Throughout this paper, \( A \) denotes a (left and right) artinian ring with identity 1, \( J \) its Jacobson radical and all modules are (unital and) finitely generated.

Let \( n \) be any natural number. Then we say that \( A \) is of right \( n \)-th local type in case for every indecomposable right \( A \)-module \( M \), the \( n \)-th top \( \top^n M = M/MJ^n \) of \( M \) is indecomposable. (Note that if \( \top^n M \) is indecomposable, then so is \( M \) since \( A \) is artinian and \( M \) is finitely generated.) Hence for such a ring \( A \), the question of indecomposability of right \( A \)-modules can be reduced to the corresponding problem of right \( A/J^n \)-modules. In [11] H. Tachikawa has studied the case \( n = 1 \) and obtained a necessary and sufficient condition for algebras (by algebra we always mean a finite dimensional algebra over a field \( k \)) to be of this type. Further the representation theory of algebras with square-zero radical is well known [5], [6], [7]. So in this paper, we examine the case \( n = 2 \) and give some necessary conditions for rings with selfduality to be of this type. Further in particular for \( QF (=\text{quasi-Frobenius}) \) rings, we give necessary and sufficient conditions to be of this type. More precisely, we show the following two theorems:

**Theorem 1.** Let \( A \) be a ring with selfduality which is of right 2nd local type and \( e \) any primitive idempotent in \( A \). Then

1. \( J^2e \) is a uniserial waist in \( Ae \) if \( J^2e \neq 0 \) (see section 2 for definition of a waist),
2. \( ef^m \) is a direct sum of local modules for every \( m \geq 2 \),
3. for each local direct summand \( L \) of \( ef^2 \), \( LJ^2 \) is uniserial (thus \( ef^4 \) is a direct sum of uniserial modules).

Further if \( A \) is an algebra, we have

4. \( Ae \) is uniserial if \( h(Ae) \geq 5 \).
   In particular if the base field \( k \) is, in addition, an algebraically closed field, then
5. \( Ae \) is uniserial if \( h(Ae) \geq 4 \),
   and then
6. \( ej^2 \) is a direct sum of uniserial modules.
Theorem 2. Let $A$ be a QF ring. Then the following statements are equivalent:

1. $A$ is of right 2nd local type.
2. $A$ is of right 2nd colocal type (see section 1 for definition).
3. For any primitive idempotent $e$ in $A$, $eA$ is uniserial if $h(eA) \geq 4$.
4. $A/J^t$ is QF for every $t \geq 3$.
5. For each $M_A$ indecomposable with $h(M) \geq 3$, there is a primitive idempotent $e$ in $A$ such that $M = eA[eJ^h(M)]$.
6. $A = A_1 \times A_2$ for some QF rings $A_1$ and $A_2$ such that $A_1$ has cube-zero radical and $A_2$ is a serial ring.

Furthermore, each of these conditions are equivalent to the corresponding left side version.

In the theorems above $h(M)$ denotes the height (=Loewy length) of $M$, namely $h(M) := \min \{n \in \mathbb{N} \cup \{0\} \mid MJ^n = 0\}$. We remark that Theorem 1 (5) and (6) remain valid also in the case where $k$ is a splitting field for $A$.

In section 1, we introduce the basic tools used in the following sections. Section 2 is devoted to the structure of an indecomposable projective left module and in section 3, we examine the structure of an indecomposable projective right module mainly using the technique of Sumioka [10]. In section 4, we give the proof of Theorem 2. Finally in section 5, we give some examples.

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1. Preliminaries

1.1. Throughout the paper, we write homomorphisms on the opposite side to scalar multiplications, and for homomorphisms $p: K \to L$ and $q: L \to M$ of left $A$-modules and for a decomposition $D: L = \bigoplus L_i$ of $L$, $(p, D) = (p_i)_{i=1}^n$ and $(D, q) = (q_i)_{i=1}^n$ are matrix expressions of $p$ and $q$ relative to $D$, respectively (for homomorphisms of right $A$-modules, we write as $(p, D) = (p_i)_{i=1}^n$ and $(D, q) = (q_i)_{i=1}^n$). In addition to the definition of right $n$-th local type for $n$ any natural number, we define the dual notion: $A$ is called to be of left $n$-th colocal type in case for every indecomposable left $A$-module $M$, the $n$-th socle $\text{soc}^n M := (\text{the right annihilator of } J^n \text{ in } M)$ of $M$ is indecomposable. It should be noted that if $A$ has a selfduality, then $A$ is of right $n$-th local type iff $A$ is of left $n$-th colocal type. Further noting that the composition lengths of the projective covers (over $A$) of all indecomposable right $A/J^n$-modules have a bound if $A/J^n$ is of finite representation type (i.e. it has only finitely many isomorphism classes of indecomposable right modules), we see easily that when $A$ is of right $n$-th local type, $A$ is of finite representation type iff so is $A/J^n$ (See
Auslander [3]).

Since the property to be of \( n \)-th local (colocal) type is Morita invariant, we may assume that \( A \) is a basic ring. We put \( \pi(A) = \{ e_i, \ldots, e_j \} \) to be a basic set of primitive idempotents of \( A \).

**Definition 1.2** ([2]). Let \( D: L = \bigoplus_{i=1}^{n} L_i \) be a decomposition of a right \( A \)-module \( L \) and \( \rho: K \to L \) be a homomorphism, and \( j \) in \( \{ 1, \ldots, n \} \). Then the pair \( (\rho, D) \) (or simply \( \rho: K \to \bigoplus L_i \)) is called \( j \)-fusible in case there is a homomorphism \( q: \bigoplus L_i \to L_j \) such that the diagram

\[
\begin{array}{ccc}
K & \to & \bigoplus_{i=1}^{n} L_i \\
\downarrow & & \downarrow q \\
K & \to & L_j \\
\rho_i & \to & \rho_j
\end{array}
\]

commutes where \( (\rho, D) = (\rho_i)_{i=1}^{n} \). The pair \( (\rho, D) \) is called fusible in case \( (\rho, D) \) is \( j \)-fusible for some \( j = 1, \ldots, n \). Finally \( (\rho, D) \) is called infusible in case \( (\rho, D) \) is not fusible.

**Corollary 1.2.1** ([2, Corollary 1.4]). Let \( K_i \cong L_i \) for each \( i = 1, 2 \) and \( h: K_1 \to K_2 \) be an isomorphism. Define \( p_i = k_i, p_2 = k_2 h \) where \( k_i: K_i \to L_i \) is the inclusion map for each \( i \). Then \( h \) or \( h^{-1} \) is extendable to a homomorphism \( L_1 \to L_2 \) or \( L_2 \to L_1 \), respectively iff \( \rho: K_1 \to L_1 \oplus L_2 \) is fusible. //

**Proposition 1.2.2** ([2, Proposition 1.1]). Consider an exact sequence \( 0 \to K \to L \to M \to 0 \) of right \( A \)-modules and let \( D: L = \bigoplus_{i=1}^{n} L_i \) be a decomposition of \( L \), \( (\rho, D) = (\rho_i)_{i=1}^{n} \), \( (D, q) = (q_i)_{i=1}^{n} \) and \( j \) in \( \{ 1, \ldots, n \} \). Then the following statements are equivalent:

1. \( (\rho, D) \) is \( j \)-fusible.
2. There is a homomorphism \( r=(r_i)_{i=1}^{n}: \bigoplus_{i=1}^{n} L_i \to X \) such that \( rp = 0 \) and \( r_j \) is an isomorphism.
3. \( q_i \) is a split monomorphism.

Proof. See [2]. //

**Remark.** In [2] the fusible maps were defined by the condition (2) above.

**Proposition 1.3** Let \( 0 \to K \to L \to M \to 0 \) be a nonsplit exact sequence of right \( A \)-modules and \( D: L = \bigoplus_{i=1}^{n} L_i \) be a decomposition of \( L \) (\( n \geq 2 \)). Then we have

1. if \( M \) is indecomposable, then \( (\rho, D) \) is infusible,
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(2) if $K$ is simple, each $L_i$ is local and $(p, D)$ is infusible, then $M$ is indecomposable.

Proof. See [1] or [2]. //

1.4. Let $I$ be a two-sided ideal of $A$ and $e$ and $f$ in $\text{pi}(A)$. Then we have the canonical isomorphisms $\text{Hom}_A(fA, eA|eI) \cong eAf/eIf \cong \text{Hom}_A(Ae, Af|Ie)$. We denote by $p^*$ the image of every $p$ in $\text{Hom}_A(fA, eA|eI)$ or the inverse image of every $p$ in $\text{Hom}_A(Ae, Af|Ie)$ under the composition of these isomorphisms.

Proposition 1.4.1 Let $e, f_1, \cdots, f_n$ be in $\text{pi}(A)$, $l>m$, $j$ in $\{1, \cdots, n\}$ and $p=(p_i)_{i=1}^n: \bigoplus_{i=1}^n f_i A \rightarrow eJ^m|eJ^1$ be a homomorphism. Then the following statements are equivalent:

(1) $p(f_i A) \leq \sum_{i \neq j} p(f_i A)$.

(2) $p^*: Ae|J^m\rightarrow \bigoplus_{i=1}^n Af_i|J^1f_i$ is j-fusible, where $p^*$ is the map induced by the homomorphism $(p_i)_{i=1}^n$.

Proof. There is some $u_i$ in $eJ^m f_i$ such that each $p^*_i$ is the left multiplication by $u_i$. Then $p$ has the property stated in (1) iff $(u_j f_i A + eJ^1)/eJ^1 \leq (\sum_{i \neq j} u_i f_i A + eJ^1)/eJ^1$ iff $u_j A \leq \sum_{i \neq j} u_i A + eJ^1$

iff $u_j = \sum_{i \neq j} u_i a_i + b$, for some $a_i$ in $f_i A$ and $b$ in $eJ^1$

iff $u_j = \sum_{i \neq j} u_i a_i + b$, for some $a_i$ in $f_i Af_j$ and $b$ in $eJ^1 f_j$

iff $u_j = \sum_{i \neq j} u_i a_i + b$, for some $a_i$ in $f_i Af_j$ and $b$ in $J^1 f_j$

iff $p^*$ is j-fusible. //

In future $p^*$ shall always mean the above induced homomorphism when the domain of $p$ is of the form as above.

Corollary 1.4.2. Under the same situation as above but $l=m+1$, the following are equivalent:

(1) $\tilde{p}: \bigoplus_{i=1}^n f_i A|f_i J \rightarrow eJ^m|eJ^{m+1}$ (the induced map) is a monomorphism.

(2) $p^*: Ae|J e \rightarrow \bigoplus_{i=1}^n Af_i|J^{m+1}f_i$ is infusible.

In particular if $p: \bigoplus_{i=1}^n f_i A \rightarrow eJ^m$ is a projective cover of $eJ^m$, then $p^*: Ae|J e \rightarrow \bigoplus_{i=1}^n Af_i|J^{m+1}f_i$ is infusible. //

Corollary 1.4.3. Let $p: \bigoplus_{i=1}^n f_i A \rightarrow eJ^m$ be a projective cover of $eJ^m$ and $0 \rightarrow$
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Let \( p^* : A \rightarrow \oplus A_{i+1} f_i \rightarrow M \rightarrow 0 \) be an exact sequence. Then \( M \) is indecomposable.

Proof. Clear from (1.4.2) and (1.3).

2. Structure of an indecomposable projective left module

For an \( A \)-module \( M \), we put \( |M| := \text{the composition length of } M \).

Proposition 2.1. Let \( A \) be of right \( n \)-th local type, \( n \) any natural number and \( e \in \rho_i(-4) \). Then \( J^2 e \) is uniserial.

Proof. It is sufficient to prove that \( |J^m e|/|J^{m+1} e| \leq 1 \) for every \( m \geq n \). Suppose \( |J^m e|/|J^{m+1} e| \geq 2 \) for some \( m \geq n \). Then we have a homomorphism \( p : A_{f_1} \oplus A_{f_2} \rightarrow J^m e/J^{m+1} e \); \( f_1, f_2 \) in \( \rho_i(A) \) such that the induced map \( \overline{p} : (A_{f_1} J_{f_1}) \oplus (A_{f_2} J_{f_2}) \rightarrow J^m e/J^{m+1} e \) is a monomorphism. Putting \( L = (f_1 A_{f_1} J_{f_2}) \oplus (f_2 A_{f_2} J_{f_1}) \), we have an exact sequence \( 0 \rightarrow eA/eJ^p \rightarrow L \rightarrow M \rightarrow 0 \) where \( M \) is indecomposable by (1.4.2) and (1.3). But since \( p(eA/eJ) \leq LB \leq LJ^p \), \( \text{top} M \approx \text{top} L \) is decomposable. This is a contradiction.

Definition 2.2 ([4]). Let \( \lambda L \leq \lambda M \). Then \( L \) is called to be a waist in \( M \) in case \( O \neq L \neq M \) and for each \( \lambda N \leq \lambda M \), it holds that \( L \leq N \) or \( N \leq L \).

Proposition 2.2.1. Let \( A \) be a ring with self-duality which is of right 2nd local type and \( e \in \rho_i(A) \). Then \( J^2 e \) is a waist in \( A e \) if \( J^2 e \neq 0 \).

Proof. Deduced from the following three lemmas for an artinian ring \( A \):

Lemma 2.2.2 ([9, Lemma 1.2]). Let \( \lambda M \) be nonsimple indecomposable. Then \( \text{soc}(JM) = \text{soc} M \).

Proof. Let \( S \) be any simple submodule of \( M \) and \( X \) be any proper submodule of \( M \). If \( S + X = M \) then \( S \) is not contained in \( X \). Thus \( S \cap X = 0 \). Hence \( S = M \), a contradiction. Therefore \( S \) is small in \( M \) i.e. \( S \leq JM \). Hence \( \text{soc} M \leq JM \) and \( \text{soc} M = \text{soc}(JM) \).

Lemma 2.2.3. Let \( \lambda M \) be local and \( \text{soc}^2 M \) indecomposable. Then \( \text{soc}(J^2 M) = \text{soc} M \) if \( J^2 M \neq 0 \).

Proof. Clear from (2.2.2) nothing that \( JM \) is nonsimple indecomposable since \( J^2 M \neq 0 \) and \( \text{soc}^2 M \leq JM \).

Lemma 2.2.4. Let \( A \) be a ring of left 2nd colocal type, \( \lambda M \) be local and \( J^2 M \) be a nonzero uniserial module. Then \( J^2 M \) is a waist in \( M \).

Proof. Suppose that \( J^2 M \) is not a waist in \( M \). Then for some \( X \leq M \),
\( J^t M \triangleleft X \) and \( X \triangleleft J^t M \). And, \( J^t M \cap X = J^t M \) for some \( t \geq 3 \). Hence \( M/J^t M \geq (J^t M/J^t M) \oplus (X/J^t M) \) where \( J^t M/J^t M \neq 0 \) and \( X/J^t M \neq 0 \). On the other hand since \( \text{soc}(M/J^t M) \) is indecomposable and \( J^t (M/J^t M) \neq 0 \), we have that \( \text{soc}(M/J^t M) = \text{soc}(J^t M/J^t M) \) is simple by (2.2.3). This is a contradiction. \( \Box \)

We get Theorem 1 (1) from Propositions 2.1 and 2.2.1.

Corollary 2.2.5. Let \( A \) be a ring with selfduality which is of right 2nd local type, \( e \) in \( \pi(A) \) and \( h = h(Ae) \). Then we have \( \text{soc}^h(Ae) = J^e \) for every \( t = 0, \cdots, h \).

Proof. It is clear from Theorem 1 (1) in case \( t^2 \). The other cases \( (t = 0, 1) \) are trivial.

Lemma 2.3.1. Let \( _AA L_1 \) and \( _AA L_2 \) be local of height \( \geq 3 \) such that for each \( i = 1, 2 \), \( \text{soc}^2 L_i \) is uniserial and \( J^2 e_i \) is a uniserial waist in \( Ae_i \) where \( Ae_i \) is the projective cover of \( \text{soc}^3 L_i \). Suppose that \( _AA K \) is simple and there exists an isomorphism \( p_i : K \twoheadrightarrow \text{soc} L_i \) for each \( i = 1, 2 \). Consider an exact sequence:

\[
0 \to K \xrightarrow{p=(p_1, p_2)} L_1 \oplus L_2 \xrightarrow{q=\begin{bmatrix} q_1 \\ -q_2 \end{bmatrix}} M \to 0.
\]

Then \( \text{soc}^2 M \) is decomposable if \( p : K \twoheadrightarrow \text{soc}^2 L_1 \oplus \text{soc}^2 L_2 \) is fusible.

Proof. Assume that \( p : K \twoheadrightarrow \text{soc}^2 L_1 \oplus \text{soc}^2 L_2 \) is fusible, say \( 2 \)-fusible. Then we have a commutative diagram

\[
\begin{array}{ccc}
K & \xrightarrow{p_1} & \text{soc}^2 L_1 \\
\| & & \| \\
K & \xrightarrow{p_2} & \text{soc}^2 L_2.
\end{array}
\]

And, \( M \geq (\text{soc}^2 L_1) q_1 + L_2 q_2 = U \oplus L_2 q_2 \) where \( U = (\text{soc}^2 L_1) (q_1 - r q_2) \neq 0 \). Now for each \( x \) in \( \text{soc}^2 M \), \( x = l_1 q_1 + l_2 q_2 \) for some \( (l_1, l_2) \) in \( L_1 \oplus L_2 \). Since \( u x = 0 \) for each \( u \) in \( J^2 \), we have \( ul_1 q_1 = -ul_2 q_2 \subseteq L_1 q_1 \cap L_2 q_2 = K p_i q_1 (= : S) \). Hence \( J^2 l_1 q_1 = J^2 l_2 q_2 \leq S \) where \( S \) is simple. In particular, \( \text{soc}^2 M \leq \text{soc}^2 L_1 q_1 + \text{soc}^2 L_2 q_2 \).

i) In case for each \( x \) in \( \text{soc}^2 M \), there are \( l_1, l_2 \) with \( x = l_1 q_1 + l_2 q_2 \) such that \( J^2 l_1 q_1 = J^2 l_2 q_2 = 0 \). Then we have \( J^2 l_1 = 0 \) for \( q_1 \) is monic. Thus \( l_1 \) is in \( \text{soc}^2 L_1 \) and \( x \) is in \( U \oplus L_2 q_2 \). Therefore \( \text{soc}^2 M \leq U \oplus L_2 q_2 \). Hence \( \text{soc}^2 M \) is decomposable.

ii) In case for some \( x \) in \( \text{soc}^2 M \), there are \( l_1, l_2 \) with \( x = l_1 q_1 + l_2 q_2 \) such that \( J^2 l_1 q_1 = J^2 l_2 q_2 = S \). We may assume that \( x = e x \) for some \( e \) in \( \pi(A) \). Since \( S \) is simple and \( q_i \) are monic, \( J^2 l_1 = J^2 l_2 = 0 \). Thus \( l_i \) is in \( \text{soc}^2 L_i \setminus \text{soc}^2 L_i \) for each \( i \). Also, we may assume that \( l_i = e l_i \) for each \( i \) since \( x = e x \). Further we have \( \text{soc}^2 L_i = A e l_i \) for each \( i = 1, 2 \) since \( \text{soc}^2 L_i \) are uniserial. Hence we
may assume that $e = e_1 = e_2$. Define a homomorphism $s: \text{soc}^3L_1 \to \text{soc}^3L_2$ by $a \in L_1 \mapsto a\in L_2$ for each $a$ in $A$. Then $s$ is well-defined. In fact, if $t$ is in $Ae$ and $t_1 = 0$, then $t$ is in $\text{Ann}_{Ae}(l_1)$, the annihilator of $l_1$ in $Ae$. On the other hand, by the fact that $J^2e_1 = 0$, we see $\text{Ann}_{Ae}(l_1)$ does not contain $J^2e$ which is a uniserial waist in $Ae$. Hence $\text{Ann}_{Ae}(l_1)$ is contained in $J^2e$ and $t$ is in $J^2e$. Thus $t_2$ is in $J^2e = 0$.

Further the diagram

$$
\begin{array}{c}
K \xrightarrow{p_1} \text{soc}^3L_1 \\
\downarrow s \\
K \xrightarrow{p_2} \text{soc}^3L_2
\end{array}
$$

is commutative. For, $J^2(l_1, l_2) (\neq 0)$ is contained in the simple module $\text{Im} \ p$ since $J^2(l_1, l_2)q = 0$. Hence $J^2(l_1, l_2) = \text{Im} \ p$. Let $c$ be a nonzero element in $K$. Then $K = A e$ and $c p = (u_1, u_2)$ for some $u$ in $J^3$. Therefore $c(p, s) = u_1 s = u_2 = c p_2$. Thus $p_2 s = p_2$.

Then putting $V := (\text{soc}^3L_1)(q_1 - q_2)$, the same argument as in i) shows that $\text{soc}^3M \leq V \oplus L_2 q_2$ and $\text{soc}^3M$ is decomposable. //

**Proposition 2.3.2.** Let $A$ be a ring with self-duality which is of right 2nd local type and $L_1, L_2$ be local of height $\geq 3$ such that $\text{soc}^3L_1$ are uniserial and $|L_1| \leq |L_2|$. Then for every isomorphism $r: \text{soc}^3L_1 \to \text{soc}^3L_2$, $r$ is extendable to a monomorphism $L_1 \to L_2$ if $r$ is extendable to a homomorphism $\text{soc}^3L_1 \to \text{soc}^3L_2$.

Proof. Put $K = \text{soc}^3L_1$, $p_1 =$ identity map of $\text{soc}^3L_1$ and $p_2 = r$. Consider an exact sequence $0 \to K \xrightarrow{\hat{p} = (p_1, p_2)} L_1 \oplus L_2 \xrightarrow{q} M \to 0$. If $r$ is extendable to a homomorphism $\text{soc}^3L_1 \to \text{soc}^3L_2$, then $p: K \to \text{soc}^3L_1 \oplus \text{soc}^3L_2$ is fusible. Hence by (2.3.1), $\text{soc}^3M$ is decomposable. Therefore $p: K \to L_1 \oplus L_2$ is fusible by (1.3). Hence by (1.2.1), $r$ is extendable to a homomorphism $q: L_1 \to L_2$ since $|L_1| \leq |L_2|$ where $q$ is monic since $\text{soc}^3L_1$ is simple. //

2.4. Throughout the rest of this section, $A$ is a ring with self-duality which is of right 2nd local type. Here, we examine indecomposable projective left $A$-modules of height $\geq 4$.

**Proposition 2.4.1.** Let $e$ and $f$ be in $\text{pi}(A)$ and $fJ^2e \neq 0$. Then $Af$ is uniserial if $h(Ae) \geq 4$.

Proof. Take some $u$ in $fJ^2e \setminus fJ^2e$ and define $p: Af \to Je$ by the right multiplication by $u$. Then $\ker p \leq J^2f$ or $\ker p \geq J^2f$ since $J^2f$ is a waist in $Af$ (if $J^2f \neq 0$). Assume that $\ker p \geq J^2f$. Then $h(\text{Im} \ p) \leq 2$ since $\text{Im} \ p = Af/\ker p$ is an epimorph of $Af/J^2f$. Hence $\text{Im} \ p \leq \text{soc}^3(Ae) \leq J^2e$ for $h(Ae) \geq 4$ and $\text{soc}^3(Ae) = J^2((Ae) - 2e)$. But by the definition of $p$ we have $\text{Im} \ p \leq J^2e$, a contradiction.
Accordingly, Ker \( p \leq J^2f \). Then Ker \( p = J^t f \) for some \( t \geq 2 \) and \( Af/J^t f \) is embedded into \( Je \). Therefore \( |Jf/J^2 f| = 1 \) since \( Jf/J^t f \) is embedded into \( J^2 e \) which is uniserial. Hence \( Af \) is uniserial. 

**Proposition 2.4.2.** Assume that \( e \) is in \( \text{pi}(A) \), \( h(Ae) \geq 4 \) and \( Ae \) is not uniserial. Then

1. all simple submodules of \( Je/J^2 e \) are pairwise isomorphic, and
2. \( J^3 e/J^2 e \cong J^3 e/J^4 e \).

**Proof.** Let \( u: \bigoplus_{i=1}^n Af_i \to Je/J^t e \) be a projective cover of \( Je/J^t e \). Then \( n \geq 2 \) since \( Ae \) is not uniserial. Putting \( L_i := (Af_i)u \), we have \( L_i \cap L_j = J^2 e/J^t e \) for each \( i \neq j \) in \( \{1, \ldots, n\} \). By (2.4.1), each \( L_i \) is uniserial and \( h(L_i) = 3 \). Further \( \text{soc} L_i = J^2 e/J^t e \) is simple and \( \text{soc}^2 L_i = J^2 e/J^t e \) for each \( i = 1, \ldots, n \).

1. For any \( i \neq j \) in \( \{1, \ldots, n\} \), the identity map \( p: \text{soc} L_i \to \text{soc} L_j \) is extendable to a homomorphism \( \text{soc}^2 L_i \to \text{soc}^2 L_j \) since \( L_i \cap L_j = J^2 e/J^t e = \text{soc}^2 L_i = \text{soc}^2 L_j \). Hence by (2.3.2), \( p \) is extendable to an isomorphism \( L_i \to L_j \). Thus all simple submodules of \( Je/J^t e \) are pairwise isomorphic.

2. Putting \( p_i: J^2 e/J^t e \to L_i \) and \( q_i: L_i \to L_i + L_2 \) to be inclusion maps for \( i = 1, 2 \), we have an exact sequence

\[
0 \to J^2 e/J^t e \xrightarrow{(p_1, p_2)} L_i \oplus L_2 \xrightarrow{q_1 - q_2} L_1 + L_2 \to 0
\]

where \( L_1 + L_2 \) is colocal. Hence the identity map \( r: \text{soc}^2 L_i \to \text{soc}^2 L_2 \) is not extendable to any isomorphism \( L_1 \to L_2 \). On the other hand, the identity map \( p: \text{soc} L_1 \to \text{soc} L_2 \) is extendable to an isomorphism \( s: L_1 \to L_2 \) since \( r|\text{soc} L_1 = p \). As a consequence, \( s|\text{soc}^2 L_1 \neq r \). But if \( J^2 e/J^t e \cong J^3 e/J^4 e \), then the restriction map

\[
\text{Hom}_A(\text{soc}^2 L_1, \text{soc}^2 L_2) \to \text{Hom}_A(\text{soc} L_1, \text{soc} L_2)
\]

is an injection. This implies that \( s|\text{soc}^2 L_1 = r \) since both \( s|\text{soc}^2 L_1 \) and \( r \) are extensions of \( p \). This is a contradiction. 

**Proposition 2.4.3.** Assume that \( e, f \) and \( g \) are in \( \text{pi}(A) \), \( h(Ae) \geq 5 \) and \( Ae \) is not uniserial, \( Jf/J^2 e \neq 0 \) and \( J^2 e/J^3 e \cong Ag/Jg \). Then \( fAf/JfJf \cong gAg/gJg \) as rings.

**Proof.** There exists a submodule \( L \) of \( Je/J^t e \) such that \( L \) is uniserial of height 3 and top \( L = Af/Jf \), top \( JL = Ag/Jg \). We identify these isomorphic modules. Further \( Af \) and \( Ag \) are both uniserial by (2.4.1) and the fact that \( h(Ae) \geq 5 \) and also \( h(Af) \geq 4 \). Then we can define a homomorphism \( t: \text{End}_A(Af/Jf) \to \text{End}_A(Ag/Jg) \) by \( t(p) := (q|Jf|J^2 f) \) for each \( p \) in \( \text{End}_A(Af/Jf) \) where \( p \) is induced by some \( q \) in \( \text{End}_A(Af/J^2 f) \) and \( r \) is the map in \( \text{End}_A(Jf/J^2 f) \) induced by \( r \) for every \( r \) in \( \text{End}_A(Jf/J^2 f) \). (We identified \( \text{End}_A(Jf/J^2 f) = \text{End}_A(Ag/Jg) \).)
Then $t$ is well-defined and injective since for each $q$ in $\text{End}_A(Af|J^3f)$, $(Af|J^3f)q \leq Jf|J^3f$ iff $(Jf|J^3f)q \leq Jf|J^3f$ (See [10, section 3]). Further by (2.3.2), every automorphism $p$ of $\text{soc} L$ is extendable to an automorphism of $L$ if $p$ is extendable to an automorphism of $\text{soc}^2 L$. Thus $t$ is surjective. (Note that both $Af|J^3f$ and $Jf|J^3f$ are quasi-projective since we have $Jf|J^3f \approx Ag|J^2g$ from the fact that $Ag$ is uniserial.) Hence $fAf/Jf \approx \text{End}_A(Af|Jf) \approx \text{End}_A(Ag|Jg) \approx gAg|Jg$ as rings.

**Remark.** In the above, if $A$ is a $k$-algebra, then the isomorphism defined as above is a $k$-algebra isomorphism.

2.4.4. Proof of Theorem 1 (4) and (5). Assume that $A$ is an algebra and suppose that $Ae$ is not uniserial, and $h(Ae) \geq 4$. Let $p: \bigoplus_{i=1}^n P_i \to Je|J^3e$ be a projective cover of $Je|J^3e$ where each $P_i$ is indecomposable. Then $n \geq 2$. By (2.4.2), there is an $f$ in $\pi(A)$ such that every $P_i$ is isomorphic to $Af$. And, $J^2e|J^3e \approx Ag|Jg$ for some $g$ in $\pi(A)$. If we put $L_i := (P_i)p$ for $i = 1$, 2, then $L_i \approx Af|J^3f$, $J^2e|J^3e \leq L_i \leq Je|J^3e$, $L_i \cap L_2 = J^2e|J^3e$ and top $L_i \approx Af|Jf$ for each $i = 1$, 2. Since we have an exact sequence

$$0 \to J^2e|J^3e \to L_1 \oplus L_2 \to L_1 + L_2 \to 0$$

where $J^2e|J^3e \approx Ag|Jg$, $L_1 \oplus L_2 \approx (Af|J^3f)$ and $L_1 + L_2$ is colocal, there exists an infusible homomorphism $Ag|Jg \to (Af|J^3f)$ by (1.3; 1). Therefore $(fA|Jf)$ is isomorphic to a direct summand of $gJ|J^2$ by (1.4.2). Hence dim $(gJ|J^2) \geq 2$. If $h(Ae) > 5$ or $k$ is algebraically closed, then by (2.4.3), $d := \dim_{Ae} (gJ|J^2) = \dim (gJ|J^2)_{fAf|Jf} \geq 2$. Hence $(Ag|Jg)$ is isomorphic to a direct summand of $Jf|J^3f$ and $d \geq 2$. Thus $|Jf|J^3f| \geq 2$. This contradicts the uniseriality of $Af$. Hence $Ae$ must be uniserial. \\

3. Structure of an indecomposable projective right module

**Lemma 3.1.** Let $0 \to K \xrightarrow{\beta} L \xrightarrow{\gamma} M \to 0$ be an exact sequence of left $A$-modules such that $K$ is simple, $D: L = \bigoplus_{i=1}^n L_i$ is a decomposition of $L$ ($n \geq 2$) and for each $i = 1, \ldots, n$, $L_i \approx Ae_i|I_i$ for some $e_i$ in $\pi(A)$ and $J^{m+1}e_i \leq I_i \leq J^m e_i$ ($m \geq 1$). Then $JM = \text{soc}^n M$ if $(p, D)$ is infusible.

Proof. Put $l_i := e_i + I_i$, $l_i = l_i + JL$, $m_i := l_iq$, $m_i := m_i + JM$ and $m_i := m_i + \text{soc}^n M$. Then we have $\bigoplus_{i=1}^n A\bar{l}_i = L/JL \approx M/JM = \bigoplus_{i=1}^n A\bar{m}_i$ where each $A\bar{m}_i$ is simple. It follows from $h(M) \leq m + 1$ that $JM \leq \text{soc}^n M$. Assume that $JM \leq \text{soc}^n M$. Then we show that $(p, D)$ is fusible. (Clearly, we may assume that each $p_i \neq 0$ i.e. each $p_i$ is a monomorphism where $(p, D) = (p_i)_{i=1}^n$.) By
assumption the sum \( M/\text{soc}^2 M = \sum_{i=1}^{n} Am_i \) is redundant i.e. \( Am_i \leq \sum_{i=1}^{n} Am_i \) for some \( j \), say \( j=1 \). So \( m_1 = \sum_{i=1}^{n} -a_i m_i \) for some \( a_i \) in \( A \). By putting \( a_1 = 1 \), we have \( \sum_{i=1}^{n} a_i m_i \in \text{soc}^2 M \) and \( J^m(a_i l_i)_{i=1}^n \cdot q = 0 \). Thus \( J^m(a_i l_i)_{i=1}^n \leq \text{Im} \ p \). Further putting \( e := e_1 \) we may assume that \( a_i = e a_i \) for each \( i \neq 1 \). Put \( l_i := (a_i l_i)_{i=1}^n \). Then we have \( l_i \in L_i \), \( l \in \oplus_{i=1}^{n} L_i \), \( l = e l_i \), \( l = e l \) and \( J^m(l_i, l) \leq \text{Im} \ p \). On the other hand, it holds that \( J^m(l_i, l) \neq 0 \) since we have \( J^m l_i \neq 0 \) by the assumption \( I_i \leq \text{soc} \). Accordingly, \( J^m(l_i, l) = \text{Im} \ p \) since \( \text{Im} \ p \) is simple. Define a map \( r : \mathcal{E} \rightarrow \mathcal{E} \) by \( x l_j \rightarrow x l \) for each \( x l_j \in L_j \). Then \( r \) is well-defined. In fact, if \( x = 0 \), then \( x e \in \mathcal{E} \) and then \( x e (l_i, l) \in \text{Im} \ p \). Thus \( x e (l_i, l) = s p \) for some \( s \) in \( K \). Therefore \( sp_1 = x e l_i = x l_i = 0 \) and \( s (p_i)_{i=1}^n = x e l_i \). But since \( p_i \) is a monomorphism, we have \( s = 0 \) and \( x l = x e l_i = 0 \). Further by the similar argument as in (2.3.1), \( p r = (p_i)_{i=1}^n \) i.e. \( (p, D) \) is fusible. //

**Proposition 3.2.** Let \( A \) be a ring with selfduality which is of right 2nd local type, \( m \geq 2 \), \( e, f, \cdots, f_n (n \geq 2) \) in \( \text{pi}(A) \) and \( p : \bigoplus_{i=1}^{n} f_i A \rightarrow e J^m |e J^{m+1} \) be a projective cover of \( e J^m |e J^{m+1} \). Then \( p^* : A e |J e \rightarrow \bigoplus_{i=1}^{n} f_i j J f_i^{m+1} f_i \) is infusible.

Proof. Let \( 0 \rightarrow A e |J e \rightarrow \bigoplus_{i=1}^{n} f_i j J f_i \rightarrow M \rightarrow 0 \) be an exact sequence. Then \( M \) is indecomposable by (1.4.3). By (3.1), \( J M = \text{soc}^n M \). Accordingly, \( J M \) is indecomposable since \( J M \geq \text{soc}^2 M \) and \( \text{soc}^2 M \) is indecomposable. Then from the exact sequence \( 0 \rightarrow A e |J e \rightarrow \bigoplus_{i=1}^{n} f_i j J f_i \rightarrow J M \rightarrow 0 \), we obtain that \( p^* : A e |J e \rightarrow \bigoplus_{i=1}^{n} f_i j J f_i^{m+1} f_i \) is infusible by (1.3). ///

3.3. Proof of Theorem 1 (2). Let \( p : \bigoplus_{i=1}^{n} f_i A \rightarrow e J^m \) be a projective cover of \( e J^m \) and \( f_i \) in \( \text{pi}(A) \) for each \( i = 1, \cdots, n \). If \( n = 1 \), then the assertion is trivial. So we may assume that \( n \geq 2 \). There is some \( u_i \) in \( e J^m \setminus e J^{m+1} f_i \) such that the \( i \)-th coordinate map of \( p \) is the right multiplication by \( u_i \) for each \( i = 1, \cdots, n \). Put \( u_i := u_i + e J^{m+1}, u'_i := u_i + e J^{m+1} f_i \) and \( e' := e + J e \). Then \( e J^m = \bigoplus_{i=1}^{n} u_i A \) where each \( u_i A \) is local. Suppose that \( e J^m \) is not a direct sum of local modules. Then \( \sum_{i=1}^{n} u_i a_i = 0 \) for some \( a_i \) in \( A \) and \( u_i a_j \neq 0 \) for some \( j = 1, \cdots, n \). We may assume that there is some \( g \) in \( \text{pi}(A) \) such that \( u_i a_j \neq 0 \) and \( a_i = f_i a_i g \) for each \( i = 1, \cdots, n \). Then it holds that \( a_i \) is in \( f_i J g \) for each \( i \). In fact, if \( f_i \neq g \), then \( a_i \in f_i A g = f_i J g \). And, in case \( f_i = g \), we have \( f_i A g = f_i A J f_i J f_i = f_i J g \) is a division ring. Furthermore, \( \sum_{i=1}^{n} u_i a_i = 0 \) implies \( \sum_{i=1}^{n} u_i a_i = 0 \) and hence each \( u_i a_i = 0 \), since \( u_i A \) are independent. Then putting \( a_i := a_i + f_i J g \), we have that \( u_i a_i \) is defined and is zero. Hence if \( a_i \) is not in \( f_i J g \), then \( u_i = (u_i a_i) a_i^{-1} = 0 \), a con-
tradiction. Further $Au_i = J^m f_i$ since $J^m f_i$ is uniserial for $m \geq 2$. Therefore we may assume that $Au_i a_i = AU_i a_n$ for each $i$ and $Au_2 a_n = J^s g$ for some $s \geq m + 1 \geq 3$. Define a homomorphism $q_i: A f_i | J^{m+1} f_i \to Ag | J^{s+1} g$ by $x \mapsto xa_i$ for each $i = 1, \ldots, n$. Then $q_n$ is a monomorphism since $soc(A f_n | J^{m+1} f_n) = J^m f_n | J^{m+1} f_n$ is simple and is mapped by $q_n$ onto the simple module $J^s g | J^{s+1} g$. Further putting $q_i := q_i | (f_i | J^{m+1} f_i)$, we have $\text{Im} q_i \subseteq soc^m (J g | J^{s+1} g) = J^{m+1} g | J^{s+1} g = \text{Im} q_n$ for each $i = 1, \ldots, n$. Hence if we put $q_i := q_i | J f_i | J^{m+1} f_i$, then $p_i : Ag | J e \to \bigoplus_{i=1}^n J f_i | J^{m+1} f_i$ is fusible since $e' p^* q = 0$ and $q_i'$ is an isomorphism. This contradicts (3.2). Hence $J^{m}$ must be a direct sum of local modules.

3.4. Proof of Theorem 1 (3) and (6). Suppose that $|L J' / L J^{s+1}| \geq 2$ for some $s \geq 1$. $L J'$ is a direct sum of local modules for $L J'$ is a direct summand of $e f^{2+s}$. Further $L = a A$ for some $v$ in $e f^2 g \setminus e f^3 g$ and for some $g$ in $\pi(A)$. Hence $L J' = v f^2 g = u_1 A \oplus u_2 A \oplus \cdots$ for some $u_i$ in $e f^{2+s} f_i \setminus e f^{3+s} f_i$ where $f_i$ are in $\pi(A)$. Then for each $i = 1, 2$, there is some $a_i$ in $g f^2 f_i$ such that $u_i = v a_i$. Define a map $p_i : Ag | J^s g \to A f_i | J^{s+1} f_i$ by $x \mapsto xa_i$ for each $i = 1, 2$. Then $p_1$ and $p_2$ are both monomorphisms since putting $v' := v + J^s g$ and $u_i := u_i + J^{s+1} f_i$, $soc(A g | J^3 g) - J^s g | J^3 g = Av'$ and $soc(J f_i | J^{s+1} f_i) = J^{s+1} f_i | J^{s+1} f_i = A u_i$ are simple modules and $(Av') p_i = A u_i$ for each $i = 1, 2$. In particular, $Ag$ is uniserial by Theorem 1 (1).

i) In case $s \geq 2$. By the above,

$$Av' \xrightarrow{(p_1, p_2)} (J f_i | J^{s+1} f_i) \oplus (J f_i | J^{s+1} f_i)$$

is fusible. Also, $soc^3 (A f_i | J^{s+1} f_i) = J^{s+1} f_i | J^{s+1} f_i$ is uniserial. Hence

$$Av' \xrightarrow{(p_1, p_2)} (A f_i | J^{s+1} f_i) \oplus (A f_i | J^{s+1} f_i)$$

is fusible by (2.3.2), say 2-fusible. Then for some $a$ in $f_i A f_i$, the diagram

$$\begin{array}{c}
Av' \xrightarrow{p_1} A f_i | J^{s+3} f_i \\
\downarrow \\
Av' \xrightarrow{p_2} A f_i | J^{s+2} f_i
\end{array}$$

is commutative. Therefore $u_i := u_i + e J^{s+3}$ for each $i = 1, 2$, we have $u_2 = u_1 a$ since $u_2$ is in $u_1 a + e J^{s+3} f_2$. Thus $u_2 A \subseteq u_1 A$. This contradicts the linear independency of $u_1 A$ and $u_2 A$.

ii) In case the base field $k$ is algebraically closed. It remains only the case $s = 1$. Similarly, it holds that

$$Av' \xrightarrow{(p_1, p_2)} (J f_i | J^s f_i) \oplus (J f_i | J^s f_i)$$
is fusible. But since $0 \neq u_i \leq eJ^i f_i \leq J^i f_i$ for each $i=1, 2, h(Af_i) \geq 4$ and then $Af_i = J^i f_i$ is uniserial of length 4 and $J f_i | J^i f_i = \text{soc}^2(Af_i | J^i f_i)$ by Theorem 1 (5). Then

$$Ae' \supseteq \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \left( Af_i | J^i f_i \right) \oplus \left( Af_i | J^i f_i \right)$$

is fusible by (2.3.2). Hence by the same argument as in i) we have a contradiction.

4. QF rings of right 2nd local type

Lemma 4.1. Let $A$ be a QF ring and $e$ and $f$ be in $\pi(A)$ such that $fJ e | fJ^2 e \neq 0$. Then

(a) If $fJ e | fJ^2 e$ is simple, then $h(Af) > h(Ae)$; and
(b) If $fJ e | fJ^2 e$ is simple, then $h(eA) > h(fA)$.

Proof. (a). It follows from the fact that $fJ e | fJ^2 e$ is simple and $fJ e | fJ^2 e \neq 0$ that there is an epimorphism $p: Af \to Je$. If $p$ is a monomorphism, then $Je$ is injective and is a direct summand of $Ae$. Thus $Je = 0$ and $Je$ is small in $Ae$. But this is impossible since $Je | fJ^2 e$ is simple. Therefore $\text{Ker } p \supseteq \text{soc } Af = J^{h(Af)-1} f$ since $Af$ is colocal. Hence $h(Af) > h(Je) + 1 = h(Ae)$.

(b) Similar.

4.2. Proof of Theorem 2. Let $(x)'$ be the left side version of $(x)$ for each $x = 1, 3$. We show the following implications: $(1) \Rightarrow (3)' \Leftrightarrow (3) \Rightarrow (6) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$. Note that $(2) \Rightarrow (1)'$ is clear since $A$ has a selfduality. Denote by $D$ the selfduality $\text{Hom}_A(\cdot, A)$ of $A$.

(1) $\Rightarrow$ (3)'. Let $e$ be in $\pi(A)$ and $h: = h(Ae) \geq 4$. Then $fJ^2 e$ is a uniserial waist in $Ae$. Hence $\text{soc}^2 eA = D(Ae | fJ^2 e)$ is a waist in $eA = D(Ae)$ and $\text{soc}^2 eA = eJ^{h-2}$ is a direct sum of local modules for $h-2 \geq 2$. But since $eJ^{h-2} \leq eA$ and $eA$ is colocal, $eJ^{h-2}$ is local. Hence $|Je | fJ^2 e| = |\text{soc}^2 eA | \text{soc} (eA)| = 1$ and $Ae$ is uniserial.

(3)' $\Leftrightarrow$ (3). Clear from the fact that both height and uniseriality are preserved by $D$.

(3) $\Rightarrow$ (6). By the equivalence $(3) \Leftrightarrow (3)'$ and left-right symmetry, it is sufficient to prove that under the assumption (3)', if $A$ is an indecomposable ring and $J^2 \neq 0$, then $A$ is a left serial ring. Let $Q$ be the left quiver of $A$, namely the oriented graph with vertex set $\{1, \ldots, p\}$ where $\pi(A) = \{e_1, \ldots, e_p\}$ and with $n_{ji}$ arrows $i \to j$ iff $\dim e_i A e_j < e_j A e_i = e_i A e_j = n_{ij}$. Note that $A$ is an indecomposable ring iff $Q$ is connected. It follows from $J^2 \neq 0$ that $h(Ae_i) \geq 4$ for some $i = 1, \ldots, p$ and then $Ae_i$ is uniserial by (3)'. By 4.1 and the selfduality $D$, we have $h(Ae_i) \geq h(Ae_i) (\geq 4)$ if either

(a) there is an arrow $i \to j$; or
(b)  there is an arrow \( j \rightarrow i \).

Hence \( A \alpha_i \) is uniserial of height \( \geq 4 \) for any \( j = 1, \ldots, p \) by (4.1), (3)' and the fact that \( Q \) is connected. Thus \( A \) is a left serial ring.

(6) \( \Rightarrow \) (4). Clear from the fact that for a serial ring \( A, A \) is \( QF \) iff the admissible sequence of \( A \) is constant.

(4) \( \Rightarrow \) (5). Let \( M_A \) be indecomposable of height \( h \geq 3 \). Then \( A/J^h \) is \( QF \) by (4). Let \( 0 \rightarrow K \leftarrow \bigoplus P_i \rightarrow M \rightarrow 0 \) be a projective cover of \( M \) over \( A/J^h \) with each \( P_i \) indecomposable. Then \( \text{soc} \left( \bigoplus P_i \right) \subseteq K \) implies that \( \text{soc} \ P_i \subseteq K \) for some \( i = 1, \ldots, m \) and then \( P_i \cap K = 0 \) since \( P_i \) is colocal. Hence \( P_i \) is embedded into \( M \). But since \( P_i \) is injective, \( P_i \) is isomorphic to a direct summand of \( M \). Hence \( P_i \cong M \) for \( M \) is indecomposable. Further \( P_i \cong eA/eJ^h \) for some \( e \) in \( \text{pi}(A) \).

(5) \( \Rightarrow \) (1). Clear. //

5. Examples

In this section, we give some examples using bounden quiver algebras over an algebraically closed field \( k \). (See Gabriel [8] for details concerning bounden quiver algebras.)

**Example 1.** Let \( A \) be the algebra defined by the following bounden quiver:

\[
\begin{array}{c}
1 \xrightarrow{\beta} 2; \\
\gamma \end{array}
\]

\( \alpha \)

namely, the algebra having \( \{e_1, e_2, \alpha, \beta, \gamma, \gamma \beta, \beta \gamma \} \) as \( k \)-basis and with multiplication given by the following table:

<table>
<thead>
<tr>
<th></th>
<th>right left</th>
<th>( e_1 )</th>
<th>( e_2 )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \alpha \beta )</th>
<th>( \beta \gamma )</th>
</tr>
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<tbody>
<tr>
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<td>( e_1 )</td>
<td>( \alpha )</td>
<td>( \beta )</td>
<td>( \gamma )</td>
<td>( \gamma \beta )</td>
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<td>( e_2 )</td>
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<td>( \beta )</td>
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(each blank is zero).
Then $A$ is weakly symmetric and hence $QF$. Further as easily seen, $A$ has cube-zero radical. Therefore $A$ is of right (and left) 2nd local type by Theorem 2. But since $A$ is not a serial ring, $A$ is neither of right (1st) local type nor of left (1st) local type.

**Example 2.** Let $A$ be the algebra defined by the following quiver $Q$:

```
\[
\begin{array}{c}
  5 \\
  \downarrow \delta \\
  4 \xrightarrow{\gamma} 1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3,
\end{array}
\]
```

namely, the algebra having \{\(e_1, e_2, e_3, e_4, e_5, \alpha, \beta, \gamma, \delta, \alpha\beta\} as $k$-basis with multiplication given by the following table:

<table>
<thead>
<tr>
<th>left</th>
<th>right</th>
<th>$e_1$</th>
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<th>$e_3$</th>
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<th>$e_5$</th>
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<th>$\gamma$</th>
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<td>$\beta$</td>
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</tr>
<tr>
<td>$e_3$</td>
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<td></td>
<td></td>
<td>$e_3$</td>
<td></td>
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<td>$\gamma$</td>
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<tr>
<td>$e_4$</td>
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<td></td>
<td>$e_4$</td>
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<td>$\delta$</td>
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<tr>
<td>$e_5$</td>
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<td>$e_5$</td>
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<td></td>
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<td>$\alpha\beta$</td>
</tr>
</tbody>
</table>

(each blank is zero).

Then as easily verified, $A$ satisfies all the conditions stated in Theorem 1. But it is not of right 2nd local type. For instance, let $M$ be the right $A$-module corresponding to the following $k$-representation of $Q^{op}$ (the opposite quiver of $Q$, with all arrows reversed)

```
\[
\begin{array}{c}
k \\
\downarrow (1,0) \\
k \oplus k \\
(0,1) \xleftarrow{k} (1,1) \xrightarrow{k} k \\
\end{array}
\]
```

namely, the module having \{\(m_1, m_2, m_3, m_4, m_5, m_6\) as $k$-basis and with right $A$-action given by the following table:
Then $M$ is indecomposable but $\text{top}^2 M$ is decomposable:

$$\text{top}^2 M = \left[ \begin{array}{c} 0 \\ k \leftrightarrow k \rightarrow k \rightarrow 0 \end{array} \right] \oplus \left[ \begin{array}{c} k \\ 1 \end{array} \right].$$

Hence the conditions stated in Theorem 1 are not sufficient for algebras (even if $k$ is algebraically closed) to be of right 2nd local type.

**Example 3.** Let $A$ be the algebra defined by the following bounded quiver:

![Diagram](https://example.com/quiver.png)

Then we can see that $A$ has just 13 indecomposable left modules (up to isomorphism), all of which have indecomposable second tops and second socles since the indecomposable left $A$-modules of height $\geq 3$ are both projective and injective. Hence $A$ is of right and left 2nd local type.\(^1\) But it is neither of right (1st) local type nor of left (1st) local type. For instance, let $M_1$ and $M_2$ be the left $A$-modules corresponding to the following $k$-representations of the bounded quiver:

\[^1\] In Part II of this series of papers, we shall give some necessary and sufficient conditions for artinian rings to be of right and left $n$-th local type for any natural number $n$. Using this result, it is clear that the algebra defined in Example 3 is of right and left 2nd local type.
Then $M_1$ and $M_2$ are indecomposable but $M_1$ is not colocal and $M_2$ is not local.

References


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