xel=0$. Further by the similar argument as in (2.3.1), $p_1 r=(p_i)_{i\neq 1}$ i.e. $(p, D)$ is fusible. //

**Proposition 3.2.** *Let $A$ be a ring with selfduality which is of right 2nd local type, $m\geqslant 2$, $e, f_1, \cdots, f_n (n\geqslant 2)$ in* $\mathrm{pi}(A)$ *and $p$: $\bigoplus_{i=1}^{n} f_i A\to eJ^m/eJ^{m+1}$ be a projective cover of $eJ^m/eJ^{m+1}$. Then $p^*$: $Ae/Je\to \bigoplus_{i=1}^{n} Jf_i/J^{m+1}f_i$ is infusible.*

Proof. Let $0\to Ae/Je \xrightarrow{p^*} \bigoplus_{i+1}^{n} Af_i/J^{m+1}f_i\to M\to 0$ be an exact sequence. Then $M$ is indecomposable by (1.4.3). By (3.1), $JM=\mathrm{soc}^m M$. Accordingly, $JM$ is indecomposable since $JM\geqslant \mathrm{soc}^2 M$ and $\mathrm{soc}^2 M$ is indecomposable. Then from the exact sequence $0\to Ae/Je \xrightarrow{p^*} \bigoplus_{i=1}^{n} Jf_i/J^{m+1}f_i\to JM\to 0$, we obtain that $p^*$: $Ae/Je \to \bigoplus_{i=1}^{n} Jf_i/J^{m+1}f_i$ is infusible by (1.3). //

**3.3.** Proof of Theorem 1 (2). Let $p$: $\bigoplus_{i=1}^{n} f_i A\to eJ^m$ be a projective cover of $eJ^m$ and $f_i$ in $\mathrm{pi}(A)$ for each $i=1, \cdots, n$. If $n=1$, then the assertion is trivial. So we may assume that $n\geqslant 2$. There is some $u_i$ in $eJ^m f_i\backslash eJ^{m+1}f_i$ such that the $i$-th coordinate map of $p$ is the right multiplication by $u_i$ for each $i=1, \cdots, n$. Put $\bar{u}_i:=u_i+eJ^{m+1}$, $u'_i:=u_i+J^{m+1}f_i$ and $e':=e+Je$. Then $eJ^m=\sum_{i=1}^{n} u_i A$ where each $u_i A$ is local. Suppose that $eJ^m$ is not a direct sum of local modules. Then $\sum_{i=1}^{n} u_i a_i=0$ for some $a_i$ in $A$ and $u_j a_j\neq 0$ for some $j=1, \cdots, n$. We may assume that there is some $g$ in $\mathrm{pi}(A)$ such that $u_j a_j g\neq 0$ and $a_i=f_i a_i g$ for each $i=1, \cdots, n$. Then it holds that $a_i$ is in $f_i Jg$ for each $i$. In fact, if $f_i\neq g$, then $a_i\in f_i Ag=f_i Jg$. And, in case $f_i=g$, we have $f_i Ag/f_i Jg=f_i Af_i/f_i Jf_i$ is a division ring. Furthermore, $\sum_{i=1}^{n} u_i a_i=0$ implies $\sum_{i=1}^{n} \bar{u}_i a_i=0$ and hence each $\bar{u}_i a_i=0$, since $\bar{u}_i A$ are independent. Then putting $\bar{a}_i:=a_i+f_i Jg$, we have that $\bar{u}_i \bar{a}_i$ is defined and is zero. Hence if $a_i$ is not in $f_i Jg$, then $\bar{u}_i=(\bar{u}_i \bar{a}_i)\bar{a}_i^{-1}=0$, a con-

tradiction. Further $Au_i=J^m f_i$ since $J^m f_i$ is uniserial for $m\geqslant 2$. Therefore we may assume that $Au_i a_i\leqslant Au_n a_n$ for each $i$ and $Au_n a_n=J^s g$ for some $s\geqslant m+1\geqslant 3$. Define a homomorphism $q_i$: $Af_i/J^{m+1}f_i\to Ag/J^{s+1}g$ by $x\mapsto xa_i$ for each $i=1, \cdots, n$. Then $q_n$ is a monomorphism since $\mathrm{soc}(Af_n/J^{m+1}f_n)=J^m f_n/J^{m+1}f_n$ is simple and is mapped by $q_n$ onto the simple module $J^s g/J^{s+1}g$. Further putting $q_i':=q_i|(Jf_i/J^{m+1}f_i)$, we have $\mathrm{Im}\ q_i'\leqslant \mathrm{soc}^m(Jg/J^{s+1}g)=J^{s+1-m}g/J^{s+1}g=\mathrm{Im}\ q_n'$ for each $i=1, \cdots, n$. Hence if we put $q_i'':=q_i'$: $Jf_i/J^{m+1}f_i\to J^{s+1-m}g/J^{s+1}g$ and $q:=(q_i'')_{i=1}^{T\ n}$, then $p^*$: $Ae/Je\to\bigoplus_{i=1}^{n} Jf_i/J^{m+1}f_i$ is fusible since $e'p^*q=0$ and $q_n''$ is an isomorphism. This contradicts (3.2). Hence $eJ^m$ must be a direct sum of local modules.

**3.4.** Proof of Theorem 1 (3) and (6). Suppose that $|LJ^s/LJ^{s+1}|\geqslant 2$ for some $s\geqslant 1$. $LJ^s$ is a direct sum of local modules for $LJ^s$ is a direct summand of $eJ^{2+s}$. Further $L=vA$ for some $v$ in $eJ^2g\backslash eJ^3g$ and for some $g$ in $\mathrm{pi}(A)$. Hence $LJ^s=vJ^s=u_1A\oplus u_2A\oplus\cdots$ for some $u_i$ in $eJ^{2+s}f_i\backslash eJ^{3+s}f_i$ where $f_i$ are in $\mathrm{pi}(A)$. Then for each $i=1, 2$, there is some $a_i$ in $gJ^s f_i$ such that $u_i=va_i$. Define a map $p_i$: $Ag/J^3g\to Af_i/J^{s+3}f_i$ by $x\mapsto xa_i$ for each $i=1, 2$. Then $p_1$ and $p_2$ are both monomorphisms since putting $v':=v+J^3g$ and $u_i':=u_i+J^{s+3}f_i$, $\mathrm{soc}(Ag/J^3g)=J^2g/J^3g=Av'$ and $\mathrm{soc}(J^s f_i/J^{s+3}f_i)=J^{s+2}f_i/J^{s+3}f_i=Au_i'$ are simple modules and $(Av')p_i=Au_i'$ for each $i=1, 2$. In particular, $Ag$ is uniserial by Theorem 1 (1).

i) In case $s\geqslant 2$. By the above,

$$Av'\xrightarrow{(p_1,\,p_2)}(J^s f_1/J^{s+3}f_1)\oplus(J^s f_2/J^{s+3}f_2)$$

is fusible. Also, $\mathrm{soc}^3(Af_i/J^{s+3}f_i)=J^s f_i/J^{s+3}f_i$ is uniserial. Hence

$$Av'\xrightarrow{(p_1,\,p_2)}(Af_1/J^{s+3}f_1)\oplus(Af_2/J^{s+3}f_2)$$

is fusible by (2.3.2), say 2-fusible. Then for some $a$ in $f_1Af_2$, the diagram

$$\begin{array}{ccc} Av' & \xrightarrow{p_1} & Af_1/J^{s+3}f_1 \\ \downarrow & & \downarrow \text{right multiplication by } a \\ Av' & \xrightarrow{p_2} & Af_2/J^{s+3}f_2 \end{array}$$

is commutative. Therefore $u_2'=u_1'a$. Putting $\bar{u}_i:=u_i+eJ^{s+3}$ for each $i=1, 2$, we have $\bar{u}_2=\bar{u}_1 a$ since $u_2$ is in $u_1a+eJ^{s+3}f_2$. Thus $\bar{u}_2A\leqslant\bar{u}_1A$. This contradicts the linear independency of $\bar{u}_1A$ and $\bar{u}_2A$.

ii) In case the base field $k$ is algebraically closed. It remains only the case $s=1$. Similarly, it holds that

$$Av'\xrightarrow{(p_1,\,p_2)}(Jf_1/J^4f_1)\oplus(Jf_2/J^4f_2)$$

is fusible. But since $0 \neq u_i \in eJ^3 f_i \leqslant J^3 f_i$ for each $i=1, 2$, $h(Af_i) \geqslant 4$ and then $Af_i/J^4 f_i$ is uniserial of length 4 and $Jf_i/J^4 f_i = \mathrm{soc}^3(Af_i/J^4 f_i)$ by Theorem 1 (5). Then

$$Av' \xrightarrow{(p_1, p_2)} (Af_1/J^4 f_1) \oplus (Af_2/J^4 f_2)$$

is fusible by (2.3.2). Hence by the same argument as in i) we have a contradiction. //

## 4. *QF* rings of right 2nd local type

**Lemma 4.1.** *Let $A$ be a QF ring and $e$ and $f$ be in* $\mathrm{pi}(A)$ *such that $fJe/fJ^2e \neq 0$. Then*

(a) *If $Je/J^2e$ is simple, then $h(Af) \geqslant h(Ae)$; and*

(b) *If $fJ/fJ^2$ is simple, then $h(eA) \geqslant h(fA)$.*

Proof. (a). It follows from the fact that $Je/J^2e$ is simple and $fJe/fJ^2e \neq 0$ that there is an epimorphism $p: Af \to Je$. If $p$ is a monomorphism, then $Je$ is injective and is a direct summand of $Ae$. Thus $Je=0$ for $Je$ is small in $Ae$. But this is impossible since $Je/J^2e$ is simple. Therefore $\mathrm{Ker}\, p \geqslant \mathrm{soc}\, Af = J^{h(Af)-1} f$ since $Af$ is colocal. Hence $h(Af) \geqslant h(Je)+1 = h(Ae)$.

(b) Similar. //

**4.2.** Proof of Theorem 2. Let $(x)'$ be the left side version of $(x)$ for each $x=1, 3$. We show the following implications: $(1) \Rightarrow (3)' \Leftrightarrow (3) \Rightarrow (6) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$. Note that $(2) \Leftrightarrow (1)'$ is clear since $A$ has a selfduality. Denote by $D$ the selfduality $\mathrm{Hom}_A(?, A)$ of $A$.

$(1) \Rightarrow (3)'$. Let $e$ be in $\mathrm{pi}(A)$ and $h := h(Ae) \geqslant 4$. Then $J^2e$ is a uniserial waist in $Ae$. Hence $\mathrm{soc}^2 eA = D(Ae/J^2e)$ is a waist in $eA = D(Ae)$ and $\mathrm{soc}^2 eA = eJ^{h-2}$ is a direct sum of local modules for $h-2 \geqslant 2$. But since $eJ^{h-2} \leqslant eA$ and $eA$ is colocal, $eJ^{h-2}$ is local. Hence $|Je/J^2e| = |\mathrm{soc}^2(eA)/\mathrm{soc}(eA)| = 1$ and $Ae$ is uniserial.

$(3)' \Leftrightarrow (3)$. Clear from the fact that both height and uniseriality are preserved by $D$.

$(3) \Rightarrow (6)$. By the equivalence $(3) \Leftrightarrow (3)'$ and left-right symmetry, it is sufficient to prove that under the assumption $(3)'$, if $A$ is an indecomposable ring and $J^3 \neq 0$, then $A$ is a left serial ring. Let $Q$ be the left quiver of $A$, namely the oriented graph with vertex set $\{1, \cdots, p\}$ where $\mathrm{pi}(A) = \{e_1, \cdots, e_p\}$ and with $n_{ji}$ arrows $i \to j$ iff $\dim_{(e_jAe_j/e_jJe_j)}(e_jJe_i/e_jJ^2e_i) = n_{ji}$. Note that $A$ is an indecomposable ring iff $Q$ is connected. It follows from $J^3 \neq 0$ that $h(Ae_i) \geqslant 4$ for some $i=1, \cdots, p$ and then $Ae_i$ is uniserial by $(3)'$. By 4.1 and the selfduality $D$, we have $h(Ae_j) \geqslant h(Ae_i)$ $(\geqslant 4)$ if either

(a) there is an arrow $i \to j$; or

(b) there is an arrow $j \to i$.

Hence $Ae_j$ is uniserial of height $\geqslant 4$ for any $j=1, \cdots, p$ by (4.1), (3)′ and the fact that $Q$ is connected. Thus $A$ is a left serial ring.

(6)$\Rightarrow$(4). Clear from the fact that for a serial ring $A$, $A$ is $QF$ iff the admissible sequence of $A$ is constant.

(4)$\Rightarrow$(5). Let $M_A$ be indecomposable of height $h \geqslant 3$. Then $A/J^h$ is $QF$ by (4). Let $0 \to K \hookrightarrow \bigoplus_{i=1}^{m} P_i \to M \to 0$ be a projective cover of $M$ over $A/J^h$ with each $P_i$ indecomposable. Then $\mathrm{soc}(\bigoplus_{i=1}^{m} P_i) \not\leqslant K$ implies that $\mathrm{soc}\, P_i \not\leqslant K$ for some $i=1, \cdots, m$ and then $P_i \cap K = 0$ since $P_i$ is colocal. Hence $P_i$ is embedded into $M$. But since $P_i$ is injective, $P_i$ is isomorphic to a direct summand of $M$. Hence $P_i \cong M$ for $M$ is indecomposable. Further $P_i \cong eA/eJ^h$ for some $e$ in $\mathrm{pi}(A)$.

(5)$\Rightarrow$(1). Clear. //

## 5. Examples

In this section, we give some examples using bounden quiver algebras over an algebraically closed field $k$. (See Gabriel [8] for details concerning bounden quiver algebras.)

EXAMPLE 1. Let $A$ be the algebra defined by the following bounden quiver:

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & 2 \arrow[l, shift left, "\gamma"] \end{tikzcd}; \quad \beta\alpha = \alpha\gamma = 0\,,\ \alpha^2 = \gamma\beta\,,$$

namely, the algebra having $\{e_1, e_2, \alpha, \beta, \gamma, \gamma\beta, \beta\gamma\}$ as $k$-basis and with multiplication given by the following table:

| left \ right | $e_1$ | $e_2$ | $\alpha$ | $\beta$ | $\gamma$ | $\gamma\beta$ | $\beta\gamma$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $e_1$ | | $\alpha$ | | $\gamma$ | $\gamma\beta$ | |
| $e_2$ | | $e_2$ | | $\beta$ | | | $\beta\gamma$ |
| $\alpha$ | $\alpha$ | | $\gamma\beta$ | | | | |
| $\beta$ | $\beta$ | | | | $\beta\gamma$ | | |
| $\gamma$ | | $\gamma$ | | $\gamma\beta$ | | | |
| $\gamma\beta$ | $\gamma\beta$ | | | | | | |
| $\beta\gamma$ | | $\beta\gamma$ | | | | | |

(each blank is zero).

Then $A$ is weakly symmetric and hence $QF$. Further as easily seen, $A$ has cube-zero radical. Therefore $A$ is of right (and left) 2nd local type by Theorem 2. But since $A$ is not a serial ring, $A$ is neither of right (1st) local type nor of left (1st) local type.

EXAMPLE 2. Let $A$ be the algebra defined by the following quiver $Q$:

$$\begin{tikzcd} & 5 \arrow[d, "\delta"] & & \\ 4 \arrow[r, "\gamma"] & 1 & 2 \arrow[l, "\alpha"'] & 3 \arrow[l, "\beta"'] \end{tikzcd},$$

namely, the algebra having $\{e_1, e_2, e_3, e_4, e_5, \alpha, \beta, \gamma, \delta, \alpha\beta\}$ as $k$-basis with multiplication given by the following table:

| left \ right | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\alpha\beta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | $e_1$ | | | | | $\alpha$ | | $\gamma$ | $\delta$ | $\alpha\beta$ |
| $e_2$ | | $e_2$ | | | | | $\beta$ | | | |
| $e_3$ | | | $e_3$ | | | | | | | |
| $e_4$ | | | | $e_4$ | | | | | | |
| $e_5$ | | | | | $e_5$ | | | | | |
| $\alpha$ | | $\alpha$ | | | | | $\alpha\beta$ | | | |
| $\beta$ | | | $\beta$ | | | | | | | |
| $\gamma$ | | | | $\gamma$ | | | | | | |
| $\delta$ | | | | | $\delta$ | | | | | |
| $\alpha\beta$ | | | $\alpha\beta$ | | | | | | | |

(each blank is zero).

Then as easily verified, $A$ satisfies all the conditions stated in Theorem 1. But it is not of right 2nd local type. For instance, let $M$ be the right $A$-module corresponding to the following $k$-representation of $Q^{op}$ (the opposite quiver of $Q$, with all arrows reversed)

$$\begin{tikzcd} & k & & \\ k & k \oplus k \arrow[l, "(0,1)"] \arrow[u, "(1,0)"] \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"'] & k \oplus k \arrow[r, "(1,1)"'] & k \end{tikzcd}$$

namely, the module having $\{m_1, m_1', m_2, m_2', m_3, m_4, m_5\}$ as $k$-basis and with right $A$-action given by the following table:

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\alpha\beta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $m_1$ | $m_1$ | | | | | $m_2$ | | | $m_4$ | $m_3$ |
| $m_1'$ | $m_1'$ | | | | | $m_2'$ | | $m_4$ | | $m_3$ |
| $m_2$ | | $m_2$ | | | | | $m_3$ | | | |
| $m_2'$ | | $m_2'$ | | | | | $m_3$ | | | |
| $m_3$ | | | $m_3$ | | | | | | | |
| $m_4$ | | | | $m_4$ | | | | | | |
| $m_5$ | | | | | $m_5$ | | | | | |

(each blank is zero).

Then $M$ is indecomposable but $\mathrm{top}^2 M$ is decomposable:

$$\mathrm{top}^2 M = \left[\begin{tikzcd} & 0 & & \\ k & k \arrow[l, "1"'] \arrow[u] \arrow[r, "1"'] & k \arrow[r] & 0 \end{tikzcd}\right] \oplus \left[\begin{tikzcd} & k & & \\ 0 & k \arrow[l] \arrow[u, "1"'] \arrow[r, "1"'] & k \arrow[r] & 0 \end{tikzcd}\right].$$

Hence the conditions stated in Theorem 1 are not sufficient for algebras (even if $k$ is algebraically closed) to be of right 2nd local type.

EXAMPLE 3. Let $A$ be the algebra defined by the following bounded quiver:

$$\begin{tikzcd} 1 \arrow[dr, "\alpha"] & & 2 \arrow[dl, "\gamma"'] \\ & 3 \arrow[dl, "\beta"'] \arrow[dr, "\delta"] & \\ 4 & & 5 \end{tikzcd} \quad ; \quad \beta\alpha = \delta\gamma = 0 .$$

Then we can see that $A$ has just 13 indecomposable left modules (up to isomorphism), all of which have indecomposable second tops and second socles since the indecomposable left $A$-modules of height $\geqslant 3$ are both projective and injective. Hence $A$ is of right and left 2nd local type.[1] But it is neither of right (1st) local type nor of left (1st) local type. For instance, let $M_1$ and $M_2$ be the left $A$-modules corresponding to the following $k$-representations of the bounden quiver:

1) In Part II of this series of papers, we shall give some necessary and sufficient conditions for artinian rings to be of right and left $n$-th local type for any natural number $n$. Using this result, it is clear that the algebra defined in Example 3 is of right and left 2nd local type.

$$\begin{tikzcd} 0 \arrow[dr] & & 0 \arrow[dl] \\ & k \arrow[dl, "1"'] \arrow[dr, "1"] & \\ k & & k \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} k \arrow[dr, "1"] & & k \arrow[dl, "1"'] \\ & k \arrow[dl] \arrow[dr] & \\ 0 & & 0 \end{tikzcd} \text{, respectively.}$$

Then $M_1$ and $M_2$ are indecomposable but $M_1$ is not colocal and $M_2$ is not local.

---

## References


[1] H. Asashiba and T. Sumioka: *On Krull-Schmidt's theorem and the indecomposability of amalgamated sums,* Osaka J. Math. **20** (1983), 321–329.

[2] H. Asashiba: *On the indecomposability of amalgamated sums,* Osaka J. Math. **20** (1983), 701–711.

[3] M. Auslander: *Representation theory of artin algebras* II, Comm. Algebra **1** (1974), 269–310.

[4] M. Auslander, E.L. Green and I. Reiten: *Modules with waists,* Illinois J. Math. **19** (1975), 467–478.

[5] V. Dlab and C.M. Ringel: *On algebras of finite representation type,* J. Algebra **33** (1975), 306–394.

[6] V. Dlab and C.M. Ringel: *Indecomposable representations of graphs and algebras,* Mem. Amer. Math. Soc. **173** (1976).

[7] P. Gabriel: *Unzerlegbare Darstellungen* I, Manuscripta Math. **6** (1972), 71–103.

[8] P. Gabriel: *Auslander-Reiten sequences and representation-finite algebras,* Proc. ICRA II Springer Lecture Notes 831 (1980), 1–71.

[9] W. Müller: *Unzerlegbare Moduln über artinschen Ringen,* Math. Z. **137** (1974), 197–226.

[10] T. Sumioka: *Tachikawa's theorem on algebras of left co-local type,* Osaka J. Math. **21** (1984), to appear.

[11] H. Tachikawa: *On rings for which every indecomposable right module has a unique maximal submodule,* Math. Z. **71** (1959), 200–222.



Department of Mathematics
Osaka City University
Osaka 558, Japan.