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ON ALGEBRAS OF SECOND LOCAL TYPE, I

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Throughout this paper, A denotes a (left and right) artinian ring with identity 1, J its Jacobson radical and all modules are (unital and) finitely generated.

Let n be any natural number. Then we say that A is of *right n -th local type* in case for every indecomposable right A -module M , the n -th top $\text{top}^n M := M/MJ^n$ of M is indecomposable. (Note that if $\text{top}^n M$ is indecomposable, then so is M since A is artinian and M is finitely generated.) Hence for such a ring A , the question of indecomposability of right A -modules can be reduced to the corresponding problem of right A/J^n -modules. In [11] H. Tachikawa has studied the case $n=1$ and obtained a necessary and sufficient condition for algebras (by algebra we always mean a finite dimensional algebra over a field k) to be of this type. Further the representation theory of algebras with square-zero radical is well known [5], [6], [7]. So in this paper, we examine the case $n=2$ and give some necessary conditions for rings with selfduality to be of this type. Further in particular for QF (=quasi-Frobenius) rings, we give necessary and sufficient conditions to be of this type. More precisely, we show the following two theorems:

Theorem 1. *Let A be a ring with selfduality which is of right 2nd local type and e any primitive idempotent in A . Then*

(1) *J^2e is a uniserial waist in Ae if $J^2e \neq 0$ (see section 2 for definition of a waist),*

(2) *eJ^m is a direct sum of local modules for every $m \geq 2$,*

(3) *for each local direct summand L of eJ^2 , LJ^2 is uniserial (thus eJ^4 is a direct sum of uniserial modules).*

Further if A is an algebra, we have

(4) *Ae is uniserial if $h(Ae) \geq 5$.*

In particular if the base field k is, in addition, an algebraically closed field, then

(5) *Ae is uniserial if $h(Ae) \geq 4$,*

and then

(6) *eJ^2 is a direct sum of uniserial modules.*

Theorem 2. *Let A be a QF ring. Then the following statements are equivalent:*

- (1) A is of right 2nd local type.
- (2) A is of right 2nd colocal type (see section 1 for definition).
- (3) For any primitive idempotent e in A , eA is uniserial if $h(eA) \geq 4$.
- (4) A/J^t is QF for every $t \geq 3$.
- (5) For each M_A indecomposable with $h(M) \geq 3$, there is a primitive idempotent e in A such that $M \cong eA|eJ^{h(M)}$.
- (6) $A = A_1 \times A_2$ for some QF rings A_1 and A_2 such that A_1 has cube-zero radical and A_2 is a serial ring.

Furthermore, each of these conditions are equivalent to the corresponding left side version.

In the theorems above $h(M)$ denotes the height (=Loewy length) of M , namely $h(M) := \min\{n \in \mathbb{N} \cup \{0\} \mid MJ^n = 0\}$. We remark that Theorem 1 (5) and (6) remain valid also in the case where k is a splitting field for A .

In section 1, we introduce the basic tools used in the following sections. Section 2 is devoted to the structure of an indecomposable projective left module and in section 3, we examine the structure of an indecomposable projective right module mainly using the technique of Sumioka [10]. In section 4, we give the proof of Theorem 2. Finally in section 5, we give some examples.

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1. Preliminaries

1.1. Throughout the paper, we write homomorphisms on the opposite side to scalar multiplications, and for homomorphisms $p: K \rightarrow L$ and $q: L \rightarrow M$ of left A -modules and for a decomposition $D: L = \bigoplus_{i=1}^n L_i$ of L , $(p, D) = (p_i)_{i=1}^n$ and $(D, q) = (q_i)_{i=1}^n$ are matrix expressions of p and q relative to D , respectively (for homomorphisms of right A -modules, we write as $(p, D) = (p_i)_{i=1}^n$ and $(D, q) = (q_i)_{i=1}^n$). In addition to the definition of right n -th local type for n any natural number, we define the dual notion: A is called to be of *left n -th colocal type* in case for every indecomposable left A -module M , the n -th socle $\text{soc}^n M := (\text{the right annihilator of } J^n \text{ in } M)$ of M is indecomposable. It should be noted that if A has a selfduality, then A is of right n -th local type iff A is of left n -th colocal type. Further noting that the composition lengths of the projective covers (over A) of all indecomposable right A/J^n -modules have a bound if A/J^n is of finite representation type (i.e. it has only finitely many isomorphism classes of indecomposable right modules), we see easily that when A is of right n -th local type, A is of finite representation type iff so is A/J^n (See

Auslander [3]).

Since the property to be of n -th local (colocal) type is Morita invariant, we may assume that A is a basic ring. We put $\text{pi}(A) := \{e_1, \dots, e_p\}$ to be a basic set of primitive idempotents of A .

DEFINITION 1.2 ([2]). Let $D: L = \bigoplus_{i=1}^n L_i$ be a decomposition of a right A -module L and $p: K \rightarrow L$ be a homomorphism, and j in $\{1, \dots, n\}$. Then the pair (p, D) (or simply $p: K \rightarrow \bigoplus_{i=1}^n L_i$) is called j -fusible in case there is a homomorphism $q: \bigoplus_{i \neq j} L_i \rightarrow L_j$ such that the diagram

$$\begin{array}{ccc} K & \xrightarrow{(p_i)_{i \neq j}} & \bigoplus_{i \neq j} L_i \\ \parallel & & \downarrow q \\ K & \xrightarrow{p_j} & L_j \end{array}$$

commutes where $(p, D) = (p_i)_{i=1}^n$. The pair (p, D) is called *fusible* in case (p, D) is j -fusible for some $j=1, \dots, n$. Finally (p, D) is called *infusible* in case (p, D) is not fusible.

Corollary 1.2.1 ([2, Corollary 1.4]). Let $K_i \leq L_i$ for each $i=1, 2$ and $h: K_1 \rightarrow K_2$ be an isomorphism. Define $p_1 = k_1$, $p_2 = k_2 h$ where $k_i: K_i \rightarrow L_i$ is the inclusion map for each i . Then h or h^{-1} is extendable to a homomorphism $L_1 \rightarrow L_2$ or $L_2 \rightarrow L_1$, respectively iff $p: K_1 \rightarrow L_1 \oplus L_2$ is fusible. //

Proposition 1.2.2 ([2, Proposition 1.1]). Consider an exact sequence $K \xrightarrow{p} L \xrightarrow{q} M \rightarrow 0$ of right A -modules and let $D: L = \bigoplus_{i=1}^n L_i$ be a decomposition of L , $(p, D) = (p_i)_{i=1}^n$, $(D, q) = (q_i)_{i=1}^n$ and j in $\{1, \dots, n\}$. Then the following statements are equivalent:

- (1) (p, D) is j -fusible.
- (2) There is a homomorphism $r = (r_i)_{i=1}^n: \bigoplus_{i=1}^n L_i \rightarrow X$ such that $rp=0$ and r_j is an isomorphism.
- (3) q_j is a split monomorphism.

Proof. See [2]. //

REMARK. In [2] the fusible maps were defined by the condition (2) above.

Proposition 1.3 Let $0 \rightarrow K \xrightarrow{p} L \xrightarrow{q} M \rightarrow 0$ be a nonsplit exact sequence of right A -modules and $D: L = \bigoplus_{i=1}^n L_i$ be a decomposition of L ($n \geq 2$). Then we have

- (1) if M is indecomposable, then (p, D) is infusible,

(2) if K is simple, each L_i is local and (p, D) is infusible, then M is indecomposable.

Proof. See [1] or [2]. //

1.4. Let I be a two-sided ideal of A and e and f in $\text{pi}(A)$. Then we have the canonical isomorphisms $\text{Hom}_A(fA, eA/eI) \xrightarrow{\sim} eAf/eIf \xrightarrow{\sim} \text{Hom}_A(Ae, Af/If)$. We denote by p^* the image of every p in $\text{Hom}_A(fA, eA/eI)$ or the inverse image of every p in $\text{Hom}_A(Ae, Af/If)$ under the composition of these isomorphisms.

Proposition 1.4.1 Let e, f_1, \dots, f_n be in $\text{pi}(A)$, $l > m$, j in $\{1, \dots, n\}$ and $p = (p_i)_{i=1}^n: \bigoplus_{i=1}^n f_i A \rightarrow eJ^m/eJ^l$ be a homomorphism. Then the following statements are equivalent:

$$(1) \quad p(f_j A) \leq \sum_{i \neq j} p(f_i A).$$

(2) $p^*: Ae/J^{l-m}e \rightarrow \bigoplus_{i=1}^n Af_i/J^l f_i$ is j -fusible, where p^* is the map induced by the homomorphism $(p_i^*)_{i=1}^n$.

Proof. There is some u_i in $eJ^m f_i$ such that each p_i^* is the left multiplication by u_i . Then p has the property stated in (1) iff $(u_j f_j A + eJ^l)/eJ^l \leq (\sum_{i \neq j} u_i f_i A + eJ^l)/eJ^l$
iff $u_j A \leq \sum_{i \neq j} u_i A + eJ^l$
iff $u_j = \sum_{i \neq j} u_i a_i + b$, for some a_i in $f_i A$ and b in eJ^l
iff $u_j = \sum_{i \neq j} u_i a_i + b$, for some a_i in $f_i A f_j$ and b in $eJ^l f_j$
iff $u_j = \sum_{i \neq j} u_i a_i + b$, for some a_i in $f_i A f_j$ and b in $J^l f_j$
iff p^* is j -fusible. //

In future p^* shall always mean the above induced homomorphism when the domain of p is of the form as above.

Corollary 1.4.2. Under the same situation as above but $l = m + 1$, the following are equivalent:

(1) $\bar{p}: \bigoplus_{i=1}^n f_i A / f_i J \rightarrow eJ^m / eJ^{m+1}$ (the induced map) is a monomorphism.

(2) $p^*: Ae/Je \rightarrow \bigoplus_{i=1}^n Af_i / J^{m+1} f_i$ is infusible.

In particular if $p: \bigoplus_{i=1}^n f_i A \rightarrow eJ^m$ is a projective cover of eJ^m , then $p^*: Ae/Je \rightarrow \bigoplus_{i=1}^n Af_i / J^{m+1} f_i$ is infusible. //

Corollary 1.4.3. Let $p: \bigoplus_{i=1}^n f_i A \rightarrow eJ^m$ be a projective cover of eJ^m and $0 \rightarrow$

$Ae/Je \xrightarrow{p^*} \bigoplus_{i=1}^n Af_i/J^{m+1}f_i \rightarrow M \rightarrow 0$ be an exact sequence. Then M is indecomposable.

Proof. Clear from (1.4.2) and (1.3). //

2. Structure of an indecomposable projective left module

For an A -module M , we put $|M| :=$ the composition length of M .

Proposition 2.1. *Let A be of right n -th local type, n any natural number and e in $\text{pi}(A)$. Then $J^n e$ is uniserial.*

Proof. It is sufficient to prove that $|J^m e/J^{m+1}e| \leq 1$ for every $m \geq n$. Suppose $|J^m e/J^{m+1}e| \geq 2$ for some $m \geq n$. Then we have a homomorphism $p: Af_1 \oplus Af_2 \rightarrow J^m e/J^{m+1}e$; f_1, f_2 in $\text{pi}(A)$ such that the induced map $\bar{p}: (Af_1/Jf_1) \oplus (Af_2/Jf_2) \rightarrow J^m e/J^{m+1}e$ is a monomorphism. Putting $L = (f_1 A/Jf_1 J^{m+1}) \oplus (f_2 A/Jf_2 J^{m+1})$, we have an exact sequence $0 \rightarrow eA/eJ \xrightarrow{p^*} L \rightarrow M \rightarrow 0$ where M is indecomposable by (1.4.2) and (1.3). But since $p^*(eA/eJ) \leq LJ^m \leq LJ^n$, $\text{top}^n M \cong \text{top}^n L$ is decomposable. This is a contradiction. //

DEFINITION 2.2 ([4]). Let ${}_A L \leq {}_A M$. Then L is called to be a *waist* in M in case $0 \neq L \neq M$ and for each ${}_A N \leq {}_A M$, it holds that $L \leq N$ or $N \leq L$.

Proposition 2.2.1. *Let A be a ring with selfduality which is of right 2nd local type and e in $\text{pi}(A)$. Then $J^2 e$ is a waist in Ae if $J^2 e \neq 0$.*

Proof. Deduced from the following three lemmas for an artinian ring A :

Lemma 2.2.2 ([9, Lemma 1.2]). *Let ${}_A M$ be nonsimple indecomposable. Then $\text{soc}(JM) = \text{soc } M$.*

Proof. Let S be any simple submodule of M and X be any proper submodule of M . If $S + X = M$ then S is not contained in X . Thus $S \cap X = 0$. Hence $S = M$, a contradiction. Therefore S is small in M i.e. $S \leq JM$. Hence $\text{soc } M \leq JM$ and $\text{soc } M = \text{soc}(JM)$. //

Lemma 2.2.3. *Let ${}_A M$ be local and $\text{soc}^2 M$ indecomposable. Then $\text{soc}(J^2 M) = \text{soc } M$ if $J^2 M \neq 0$.*

Proof. Clear from (2.2.2) nothing that JM is nonsimple indecomposable since $J^2 M \neq 0$ and $\text{soc}^2 M \leq JM$. //

Lemma 2.2.4. *Let A be a ring of left 2nd colocal type, ${}_A M$ be local and $J^2 M$ be a nonzero uniserial module. Then $J^2 M$ is a waist in M .*

Proof. Suppose that $J^2 M$ is not a waist in M . Then for some $X \leq M$,

$J^2M \not\leq X$ and $X \not\leq J^2M$. And, $J^2M \cap X = J^tM$ for some $t \geq 3$. Hence $M/J^t \geq (J^2M/J^tM) \oplus (X/J^tM)$ where $J^2M/J^tM \neq 0$ and $X/J^tM \neq 0$. On the other hand since $\text{soc}^2(M/J^tM)$ is indecomposable and $J^2(M/J^tM) \neq 0$, we have that $\text{soc}(M/J^tM) = \text{soc}(J^2M/J^tM)$ is simple by (2.2.3). This is a contradiction. //

We get Theorem 1 (1) from Propositions 2.1 and 2.2.1.

Corollary 2.2.5. *Let A be a ring with selfduality which is of right 2nd local type, e in $\text{pi}(A)$ and $h = h(Ae)$. Then we have $\text{soc}^{h-t}(Ae) = J^t e$ for every $t = 0, \dots, h$.*

Proof. It is clear from Theorem 1 (1) in case $t \geq 2$. The other cases ($t = 0, 1$) are trivial.

Lemma 2.3.1. *Let ${}_A L_1$ and ${}_A L_2$ be local of height ≥ 3 such that for each $i = 1, 2$, $\text{soc}^3 L_i$ is uniserial and $J^2 e_i$ is a uniserial waist in Ae_i where Ae_i is the projective cover of $\text{soc}^3 L_i$. Suppose that ${}_A K$ is simple and there exists an isomorphism $p_i: K \rightarrow \text{soc} L_i$ for each $i = 1, 2$. Consider an exact sequence:*

$$0 \rightarrow K \xrightarrow{p=(p_1, p_2)} L_1 \oplus L_2 \xrightarrow{q=\begin{bmatrix} q_1 \\ -q_2 \end{bmatrix}} M \rightarrow 0.$$

Then $\text{soc}^2 M$ is decomposable if $p: K \rightarrow \text{soc}^2 L_1 \oplus \text{soc}^2 L_2$ is fusible.

Proof. Assume that $p: K \rightarrow \text{soc}^2 L_1 \oplus \text{soc}^2 L_2$ is fusible, say 2-fusible. Then we have a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{p_1} & \text{soc}^2 L_1 \\ \parallel & & \downarrow r \\ K & \xrightarrow{p_2} & \text{soc}^2 L_2. \end{array}$$

And, $M \geq (\text{soc}^2 L_1)q_1 + L_2 q_2 = U \oplus L_2 q_2$ where $U = (\text{soc}^2 L_1)(q_1 - r q_2) \neq 0$. Now for each x in $\text{soc}^2 M$, $x = l_1 q_1 + l_2 q_2$ for some (l_1, l_2) in $L_1 \oplus L_2$. Since $ux = 0$ for each u in J^2 , we have $ul_1 q_1 = -ul_2 q_2 \in L_1 q_1 \cap L_2 q_2 = K p_1 q_1 (= : S)$. Hence $J^2 l_1 q_1 = J^2 l_2 q_2 \leq S$ where S is simple. In particular, $\text{soc}^2 M \leq \text{soc}^3 L_1 q_1 + \text{soc}^3 L_2 q_2$.

i) In case for each x in $\text{soc}^2 M$, there are l_1, l_2 with $x = l_1 q_1 + l_2 q_2$ such that $J^2 l_1 q_1 = J^2 l_2 q_2 = 0$. Then we have $J^2 l_1 = 0$ for q_1 is monic. Thus l_1 is in $\text{soc}^2 L_1$ and x is in $U \oplus L_2 q_2$. Therefore $\text{soc}^2 M \leq U \oplus L_2 q_2$. Hence $\text{soc}^2 M$ is decomposable.

ii) In case for some x in $\text{soc}^2 M$, there are l_1, l_2 with $x = l_1 q_1 + l_2 q_2$ such that $J^2 l_1 q_1 = J^2 l_2 q_2 = S$. We may assume that $x = ex$ for some e in $\text{pi}(A)$. Since S is simple and q_i are monic, $J^3 l_1 = J^3 l_2 = 0$. Thus l_i is in $\text{soc}^3 L_i \setminus \text{soc}^2 L_i$ for each i . Also, we may assume that $l_i = e l_i$ for each i since $x = ex$. Further we have $\text{soc}^3 L_i = Ae l_i$ for each $i = 1, 2$ since $\text{soc}^3 L_i$ are uniserial. Hence we

may assume that $e=e_1=e_2$. Define a homomorphism $s: \text{soc}^3 L_1 \rightarrow \text{soc}^3 L_2$ by $a e l_1 \mapsto a e l_2$ for each a in A . Then s is well-defined. In fact, if t is in Ae and $t l_1 = 0$, then t is in $\text{Ann}_{Ae}(l_1)$, the annihilator of l_1 in Ae . On the other hand, by the fact that $J^2 e l_1 \neq 0$, we see $\text{Ann}_{Ae}(l_1)$ does not contain $J^2 e$ which is a uniserial waist in Ae . Hence $\text{Ann}_{Ae}(l_1)$ is contained in $J^3 e$ and t is in $J^3 e$. Thus $t l_2$ is in $J^3 l_2 = 0$.

Further the diagram

$$\begin{array}{ccc} K & \xrightarrow{p_1} & \text{soc}^3 L_1 \\ \parallel & & \downarrow s \\ K & \xrightarrow{p_2} & \text{soc}^3 L_2 \end{array}$$

is commutative. For, $J^2(l_1, l_2) (\neq 0)$ is contained in the simple module $\text{Im } p$ since $J^2(l_1, l_2)q = 0$. Hence $J^2(l_1, l_2) = \text{Im } p$. Let c be a nonzero element in K . Then $K = Ac$ and $cp = (ul_1, ul_2)$ for some u in J^2 . Therefore $c(p_1 s) = ul_1 s = ul_2 = cp_2$. Thus $p_1 s = p_2$.

Then putting $V := (\text{soc}^3 L_1) (q_1 - s q_2)$, the same argument as in i) shows that $\text{soc}^2 M \leq V \oplus L_2 q_2$ and $\text{soc}^2 M$ is decomposable. //

Proposition 2.3.2. *Let A be a ring with selfduality which is of right 2nd local type and ${}_A L_1, {}_A L_2$ be local of height ≥ 3 such that $\text{soc}^3 L_i$ are uniserial and $|L_1| \leq |L_2|$. Then for every isomorphism $r: \text{soc} L_1 \rightarrow \text{soc} L_2$, r is extendable to a monomorphism $L_1 \rightarrow L_2$ if r is extendable to a homomorphism $\text{soc}^2 L_1 \rightarrow \text{soc}^2 L_2$.*

Proof. Put $K = \text{soc } L_1$, $p_1 = \text{identity map of } \text{soc } L_1$ and $p_2 = r$. Consider an exact sequence $0 \rightarrow K \xrightarrow{p=(p_1, p_2)} L_1 \oplus L_2 \xrightarrow{q} M \rightarrow 0$. If r is extendable to a homomorphism $\text{soc}^2 L_1 \rightarrow \text{soc}^2 L_2$, then $p: K \rightarrow \text{soc}^2 L_1 \oplus \text{soc}^2 L_2$ is fusible. Hence by (2.3.1), $\text{soc}^2 M$ is decomposable, thus M is decomposable. Therefore $p: K \rightarrow L_1 \oplus L_2$ is fusible by (1.3). Hence by (1.2.1), r is extendable to a homomorphism $q: L_1 \rightarrow L_2$ since $|L_1| \leq |L_2|$ where q is monic since $\text{soc } L_1$ is simple. //

2.4. Throughout the rest of this section, A is a ring with selfduality which is of right 2nd local type. Here, we examine indecomposable projective left A -modules of height ≥ 4 .

Proposition 2.4.1. *Let e and f be in $\text{pi}(A)$ and $f J e / f J^2 e \neq 0$. Then $A f$ is uniserial if $h(Ae) \geq 4$.*

Proof. Take some u in $f J e \setminus f J^2 e$ and define $p: A f \rightarrow J e$ by the right multiplication by u . Then $\text{Ker } p \leq J^2 f$ or $\text{Ker } p \geq J^2 f$ since $J^2 f$ is a waist in $A f$ (if $J^2 f \neq 0$). Assume that $\text{Ker } p \geq J^2 f$. Then $h(\text{Im } p) \leq 2$ since $\text{Im } p \cong A f / \text{Ker } p$ is an epimorph of $A f / J^2 f$. Hence $\text{Im } p \leq \text{soc}^2(Ae) \leq J^2 e$ for $h(Ae) \geq 4$ and $\text{soc}^2(Ae) = J^{h(Ae)-2} e$. But by the definition of p we have $\text{Im } p \leq J^2 e$, a contradiction.

Accordingly, $\text{Ker } p \leq J^2 f$. Then $\text{Ker } p = J^t f$ for some $t \geq 2$ and $Af/J^t f$ is embedded into Je . Therefore $|Jf/J^2 f| = 1$ since $Jf/J^t f$ is embedded into $J^2 e$ which is uniserial. Hence Af is uniserial. //

Proposition 2.4.2. *Assume that e is in $\text{pi}(A)$, $h(Ae) \geq 4$ and Ae is not uniserial. Then*

- (1) *all simple submodules of $Je/J^2 e$ are pairwise isomorphic, and*
- (2) *$J^2 e/J^3 e \cong J^3 e/J^4 e$.*

Proof. Let $u: \bigoplus_{i=1}^n Af_i \rightarrow Je/J^4 e$ be a projective cover of $Je/J^4 e$. Then $n \geq 2$ since Ae is not uniserial. Putting $L_i := (Af_i)u$, we have $L_i \cap L_j = J^2 e/J^4 e$, $L_i \not\cong J^2 e/J^4 e$ for each $i \neq j$ in $\{1, \dots, n\}$. By (2.4.1), each L_i is uniserial and $h(L_i) = 3$. Further $\text{soc } L_i = J^3 e/J^4 e$ is simple and $\text{soc}^2 L_i = J^2 e/J^4 e$ for each $i = 1, \dots, n$.

(1) For any $i \neq j$ in $\{1, \dots, n\}$, the identity map $p: \text{soc } L_i \rightarrow \text{soc } L_j$ is extendable to a homomorphism $\text{soc}^2 L_i \rightarrow \text{soc}^2 L_j$ since $L_i \cap L_j = J^2 e/J^4 e = \text{soc}^2 L_i = \text{soc}^2 L_j$. Hence by (2.3.2), p is extendable to an isomorphism $L_i \rightarrow L_j$. Thus all simple submodules of $Je/J^2 e$ are pairwise isomorphic.

(2) Putting $p_i: J^2 e/J^4 e \rightarrow L_i$ and $q_i: L_i \rightarrow L_1 + L_2$ to be inclusion maps for $i = 1, 2$, we have an exact sequence

$$0 \rightarrow J^2 e/J^4 e \xrightarrow{(p_1, p_2)} L_1 \oplus L_2 \xrightarrow{\begin{bmatrix} q_1 \\ -q_2 \end{bmatrix}} L_1 + L_2 \rightarrow 0$$

where $L_1 + L_2$ is colocal. Hence the identity map $r: \text{soc}^2 L_1 \rightarrow \text{soc}^2 L_2$ is not extendable to any isomorphism $L_1 \rightarrow L_2$. On the other hand, the identity map $p: \text{soc } L_1 \rightarrow \text{soc } L_2$ is extendable to an isomorphism $s: L_1 \rightarrow L_2$ since $r|(\text{soc } L_1) = p$. As a consequence, $s|(\text{soc}^2 L_1) \neq r$. But if $J^2 e/J^3 e \cong J^3 e/J^4 e$, then the restriction map

$$\text{Hom}_A(\text{soc}^2 L_1, \text{soc}^2 L_2) \rightarrow \text{Hom}_A(\text{soc } L_1, \text{soc } L_2)$$

is an injection. This implies that $s|(\text{soc}^2 L_1) = r$ since both $s|(\text{soc}^2 L_1)$ and r are extensions of p . This is a contradiction. //

Proposition 2.4.3. *Assume that e, f and g are in $\text{pi}(A)$, $h(Ae) \geq 5$, Ae is not uniserial, $fJe/fJ^2 e \neq 0$ and $J^2 e/J^3 e \cong Ag/Jg$. Then $fAf/fJf \cong gAg/gJg$ as rings.*

Proof. There exists a submodule L of $Je/J^4 e$ such that L is uniserial of height 3 and $\text{top } L \cong Af/Jf$, $\text{top } JL \cong Ag/Jg$. We identify these isomorphic modules. Further Af and Ag are both uniserial by (2.4.1) and the fact that $h(Ae) \geq 5$ and also $h(Af) \geq 4$. Then we can define a homomorphism $t: \text{End}_A(Af/Jf) \rightarrow \text{End}_A(Ag/Jg)$ by $t(p) := (\bar{q}|Jf/J^3 f)$ for each p in $\text{End}_A(Af/Jf)$ where p is induced by some q in $\text{End}_A(Af/J^3 f)$ and \bar{r} is the map in $\text{End}_A(Jf/J^2 f)$ induced by r for every r in $\text{End}_A(Jf/J^3 f)$. (We identified $\text{End}_A(Jf/J^2 f) = \text{End}_A(Ag/Jg)$.)

Then t is well-defined and injective since for each q in $\text{End}_A(Af/J^3f)$, $(Af/J^3f)q \leq Jf/J^3f$ iff $(Jf/J^3f)q \leq J^2f/J^3f$ (See [10, section 3]). Further by (2.3.2), every automorphism p of $\text{soc } L$ is extendable to an automorphism of L if p is extendable to an automorphism of $\text{soc}^2 L$. Thus t is surjective. (Note that both Af/J^3f and Jf/J^3f are quasi-projective since we have $Jf/J^3f \cong Ag/J^2g$ from the fact that Ag is uniserial.) Hence $fAf/Jf \cong \text{End}_A(Af/Jf) \cong \text{End}_A(Ag/Jg) \cong gAg/gJg$ as rings.

REMARK. In the above, if A is a k -algebra, then the isomorphism defined as above is a k -algebra isomorphism.

2.4.4. Proof of Theorem 1 (4) and (5). Assume that A is an algebra and suppose that Ae is not uniserial, and $h(Ae) \geq 4$. Let $p: \bigoplus_{i=1}^n P_i \rightarrow Je/J^3e$ be a projective cover of Je/J^3e where each ${}_A P_i$ is indecomposable. Then $n \geq 2$. By (2.4.2), there is an f in $\text{pi}(A)$ such that every P_i is isomorphic to Af . And, $J^2e/J^3e \cong Ag/Jg$ for some g in $\text{pi}(A)$. If we put $L_i := (P_i)p$ for $i=1, 2$, then $L_i \cong Af/J^2f$, $J^2e/J^3e \cong L_i \leq Je/J^3e$, $L_1 \cap L_2 = J^2e/J^3e$ and $\text{top } L_i \cong Af/Jf$ for each $i=1, 2$. Since we have an exact sequence

$$0 \rightarrow J^2e/J^3e \rightarrow L_1 \oplus L_2 \rightarrow L_1 + L_2 \rightarrow 0$$

where $J^2e/J^3e \cong Ag/Jg$, $L_1 \oplus L_2 \cong (Af/J^2f)^{(2)}$ and $L_1 + L_2$ is colocal, there exists an infusible homomorphism $Ag/Jg \rightarrow (Af/J^2f)^{(2)}$ by (1.3; 1). Therefore $(fAf/Jf)^{(2)}$ is isomorphic to a direct summand of gJ/gJ^2 by (1.4.2). Hence $\dim(gJf/gJ^2f)_{fAf/Jf} \geq 2$. If $h(Ae) \geq 5$ or k is algebraically closed, then by (2.4.3), $d := \dim_{gAg/gJg}(gJf/gJ^2f) = \dim(gJf/gJ^2f)_{fAf/Jf} \geq 2$. Hence $(Ag/Jg)^{(d)}$ is isomorphic to a direct summand of Jf/J^2f and $d \geq 2$. Thus $|Jf/J^2f| \geq 2$. This contradicts the uniseriality of Af . Hence Ae must be uniserial. //

3. Structure of an indecomposable projective right module

Lemma 3.1. Let $0 \rightarrow K \xrightarrow{p} L \xrightarrow{q} M \rightarrow 0$ be an exact sequence of left A -modules such that K is simple, $D: L = \bigoplus_{i=1}^n L_i$ is a decomposition of L ($n \geq 2$) and for each $i=1, \dots, n$, $L_i = Ae_i/I_i$ for some e_i in $\text{pi}(A)$ and $J^{m+1}e_i \leq I_i \leq J^m e_i$ ($m \geq 1$). Then $JM = \text{soc}^m M$ if (p, D) is infusible.

Proof. Put $l_i := e_i + I_i$, $\bar{l}_i = l_i + JL$, $m_i := l_i q$, $\bar{m}_i := m_i + JM$ and $m'_i := m_i + \text{soc}^m M$. Then we have $\bigoplus_{i=1}^n A\bar{l}_i = L/JL \cong M/JM = \bigoplus_{i=1}^n A\bar{m}_i$ where each $A\bar{m}_i$ is simple. It follows from $h(M) \leq m+1$ that $JM \leq \text{soc}^m M$. Assume that $JM \leq \text{soc}^m M$. Then we show that (p, D) is fusible. (Clearly, we may assume that each $p_i \neq 0$ i.e. each p_i is a monomorphism where $(p, D) = (p_i)_{i=1}^n$.) By

assumption the sum $M/\text{soc}^m M = \sum_{i=1}^n A m'_i$ is redundant i.e. $A m'_j \leq \sum_{i \neq j} A m'_i$ for some j , say $j=1$. So $m'_1 = \sum_{i \neq 1} -a_i m'_i$ for some a_i in A . By putting $a_1=1$, we have $\sum_{i=1}^n a_i m_i \in \text{soc}^m M$ and $J^m(a_i l_i)_{i=1}^n \cdot q = 0$. Thus $J^m(a_i l_i)_{i=1}^n \leq \text{Im } p$. Further putting $e := e_1$ we may assume that $a_i = e a_i$ for each $i \neq 1$. Put $l := (a_i l_i)_{i \neq 1}$. Then we have $l_1 \in L_1$, $l \in \bigoplus_{i \neq 1} L_i$, $l_1 = e l_1$, $l = e l$ and $J^m(l_1, l) \leq \text{Im } p$. On the other hand, it holds that $J^m(l_1, l) \neq 0$ since we have $J^m l_1 \neq 0$ by the assumption $I_i \not\leq J^m e_i$. Accordingly, $J^m(l_1, l) = \text{Im } p$ since $\text{Im } p$ is simple. Define a map $r: L_1 \rightarrow \bigoplus_{i \neq 1} L_i$ by $x l_1 \mapsto x l$ for each $x l_1 \in L_1$. Then r is well-defined. In fact, if $x l_1 = 0$, then $x e \in I_1 \leq J^m$ and then $x e(l_1, l) \in \text{Im } p$. Thus $x e(l_1, l) = s p$ for some s in K . Therefore $s p_1 = x e l_1 = x l_1 = 0$ and $s(p_i)_{i \neq 1} = x e l$. But since p_1 is a monomorphism, we have $s = 0$ and $x l = x e l = 0$. Further by the similar argument as in (2.3.1), $p_1 r = (p_i)_{i \neq 1}$ i.e. (p, D) is fusible. //

Proposition 3.2. *Let A be a ring with selfduality which is of right 2nd local type, $m \geq 2$, e, f_1, \dots, f_n ($n \geq 2$) in $\text{pi}(A)$ and $p: \bigoplus_{i=1}^n f_i A \rightarrow e J^m / e J^{m+1}$ be a projective cover of $e J^m / e J^{m+1}$. Then $p^*: A e / J e \rightarrow \bigoplus_{i=1}^n J f_i / J^{m+1} f_i$ is infusible.*

Proof. Let $0 \rightarrow A e / J e \xrightarrow{p^*} \bigoplus_{i=1}^n A f_i / J^{m+1} f_i \rightarrow M \rightarrow 0$ be an exact sequence. Then M is indecomposable by (1.4.3). By (3.1), $J M = \text{soc}^m M$. Accordingly, $J M$ is indecomposable since $J M \geq \text{soc}^2 M$ and $\text{soc}^2 M$ is indecomposable. Then from the exact sequence $0 \rightarrow A e / J e \xrightarrow{p^*} \bigoplus_{i=1}^n J f_i / J^{m+1} f_i \rightarrow J M \rightarrow 0$, we obtain that $p^*: A e / J e \rightarrow \bigoplus_{i=1}^n J f_i / J^{m+1} f_i$ is infusible by (1.3). //

3.3. Proof of Theorem 1 (2). Let $p: \bigoplus_{i=1}^n f_i A \rightarrow e J^m$ be a projective cover of $e J^m$ and f_i in $\text{pi}(A)$ for each $i=1, \dots, n$. If $n=1$, then the assertion is trivial. So we may assume that $n \geq 2$. There is some u_i in $e J^m f_i \setminus e J^{m+1} f_i$ such that the i -th coordinate map of p is the right multiplication by u_i for each $i=1, \dots, n$. Put $\bar{u}_i := u_i + e J^{m+1}$, $\bar{u}'_i := u_i + J^{m+1} f_i$ and $e' := e + J e$. Then $e J^m = \sum_{i=1}^n u_i A$ where each $u_i A$ is local. Suppose that $e J^m$ is not a direct sum of local modules. Then $\sum_{i=1}^n u_i a_i = 0$ for some a_i in A and $u_j a_j \neq 0$ for some $j=1, \dots, n$. We may assume that there is some g in $\text{pi}(A)$ such that $u_j a_j g \neq 0$ and $a_i = f_i a_i g$ for each $i=1, \dots, n$. Then it holds that a_i is in $f_i J g$ for each i . In fact, if $f_i \neq g$, then $a_i \in f_i A g = f_i J g$. And, in case $f_i = g$, we have $f_i A g / f_i J g = f_i A f_i / f_i J f_i$ is a division ring. Furthermore, $\sum_{i=1}^n u_i a_i = 0$ implies $\sum_{i=1}^n \bar{u}_i a_i = 0$ and hence each $\bar{u}_i a_i = 0$, since $\bar{u}_i A$ are independent. Then putting $\bar{a}_i := a_i + f_i J g$, we have that $\bar{u}_i \bar{a}_i$ is defined and is zero. Hence if a_i is not in $f_i J g$, then $\bar{u}_i = (\bar{u}_i \bar{a}_i) \bar{a}_i^{-1} = 0$, a con-

tradiction. Further $Au_i = J^m f_i$ since $J^m f_i$ is uniserial for $m \geq 2$. Therefore we may assume that $Au_i a_i \leq Au_n a_n$ for each i and $Au_n a_n = J^s g$ for some $s \geq m+1 \geq 3$. Define a homomorphism $q_i: Af_i/J^{m+1}f_i \rightarrow Ag/J^{s+1}g$ by $x \mapsto xa_i$ for each $i=1, \dots, n$. Then q_n is a monomorphism since $\text{soc}(Af_n/J^{m+1}f_n) = J^m f_n/J^{m+1}f_n$ is simple and is mapped by q_n onto the simple module $J^s g/J^{s+1}g$. Further putting $q'_i := q_i|_{(Jf_i/J^{m+1}f_i)}$, we have $\text{Im } q'_i \leq \text{soc}^m(Jg/J^{s+1}g) = J^{s+1-m}g/J^{s+1}g = \text{Im } q'_n$ for each $i=1, \dots, n$. Hence if we put $q''_i := q'_i: Jf_i/J^{m+1}f_i \rightarrow J^{s+1-m}g/J^{s+1}g$ and $q := (q''_i)_{i=1}^n$, then $p^*: Ae/Je \rightarrow \bigoplus_{i=1}^n Jf_i/J^{m+1}f_i$ is fusible since $e'p^*q=0$ and q''_n is an isomorphism. This contradicts (3.2). Hence eJ^m must be a direct sum of local modules.

3.4. Proof of Theorem 1 (3) and (6). Suppose that $|LJ^s/LJ^{s+1}| \geq 2$ for some $s \geq 1$. LJ^s is a direct sum of local modules for LJ^s is a direct summand of eJ^{2+s} . Further $L = vA$ for some v in $eJ^2g \setminus eJ^3g$ and for some g in $\text{pi}(A)$. Hence $LJ^s = vJ^s = u_1A \oplus u_2A \oplus \dots$ for some u_i in $eJ^{2+s}f_i \setminus eJ^{3+s}f_i$ where f_i are in $\text{pi}(A)$. Then for each $i=1, 2$, there is some a_i in $gJ^s f_i$ such that $u_i = va_i$. Define a map $p_i: Ag/J^3g \rightarrow Af_i/J^{s+3}f_i$ by $x \mapsto xa_i$ for each $i=1, 2$. Then p_1 and p_2 are both monomorphisms since putting $v' := v + J^3g$ and $u'_i := u_i + J^{s+3}f_i$, $\text{soc}(Ag/J^3g) = J^2g/J^3g = Av'$ and $\text{soc}(J^s f_i/J^{s+3}f_i) = J^{s+2}f_i/J^{s+3}f_i = Au'_i$ are simple modules and $(Av')p_i = Au'_i$ for each $i=1, 2$. In particular, Ag is uniserial by Theorem 1 (1).

i) In case $s \geq 2$. By the above,

$$Av' \xrightarrow{(p_1, p_2)} (J^s f_1/J^{s+3}f_1) \oplus (J^s f_2/J^{s+3}f_2)$$

is fusible. Also, $\text{soc}^3(Af_i/J^{s+3}f_i) = J^s f_i/J^{s+3}f_i$ is uniserial. Hence

$$Av' \xrightarrow{(p_1, p_2)} (Af_1/J^{s+3}f_1) \oplus (Af_2/J^{s+3}f_2)$$

is fusible by (2.3.2), say 2-fusible. Then for some a in $f_1 Af_2$, the diagram

$$\begin{array}{ccc} Av' & \xrightarrow{p_1} & Af_1/J^{s+3}f_1 \\ \downarrow & & \downarrow \text{right multiplication by } a \\ Av' & \xrightarrow{p_2} & Af_2/J^{s+3}f_2 \end{array}$$

is commutative. Therefore $u'_2 = u'_1 a$. Putting $\bar{u}_i := u_i + eJ^{s+3}$ for each $i=1, 2$, we have $\bar{u}_2 = \bar{u}_1 a$ since u_2 is in $u_1 a + eJ^{s+3}f_2$. Thus $\bar{u}_2 A \leq \bar{u}_1 A$. This contradicts the linear independency of $\bar{u}_1 A$ and $\bar{u}_2 A$.

ii) In case the base field k is algebraically closed. It remains only the case $s=1$. Similarly, it holds that

$$Av' \xrightarrow{(p_1, p_2)} (Jf_1/J^4f_1) \oplus (Jf_2/J^4f_2)$$

is fusible. But since $0 \neq u_i \in eJ^3f_i \leq J^3f_i$ for each $i=1, 2$, $h(Af_i) \geq 4$ and then Af_i/J^4f_i is uniserial of length 4 and $Jf_i/J^4f_i = \text{soc}^3(Af_i/J^4f_i)$ by Theorem 1 (5). Then

$$Av' \xrightarrow{(\rho_1, \rho_2)} (Af_1/J^4f_1) \oplus (Af_2/J^4f_2)$$

is fusible by (2.3.2). Hence by the same argument as in i) we have a contradiction. //

4. QF rings of right 2nd local type

Lemma 4.1. *Let A be a QF ring and e and f be in $\text{pi}(A)$ such that $fJe/fJ^2e \neq 0$. Then*

- (a) *If Je/J^2e is simple, then $h(Af) \geq h(Ae)$; and*
- (b) *If fJ/fJ^2 is simple, then $h(eA) \geq h(fA)$.*

Proof. (a). It follows from the fact that Je/J^2e is simple and $fJe/fJ^2e \neq 0$ that there is an epimorphism $p: Af \rightarrow Je$. If p is a monomorphism, then Je is injective and is a direct summand of Ae . Thus $Je=0$ for Je is small in Ae . But this is impossible since Je/J^2e is simple. Therefore $\text{Ker } p \geq \text{soc } Af = J^{h(Af)-1}f$ since Af is colocal. Hence $h(Af) \geq h(Je) + 1 = h(Ae)$.

(b) Similar. //

4.2. Proof of Theorem 2. Let $(x)'$ be the left side version of (x) for each $x=1, 3$. We show the following implications: $(1) \Rightarrow (3)' \Leftrightarrow (3) \Rightarrow (6) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$. Note that $(2) \Leftrightarrow (1)'$ is clear since A has a selfduality. Denote by D the selfduality $\text{Hom}_A(?, A)$ of A .

$(1) \Rightarrow (3)'$. Let e be in $\text{pi}(A)$ and $h := h(Ae) \geq 4$. Then J^2e is a uniserial waist in Ae . Hence $\text{soc}^2eA = D(Ae/J^2e)$ is a waist in $eA = D(Ae)$ and $\text{soc}^2eA = eJ^{h-2}$ is a direct sum of local modules for $h-2 \geq 2$. But since $eJ^{h-2} \leq eA$ and eA is colocal, eJ^{h-2} is local. Hence $|Je/J^2e| = |\text{soc}^2(eA)/\text{soc}(eA)| = 1$ and Ae is uniserial.

$(3)' \Leftrightarrow (3)$. Clear from the fact that both height and uniseriality are preserved by D .

$(3) \Rightarrow (6)$. By the equivalence $(3) \Leftrightarrow (3)'$ and left-right symmetry, it is sufficient to prove that under the assumption $(3)'$, if A is an indecomposable ring and $J^3 \neq 0$, then A is a left serial ring. Let Q be the left quiver of A , namely the oriented graph with vertex set $\{1, \dots, p\}$ where $\text{pi}(A) = \{e_1, \dots, e_p\}$ and with n_{ji} arrows $i \rightarrow j$ iff $\dim_{(e_jAe_j/e_jJ^2e_j)}(e_jJe_i/e_jJ^2e_i) = n_{ji}$. Note that A is an indecomposable ring iff Q is connected. It follows from $J^3 \neq 0$ that $h(Ae_i) \geq 4$ for some $i=1, \dots, p$ and then Ae_i is uniserial by $(3)'$. By 4.1 and the selfduality D , we have $h(Ae_j) \geq h(Ae_i) (\geq 4)$ if either

- (a) there is an arrow $i \rightarrow j$; or

(b) there is an arrow $j \rightarrow i$.

Hence Ae_j is uniserial of height ≥ 4 for any $j=1, \dots, p$ by (4.1), (3)' and the fact that Q is connected. Thus A is a left serial ring.

(6) \Rightarrow (4). Clear from the fact that for a serial ring A , A is QF iff the admissible sequence of A is constant.

(4) \Rightarrow (5). Let M_A be indecomposable of height $h \geq 3$. Then A/J^h is QF by (4). Let $0 \rightarrow K \hookrightarrow \bigoplus_{i=1}^m P_i \rightarrow M \rightarrow 0$ be a projective cover of M over A/J^h with each P_i indecomposable. Then $\text{soc}(\bigoplus_{i=1}^m P_i) \not\leq K$ implies that $\text{soc } P_i \not\leq K$ for some $i=1, \dots, m$ and then $P_i \cap K = 0$ since P_i is colocal. Hence P_i is embedded into M . But since P_i is injective, P_i is isomorphic to a direct summand of M . Hence $P_i \cong M$ for M is indecomposable. Further $P_i \cong eA/eJ^h$ for some e in $\text{pi}(A)$.

(5) \Rightarrow (1). Clear. //

5. Examples

In this section, we give some examples using bounden quiver algebras over an algebraically closed field k . (See Gabriel [8] for details concerning bounden quiver algebras.)

EXAMPLE 1. Let A be the algebra defined by the following bounden quiver:

$$\alpha \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} 1 \begin{array}{c} \xrightarrow{\beta} \\ \xleftarrow{\gamma} \end{array} 2; \quad \beta\alpha = \alpha\gamma = 0, \quad \alpha^2 = \gamma\beta,$$

namely, the algebra having $\{e_1, e_2, \alpha, \beta, \gamma, \gamma\beta, \beta\gamma\}$ as k -basis and with multiplication given by the following table:

right left	e_1	e_2	α	β	γ	$\gamma\beta$	$\beta\gamma$
	e_1	e_2	α		γ	$\gamma\beta$	
e_2		e_2		β			$\beta\gamma$
α	α		$\gamma\beta$				
β	β				$\beta\gamma$		
γ		γ		$\gamma\beta$			
$\gamma\beta$	$\gamma\beta$						
$\beta\gamma$		$\beta\gamma$					

(each blank is zero).

Then A is weakly symmetric and hence QF . Further as easily seen, A has cube-zero radical. Therefore A is of right (and left) 2nd local type by Theorem 2. But since A is not a serial ring, A is neither of right (1st) local type nor of left (1st) local type.

EXAMPLE 2. Let A be the algebra defined by the following quiver Q :

$$\begin{array}{ccccc} & & 5 & & \\ & & \downarrow \delta & & \\ 4 & \xrightarrow{\gamma} & 1 & \xleftarrow{\alpha} & 2 \xleftarrow{\beta} 3, \end{array}$$

namely, the algebra having $\{e_1, e_2, e_3, e_4, e_5, \alpha, \beta, \gamma, \delta, \alpha\beta\}$ as k -basis with multiplication given by the following table:

right left	e_1	e_2	e_3	e_4	e_5	α	β	γ	δ	$\alpha\beta$
e_1	e_1					α		γ	δ	$\alpha\beta$
e_2		e_2					β			
e_3			e_3							
e_4				e_4						
e_5					e_5					
α		α					$\alpha\beta$			
β			β							
γ				γ						
δ					δ					
$\alpha\beta$			$\alpha\beta$							

(each blank is zero).

Then as easily verified, A satisfies all the conditions stated in Theorem 1. But it is not of right 2nd local type. For instance, let M be the right A -module corresponding to the following k -representation of Q^{op} (the opposite quiver of Q , with all arrows reversed)

$$\begin{array}{ccccc} & & k & & \\ & & \uparrow (1,0) & & \\ k & \xleftarrow{(0,1)} & k \oplus k & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} & k \oplus k \xrightarrow{(1,1)} k \end{array}$$

namely, the module having $\{m_1, m'_1, m_2, m'_2, m_3, m_4, m_5\}$ as k -basis and with right A -action given by the following table:

	e_1	e_2	e_3	e_4	e_5	α	β	γ	δ	$\alpha\beta$
m_1	m_1					m_2			m_4	m_3
m'_1	m'_1					m'_2		m_4		m_3
m_2		m_2					m_3			
m'_2		m'_2					m_3			
m_3			m_3							
m_4				m_4						
m_5					m_5					

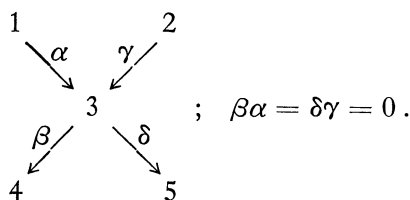
(each blank is zero).

Then M is indecomposable but $\text{top}^2 M$ is decomposable:

$$\text{top}^2 M = \left[\begin{array}{c} 0 \\ \uparrow \\ k \leftarrow k \rightarrow k \rightarrow 0 \\ \downarrow \\ 1 \end{array} \right] \oplus \left[\begin{array}{c} k \\ \uparrow \\ 0 \leftarrow k \rightarrow k \rightarrow 0 \\ \downarrow \\ 1 \end{array} \right].$$

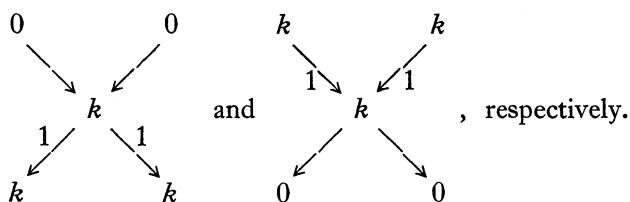
Hence the conditions stated in Theorem 1 are not sufficient for algebras (even if k is algebraically closed) to be of right 2nd local type.

EXAMPLE 3. Let A be the algebra defined by the following bounded quiver:



Then we can see that A has just 13 indecomposable left modules (up to isomorphism), all of which have indecomposable second tops and second socles since the indecomposable left A -modules of height ≥ 3 are both projective and injective. Hence A is of right and left 2nd local type.¹⁾ But it is neither of right (1st) local type nor of left (1st) local type. For instance, let M_1 and M_2 be the left A -modules corresponding to the following k -representations of the bounden quiver:

1) In Part II of this series of papers, we shall give some necessary and sufficient conditions for artinian rings to be of right and left n -th local type for any natural number n . Using this result, it is clear that the algebra defined in Example 3 is of right and left 2nd local type.



Then M_1 and M_2 are indecomposable but M_1 is not colocal and M_2 is not local.

References

- [1] H. Asashiba and T. Sumioka: *On Krull-Schmidt's theorem and the indecomposability of amalgamated sums*, Osaka J. Math. **20** (1983), 321–329.
- [2] H. Asashiba: *On the indecomposability of amalgamated sums*, Osaka J. Math. **20** (1983), 701–711.
- [3] M. Auslander: *Representation theory of artin algebras II*, Comm. Algebra **1** (1974), 269–310.
- [4] M. Auslander, E.L. Green and I. Reiten: *Modules with waists*, Illinois J. Math. **19** (1975), 467–478.
- [5] V. Dlab and C.M. Ringel: *On algebras of finite representation type*, J. Algebra **33** (1975), 306–394.
- [6] V. Dlab and C.M. Ringel: *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc. **173** (1976).
- [7] P. Gabriel: *Unzerlegbare Darstellungen I*, Manuscripta Math. **6** (1972), 71–103.
- [8] P. Gabriel: *Auslander-Reiten sequences and representation-finite algebras*, Proc. ICRA II Springer Lecture Notes 831 (1980), 1–71.
- [9] W. Müller: *Unzerlegbare Moduln über artinschen Ringen*, Math. Z. **137** (1974), 197–226.
- [10] T. Sumioka: *Tachikawa's theorem on algebras of left co-local type*, Osaka J. Math. **21** (1984), to appear.
- [11] H. Tachikawa: *On rings for which every indecomposable right module has a unique maximal submodule*, Math. Z. **71** (1959), 200–222.

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