DISSERTATION IN PHYSICS

New Description of Spin 3/2 Particle

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Abstract

A consistent theory of the interacting spin 3/2 particle is presented. Instead of the constraint conditions, which are the origion of troubles in a theory of interacting higher spin particles, we suppress redundant particles by making their masses infinite. Our theory is general enough to extend to particles with s > 3/2.

§1. Introduction

The developments of high energy experiments have revealed the many new particles with spins greater than unity. This situation enforces us to construct a consistent theory for higher spin particles as a practical task.

The relativistic equations for higher spin fields were originated by Dirac[1] and followed up to the quantized form by Fierz[2]. Lagrangian, whose Euler equations yield both equations of motion and constraint conditions, was proposed by Fierz and Pauli[3]. It was further developed and simplified by the works of Rarita and Schwinger[4], Moldaur and Case[5].

These theories stand on the same footing in the sense that field variables are the realizations of the irreducible representations of the spatial rotation group in the rest system and express only one spin state. They give a consistent and equivalent description of free particle. However in the presseence of interaction all the conventional theories* have failed to describe the system of higher spin particle in consistent ways.

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* In parallel with the articles mentioned above different types of formulation of higher spin particle were performed[6]. However these formulations also can not introduce an interaction in a consistent manner.

Lagrangian of charged spin 2 particle, with the minimally coupled electromagnetic field, loses two of the constraints and gives the spin 2 particle undesired degrees of freedom, 6, instead of required 5: This is a firstly appeared disease concerned with interacting higher spin particles and still remains as an open question. In addition to this desease there are several inconsistencies in the theory of interacting higher spin particles. During the last decade these inconsistencies have been examined extensively.

Among higher spin particles spin 3/2 particle is most important and the simplest example. The troubles occuring in the system of interacting spin 3/2 particles are as follows:

A) In the presence of the external electromagnetic field, the equal-time anticommutator of spin 3/2 particle must be a function of a coupled field and can not be positive in the region

$$\left(\frac{2}{3} e m^{-2} \right)^2 \underset{\sim}{H}^2 > 1, \tag{1.1}$$

where $\underset{\sim}{H}$ means a magnetic field strength[8]. This disease appears even in more general types of interactions[9].

B) The wave fronts of the classical solutions in an electromagnetic field propagate faster than light in the same critical value as (1.1)[10].

C) The time ordered product is non-covariant. Because of this non-covariance there is a difference between vacuum expectation value of the T product and the covariant propagator. This difference must be explained by the complicated normal dependent interaction Hamiltonian. When we consider interaction Hamiltonian as power series of the coupling constant, for the particle with $s \geqslant 3/2$, this series may be infinite. Therefore, we do not have field theoretical basis for Feynman rules for the system including higher spin particles. Moreover we have no clear outlook for the Lehmann-Symanzik-Zimmermann reduction formula.

These three difficulties are very serious. Some people even suggest that higher spin particle can not be elementary. These come from the constraint conditions introduced to suppress redundant fields.

In this article we propose non-constraint theory of higher spin field and show that our theory is free from the difficulties A), B), C). We suppress redundant fields not by constraints but by making the masses of the corresponding particles infinite. This is the extension of the ξ-limiting theory proposed by Lee and Yang[11)] for the interacting spin one particle.

In §2 we give Lagrangian in our theory and discuss the canonical quantization procedure. The equal time commutators of the field operators are quite different from those of usual theory, so that the difficulty A) disappears.

In §3 we study the quantization based on the idea of Peierls[12] and Takahashi-Umezawa[13]. This method of quantization is useful to develop the theory manifestly covariant and to separate covariantly the contributions of the redundant fields. We find that the special suppression procedure is necessary to get a causal theory. This fact seems to relate to the difficulty B) in usual theory. The time ordered product is covariant. Then, our theory is free from difficulty C). In §3 we also obtain the various Green functions of the respective fields. Through the Green functions we see that the redundant particles play a role of regulator. When the masses of the redundant particles are finite our theory is renormalizable. By taking the limit the divergence appears for some Feynman diagrams. This divergence is nothing but the ultra-violet divergence in higher spin field.

In §4 we give explicit forms of the wave functions of the respective fields, their orthonormality relations and completeness conditions.

In §5 we give Fourier expansion of field operator and see that the redundant spin 1/2 particles are both ghosts. The difficulty A) may be due to the incomplete separation of the redundant fields. We also write down the Hamiltonian operator in Fock representation and LSZ reduction formulas. We shall discuss only spin 3/2 field in detail. However our method is also applicable to the general cases without significant change.

§2. Canonical Formulation.

We consider the general spinor-vector field. The most general equation of motion consisting of at most first-order derivative is the following:

$$\Lambda_{\mu\nu}(\partial)\psi_\nu(x) = 0, \tag{2.1}$$

where

$$\Lambda_{\mu\nu}(\partial) = -(\gamma_\lambda\partial_\lambda + m)\delta_{\mu\nu} - A(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)$$
$$- B\gamma_\mu\gamma_\lambda\partial_\lambda\gamma_\nu - Cm\gamma_\mu\gamma_\nu. \tag{2.2}$$

Eq.(2.2) can be rewritten as

$$\Lambda_{\mu\nu}(\partial) = -(\Gamma_\lambda)_{\mu\nu}\partial_\lambda - M_{\mu\nu} \tag{2.3}$$

with

$$(\Gamma_\lambda)_{\mu\nu} = \gamma_\lambda\delta_{\mu\nu} + A(\gamma_\mu\delta_{\nu\lambda} + \gamma_\nu\delta_{\mu\lambda}) + B\gamma_\mu\gamma_\lambda\gamma_\nu \tag{2.4}$$

and

$$M_{\mu\nu} = m\delta_{\mu\nu} + Cm\gamma_\mu\gamma_\nu. \tag{2.5}$$

In (2.2) we put the coefficients of $\gamma_\mu\partial_\nu$ and $\gamma_\nu\partial_\mu$ equal so as to make Lagrangian,

$$L = \overline{\psi}_\mu \Lambda_{\mu\nu} \psi_\nu$$

$$= -\overline{\psi}_\mu \{(\Gamma_\lambda)_{\mu\nu} \partial_\lambda + M_{\mu\nu}\} \psi_\nu \tag{2.6}$$

with

$$\overline{\psi}_\mu = \psi_\mu^* \gamma_4, \tag{2.7}$$

real. In Eq.(2.1) if,

$$A \neq -1/2, \tag{2.8a}$$

$$B = (3/2)A^2 + A + 1/2, \tag{2.8b}$$

$$C = -(3A^2 + 3A + 1), \tag{2.8c}$$

we immediately get

$$(\gamma_\nu \partial_\nu + m)\psi_\mu = 0, \tag{2.9a}$$

$$\gamma_\mu \psi_\mu = 0, \tag{2.9b}$$

$$\partial_\mu \psi_\mu = 0. \tag{2.9c}$$

This is the Rarita-Schwinger equation describing irreducible spin 3/2 field. The parameter A comes from the ambiguity of the wave function in the point transformation,

$$\psi'_\mu = (\delta_{\mu\nu} + a\gamma_\mu\gamma_\nu)\psi_\nu. \tag{2.10}$$

Now we discuss, the general case where the parameters do not satisfy (2.8). We treat all ψ_μ as dynamical variables. The canonically conjugate momenta π_μ of ψ_μ are

$$\pi_\mu \equiv \partial L/\partial(\partial\psi_\mu/\partial x_0) = i\overline{\psi}_\nu(\Gamma_4)_{\nu\mu}. \tag{2.11}$$

From the standard commutation relations

$$\{\psi_\mu(x),\ \pi_\nu(x')\}_{x_0=x_0'} = i\delta^3(\underset{\sim}{x} - \underset{\sim}{x}')\delta_{\mu\nu}, \tag{2.12}$$

we get

$$\{\psi_\mu(x),\ \overline{\psi}_\nu(x')\}_{x_0=x_0'} = (\Gamma_4)^{-1}_{\mu\nu}\ \delta^3(\underset{\sim}{x}-\underset{\sim}{x}'). \tag{2.13}$$

Here Γ_4^{-1} means the inverse of Γ_4:

$$(\Gamma_4)^{-1}_{\mu\nu} = \{(1+2A+3A^2-2B)\gamma_4\delta_{\mu\nu} - (A^2-A-2B)(\gamma_\mu\delta_{4\nu}+\gamma_\nu\delta_{4\mu})$$
$$- 2(2A+A^2+2B)\gamma_4\delta_{4\mu}\delta_{4\nu} + (A^2-B)\gamma_\mu\gamma_4\gamma_\nu\}/(1+2A+3A^2-2B) \tag{2.14}$$

Although Γ_4 is singular in the case of (2.8), as will be seen later in our theory, Γ_4 is regular. Following the standard method, we obtain the Hamiltonian

$$H = \pi_\mu(\partial\psi_\mu/\partial x_0) - L$$
$$= \overline{\psi}_\mu\{(\Gamma_i)_{\mu\nu}\partial_i + M_{\mu\nu}\}\psi_\nu, \tag{2.15}$$

and the energy momentum tensor, where Latin indicies denote spatial components. The symmetrical energy momentum tensor in our theory satisfies Schwinger's commutator condition in its simplest form contrary to usual theory[14].

§3. Extended Method of Peierls' Quantization.

We discuss Peierls' quantization for the free reducible spinor-vector field ψ_μ. We start from the investigation of $\Lambda^{-1}_{\mu\nu}$ defined by

$$\Lambda_{\mu\nu}(ip)\Lambda^{-1}_{\nu\lambda}(ip) = \Lambda^{-1}_{\mu\nu}(ip)\Lambda_{\nu\lambda}(ip) = \delta_{\mu\lambda}. \qquad (3.1)$$

$\Lambda^{-1}_{\mu\nu}(ip)$ should have the following form.

$$\Lambda^{-1}_{\mu\nu}(ip) = I_1(p^2)\delta_{\mu\nu} + I_2(p^2)\gamma_\mu\gamma_\nu + I_3(p^2)(i\gamma_\mu p_\nu + i\gamma_\nu p_\mu)$$
$$- I_4(p^2)p_\mu p_\nu + I_5(p^2)i\gamma_\lambda p_\lambda\delta_{\mu\nu} + I_6(p^2)\gamma_\mu i\gamma_\lambda p_\lambda\gamma_\nu$$
$$- I_7(p^2)(\gamma_\mu\gamma_\lambda p_\lambda p_\nu + p_\mu\gamma_\lambda p_\lambda\gamma_\nu) - I_8(p^2)p_\mu p_\nu i\gamma_\lambda p_\lambda. \qquad (3.2)$$

Substituting (3.2) into (3.1), we get

$$I_1 = -m/(p^2+m^2) \qquad (3.3a)$$

$$I_2 = m[m^4C(1+4C)+m^2p^2(6CA+4B^2+A^2+7CA^2+2BA-2B)$$
$$+p^4\{(A^2+A)(1+2A+3A^2-2B)-(1+A)^2(A+2B+C)\}]/D(p^2) \qquad (3.3b)$$

$$I_3 = [2m^2p^2(A+2B+C)(A^2-2B-2C)-\{(2C-A)m^2+(A^2-A-2B)p^2\}$$
$$\times\{(1+4C)m^2+(1+2A+3A^2-2B)p^2\}]/D(p^2) \qquad (3.3c)$$

$$I_4 = -4m[(A^2+A-C)(1+4C)m^2+\{(1+2A+3A^2-2B)(A^2+A-C)-(A+2B+C)$$
$$\times (A^2+2A+2B)\}p^2]/D(p^2) \qquad (3.3d)$$

$$I_5 = 1/(p^2+m^2) \qquad (3.3e)$$

$$I_6 = [(1+2A+3A^2-2B)(A^2-B)p^4+m^2p^2(-2A^3+A^2+6AC+8A^2C-BA^2+6AB$$
$$+2C+6B^2+2C^2)+(m^4/2)\{(A+2B+C)+(1+4C)(3C-A)\}]/D(p^2) \qquad (3.3f)$$

$$I_7 = m[m^2\{2(A+2B+C)(2C-A)+(1+4C)(A^2-2B-2C)\}$$
$$+p^2\{2(A+2B+C)(A^2-A-2B)+(A^2-2B-2C)(1+2A+3A^2-2B)\}]/D(p^2)$$
$$(3.3g)$$

$$I_8 = 2[\{(1+4C)(A^2+A-C)+(A+2B+C)(1+2A)^2\}m^2$$
$$+(1+2A+3A^2-2B)(A^2+2A+2B)p^2]/D(p^2), \qquad (3.3h)$$

where

$$D(p^2) = [\{(1+4C)m^2+(1+2A+3A^2-2B)p^2\}^2+4m^2p^2(A+2B+C)^2](p^2+m^2). \qquad (3.3i)$$

That is, Λ^{-1} have three poles at

$$p^2 = -m^2 \qquad (3.4a)$$

and

$$p^2 = -M^{(\pm)2}$$
$$= -m^2[\{A+2B+C\pm\{(A+2B+C)^2+(1+2A+3A^2-2B)(1+4C)\}^{1/2}\}/(1+2A+3A^2-2B)]^2 \qquad (3.4b)$$

Our spinor-vector field ψ_μ describes three kinds of particles whose masses are m, $M^{(+)}$, $M^{(-)}$. The particle with mass m is the spin 3/2 particle which we want to have, whereas the particles with masses $M^{(\pm)}$ are the redundant spin 1/2 particles which we must suppress. For simplicity we impose the following condition among the parameters:

$$A + 2B + C = 0. \qquad (3.5)$$

Then

$$M^{(+)2} = M^{(-)2} = \frac{1+4C}{1+3A+3A^2+C} m^2 \equiv M^2 \qquad (3.6)$$

and

$$\Lambda^{-1}_{\mu\nu} = -\frac{d_{\mu\nu}(ip)}{(p^2+m^2)} - \frac{d^{(+)}_{\mu\nu}(ip)}{(p^2+M^2)} - \frac{d^{(-)}_{\mu\nu}(ip)}{(p^2+M^2)}, \tag{3.7}$$

where

$$d_{\mu\nu}(ip) = m\delta_{\mu\nu} - (m/3)\gamma_\mu\gamma_\nu + (1/3)(i\gamma_\mu p_\nu + i\gamma_\nu p_\mu) + (4/3m)p_\mu p_\nu$$
$$-i\gamma_\lambda p_\lambda\delta_{\mu\nu} - (1/3)\gamma_\mu i\gamma_\lambda p_\lambda\gamma_\nu - (1/3m)(\gamma_\mu\gamma_\lambda p_\lambda p_\nu + p_\mu\gamma_\lambda p_\lambda\gamma_\nu)$$
$$-(2/3m^2)p_\mu p_\nu i\gamma_\lambda p_\lambda, \tag{3.8a}$$

$$d^{(\pm)}_{\mu\nu}(ip) = \frac{m}{12(1+3A+3A^2+C)}\{2(1+C)\pm(\frac{1+4C}{1+3A+3A^2+C})^{1/2}(2+3A-C)\}\gamma_\mu\gamma_\nu$$
$$-\frac{1}{6(1+3A+3A^2+C)}\{(1+3A-2C)\pm(\frac{1+4C}{1+3A+3A^2+C})^{1/2}\}(i\gamma_\mu p_\nu + i\gamma_\nu p_\mu)$$
$$+\frac{1}{3m}\{-2\pm(\frac{1+4C}{1+3A+3A^2+C})^{1/2}\}p_\mu p_\nu + \frac{1}{12(1+3A+3A^2+C)}\{(2+3A-C)$$
$$\pm 2(\frac{1+4C}{1+3A+3A^2+C})^{1/2}(1+3A+3A^2+C)\}i\gamma_\mu\gamma_\lambda p_\lambda\gamma_\nu - \frac{1}{6m(1+4C)}\{-(1+4C)$$
$$\pm(\frac{1+4C}{1+3A+3A^2+C})^{1/2}(2C-3A-1)\}(\gamma_\mu\gamma_\lambda p_\lambda p_\nu + p_\mu\gamma_\lambda p_\lambda\gamma_\nu)$$
$$-\frac{1}{3m^2(1+4C)}\{-(1+4C)\pm 2(\frac{1+4C}{1+3A+3A^2+C})^{1/2}(1+3A+3A^2+C)\}p_\mu p_\nu i\gamma_\lambda p_\lambda. \tag{3.8b}$$

In Eq.(3.7) the separation to $d^{(+)}(ip)$ and $d^{(-)}(ip)$ in the mumerator of $1/(p^2+m^2)$ is made before putting (3.6).

Using the new parameter,

$$\xi = -(1+3A+3A^2+C). \tag{3.9}$$

instead of C, (3.6) is rewritten as

$$M^2 = m^2\{4+3(1+2A)^2/\xi\} \tag{3.10}$$

Then, to make the mass M of redundant particles infinite, we must take

$$2A + 1 \neq 0 \tag{3.11a}$$

and

$$\xi \to +0. \tag{3.11b}$$

Since we already have (3.5), (3.11a) and (3.11b) mean that the irreducible condition (2.8) is satisfied in the limit. However as is specified in (3.11b), we must take a limit from a positive value of ξ. If we take a limit from a negative value, our equation of motion becomes accausal. Then, it is not sure that whether the equation of motion (2.1) with (2.8) is causal. In order that (2.1) is obviously causal we must assume the limit (3.11b). This fact may be the reason of the difficulty B) in usual theory. The same situation exists in the spin one particle[11]. The difficulty B) for the spin one particle has been reported by Minkowski and Seiler[15].

According to Peierls' quantization procedure, we assume

$$\{\psi_\mu(x), \overline{\psi}_\nu(x')\} = id_{\mu\nu}(\partial)\Delta(x-x';m)$$
$$+ id^{(+)}_{\mu\nu}(\partial)\Delta(x-x';M) + id^{(-)}_{\mu\nu}(\partial)\Delta(x-x';M), \tag{3.12}$$

where $\Delta(x-x';m)$ is the invariant delta function with mass m. In (3.12) if we put $x_0 = x'_0$ we get exactly the same equation as (2.12).

We also get

$$\langle 0|T\{\psi_\mu(x),\ \bar{\psi}_\nu(x')\}|0\rangle = id_{\mu\nu}(\partial)\Delta_c(x-x';\ m)$$
$$+\ id^{(+)}_{\mu\nu}(\partial)\Delta_c(x-x';\ M) + id^{(-)}_{\mu\nu}(\partial)\Delta_c(x-x';\ M)$$
$$+\ i/2[\varepsilon(x_0-x_0'),d_{\mu\nu}(\partial)]\Delta(x-x';\ m) + i/2[\varepsilon(x_0-x_0'),d^{(+)}_{\mu\nu}(\partial)]\Delta(x-x';M)$$
$$+\ i/2[\varepsilon(x_0-x_0'),d^{(-)}_{\mu\nu}(\partial)]\Delta(x-x';\ M), \qquad (3.13)$$

where $\Delta_c = -(i/2)\Delta_F$ is the causal delta function. Inserting the expression given by (3.8) into (3.13) we easily see that the last three normal dependent terms cancel out. Then

$$\langle 0|T\{\psi_\mu(x),\ \bar{\psi}_\nu(x')\}|0\rangle = id_{\mu\nu}(\partial)\Delta_c(x-x';\ m)$$
$$+\ id^{(+)}_{\mu\nu}(\partial)\Delta_c(x-x';\ M) + id^{(-)}_{\mu\nu}(\partial)\Delta_c(x-x';\ M). \qquad (3.14)$$

The T-product is covariant in spite of the fact that each $d(\partial)$ includes third-order derivatives. Thus the correspondence of the canonical formulation and the covariant formulation is established.

So far we have discussed only free field equations. However it is obvious that our theory does not give rise to any difficulty in the presence of the interaction. For example, normal dependent Hamiltonian stated in C) is not required to get a covariant theory[16]. There exists the unitary transformation which connects operators of the Heisenberg representation with those of the interaction representation in one-to-one correspondent menner. That is, in Takahashi-Umezawa's notation[17], we have

$$\underset{\sim}{\psi}(x) = \psi(x/\sigma) = S^{-1}[\sigma]\,\psi(x)S[\sigma]. \qquad (3.15)$$

Furthermore we see from (3.8) that p^2 and p^3 terms (leading terms at $p\to\infty$) of the propagator are supressed by means of the presence of the redundant spin 1/2 fields. The redundant spin 1/2 fields play a role of regulator. Therefore, when ξ is finite, our theory is renormalizable. In a calculation of certain Feynman diagrams, we happen to have terms such as $\log\xi$, ξ^{-1}, ξ^{-2} etc., which cannot be removed by the renormalization procedure. Then our theory is unrenormalizable in the limit. However this does not mean that we never be able to calculate higher-order corrections. Finite results may be obtained by rearranging the perturbation expansion as was done by Lee[18).

§4. Wave Functions, Their Orthonormalities and Completeness.

In this section, we give explicit forms of the respective wave functions. We denote spin 3/2 wave function and two kinds of spin 1/2 wave functions with helicity r by $U_{r,\mu}(\underset{\sim}{p},m)$ and $U^{(\pm)}_{r,\mu}(\underset{\sim}{p},M)$ respectively. We construct then, from the helicity diagonalized wave functions of spin 1/2 and spin 1 particles, using the usual composition law:

$$U_{3/2,\mu}(\underset{\sim}{p},m) = u^{+}(\underset{\sim}{p},m)e^{+}_{\mu}(\underset{\sim}{p},m) \tag{4.1a}$$

$$U_{1/2,\mu} = (1/3)^{1/2}(2^{1/2}u^{+}e^{0}_{\mu} + u^{-}e^{+}_{\mu}) \tag{4.1b}$$

$$U_{-1/2,\mu} = (1/3)^{1/2}(u^{+}e^{-}_{\mu} + 2^{1/2}u^{-}e^{0}_{\mu}) \tag{4.1c}$$

$$U_{-3/2,\mu} = u^{-}e^{-}_{\mu} \tag{4.1d}$$

$$U^{(+)}_{1/2,\mu}(\underset{\sim}{p},M) = (1/3)^{1/2}[-u^{+}(\underset{\sim}{p},M)e^{0}_{\mu}(\underset{\sim}{p},M)+2^{1/2}u^{-}(\underset{\sim}{p},M)e^{+}_{\mu}(\underset{\sim}{p},M)] \tag{4.2a}$$

$$U^{(+)}_{-1/2,\mu} = (1/3)^{1/2}[-2^{1/2}u^+e^-_\mu + u^-e^0_\mu] \qquad (4.2b)$$

$$U^{(-)}_{1/2,\mu}(\underset{\sim}{p},M) = u^+(\underset{\sim}{p},M)\,e_\mu(\underset{\sim}{p},M) \qquad (4.3a)$$

$$U^{(-)}_{-1/2,\mu} = u^-e_\mu \qquad (4.3b)$$

Using the angular variables defined as

$$\underset{\sim}{p} = p(\cos\phi\sin\theta,\ \sin\phi\sin\theta,\ \cos\theta), \qquad (4.4)$$

$u^\pm$, $e^{\pm 0}_\mu$ and e_μ we have used are given in the forms:

$$u^+(\underset{\sim}{p},m) = \left[\frac{(\underset{\sim}{p}^2+m^2)^{1/2}+m}{2(\underset{\sim}{p}^2+m^2)^{1/2}}\right]^{1/2}\begin{pmatrix}\cos(\theta/2)\\ e^{i\phi}\sin(\theta/2)\\ \cos(\theta/2)p/[(\underset{\sim}{p}^2+m^2)^{1/2}+m]\\ e^{i\phi}\sin(\theta/2)p/[(\underset{\sim}{p}^2+m^2)^{1/2}+m]\end{pmatrix} \qquad (4.5a)$$

$$u^-(\underset{\sim}{p},m) = \left[\frac{(\underset{\sim}{p}^2+m^2)^{1/2}+m}{2(\underset{\sim}{p}^2+m^2)^{1/2}}\right]^{1/2}\begin{pmatrix}-e^{-i\phi}\sin(\theta/2)\\ \cos(\theta/2)\\ e^{-i\phi}\sin(\theta/2)p/[(\underset{\sim}{p}^2+m^2)^{1/2}+m]\\ -\cos(\theta/2)p/[(\underset{\sim}{p}^2+m^2)^{1/2}+m]\end{pmatrix} \qquad (4.5b)$$

$$e^+_\mu(\underset{\sim}{p},m) = (e^{i\phi}/2^{1/2})(\cos\theta\cos\phi-i\sin\phi,\cos\theta\sin\phi+i\cos\phi,-\sin\theta,0) \qquad (4.6a)$$

$$e^-_\mu(\underset{\sim}{p},m) = (e^{-i\phi}/2^{1/2})(-\cos\theta\cos\phi-i\sin\phi,-\cos\theta\sin\phi+i\cos\phi,\sin\theta,0) \qquad (4.6b)$$

$$e^0_\mu(\underset{\sim}{p},m) = -[(\underset{\sim}{p}^2+m^2)^{1/2}/m][\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta,ip/(\underset{\sim}{p}^2+m^2)^{1/2}] \qquad (4.6c)$$

$$e_\mu(\underset{\sim}{p},m) = [\underset{\sim}{p},\ (\underset{\sim}{p}^2+m^2)^{1/2}] \qquad (4.6d)$$

Now we immediately see that $U_{r,\mu}$ satisfies the equation of motion (2.1) but $U^{(\pm)}_{r,\mu}$ not. We must modify $U^{(\pm)}_{r,\mu}$ so as to satisfy

(2.1) using the ambiguity of wave functions $U_{r,\mu}$ under the point transformation (2.10). That is, the correct wave functions for spin 1/2 parts should have the form

$$U^{(\pm)'}_{r,\mu}(\underset{\sim}{p},M) = N^{(\pm)}(\delta_{\mu\nu} + \alpha^{(\pm)}\gamma_\mu\gamma_\nu)U^{(\pm)}_{r,\nu}. \tag{4.7}$$

Using the identities

$$3^{1/2}\gamma_5 U^{(+)}_{\pm 1/2,\mu}(\underset{\sim}{p},M) + U^{(-)}_{\pm 1/2,\mu}(\underset{\sim}{p},M) = i\gamma_\mu u^{\pm}(\underset{\sim}{p},M), \tag{4.8a}$$

$$i\gamma_\mu U^{(-)}_{\pm 1/2,\mu}(\underset{\sim}{p},M) = -u^{\pm}(\underset{\sim}{p},M), \tag{4.8b}$$

$$i\gamma_\mu U^{(+)}_{\pm 1/2,\mu}(\underset{\sim}{p},M) = 3^{1/2}\gamma_5 u^{\pm}(\underset{\sim}{p},M), \tag{4.8c}$$

from the equation of motion (2.1) and the normalization condition

$$|\bar{U}^{(\pm)'}_{r,\mu}(\underset{\sim}{p},M)(\Gamma_4)_{\mu\nu}U^{(\pm)'}_{r,\nu}(\underset{\sim}{p},M)| = 1, \tag{4.9}$$

$\alpha^{(\pm)}$ and $N^{(\pm)}$ are determined. The results are

$$\alpha^{(+)} = \frac{m-(1+3A)M}{6M(1+2A)}, \tag{4.10a}$$

$$\alpha^{(-)} = \frac{m-(1+A)M}{2M(1+2A)}, \tag{4.10b}$$

$$N^{(+)} = \left[\frac{M(M-2m)}{m^2}\right]^{1/2}, \tag{4.11a}$$

and

$$N^{(-)} = \left[\frac{M(M+2m)}{3m^2}\right]^{1/2}. \tag{4.11b}$$

Instead of $U_{r,\mu}$ and $U'^{(\pm)'}_{r,\mu}$, we define $u^{(\alpha)}(\underset{\sim}{p})$ by

$$u^{(1)}_\mu(\underset{\sim}{p}) = U_{3/2,\mu}(\underset{\sim}{p},m), \tag{4.12a}$$

$$u_\mu^{(2)}(\underset{\sim}{p}) = U_{1/2,\mu}(\underset{\sim}{p},m), \tag{4.12b}$$

$$u_\mu^{(3)}(\underset{\sim}{p}) = U_{-1/2,\mu}(\underset{\sim}{p},m), \tag{4.12c}$$

$$u_\mu^{(4)}(\underset{\sim}{p}) = U_{-3/2,\mu}(\underset{\sim}{p},m), \tag{4.12d}$$

$$u_\mu^{(5)}(\underset{\sim}{p}) = U_{1/2,\mu}^{(+)'}(\underset{\sim}{p},M), \tag{4.12e}$$

$$u_\mu^{(6)}(\underset{\sim}{p}) = U_{-1/2,\mu}^{(+)'}(\underset{\sim}{p},M), \tag{4.12f}$$

$$u_\mu^{(7)}(\underset{\sim}{p}) = U_{1/2,\mu}^{(-)'}(\underset{\sim}{p},M), \tag{4.12g}$$

and

$$u_\mu^{(8)}(\underset{\sim}{p}) = U_{-1/2,\mu}^{(-)'}(\underset{\sim}{p},M). \tag{4.12h}$$

The wave functions for antiparticles are written as

$$V_{r,\mu}(\underset{\sim}{p}) = \gamma_5 U_{r,\mu}(\underset{\sim}{p}) \tag{4.13a}$$

$$V_{r,\mu}^{(\pm)'}(\underset{\sim}{p}) = \gamma_5 U_{r,\mu}^{(\pm)}(\underset{\sim}{p}) \tag{4.13b}$$

$$v_\mu^{(\pm)}(\underset{\sim}{p}) = \gamma_5 u^{(\alpha)}(\underset{\sim}{p}) \tag{4.13c}$$

These functions satisfy the orthogonality relations,

$$\bar{u}_\mu^{(\alpha)}(\underset{\sim}{p})(\Gamma_4)_{\mu\nu}u_\nu^{(\beta)}(\underset{\sim}{p}) = \varepsilon_\alpha\delta_{\alpha\beta}, \tag{4.14a}$$

$$\bar{v}_\mu^{(\alpha)}(-\underset{\sim}{p})(\Gamma_4)_{\mu\nu}u_\nu^{(\beta)}(\underset{\sim}{p}) = 0, \tag{4.14b}$$

$$\bar{u}_\mu^{(\alpha)}(\underset{\sim}{p})(\Gamma_4)_{\mu\nu}v_\nu^{(\beta)}(-\underset{\sim}{p}) = 0 \tag{4.14c}$$

$$\bar{v}_\mu^{(\alpha)}(-\underset{\sim}{p})(\Gamma_4)_{\mu\nu}v_\nu^{(\beta)}(-\underset{\sim}{p}) = \varepsilon_\alpha\delta_{\alpha\beta}, \tag{4.14d}$$

with

$$\varepsilon_\alpha = \begin{cases} +1 & \text{for } \alpha = 1,2,3,4 \\ -1 & \text{for } \alpha = 5,6,7,8. \end{cases} \tag{4.15}$$

and the completeness relations,

$$\sum_r U_{r,\mu}(\underset{\sim}{p})\overline{U}_{r,\nu}(\underset{\sim}{p}) = \int d_{\mu\nu}(ip)\Delta^{+}(p,m)dp_0, \tag{4.16a}$$

$$\sum_r V_{r,\mu}(-\underset{\sim}{p})\overline{V}_{r,\nu}(-\underset{\sim}{p}) = \int d_{\mu\nu}(ip)\Delta^{-}(p,m)dp_0, \tag{4.16b}$$

$$\sum_r U^{(\pm)'}_{r,\mu}(\underset{\sim}{p})\overline{U}^{(\pm)'}_{r,\nu}(\underset{\sim}{p}) = -\int d^{(\pm)}_{\mu\nu}(ip)\Delta^{+}(p,M)dp_0, \tag{4.16c}$$

$$\sum_r V^{(\pm)'}_{r.\mu}(-\underset{\sim}{p})\overline{V}^{(\pm)'}_{r,\nu}(-\underset{\sim}{p}) = -\int d^{(\pm)}_{\mu\nu}(ip)\Delta^{-}(p,M)dp_0, \tag{4.16d}$$

where $d_{\mu\nu}$ and $d^{(\pm)}_{\mu\nu}$ are those given by (3.8) and $\Delta^{+}(p,m)$ and $\Delta^{-}(p,m)$ are positive and negative frequency parts of $\Delta(p,m)$. From (4.16) we can say that $d_{\mu\nu}$ and $d^{(\pm)}_{\mu\nu}$ derived from $\Lambda^{-1}_{\mu\nu}(ip)$ are projection operatprs of the spin 3/2 field and the redundant spin 1/2 fields. Summing the four relations of (4.16), we have

$$\sum_\alpha \varepsilon_\alpha[u^{(\alpha)}_\mu(\underset{\sim}{p})\overline{u}^{(\alpha)}_\nu(\underset{\sim}{p}) + v^{(\alpha)}_\mu(-\underset{\sim}{p})\overline{v}^{(\alpha)}_\nu(-\underset{\sim}{p})] = \int[d_{\mu\nu}(ip)\Delta(p,m)$$
$$+ \{d^{(+)}_{\mu\nu}(ip) + d^{(-)}_{\mu\nu}(ip)\}\Delta(p,M)]dp_0 = (\Gamma_4)^{-1}_{\mu\nu}. \tag{4.17}$$

Finally, it should be remarked that, because of the identities (4.8), we have the identity,

$$(\delta_{\mu\nu} + \alpha\gamma_\mu\gamma_\nu)U^{(+)}_{r,\nu} = -(1/3)^{1/2}[\delta_{\mu\nu}-(1+3\alpha)\gamma_\mu\gamma_\nu]\gamma_5 U^{(-)}_{r,\nu}. \tag{4.18}$$

Then, we cannot say that which is which, even if we consider that the states described by $U^{(\pm)'}_{r,\mu}$ are the ones composed of the spin 1/2 state of the spinor and the spin 1, 0 states of the vector. We can say only that there are two kinds of spin 1/2 states.

§5. Fourier Expansion and LSZ Formulas

Using the orthonormality relations (4.14) and the completeness relation (4.17), we can expand $\psi_\mu(x)$ in terms of the annihilation operators $a^{(\alpha)}(\underset{\sim}{p})$ $b^{(\alpha)}(\underset{\sim}{p})$ and the creation operators $a^{(\alpha)+}(\underset{\sim}{p})$ $b^{(\alpha)+}(\underset{\sim}{p})$:

$$\psi_\mu(x) = V^{-1/2}\sum_{\underset{\sim}{p}} e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\sum_{\alpha}[a^{(\alpha)}(\underset{\sim}{p})u^{(\alpha)}(\underset{\sim}{p}) + b^{(\alpha)+}(-\underset{\sim}{p})v^{(\alpha)}(-\underset{\sim}{p})] \quad (5.1)$$

From (2.13), (4.14) and (5.1), we obtain

$$\{a^{(\alpha)+}(\underset{\sim}{p}),a^{(\beta)}(\underset{\sim}{p}')\}=\{b^{(\alpha)+}(-\underset{\sim}{p}),b^{(\beta)}(-\underset{\sim}{p}')\}=\varepsilon_\alpha\delta_{\alpha\beta}\delta_{\underset{\sim}{p},\underset{\sim}{p}'}, \quad (5.2a)$$

$$\{a^{(\alpha)+}(\underset{\sim}{p}),b^{(\beta)}(-\underset{\sim}{p}')\}=\{b^{(\alpha)+}(-\underset{\sim}{p}),a^{(\beta)}(\underset{\sim}{p}')\}=0. \quad (5.2b)$$

In (5.2) $\varepsilon_\alpha=-1$ for $\alpha=5,\ldots,8$, accordingly we should interpret ψ^+ not as a Hermitian conjugate but as an adjoint conjugate. Here we don't give an explicit relation between the adjoint conjugate and the Hermitian conjugate, contrary to the earlier work on the theory with the negative metric[19], since, if we give the explicit relation, there arises a question on the manifest covariance of the theory[20].

We define the vacuum state by

$$a^{\alpha}(\underset{\sim}{p})\,|0\rangle=0, \quad (5.3a)$$

$$b^{\alpha}(\underset{\sim}{p})\,|0\rangle=0 \quad (5.3b)$$

and one particle state by

$$a^{(\alpha)+}(\underset{\sim}{p})|0>, \tag{5.4a}$$

$$b^{(\alpha)+}(\underset{\sim}{p})|0> \tag{5.4b}$$

and so on. The bra vector, $<|$, is an adjoint congugate to the ket vector. The norm of physical state, for instance one particle state, is calculated by using (5.2) and (5.3) as

$$<0|a^{(\alpha)}(\underset{\sim}{p})a^{(\beta)+}(\underset{\sim}{p}')|0> = \epsilon_\alpha \delta_{\alpha\beta}\delta_{\underline{pp'}}. \tag{5.5}$$

Thus we see that both of lower spin particles are quantized with negative metric. Substituting (5.1) into (2.15) and integrating with respect to x, we get total Hamiltonian in Fock representation

$$\int H\, d^3x = \sum_{\underset{\sim}{p}}\sum_\alpha (\underset{\sim}{p}^2+m_\alpha^2)^{1/2}\epsilon_\alpha [a^{(\alpha)+}(\underset{\sim}{p})a^{(\alpha)}(\underset{\sim}{p}) - b^{(\alpha)}(-\underset{\sim}{p})b^{(\alpha)+}(-\underset{\sim}{p})], \tag{5.6}$$

where

$$m_\alpha = \begin{cases} m \text{ for } \alpha = 1,2,3,4, \\ M \text{ for } \alpha = 5,6,7,8. \end{cases} \tag{5.7}$$

By virtue of the covariance of the T product and the orthonormalities of wave functions discussed in the previous section, we have LSZ formulas:

$$a_{out}^{(\alpha)}(\underset{\sim}{p})T[O(x_1)....] -(-)^n T[O(x_1)...]a_{in}^{(\alpha)}(\underset{\sim}{p})$$

$$= -i\epsilon_\alpha V^{-1/2}\int d^4x e^{-ipx}\bar{u}_\mu^{(\alpha)}(\underset{\sim}{p})\Lambda_{\mu\nu}(\delta)T[\psi_\mu(x),O(x_1),...], \tag{5.8a}$$

$$(-)^n T[O(x_1)\ldots]a_{in}^{(\alpha)+}(\underset{\sim}{p}) - a_{out}^{(\alpha)+}(\underset{\sim}{p})T[O(x_1)\ldots] \qquad (5.8b)$$

$$= -i\varepsilon_\alpha V^{-1/2}\int d^4x T[\overline{\psi}_\mu(x),\ O(x_1),\ldots]\,\Lambda_{\mu\nu}(-\overleftarrow{\partial})u_\nu^{(\alpha)}(\underset{\sim}{p})e^{ipx}$$

$$b_{out}^{(\alpha)+}(-\underset{\sim}{p})T[O(x_1)\ldots] - (-)^n T[O(x_1)\ldots]b_{in}^{(\alpha)+}(-\underset{\sim}{p})$$

$$= -i\varepsilon_\alpha V^{-1/2}\int d^4x e^{ipx}\overline{v}_\mu^{(\alpha)}(\underset{\sim}{p})\Lambda_{\mu\nu}(\partial)T[\psi_\nu(x),O(x_1)\ldots] \qquad (5.8c)$$

$$(-)^n T[O(x_1)\ldots]b_{in}^{(\alpha)}(-\underset{\sim}{p}) - b_{out}^{(\alpha)}(-\underset{\sim}{p})T[O(x_1)\ldots]$$

$$= -i\varepsilon_\alpha V^{-1/2}\int d^4x T[\psi_\mu(x),O(x_1)\ldots]\Lambda_{\mu\nu}(-\overleftarrow{\partial})v_\nu^{(\alpha)}(\underset{\sim}{p})e^{-ipx} \qquad (5.8d)$$

where n is the number of ψ_μ and $\overline{\psi}_\mu$ in $O(x_1)\ldots$ and p_μ is the energy momentum vector with $p_0 = (\underset{\sim}{p}^2+m^2)^{1/2}$, $O(x_1)\ldots$ are local operators and $d^4x = dx_1dx_2dx_3dx_0$. Of course, we are interested only for (5.8) with $\alpha = 1\sim4$. We have similar formulas for Retarded products.

Acknowledgements

The author is grateful to Professor K. Yamamoto for his suggestions and useful discussions throughout this work. He is also greatly indebted to Professor R. Utiyama for comprehensive guidance and valuable discussions.

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Note added in proof

After completing this work Professor Y. Takahashi kindly pointed out us the existence of a closely related work by H. Munczek, Phys. Rev. 164 (1967) 1794.

Ours is different from his work about the treatments of two spin 1/2 particles whose masses are denoted by M_1 and M_2. In his work one of the spin 1/2 particles is quantized with positive metric and the other with negative metric, whereas, in ours both are quantized with negative metric. His propagator has double pole in the case of $M_1=M_2$. This is not the case in ours.

Let us discuss the reason why these differences occur. Introducing a and b by

$$a = (A+2B+C)/(1+2A+3A^2-2B),$$

$$b = [(A+2B+C)^2+(1+4C)(1+2A+3A^2-2B)]^{1/2}/(1+2A+3A^2-2B),$$

Eq. (3.4b) becomes $M^2=(a\pm b)^2$. Then we have following four possibilities of defining M_1 and M_2:

Case A	$M_1 = -a+b$,	$M_2 = -a-b$,
Case B	$M_1 = -a+b$,	$M_2 = a+b$,
Case C	$M_1 = a-b$,	$M_2 = a+b$,
Case D	$M_1 = a-b$,	$M_2 = -a-b$.

The spin 1/2 part of $\Lambda^{-1}_{\mu\nu}$, that is, $\Lambda^{-1}_{\mu\nu} + d_{\mu\nu}/(p^2+m^2)$ is obtained by the uses of (3.3a)∿(3.3h). When A = -1, it has the form:

Case A

$$\frac{m\gamma_\mu+2ip_\mu}{6(M_1-M_2)m^2}\left[(-M_2+2m)\frac{i\gamma\cdot p-M_1}{p^2+M_1{}^2} + (M_1-2m)\frac{i\gamma\cdot p-M_2}{p^2+M_2{}^2}\right](m\gamma_\nu+2ip_\nu)$$

Case B

$$\frac{m\gamma_\mu+2ip_\mu}{6(M_1+M_2)m^2}\left[(M_2+2m)\frac{i\gamma\cdot p-M_1}{p^2+M_1{}^2} + (M_1-2m)\frac{i\gamma\cdot p+M_2}{p^2+M_2{}^2}\right](m\gamma_\nu+2ip_\nu)$$

The expressions for the cases C and D are given by replacing M_i with $-M_i$ in those of the cases A and B respectively. Thus we find that there are two different cases. The case A is corresponding to Munczek's one and the case B is to ours. If $M_1=M_2$ in our case, the propagator becomes

$$(m\gamma_\mu+2ip_\mu)(i\gamma\cdot p-2m)(m\gamma_\nu+2ip_\nu)/[6m^2(p^2+M_1{}^2)].$$

M_1 appears only in the denominator $P^2+M_1{}^2$. This is exactly the same as what happens for vector meson[11]. The difference between two cases appears when we consider the diagram including closed loops.