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## ON ALGEBRAS OF SECOND LOCAL TYPE, I

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Throughout this paper,  $A$  denotes a (left and right) artinian ring with identity 1,  $J$  its Jacobson radical and all modules are (unital and) finitely generated.

Let  $n$  be any natural number. Then we say that  $A$  is of *right  $n$ -th local type* in case for every indecomposable right  $A$ -module  $M$ , the  $n$ -th top top $"M":=M/MJ^n$  of  $M$  is indecomposable. (Note that if top $"M"$  is indecomposable, then so is  $M$  since  $A$  is artinian and  $M$  is finitely generated.) Hence for such a ring  $A$ , the question of indecomposability of right  $A$ -modules can be reduced to the corresponding problem of right  $A/J^n$ -modules. In [11] H. Tachikawa has studied the case  $n=1$  and obtained a necessary and sufficient condition for algebras (by algebra we always mean a finite dimensional algebra over a field  $k$ ) to be of this type. Further the representation theory of algebras with square-zero radical is well known [5], [6], [7]. So in this paper, we examine the case  $n=2$  and give some necessary conditions for rings with selfduality to be of this type. Further in particular for *QF* (=quasi-Frobenius) rings, we give necessary and sufficient conditions to be of this type. More precisely, we show the following two theorems:

**Theorem 1.** *Let  $A$  be a ring with selfduality which is of right 2nd local type and  $e$  any primitive idempotent in  $A$ . Then*

- (1)  *$J^2e$  is a uniserial waist in  $Ae$  if  $J^2e \neq 0$  (see section 2 for definition of a waist),*
- (2)  *$eJ^m$  is a direct sum of local modules for every  $m \geq 2$ ,*
- (3) *for each local direct summand  $L$  of  $eJ^2$ ,  $LJ^2$  is uniserial (thus  $eJ^4$  is a direct sum of uniserial modules).*

*Further if  $A$  is an algebra, we have*

- (4)  *$Ae$  is uniserial if  $h(Ae) \geq 5$ .*

*In particular if the base field  $k$  is, in addition, an algebraically closed field, then*

- (5)  *$Ae$  is uniserial if  $h(Ae) \geq 4$ ,*

*and then*

- (6)  *$eJ^2$  is a direct sum of uniserial modules.*

**Theorem 2.** *Let  $A$  be a QF ring. Then the following statements are equivalent:*

- (1)  $A$  is of right 2nd local type.
- (2)  $A$  is of right 2nd colocal type (see section 1 for definition).
- (3) For any primitive idempotent  $e$  in  $A$ ,  $eA$  is uniserial if  $h(eA) \geq 4$ .
- (4)  $A/J^t$  is QF for every  $t \geq 3$ .
- (5) For each  $M_A$  indecomposable with  $h(M) \geq 3$ , there is a primitive idempotent  $e$  in  $A$  such that  $M \cong eA/eJ^{h(M)}$ .
- (6)  $A = A_1 \times A_2$  for some QF rings  $A_1$  and  $A_2$  such that  $A_1$  has cube-zero radical and  $A_2$  is a serial ring.

Furthermore, each of these conditions are equivalent to the corresponding left side version.

In the theorems above  $h(M)$  denotes the height (=Loewy length) of  $M$ , namely  $h(M) := \min\{n \in \mathbb{N} \cup \{0\} \mid MJ^n = 0\}$ . We remark that Theorem 1 (5) and (6) remain valid also in the case where  $k$  is a splitting field for  $A$ .

In section 1, we introduce the basic tools used in the following sections. Section 2 is devoted to the structure of an indecomposable projective left module and in section 3, we examine the structure of an indecomposable projective right module mainly using the technique of Sumioka [10]. In section 4, we give the proof of Theorem 2. Finally in section 5, we give some examples.

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## 1. Preliminaries

**1.1.** Throughout the paper, we write homomorphisms on the opposite side to scalar multiplications, and for homomorphisms  $p: K \rightarrow L$  and  $q: L \rightarrow M$  of left  $A$ -modules and for a decomposition  $D: L = \bigoplus_{i=1}^n L_i$  of  $L$ ,  $(p, D) = (p_i)_{i=1}^n$  and  $(D, q) = (q_i)_{i=1}^n$  are matrix expressions of  $p$  and  $q$  relative to  $D$ , respectively (for homomorphisms of right  $A$ -modules, we write as  $(p, D) = (p_i)_{i=1}^n$  and  $(D, q) = (q_i)_{i=1}^n$ ). In addition to the definition of right  $n$ -th local type for  $n$  any natural number, we define the dual notion:  $A$  is called to be of *left n-th colocal type* in case for every indecomposable left  $A$ -module  $M$ , the  $n$ -th socle  $\text{soc}^n M := (\text{the right annihilator of } J^n \text{ in } M)$  of  $M$  is indecomposable. It should be noted that if  $A$  has a selfduality, then  $A$  is of right  $n$ -th local type iff  $A$  is of left  $n$ -th colocal type. Further noting that the composition lengths of the projective covers (over  $A$ ) of all indecomposable right  $A/J^n$ -modules have a bound if  $A/J^n$  is of finite representation type (i.e. it has only finitely many isomorphism classes of indecomposable right modules), we see easily that when  $A$  is of right  $n$ -th local type,  $A$  is of finite representation type iff so is  $A/J^n$  (See

Auslander [3]).

Since the property to be of  $n$ -th local (colocal) type is Morita invariant, we may assume that  $A$  is a basic ring. We put  $\pi(A) := \{e_1, \dots, e_p\}$  to be a basic set of primitive idempotents of  $A$ .

**DEFINITION 1.2** ([2]). Let  $D: L = \bigoplus_{i=1}^n L_i$  be a decomposition of a right  $A$ -module  $L$  and  $p: K \rightarrow L$  be a homomorphism, and  $j$  in  $\{1, \dots, n\}$ . Then the pair  $(p, D)$  (or simply  $p: K \rightarrow \bigoplus_{i=1}^n L_i$ ) is called *j-fusible* in case there is a homomorphism  $q: \bigoplus_{i \neq j} L_i \rightarrow L_j$  such that the diagram

$$\begin{array}{ccc} K & \xrightarrow{(p_i)_{i \neq j}} & \bigoplus_{i \neq j} L_i \\ \parallel & & \downarrow q \\ K & \xrightarrow{p_j} & L_j \end{array}$$

commutes where  $(p, D) = (p_i)_{i=1}^n$ . The pair  $(p, D)$  is called *fusible* in case  $(p, D)$  is  $j$ -fusible for some  $j = 1, \dots, n$ . Finally  $(p, D)$  is called *infusible* in case  $(p, D)$  is not fusible.

**Corollary 1.2.1** ([2, Corollary 1.4]). *Let  $K_i \leq L_i$  for each  $i = 1, 2$  and  $h: K_1 \rightarrow K_2$  be an isomorphism. Define  $p_1 = k_1$ ,  $p_2 = k_2 h$  where  $k_i: K_i \rightarrow L_i$  is the inclusion map for each  $i$ . Then  $h$  or  $h^{-1}$  is extendable to a homomorphism  $L_1 \rightarrow L_2$  or  $L_2 \rightarrow L_1$ , respectively iff  $p: K_1 \rightarrow L_1 \oplus L_2$  is fusible.* //

**Proposition 1.2.2** ([2, Proposition 1.1]). *Consider an exact sequence  $K \xrightarrow{p} L \xrightarrow{q} M \rightarrow 0$  of right  $A$ -modules and let  $D: L = \bigoplus_{i=1}^n L_i$  be a decomposition of  $L$ ,  $(p, D) = (p_i)_{i=1}^n$ ,  $(D, q) = (q_i)_{i=1}^n$  and  $j$  in  $\{1, \dots, n\}$ . Then the following statements are equivalent:*

(1)  $(p, D)$  is  $j$ -fusible.

(2) *There is a homomorphism  $r = (r_i)_{i=1}^n: \bigoplus_{i=1}^n L_i \rightarrow X$  such that  $rp = 0$  and  $r_j$  is an isomorphism.*

(3)  $q_j$  is a split monomorphism.

Proof. See [2]. //

REMARK. In [2] the fusible maps were defined by the condition (2) above.

**Proposition 1.3** *Let  $0 \rightarrow K \xrightarrow{p} L \xrightarrow{q} M \rightarrow 0$  be a nonsplit exact sequence of right  $A$ -modules and  $D: L = \bigoplus_{i=1}^n L_i$  be a decomposition of  $L$  ( $n \geq 2$ ). Then we have*

(1) *if  $M$  is indecomposable, then  $(p, D)$  is infusible,*

(2) if  $K$  is simple, each  $L_i$  is local and  $(p, D)$  is infusible, then  $M$  is indecomposable.

Proof. See [1] or [2]. //

**1.4.** Let  $I$  be a two-sided ideal of  $A$  and  $e$  and  $f$  in  $\text{pi}(A)$ . Then we have the canonical isomorphisms  $\text{Hom}_A(fA, eA/eI) \xrightarrow{\sim} eAf/eIf \xrightarrow{\sim} \text{Hom}_A(Ae, Af/I)$ . We denote by  $p^*$  the image of every  $p$  in  $\text{Hom}_A(fA, eA/eI)$  or the inverse image of every  $p$  in  $\text{Hom}_A(Ae, Af/I)$  under the composition of these isomorphisms.

**Proposition 1.4.1** Let  $e, f_1, \dots, f_n$  be in  $\text{pi}(A)$ ,  $l > m, j$  in  $\{1, \dots, n\}$  and  $p = (p_i)_{i=1}^n: \bigoplus_{i=1}^n f_i A \rightarrow eJ^m/eJ^l$  be a homomorphism. Then the following statements are equivalent:

- (1)  $p(f_j A) \leq \sum_{i \neq j} p(f_i A)$ .
- (2)  $p^*: Ae/J^{l-m}e \rightarrow \bigoplus_{i=1}^n Af_i/J^l f_i$  is  $j$ -fusible, where  $p^*$  is the map induced by the homomorphism  $(p_i^*)_{i=1}^n$ .

Proof. There is some  $u_i$  in  $eJ^m f_i$  such that each  $p_i^*$  is the left multiplication by  $u_i$ . Then  $p$  has the property stated in (1) iff  $(u_j f_j A + eJ^l)/eJ^l \leq (\sum_{i \neq j} u_i f_i A + eJ^l)/eJ^l$   
iff  $u_j A \leq \sum_{i \neq j} u_i A + eJ^l$   
iff  $u_j = \sum_{i \neq j} u_i a_i + b$ , for some  $a_i$  in  $f_i A$  and  $b$  in  $eJ^l$   
iff  $u_j = \sum_{i \neq j} u_i a_i + b$ , for some  $a_i$  in  $f_i A f_j$  and  $b$  in  $eJ^l f_j$   
iff  $u_j = \sum_{i \neq j} u_i a_i + b$ , for some  $a_i$  in  $f_i A f_j$  and  $b$  in  $J^l f_j$   
iff  $p^*$  is  $j$ -fusible. //

In future  $p^*$  shall always mean the above induced homomorphism when the domain of  $p$  is of the form as above.

**Corollary 1.4.2.** Under the same situation as above but  $l = m + 1$ , the following are equivalent:

- (1)  $\bar{p}: \bigoplus_{i=1}^n f_i A / f_i J \rightarrow eJ^m / eJ^{m+1}$  (the induced map) is a monomorphism.

- (2)  $p^*: Ae/Je \rightarrow \bigoplus_{i=1}^n Af_i / J^{m+1} f_i$  is infusible.

In particular if  $p: \bigoplus_{i=1}^n f_i A \rightarrow eJ^m$  is a projective cover of  $eJ^m$ , then  $p^*: Ae/Je \rightarrow \bigoplus_{i=1}^n Af_i / J^{m+1} f_i$  is infusible. //

**Corollary 1.4.3.** Let  $p: \bigoplus_{i=1}^n f_i A \rightarrow eJ^m$  be a projective cover of  $eJ^m$  and  $0 \rightarrow$

$Ae|Je \xrightarrow{p^*} \bigoplus_{i=1}^n Af_i|J^{m+1}f_i \rightarrow M \rightarrow 0$  be an exact sequence. Then  $M$  is indecomposable.

Proof. Clear from (1.4.2) and (1.3). //

## 2. Structure of an indecomposable projective left module

For an  $A$ -module  $M$ , we put  $|M| :=$  the composition length of  $M$ .

**Proposition 2.1.** *Let  $A$  be of right  $n$ -th local type,  $n$  any natural number and  $e$  in  $\text{pi}(A)$ . Then  $J^n e$  is uniserial.*

Proof. It is sufficient to prove that  $|J^m e| \leq 1$  for every  $m \geq n$ . Suppose  $|J^m e| \geq 2$  for some  $m \geq n$ . Then we have a homomorphism  $p: Af_1 \oplus Af_2 \rightarrow J^m e|J^{m+1}e$ ;  $f_1, f_2$  in  $\text{pi}(A)$  such that the induced map  $\bar{p}: (Af_1/Jf_1) \oplus (Af_2/Jf_2) \rightarrow J^m e|J^{m+1}e$  is a monomorphism. Putting  $L = (f_1 A/f_1 J^{m+1}) \oplus (f_2 A/f_2 J^{m+1})$ , we have an exact sequence  $0 \rightarrow eA/eJ \xrightarrow{p^*} L \rightarrow M \rightarrow 0$  where  $M$  is indecomposable by (1.4.2) and (1.3). But since  $p^*(eA/eJ) \leq LJ^n \leq LJ^m$ ,  $\text{top}^n M \cong \text{top}^m L$  is decomposable. This is a contradiction. //

**DEFINITION 2.2 ([4]).** Let  $_A L \leq_A M$ . Then  $L$  is called to be a *waist* in  $M$  in case  $0 \neq L \neq M$  and for each  $_A N \leq_A M$ , it holds that  $L \leq N$  or  $N \leq L$ .

**Proposition 2.2.1.** *Let  $A$  be a ring with selfduality which is of right 2nd local type and  $e$  in  $\text{pi}(A)$ . Then  $J^2 e$  is a waist in  $Ae$  if  $J^2 e \neq 0$ .*

Proof. Deduced from the following three lemmas for an artinian ring  $A$ :

**Lemma 2.2.2** ([9, Lemma 1.2]). *Let  $_A M$  be nonsimple indecomposable. Then  $\text{soc}(JM) = \text{soc } M$ .*

Proof. Let  $S$  be any simple submodule of  $M$  and  $X$  be any proper submodule of  $M$ . If  $S+X=M$  then  $S$  is not contained in  $X$ . Thus  $S \cap X=0$ . Hence  $S=M$ , a contradiction. Therefore  $S$  is small in  $M$  i.e.  $S \leq JM$ . Hence  $\text{soc } M \leq JM$  and  $\text{soc } M = \text{soc}(JM)$ . //

**Lemma 2.2.3.** *Let  $_A M$  be local and  $\text{soc}^2 M$  indecomposable. Then  $\text{soc}(J^2 M) = \text{soc } M$  if  $J^2 M \neq 0$ .*

Proof. Clear from (2.2.2) nothing that  $JM$  is nonsimple indecomposable since  $J^2 M \neq 0$  and  $\text{soc}^2 M \leq JM$ . //

**Lemma 2.2.4.** *Let  $A$  be a ring of left 2nd colocal type,  $_A M$  be local and  $J^2 M$  be a nonzero uniserial module. Then  $J^2 M$  is a waist in  $M$ .*

Proof. Suppose that  $J^2 M$  is not a waist in  $M$ . Then for some  $X \leq M$ ,

$J^2M \leq X$  and  $X \leq J^2M$ . And,  $J^2M \cap X = J^tM$  for some  $t \geq 3$ . Hence  $M/J^t \geq (J^2M/J^tM) \oplus (X/J^tM)$  where  $J^2M/J^tM \neq 0$  and  $X/J^tM \neq 0$ . On the other hand since  $\text{soc}^2(M/J^tM)$  is indecomposable and  $J^2(M/J^tM) \neq 0$ , we have that  $\text{soc}(M/J^tM) = \text{soc}(J^2M/J^tM)$  is simple by (2.2.3). This is a contradiction. //

We get Theorem 1 (1) from Propositions 2.1 and 2.2.1.

**Corollary 2.2.5.** *Let  $A$  be a ring with selfduality which is of right 2nd local type,  $e$  in  $\text{pi}(A)$  and  $h = h(Ae)$ . Then we have  $\text{soc}^{h-t}(Ae) = J^t e$  for every  $t = 0, \dots, h$ .*

Proof. It is clear from Theorem 1 (1) in case  $t \geq 2$ . The other cases ( $t = 0, 1$ ) are trivial.

**Lemma 2.3.1.** *Let  ${}_A L_1$  and  ${}_A L_2$  be local of height  $\geq 3$  such that for each  $i = 1, 2$ ,  $\text{soc}^3 L_i$  is uniserial and  $J^2 e_i$  is a uniserial waist in  $Ae_i$ , where  $Ae_i$  is the projective cover of  $\text{soc}^3 L_i$ . Suppose that  ${}_A K$  is simple and there exists an isomorphism  $p_i: K \rightarrow \text{soc } L_i$  for each  $i = 1, 2$ . Consider an exact sequence:*

$$0 \rightarrow K \xrightarrow{p=(p_1, p_2)} L_1 \oplus L_2 \xrightarrow{q=\begin{bmatrix} -q_1 \\ -q_2 \end{bmatrix}} M \rightarrow 0.$$

*Then  $\text{soc}^2 M$  is decomposable if  $p: K \rightarrow \text{soc}^2 L_1 \oplus \text{soc}^2 L_2$  is fusible.*

Proof. Assume that  $p: K \rightarrow \text{soc}^2 L_1 \oplus \text{soc}^2 L_2$  is fusible, say 2-fusible. Then we have a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{p_1} & \text{soc}^2 L_1 \\ \parallel & & \downarrow r \\ K & \xrightarrow{p_2} & \text{soc}^2 L_2. \end{array}$$

And,  $M \geq (\text{soc}^2 L_1)q_1 + L_2 q_2 = U \oplus L_2 q_2$  where  $U = (\text{soc}^2 L_1)$  ( $q_1 - rq_2 \neq 0$ ). Now for each  $x$  in  $\text{soc}^2 M$ ,  $x = l_1 q_1 + l_2 q_2$  for some  $(l_1, l_2)$  in  $L_1 \oplus L_2$ . Since  $ux = 0$  for each  $u$  in  $J^2$ , we have  $ul_1 q_1 = -ul_2 q_2 \in L_1 q_1 \cap L_2 q_2 = Kp_1 q_1$  ( $=: S$ ). Hence  $J^2 l_1 q_1 = J^2 l_2 q_2 \leq S$  where  $S$  is simple. In particular,  $\text{soc}^2 M \leq \text{soc}^3 L_1 q_1 + \text{soc}^3 L_2 q_2$ .

i) In case for each  $x$  in  $\text{soc}^2 M$ , there are  $l_1, l_2$  with  $x = l_1 q_1 + l_2 q_2$  such that  $J^2 l_1 q_1 = J^2 l_2 q_2 = 0$ . Then we have  $J^2 l_1 = 0$  for  $q_1$  is monic. Thus  $l_1$  is in  $\text{soc}^2 L_1$  and  $x$  is in  $U \oplus L_2 q_2$ . Therefore  $\text{soc}^2 M \leq U \oplus L_2 q_2$ . Hence  $\text{soc}^2 M$  is decomposable.

ii) In case for some  $x$  in  $\text{soc}^2 M$ , there are  $l_1, l_2$  with  $x = l_1 q_1 + l_2 q_2$  such that  $J^2 l_1 q_1 = J^2 l_2 q_2 = S$ . We may assume that  $x = ex$  for some  $e$  in  $\text{pi}(A)$ . Since  $S$  is simple and  $q_i$  are monic,  $J^3 l_1 = J^3 l_2 = 0$ . Thus  $l_i$  is in  $\text{soc}^3 L_i \setminus \text{soc}^2 L_i$  for each  $i$ . Also, we may assume that  $l_i = el_i$  for each  $i$  since  $x = ex$ . Further we have  $\text{soc}^3 L_i = Ae l_i$  for each  $i = 1, 2$  since  $\text{soc}^3 L_i$  are uniserial. Hence we

may assume that  $e=e_1=e_2$ . Define a homomorphism  $s: \text{soc}^3 L_1 \rightarrow \text{soc}^3 L_2$  by  $a\text{el}_1 \mapsto a\text{el}_2$  for each  $a$  in  $A$ . Then  $s$  is well-defined. In fact, if  $t$  is in  $Ae$  and  $tl_1=0$ , then  $t$  is in  $\text{Ann}_{Ae}(l_1)$ , the annihilator of  $l_1$  in  $Ae$ . On the other hand, by the fact that  $J^2\text{el}_1 \neq 0$ , we see  $\text{Ann}_{Ae}(l_1)$  does not contain  $J^2e$  which is a uniserial waist in  $Ae$ . Hence  $\text{Ann}_{Ae}(l_1)$  is contained in  $J^3e$  and  $t$  is in  $J^3e$ . Thus  $tl_2$  is in  $J^3l_2=0$ .

Further the diagram

$$\begin{array}{ccc} K & \xrightarrow{p_1} & \text{soc}^3 L_1 \\ \parallel & & \downarrow s \\ K & \xrightarrow{p_2} & \text{soc}^3 L_2 \end{array}$$

is commutative. For,  $J^2(l_1, l_2) (\neq 0)$  is contained in the simple module  $\text{Im } p$  since  $J^2(l_1, l_2)q=0$ . Hence  $J^2(l_1, l_2)=\text{Im } p$ . Let  $c$  be a nonzero element in  $K$ . Then  $K=Ac$  and  $cp=(ul_1, ul_2)$  for some  $u$  in  $J^2$ . Therefore  $c(p_1s)=ul_1s=ul_2=c p_2$ . Thus  $p_1s=p_2$ .

Then putting  $V:=(\text{soc}^3 L_1) (q_1-sq_2)$ , the same argument as in i) shows that  $\text{soc}^2 M \leq V \oplus L_2 q_2$  and  $\text{soc}^2 M$  is decomposable. //

**Proposition 2.3.2.** *Let  $A$  be a ring with selfduality which is of right 2nd local type and  ${}_A L_1, {}_A L_2$  be local of height  $\geq 3$  such that  $\text{soc}^3 L_i$  are uniserial and  $|L_1| \leq |L_2|$ . Then for every isomorphism  $r: \text{soc} L_1 \rightarrow \text{soc} L_2$ ,  $r$  is extendable to a monomorphism  $L_1 \rightarrow L_2$  if  $r$  is extendable to a homomorphism  $\text{soc}^2 L_1 \rightarrow \text{soc}^2 L_2$ .*

Proof. Put  $K=\text{soc } L_1$ ,  $p_1=\text{identity map of } \text{soc } L_1$  and  $p_2=r$ . Consider an exact sequence  $0 \rightarrow K \xrightarrow{p=(p_1, p_2)} L_1 \oplus L_2 \xrightarrow{q} M \rightarrow 0$ . If  $r$  is extendable to a homomorphism  $\text{soc}^2 L_1 \rightarrow \text{soc}^2 L_2$ , then  $p: K \rightarrow \text{soc}^2 L_1 \oplus \text{soc}^2 L_2$  is fusible. Hence by (2.3.1),  $\text{soc}^2 M$  is decomposable, thus  $M$  is decomposable. Therefore  $p: K \rightarrow L_1 \oplus L_2$  is fusible by (1.3). Hence by (1.2.1),  $r$  is extendable to a homomorphism  $q: L_1 \rightarrow L_2$  since  $|L_1| \leq |L_2|$  where  $q$  is monic since  $\text{soc } L_1$  is simple. //

**2.4.** Throughout the rest of this section,  $A$  is a ring with selfduality which is of right 2nd local type. Here, we examine indecomposable projective left  $A$ -modules of height  $\geq 4$ .

**Proposition 2.4.1.** *Let  $e$  and  $f$  be in  $\text{pi}(A)$  and  $fJe \neq fJ^2e$ . Then  $Af$  is uniserial if  $h(Ae) \geq 4$ .*

Proof. Take some  $u$  in  $fJe \setminus fJ^2e$  and define  $p: Af \rightarrow Je$  by the right multiplication by  $u$ . Then  $\text{Ker } p \leq J^2f$  or  $\text{Ker } p \geq J^2f$  since  $J^2f$  is a waist in  $Af$  (if  $J^2f \neq 0$ ). Assume that  $\text{Ker } p \geq J^2f$ . Then  $h(\text{Im } p) \leq 2$  since  $\text{Im } p \cong Af/\text{Ker } p$  is an epimorph of  $Af/J^2f$ . Hence  $\text{Im } p \leq \text{soc}^2(Ae) \leq J^2e$  for  $h(Ae) \geq 4$  and  $\text{soc}^2(Ae) = J^{h(Ae)-2}e$ . But by the definition of  $p$  we have  $\text{Im } p \leq J^2e$ , a contradiction.

Accordingly,  $\text{Ker } p \leq J^2f$ . Then  $\text{Ker } p = J^t f$  for some  $t \geq 2$  and  $Af/J^t f$  is embedded into  $Je$ . Therefore  $|Jf/J^2f| = 1$  since  $Jf/J^t f$  is embedded into  $J^2e$  which is uniserial. Hence  $Af$  is uniserial.  $\quad //$

**Proposition 2.4.2.** *Assume that  $e$  is in  $\text{pi}(A)$ ,  $h(Ae) \geq 4$  and  $Ae$  is not uniserial. Then*

- (1) *all simple submodules of  $Je/J^2e$  are pairwise isomorphic, and*
- (2)  *$J^2e/J^3e \cong J^3e/J^4e$ .*

**Proof.** Let  $u: \bigoplus_{i=1}^n Af_i \rightarrow Je/J^4e$  be a projective cover of  $Je/J^4e$ . Then  $n \geq 2$  since  $Ae$  is not uniserial. Putting  $L_i := (Af_i)u$ , we have  $L_i \cap L_j = J^2e/J^4e$ ,  $L_i \not\cong J^2e/J^4e$  for each  $i \neq j$  in  $\{1, \dots, n\}$ . By (2.4.1), each  $L_i$  is uniserial and  $h(L_i) = 3$ . Further  $\text{soc } L_i = J^3e/J^4e$  is simple and  $\text{soc}^2 L_i = J^2e/J^4e$  for each  $i = 1, \dots, n$ .

(1) For any  $i \neq j$  in  $\{1, \dots, n\}$ , the identity map  $p: \text{soc } L_i \rightarrow \text{soc } L_j$  is extendable to a homomorphism  $\text{soc}^2 L_i \rightarrow \text{soc}^2 L_j$  since  $L_i \cap L_j = J^2e/J^4e = \text{soc}^2 L_i = \text{soc}^2 L_j$ . Hence by (2.3.2),  $p$  is extendable to an isomorphism  $L_i \rightarrow L_j$ . Thus all simple submodules of  $Je/J^2e$  are pairwise isomorphic.

(2) Putting  $p_i: J^2e/J^4e \rightarrow L_i$  and  $q_i: L_i \rightarrow L_1 + L_2$  to be inclusion maps for  $i = 1, 2$ , we have an exact sequence

$$0 \rightarrow J^2e/J^4e \xrightarrow{(p_1, p_2)} L_1 \oplus L_2 \xrightarrow{\begin{bmatrix} q_1 \\ -q_2 \end{bmatrix}} L_1 + L_2 \rightarrow 0$$

where  $L_1 + L_2$  is colocal. Hence the identity map  $r: \text{soc}^2 L_1 \rightarrow \text{soc}^2 L_2$  is not extendable to any isomorphism  $L_1 \rightarrow L_2$ . On the other hand, the identity map  $p: \text{soc } L_1 \rightarrow \text{soc } L_2$  is extendable to an isomorphism  $s: L_1 \rightarrow L_2$  since  $r|(\text{soc } L_1) = p$ . As a consequence,  $s|(\text{soc}^2 L_1) \neq r$ . But if  $J^2e/J^3e \cong J^3e/J^4e$ , then the restriction map

$$\text{Hom}_A(\text{soc}^2 L_1, \text{soc}^2 L_2) \rightarrow \text{Hom}_A(\text{soc } L_1, \text{soc } L_2)$$

is an injection. This implies that  $s|(\text{soc}^2 L_1) = r$  since both  $s|(\text{soc}^2 L_1)$  and  $r$  are extensions of  $p$ . This is a contradiction.  $\quad //$

**Proposition 2.4.3.** *Assume that  $e, f$  and  $g$  are in  $\text{pi}(A)$ ,  $h(Ae) \geq 5$ ,  $Ae$  is not uniserial,  $fJe/fJ^2e \neq 0$  and  $J^2e/J^3e \cong Ag/Jg$ . Then  $fAf/fJf \cong gAg/gJg$  as rings.*

**Proof.** There exists a submodule  $L$  of  $Je/J^4e$  such that  $L$  is uniserial of height 3 and top  $L \cong Af/Jf$ , top  $JL \cong Ag/Jg$ . We identify these isomorphic modules. Further  $Af$  and  $Ag$  are both uniserial by (2.4.1) and the fact that  $h(Ae) \geq 5$  and also  $h(Af) \geq 4$ . Then we can define a homomorphism  $t: \text{End}_A(Af/Jf) \rightarrow \text{End}_A(Ag/Jg)$  by  $t(p) := \overline{(q|Jf/J^3f)}$  for each  $p$  in  $\text{End}_A(Af/Jf)$  where  $p$  is induced by some  $q$  in  $\text{End}_A(Af/J^3f)$  and  $\overline{r}$  is the map in  $\text{End}_A(Jf/J^2f)$  induced by  $r$  for every  $r$  in  $\text{End}_A(Jf/J^3f)$ . (We identified  $\text{End}_A(Jf/J^2f) = \text{End}_A(Ag/Jg)$ .)

Then  $t$  is well-defined and injective since for each  $q$  in  $\text{End}_A(Af/J^3f)$ ,  $(Af/J^3f)q \leq Jf/J^3f$  iff  $(Jf/J^3f)q \leq J^2f/J^3f$  (See [10, section 3]). Further by (2.3.2), every automorphism  $p$  of  $\text{soc } L$  is extendable to an automorphism of  $L$  if  $p$  is extendable to an automorphism of  $\text{soc}^2 L$ . Thus  $t$  is surjective. (Note that both  $Af/J^3f$  and  $Jf/J^3f$  are quasi-projective since we have  $Jf/J^3f \cong Ag/J^2g$  from the fact that  $Ag$  is uniserial.) Hence  $fAf/fJf \cong \text{End}_A(Af/Jf) \cong \text{End}_A(Ag/Jg) \cong gAg/gJg$  as rings.

REMARK. In the above, if  $A$  is a  $k$ -algebra, then the isomorphism defined as above is a  $k$ -algebra isomorphism.

**2.4.4. Proof of Theorem 1 (4) and (5).** Assume that  $A$  is an algebra and suppose that  $Ae$  is not uniserial, and  $h(Ae) \geq 4$ . Let  $p: \bigoplus_{i=1}^n P_i \rightarrow Je/J^3e$  be a projective cover of  $Je/J^3e$  where each  ${}_A P_i$  is indecomposable. Then  $n \geq 2$ . By (2.4.2), there is an  $f$  in  $\text{pi}(A)$  such that every  $P_i$  is isomorphic to  $Af$ . And,  $J^2e/J^3e \cong Ag/Jg$  for some  $g$  in  $\text{pi}(A)$ . If we put  $L_i := (P_i)p$  for  $i = 1, 2$ , then  $L_i \cong Af/J^2f$ ,  $J^2e/J^3e \leq L_i \leq Je/J^3e$ ,  $L_1 \cap L_2 = J^2e/J^3e$  and  $\text{top } L_i \cong Af/Jf$  for each  $i = 1, 2$ . Since we have an exact sequence

$$0 \rightarrow J^2e/J^3e \rightarrow L_1 \oplus L_2 \rightarrow L_1 + L_2 \rightarrow 0$$

where  $J^2e/J^3e \cong Ag/Jg$ ,  $L_1 \oplus L_2 \cong (Af/J^2)^{(2)}$  and  $L_1 + L_2$  is colocal, there exists an infusible homomorphism  $Ag/Jg \rightarrow (Af/J^2)^{(2)}$  by (1.3; 1). Therefore  $(fAf/fJf)^{(2)}$  is isomorphic to a direct summand of  $gJ/gJ^2$  by (1.4.2). Hence  $\dim(gJf/gJ^2f)_{fAf/fJf} \geq 2$ . If  $h(Ae) \geq 5$  or  $k$  is algebraically closed, then by (2.4.3),  $d := \dim_{gAg/gJg}(gJf/gJ^2f) = \dim(gJf/gJ^2f)_{fAf/fJf} \geq 2$ . Hence  $(Ag/Jg)^{(d)}$  is isomorphic to a direct summand of  $Jf/J^2f$  and  $d \geq 2$ . Thus  $|Jf/J^2f| \geq 2$ . This contradicts the uniseriality of  $Af$ . Hence  $Ae$  must be uniserial.  $\|$

### 3. Structure of an indecomposable projective right module

**Lemma 3.1.** *Let  $0 \rightarrow K \xrightarrow{p} L \xrightarrow{q} M \rightarrow 0$  be an exact sequence of left  $A$ -modules such that  $K$  is simple,  $D: L = \bigoplus_{i=1}^n L_i$  is a decomposition of  $L$  ( $n \geq 2$ ) and for each  $i = 1, \dots, n$ ,  $L_i = Ae_i/I_i$  for some  $e_i$  in  $\text{pi}(A)$  and  $J^{m+1}e_i \leq I_i \leq J^m e_i$  ( $m \geq 1$ ). Then  $JM = \text{soc}^m M$  if  $(p, D)$  is infusible.*

Proof. Put  $l_i := e_i + I_i$ ,  $\bar{l}_i = l_i + JL$ ,  $m_i := l_i q$ ,  $\bar{m}_i := m_i + JM$  and  $m'_i := m_i + \text{soc}^m M$ . Then we have  $\bigoplus_{i=1}^n A\bar{l}_i = L/JL \cong M/JM = \bigoplus_{i=1}^n A\bar{m}_i$  where each  $A\bar{m}_i$  is simple. It follows from  $h(M) \leq m+1$  that  $JM \leq \text{soc}^m M$ . Assume that  $JM \leq \text{soc}^m M$ . Then we show that  $(p, D)$  is fusible. (Clearly, we may assume that each  $p_i \neq 0$  i.e. each  $p_i$  is a monomorphism where  $(p, D) = (p_i)_{i=1}^n$ .) By

assumption the sum  $M/\text{soc}^m M = \sum_{i=1}^n Am'_i$  is redundant i.e.  $Am'_j \leq \sum_{i \neq j} Am'_i$  for some  $j$ , say  $j=1$ . So  $m'_1 = \sum_{i \neq 1} -a_i m'_i$  for some  $a_i$  in  $A$ . By putting  $a_1=1$ , we have  $\sum_{i=1}^n a_i m'_i \in \text{soc}^m M$  and  $J^m(a_i l_i)_{i=1}^n \cdot q = 0$ . Thus  $J^m(a_i l_i)_{i=1}^n \leq \text{Im } p$ . Further putting  $e:=e_1$  we may assume that  $a_i = ea_i$  for each  $i \neq 1$ . Put  $l:=(a_i l_i)_{i \neq 1}$ . Then we have  $l_1 \in L_1$ ,  $l \in \bigoplus_{i \neq 1} L_i$ ,  $l_1 = el_1$ ,  $l = el$  and  $J^m(l_1, l) \leq \text{Im } p$ . On the other hand, it holds that  $J^m(l_1, l) \neq 0$  since we have  $J^m l_1 \neq 0$  by the assumption  $I_1 \leq J^m e_i$ . Accordingly,  $J^m(l_1, l) = \text{Im } p$  since  $\text{Im } p$  is simple. Define a map  $r: L_1 \rightarrow \bigoplus_{i \neq 1} L_i$  by  $xl_1 \mapsto xl$  for each  $xl_1 \in L_1$ . Then  $r$  is well-defined. In fact, if  $xl_1 = 0$ , then  $xe \in I_1 \leq J^m$  and then  $xe(l_1, l) \in \text{Im } p$ . Thus  $xe(l_1, l) = sp$  for some  $s$  in  $K$ . Therefore  $sp_1 = xel_1 = xl_1 = 0$  and  $s(p_i)_{i \neq 1} = xel$ . But since  $p_1$  is a monomorphism, we have  $s=0$  and  $xl=xel=0$ . Further by the similar argument as in (2.3.1),  $p_1 r = (p_i)_{i \neq 1}$  i.e.  $(p, D)$  is fusible. //

**Proposition 3.2.** *Let  $A$  be a ring with selfduality which is of right 2nd local type,  $m \geq 2$ ,  $e, f_1, \dots, f_n$  ( $n \geq 2$ ) in  $\text{pi}(A)$  and  $p: \bigoplus_{i=1}^n f_i A \rightarrow eJ^m/eJ^{m+1}$  be a projective cover of  $eJ^m/eJ^{m+1}$ . Then  $p^*: Ae/Je \rightarrow \bigoplus_{i=1}^n Jf_i/J^{m+1}f_i$  is infusible.*

Proof. Let  $0 \rightarrow Ae/Je \xrightarrow{p^*} \bigoplus_{i=1}^n Af_i/J^{m+1}f_i \rightarrow M \rightarrow 0$  be an exact sequence. Then  $M$  is indecomposable by (1.4.3). By (3.1),  $JM = \text{soc}^m M$ . Accordingly,  $JM$  is indecomposable since  $JM \geq \text{soc}^2 M$  and  $\text{soc}^2 M$  is indecomposable. Then from the exact sequence  $0 \rightarrow Ae/Je \xrightarrow{p^*} \bigoplus_{i=1}^n Jf_i/J^{m+1}f_i \rightarrow JM \rightarrow 0$ , we obtain that  $p^*: Ae/Je \rightarrow \bigoplus_{i=1}^n Jf_i/J^{m+1}f_i$  is infusible by (1.3). //

**3.3. Proof of Theorem 1 (2).** Let  $p: \bigoplus_{i=1}^n f_i A \rightarrow eJ^m$  be a projective cover of  $eJ^m$  and  $f_i$  in  $\text{pi}(A)$  for each  $i=1, \dots, n$ . If  $n=1$ , then the assertion is trivial. So we may assume that  $n \geq 2$ . There is some  $u_i$  in  $eJ^m f_i \setminus eJ^{m+1} f_i$  such that the  $i$ -th coordinate map of  $p$  is the right multiplication by  $u_i$  for each  $i=1, \dots, n$ . Put  $u_i := u_i + eJ^{m+1}$ ,  $u'_i := u_i + J^{m+1}f_i$  and  $e' := e + Je$ . Then  $eJ^m = \sum_{i=1}^n u_i A$  where each  $u_i A$  is local. Suppose that  $eJ^m$  is not a direct sum of local modules. Then  $\sum_{i=1}^n u_i a_i = 0$  for some  $a_i$  in  $A$  and  $u_j a_j \neq 0$  for some  $j=1, \dots, n$ . We may assume that there is some  $g$  in  $\text{pi}(A)$  such that  $u_j a_j g \neq 0$  and  $a_i = f_i a_i g$  for each  $i=1, \dots, n$ . Then it holds that  $a_i$  is in  $f_i Jg$  for each  $i$ . In fact, if  $f_i \neq g$ , then  $a_i \in f_i Ag = f_i Jg$ . And, in case  $f_i = g$ , we have  $f_i Ag/f_i Jg = f_i Af_i/f_i Jf_i$  is a division ring. Furthermore,  $\sum_{i=1}^n u_i a_i = 0$  implies  $\sum_{i=1}^n u_i a_i = 0$  and hence each  $u_i a_i = 0$ , since  $u_i A$  are independent. Then putting  $\bar{a}_i := a_i + f_i Jg$ , we have that  $\bar{u}_i \bar{a}_i$  is defined and is zero. Hence if  $a_i$  is not in  $f_i Jg$ , then  $\bar{u}_i = (\bar{u}_i \bar{a}_i) \bar{a}_i^{-1} = 0$ , a con-

tradiction. Further  $Au_i = J^m f_i$  since  $J^m f_i$  is uniserial for  $m \geq 2$ . Therefore we may assume that  $Au_i a_i \leqslant Au_n a_n$  for each  $i$  and  $Au_n a_n = J^s g$  for some  $s \geq m+1 \geq 3$ . Define a homomorphism  $q_i: Af_i/J^{m+1}f_i \rightarrow Ag/J^{s+1}g$  by  $x \mapsto xa_i$  for each  $i=1, \dots, n$ . Then  $q_n$  is a monomorphism since  $\text{soc}(Af_n/J^{m+1}f_n) = J^m f_n/J^{m+1}f_n$  is simple and is mapped by  $q_n$  onto the simple module  $J^s g/J^{s+1}g$ . Further putting  $q'_i := q_i|_{(Jf_i/J^{m+1}f_i)}$ , we have  $\text{Im } q'_i \leq \text{soc}^m(Jg/J^{s+1}g) = J^{s+1-m}g/J^{s+1}g = \text{Im } q'_n$  for each  $i=1, \dots, n$ . Hence if we put  $q''_i := q'_i: Jf_i/J^{m+1}f_i \rightarrow J^{s+1-m}g/J^{s+1}g$  and  $q := (q''_i)_{i=1}^n$ , then  $p^*: Ae/Je \rightarrow \bigoplus_{i=1}^n Jf_i/J^{m+1}f_i$  is fusible since  $e'p^*q=0$  and  $q''_n$  is an isomorphism. This contradicts (3.2). Hence  $eJ^m$  must be a direct sum of local modules.

**3.4. Proof of Theorem 1 (3) and (6).** Suppose that  $|LJ^s/LJ^{s+1}| \geq 2$  for some  $s \geq 1$ .  $LJ^s$  is a direct sum of local modules for  $LJ^s$  is a direct summand of  $eJ^{2+s}$ . Further  $L=vA$  for some  $v$  in  $eJ^2g \setminus eJ^3g$  and for some  $g$  in  $\text{pi}(A)$ . Hence  $LJ^s = vJ^s = u_1A \oplus u_2A \oplus \dots$  for some  $u_i$  in  $eJ^{2+s}f_i \setminus eJ^{3+s}f_i$  where  $f_i$  are in  $\text{pi}(A)$ . Then for each  $i=1, 2$ , there is some  $a_i$  in  $gJ^s f_i$  such that  $u_i = va_i$ . Define a map  $p_i: Ag/J^3g \rightarrow Af_i/J^{s+3}f_i$  by  $x \mapsto xa_i$  for each  $i=1, 2$ . Then  $p_1$  and  $p_2$  are both monomorphisms since putting  $v' := v + J^3g$  and  $u'_i := u_i + J^{s+3}f_i$ ,  $\text{soc}(Ag/J^3g) = J^2g/J^3g = Av'$  and  $\text{soc}(J^s f_i/J^{s+3}f_i) = J^{s+2}f_i/J^{s+3}f_i = Au'_i$  are simple modules and  $(Av')p_i = Au'_i$  for each  $i=1, 2$ . In particular,  $Ag$  is uniserial by Theorem 1 (1).

i) In case  $s \geq 2$ . By the above,

$$Av' \xrightarrow{(p_1, p_2)} (J^s f_1/J^{s+3}f_1) \oplus (J^s f_2/J^{s+3}f_2)$$

is fusible. Also,  $\text{soc}^3(Af_i/J^{s+3}f_i) = J^s f_i/J^{s+3}f_i$  is uniserial. Hence

$$Av' \xrightarrow{(p_1, p_2)} (Af_1/J^{s+3}f_1) \oplus (Af_2/J^{s+3}f_2)$$

is fusible by (2.3.2), say 2-fusible. Then for some  $a$  in  $f_1Af_2$ , the diagram

$$\begin{array}{ccc} Av' & \xrightarrow{p_1} & Af_1/J^{s+3}f_1 \\ \downarrow & & \downarrow \text{right multiplication by } a \\ Av' & \xrightarrow{p_2} & Af_2/J^{s+3}f_2 \end{array}$$

is commutative. Therefore  $u'_2 = u'_1a$ . Putting  $u_i := u_i + eJ^{s+3}$  for each  $i=1, 2$ , we have  $u_2 = u_1a$  since  $u_2$  is in  $u_1a + eJ^{s+3}f_2$ . Thus  $u_2A \leq u_1A$ . This contradicts the linear independency of  $u_1A$  and  $u_2A$ .

ii) In case the base field  $k$  is algebraically closed. It remains only the case  $s=1$ . Similarly, it holds that

$$Av' \xrightarrow{(p_1, p_2)} (Jf_1/J^4f_1) \oplus (Jf_2/J^4f_2)$$

is fusible. But since  $0 \neq u_i \in eJ^3f_i \leq J^3f_i$  for each  $i=1, 2$ ,  $h(Af_i) \geq 4$  and then  $Af_i/J^4f_i$  is uniserial of length 4 and  $Jf_i/J^4f_i = \text{soc}^3(Af_i/J^4f_i)$  by Theorem 1 (5). Then

$$Av' \xrightarrow{(p_1, p_2)} (Af_1/J^4f_1) \oplus (Af_2/J^4f_2)$$

is fusible by (2.3.2). Hence by the same argument as in i) we have a contradiction. //

#### 4. QF rings of right 2nd local type

**Lemma 4.1.** *Let  $A$  be a QF ring and  $e$  and  $f$  be in  $\text{pi}(A)$  such that  $fJe/fJ^2e \neq 0$ . Then*

- (a) *If  $Je/J^2e$  is simple, then  $h(Af) \geq h(Ae)$ ; and*
- (b) *If  $fJ/fJ^2$  is simple, then  $h(eA) \geq h(fA)$ .*

Proof. (a). It follows from the fact that  $Je/J^2e$  is simple and  $fJe/fJ^2e \neq 0$  that there is an epimorphism  $p: Af \rightarrow Je$ . If  $p$  is a monomorphism, then  $Je$  is injective and is a direct summand of  $Ae$ . Thus  $Je=0$  for  $Je$  is small in  $Ae$ . But this is impossible since  $Je/J^2e$  is simple. Therefore  $\text{Ker } p \geq \text{soc } Af = J^{h(Af)-1}f$  since  $Af$  is colocal. Hence  $h(Af) \geq h(Je) + 1 = h(Ae)$ .

- (b) Similar. //

**4.2.** Proof of Theorem 2. Let  $(x)'$  be the left side version of  $(x)$  for each  $x=1, 3$ . We show the following implications:  $(1) \Rightarrow (3)' \Leftrightarrow (3) \Rightarrow (6) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$ . Note that  $(2) \Leftrightarrow (1)'$  is clear since  $A$  has a selfduality. Denote by  $D$  the selfduality  $\text{Hom}_A(?, A)$  of  $A$ .

$(1) \Rightarrow (3)'$ . Let  $e$  be in  $\text{pi}(A)$  and  $h := h(Ae) \geq 4$ . Then  $J^2e$  is a uniserial waist in  $Ae$ . Hence  $\text{soc}^2eA = D(Ae/J^2e)$  is a waist in  $eA = D(Ae)$  and  $\text{soc}^2eA = eJ^{h-2}$  is a direct sum of local modules for  $h-2 \geq 2$ . But since  $eJ^{h-2} \leq eA$  and  $eA$  is colocal,  $eJ^{h-2}$  is local. Hence  $|Je/J^2e| = |\text{soc}^2(eA)/\text{soc}(eA)| = 1$  and  $Ae$  is uniserial.

$(3)' \Leftrightarrow (3)$ . Clear from the fact that both height and uniseriality are preserved by  $D$ .

$(3) \Rightarrow (6)$ . By the equivalence  $(3) \Leftrightarrow (3)'$  and left-right symmetry, it is sufficient to prove that under the assumption  $(3)'$ , if  $A$  is an indecomposable ring and  $J^3 \neq 0$ , then  $A$  is a left serial ring. Let  $Q$  be the left quiver of  $A$ , namely the oriented graph with vertex set  $\{1, \dots, p\}$  where  $\text{pi}(A) = \{e_1, \dots, e_p\}$  and with  $n_{ji}$  arrows  $i \rightarrow j$  iff  $\dim_{(e_jAe_j/e_jJe_j)}(e_jJe_i/e_jJ^2e_i) = n_{ji}$ . Note that  $A$  is an indecomposable ring iff  $Q$  is connected. It follows from  $J^3 \neq 0$  that  $h(Ae_i) \geq 4$  for some  $i=1, \dots, p$  and then  $Ae_i$  is uniserial by  $(3)'$ . By 4.1 and the selfduality  $D$ , we have  $h(Ae_j) \geq h(Ae_i)$  ( $\geq 4$ ) if either

- (a) there is an arrow  $i \rightarrow j$ ; or

(b) there is an arrow  $j \rightarrow i$ .

Hence  $Ae_j$  is uniserial of height  $\geq 4$  for any  $j=1, \dots, p$  by (4.1), (3)' and the fact that  $Q$  is connected. Thus  $A$  is a left serial ring.

(6)  $\Rightarrow$  (4). Clear from the fact that for a serial ring  $A$ ,  $A$  is  $QF$  iff the admissible sequence of  $A$  is constant.

(4)  $\Rightarrow$  (5). Let  $M_A$  be indecomposable of height  $h \geq 3$ . Then  $A/J^h$  is  $QF$  by (4). Let  $0 \rightarrow K \hookrightarrow \bigoplus_{i=1}^m P_i \rightarrow M \rightarrow 0$  be a projective cover of  $M$  over  $A/J^h$  with each  $P_i$  indecomposable. Then  $\text{soc}(\bigoplus_{i=1}^m P_i) \nleq K$  implies that  $\text{soc } P_i \nleq K$  for some  $i=1, \dots, m$  and then  $P_i \cap K = 0$  since  $P_i$  is colocal. Hence  $P_i$  is embedded into  $M$ . But since  $P_i$  is injective,  $P_i$  is isomorphic to a direct summand of  $M$ . Hence  $P_i \cong M$  for  $M$  is indecomposable. Further  $P_i \cong eA/eJ^h$  for some  $e$  in  $\text{pi}(A)$ .

(5)  $\Rightarrow$  (1). Clear. //

## 5. Examples

In this section, we give some examples using bounden quiver algebras over an algebraically closed field  $k$ . (See Gabriel [8] for details concerning bounden quiver algebras.)

EXAMPLE 1. Let  $A$  be the algebra defined by the following bounden quiver:

$$\alpha \begin{array}{c} \curvearrowright \\ 1 \end{array} \begin{array}{c} \xrightleftharpoons[\gamma]{\beta} \\ 2 \end{array}; \quad \beta\alpha = \alpha\gamma = 0, \quad \alpha^2 = \gamma\beta,$$

namely, the algebra having  $\{e_1, e_2, \alpha, \beta, \gamma, \gamma\beta, \beta\gamma\}$  as  $k$ -basis and with multiplication given by the following table:

right \ left	$e_1$	$e_2$	$\alpha$	$\beta$	$\gamma$	$\gamma\beta$	$\beta\gamma$
$e_1$	$e_1$		$\alpha$		$\gamma$	$\gamma\beta$	
$e_2$		$e_2$		$\beta$			$\beta\gamma$
$\alpha$	$\alpha$		$\gamma\beta$				
$\beta$	$\beta$				$\beta\gamma$		
$\gamma$		$\gamma$			$\gamma\beta$		
$\gamma\beta$	$\gamma\beta$						
$\beta\gamma$		$\beta\gamma$					

(each blank is zero).

Then  $A$  is weakly symmetric and hence  $QF$ . Further as easily seen,  $A$  has cube-zero radical. Therefore  $A$  is of right (and left) 2nd local type by Theorem 2. But since  $A$  is not a serial ring,  $A$  is neither of right (1st) local type nor of left (1st) local type.

EXAMPLE 2. Let  $A$  be the algebra defined by the following quiver  $Q$ :

$$\begin{array}{ccccc} & & 5 & & \\ & & \downarrow \delta & & \\ 4 & \xrightarrow{\gamma} & 1 & \xleftarrow{\alpha} & 2 \xleftarrow{\beta} 3, \end{array}$$

namely, the algebra having  $\{e_1, e_2, e_3, e_4, e_5, \alpha, \beta, \gamma, \delta, \alpha\beta\}$  as  $k$ -basis with multiplication given by the following table:

right left \ \diagdown	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$\alpha$	$\beta$	$\gamma$	$\delta$	$\alpha\beta$
$e_1$	$e_1$					$\alpha$		$\gamma$	$\delta$	$\alpha\beta$
$e_2$		$e_2$					$\beta$			
$e_3$			$e_3$							
$e_4$				$e_4$						
$e_5$					$e_5$					
$\alpha$		$\alpha$								$\alpha\beta$
$\beta$			$\beta$							
$\gamma$				$\gamma$						
$\delta$					$\delta$					
$\alpha\beta$			$\alpha\beta$							

(each blank is zero).

Then as easily verified,  $A$  satisfies all the conditions stated in Theorem 1. But it is not of right 2nd local type. For instance, let  $M$  be the right  $A$ -module corresponding to the following  $k$ -representation of  $Q^{op}$  (the opposite quiver of  $Q$ , with all arrows reversed)

$$\begin{array}{c} k \\ \uparrow (1,0) \\ k \xleftarrow{(0,1)} k \oplus k \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} k \oplus k \xrightarrow{(1,1)} k \end{array}$$

namely, the module having  $\{m_1, m'_1, m_2, m'_2, m_3, m_4, m_5\}$  as  $k$ -basis and with right  $A$ -action given by the following table:

	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$\alpha$	$\beta$	$\gamma$	$\delta$	$\alpha\beta$
$m_1$	$m_1$					$m_2$			$m_4$	$m_3$
$m'_1$	$m'_1$					$m'_2$		$m_4$		$m_3$
$m_2$		$m_2$					$m_3$			
$m'_2$		$m'_2$						$m_3$		
$m_3$			$m_3$							
$m_4$				$m_4$						
$m_5$					$m_5$					

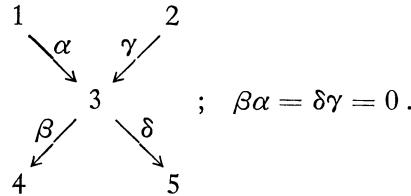
(each blank is zero).

Then  $M$  is indecomposable but  $\text{top}^2 M$  is decomposable:

$$\text{top}^2 M = \left[ \begin{array}{c} 0 \\ \uparrow \\ k \xleftarrow[1]{} k \xrightarrow[1]{} k \rightarrow 0 \end{array} \right] \oplus \left[ \begin{array}{c} k \\ \uparrow 1 \\ 0 \xleftarrow[1]{} k \xrightarrow[1]{} k \rightarrow 0 \end{array} \right].$$

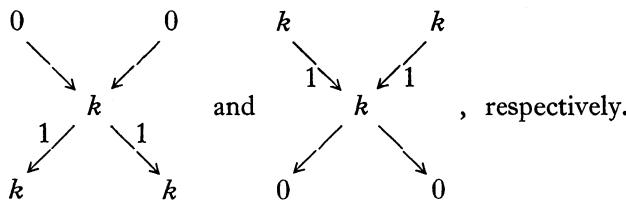
Hence the conditions stated in Theorem 1 are not sufficient for algebras (even if  $k$  is algebraically closed) to be of right 2nd local type.

EXAMPLE 3. Let  $A$  be the algebra defined by the following bounded quiver:



Then we can see that  $A$  has just 13 indecomposable left modules (up to isomorphism), all of which have indecomposable second tops and second socles since the indecomposable left  $A$ -modules of height  $\geq 3$  are both projective and injective. Hence  $A$  is of right and left 2nd local type.<sup>1)</sup> But it is neither of right (1st) local type nor of left (1st) local type. For instance, let  $M_1$  and  $M_2$  be the left  $A$ -modules corresponding to the following  $k$ -representations of the bounded quiver:

1) In Part II of this series of papers, we shall give some necessary and sufficient conditions for artinian rings to be of right and left  $n$ -th local type for any natural number  $n$ . Using this result, it is clear that the algebra defined in Example 3 is of right and left 2nd local type.



Then  $M_1$  and  $M_2$  are indecomposable but  $M_1$  is not colocal and  $M_2$  is not local.

### References

- [1] H. Asashiba and T. Sumioka: *On Krull-Schmidt's theorem and the indecomposability of amalgamated sums*, Osaka J. Math. **20** (1983), 321–329.
- [2] H. Asashiba: *On the indecomposability of amalgamated sums*, Osaka J. Math. **20** (1983), 701–711.
- [3] M. Auslander: *Representation theory of artin algebras* II, Comm. Algebra **1** (1974), 269–310.
- [4] M. Auslander, E.L. Green and I. Reiten: *Modules with waists*, Illinois J. Math. **19** (1975), 467–478.
- [5] V. Dlab and C.M. Ringel: *On algebras of finite representation type*, J. Algebra **33** (1975), 306–394.
- [6] V. Dlab and C.M. Ringel: *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc. **173** (1976).
- [7] P. Gabriel: *Unzerlegbare Darstellungen* I, Manuscripta Math. **6** (1972), 71–103.
- [8] P. Gabriel: *Auslander-Reiten sequences and representation-finite algebras*, Proc. ICRA II Springer Lecture Notes 831 (1980), 1–71.
- [9] W. Müller: *Unzerlegbare Moduln über artinschen Ringen*, Math. Z. **137** (1974), 197–226.
- [10] T. Sumioka: *Tachikawa's theorem on algebras of left co-local type*, Osaka J. Math. **21** (1984), to appear.
- [11] H. Tachikawa: *On rings for which every indecomposable right module has a unique maximal submodule*, Math. Z. **71** (1959), 200–222.

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