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# *On Harmonic Functions Representable by Poisson's Integral*

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Let $R$ be a Riemann surface with positive boundary and let $\{R_n\}$ $(n=0, 1, 2, \cdots)$ be its exhaustion with compact relative boundaries $\partial R_n$. If an open set $G$ has relative boundary consisting of at most enumerably infinite number of analytic curves which cluster nowhere in $R$, we call $G$ a domain. Let $w_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-(G\cap(R_{n+i}-R_n))$ such that $w_{n,n+i}(z)=0$ on $\partial R_{n+i}-G$ and $w_{n,n+i}(z)=1$ on $\partial(G\cap(R_{n+i}-R_n))$ and let $\omega_{n,n+i}(z)$ be a harmonic function in $R-R_0-(G\cap(R_{n+i}-R_n))$ such that $\omega_{n,n+i}(z)=0$ on $\partial R_0$, $\omega_{n,n+i}(z)=1$ on $\partial(G\cap(R_{n+i}-R_n))$ and $\frac{\partial}{\partial n}\omega_{n,n+i}(z)$ $=0$ on $\partial R_{n+i}-G$. We call $\lim_i \lim_n w_{n,n+i}(z)$ and $\lim_n \lim_i \omega_{n,n+i}(z)$[1] the *harmonic measure and the capacitary potential of the ideal boundary* $(G\cap B)$ *determined by G respectively.* We call a function $G(z)$ *a generalized Green's function,* if $G(z)$ is non negatively harmonic in $R$, the harmonic measure of $(B\cap E[z\in R: G(z)>\delta])$ is zero for $\delta>0$ and the Dirichlet integral $D(\min(M, G(z))\leqq kM$ for $M<\infty$.

We map the universal covering surface $R^\infty$ of $R$ onto $|\xi|<1$. Then

**Theorem 1.** *Let $W(z)$ be a positive harmonic in R and superharmonic in $\bar{R}$*[2]*. Then $W(z)=U(z)+V(z)$, where $U(z)$ is a harmonic function in R representable by Poisson's integral in $|\xi|<1$ and $V(z)$ is a generalized Green's function. If furthermore R has no irregular point of the Green's function, then $V(z)=0$, therefore $W(z)$ is representable by Poisson's integral.*

Let $W(z)$ be a function in Theorem 1. Then $W(z)-S(z)$ is also positively harmonic in $R-R_0$ and superharmonic in $\overline{R-R_0}$ and $W(z)-S(z)=W'(z)=0$ on $\partial R_0$, where $S(z)$ is harmonic in $R-R_0$ such that $S(z)=W(z)$ on $\partial R_0$ and $S(z)$ has M. D. I. (minimal Dirichlet integral).

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1) Z. Kuramochi: Harmonic measure and capacity of subsets of the ideal boundary, Proc. Japan Acad. 31, 1955.

2) Let $U(z)$ be a positively harmonic function which satisfies $D(\min(M, U(z))<\infty$. If $U(z)$ $>U_G(z)$ for every compact or noncompact domain $G$, we say $U(z)$ is superharmonic in $\overline{R}$, where $U_G(z)=\lim_{M=\infty} U_G^M(z)$, $U_G^M(z)=\min(M, U(z))$ on $\partial G$ and $U_G^M(z)$ has minimal Dirichlet integral over $G$.

Then $W'(z)$ is representable by a positive mass distribution as follows:[3)]

$$W'(z)=\int_{B_1} N(z, p)d\mu(p),$$

where $B_1$[4)] is the set of minimal points and the total mass $\mu_0$ is given by $ds\int_{\partial R_0}\frac{\partial}{\partial n}W'(z)$ and $D(\min(M, W(z)))\leqq 2\pi M\mu_0$.

First we shall prove for $N(z, p)$. Then

**Theorem 2.** *Let $N(z, p)$ be a minimal function*[5)]*. Then $N(z, p)= U(z, p)+V(z, p)$, where $U(z, p)$ is a positive harmonic function representable by Poisson's integral and $V(z, p)$ is a generalized Green's function. $U(z, p)$ and $V(z, p)$ are functions of at most second class of Baire's function of $p$ for fixed $z\in R-R_0$ with respect to Martin's topology.*[4)]

If $\sup N(z, p)<\infty$, our assertion is trivial and in this case by the boundedness of $V(z, p)$, $V(z, p)$ reduces to constant zero. We shall suppose $\sup N(z, p)=\infty$. Put $G_M=E[z\in R: N(z, p)>M]$. Then $G_M$ is a non compact domain. Consider a harmonic function $w_n(z)$ in $R_n-G_M-R_0$ such that $w_n(z)=0$ on $\partial R_0+\partial R_n-G_M$ and $w_n(z)=1$ on $\partial G_M$. Let $w_M(z)=\lim_n w_n(z)$. Since $N(z, p)$ has M. I. D. over $R-R_0-G_M$ among all functions with values 0 on $\partial R_0$ and $M$ on $\partial G_M$ respectively, $N(z, p)=\lim_n N_n(z, p)$, where $N_n(z, p)$ is harmonic in $R_n-R_0-G_M$ such that $N_n(z, p)=M$ on $\partial G_M$, $N_n(z, p)=0$ on $\partial R_0$ and $\frac{\partial}{\partial n}N_n(z, p)=0$ on $\partial R_n-G_M$. Hence by the maximum principle $N(z, p)\geqq Mw_M(z)$, whence $\lim_{M=\infty} w_M(z)=0$. Map the universal covering surface $(R-R_0)^\infty$ onto $|\zeta|<1$ and consider $w_M(z)$ in $|\zeta|<1$. Then $w_M(z)$ has angular limits $=0$ a. e. (almost everywhere) on a set $E_M$ on $|\zeta|=1$ where $N(z, p)$ has angular limits $<M$. To the contrary, suppose that there exists a set $E$ of positive measure such that $w_M(z)$ has angular limits $>0$ on $E$ and $N(z, p)$ has angular limits $<M$. Then there exists a closed set $E'\subset E$ such that mes $(E-E')<\varepsilon$, $N(z, p)<M-\varepsilon$ in angular domain $D_\varepsilon=[\arg|\zeta-\zeta_0|<\frac{\pi}{2}-\varepsilon, \zeta_0\in E', |\zeta|>1-\varepsilon]$ for any given positive number $\varepsilon$. Let $D'$ be one of components of $D_\varepsilon$. Then the image of

3) Z. Kuramochi: Mass distributions on the ideal boundaries, II. Osaka Math. Jour., 8, 1956.

4) See 3).

5) If $U(z)$ has no functions $V(z)$ such that both $V(z)>0$ and $U(z)-V(z)>0$ are harmonic and superharmonic in $\overline{R-R_0}$ except its own multiples, we say that $U(z)$ is a minimal function.

$G_M$ does not intersect the above $D'$. Let $H(z)$ be a harmonic function in $D'$ with values 1 on $\partial D' - E[|\zeta|=1]$ and 0 on $\partial D' \cap E[|\zeta|=1]$. Since $\partial D'$ is rectifiable, $H(z)=0$ on a. e. $\partial D' \cap E[|\zeta|=1]$. But $w_M(z) \leqq H(z)$, whence $w_M(z)=0$ a. e. on $E_M$.

Let $N_n'(z, p)$ be a harmonic function in $R_n - R_0 - G_L (= E[z \in R: N(z, p) > L])$ such that $N_n'(z, p)=0$ on $\partial R_0$, $N_n'(z, p)=L$ on $\partial G_L \cap R_n$, $N_n'(z, p)= N(z, p)$ on $\partial R_n - G_M (M<L)$ and $\frac{\partial}{\partial n} N_n'(z, p)=0$ on $\partial R_n \cap (G_M - G_L)$. Then

$$D_{R_n - G_L}(N_n'(z, p)) < D_{R_n - G_L}(N(z, p)) .$$

Since $N(z, p)$ has M. D. I. over $R - G_L$, $N_n'(z, p) \to N(z, p)$ in mean. Let $U_{M,n}(z, p)$ be a harmonic function in $R_n - R_0$ such that $U_{M,n}(z, p)=0$ on $\partial R_0$, $U_{M,n}(z, p)=N_n'(z, p)$ on $\partial R_n - G_M$ and $U_{M,n}(z, p)=M$ on $\partial R_n \cap G_M$. In $R_n - R_0 - G_M$, $0 < N_n'(z, p) - U_{M,n}(z, p) \leqq L w_n(z)$. Hence by letting $n \to \infty$, $0 < N(z, p) - U_M(z, p) < L w_M(z)$, where $U_M(z, p)$ is a limit function from a subsequence $(n_1, n_2, \cdots)$. Thus $U_M(z, p)$ has the same angular limits as $N(z, p)$ a. e. on a set $E_M$ on $|\zeta|=1$ on which $N(z, p)$ has angular limits $<M$. Next let $U'_{M,n}(z, p)$ be a harmonic function in $R_n - R_0$ such that $U'_{M,n}(z, p)=0$ on $\partial R_0$ and $U'_{M,n}(z,p)=\min(M, N(z, p))$ on $\partial R_n$. Then we have clearly $\lim_n U_{M,n}(z, p)=\lim_n U'_{M,n}(z, p)$ and $U_{M_2,n}(z, p) > U_{M_1,n}(z, p)$ for $M_2 > M_1$.

Choose a subsequence $(n_1', n_2', \cdots)$ from $(n_1, n_2, \cdots)$ such that $U_{M_2,n'}(z, p)$ converges to $U_{M_2}(z, p)$. Then $U_{M_2}(z, p) \geqq U_{M_1}(z, p)$. Let $U(z, p)=\lim_{M=\infty} U_M(z, p)$. Then $U(z, p)$ is a function representable by Poisson's integral and $U(z, p)$ has the same angular limits as $N(z, p)$ a. e. on $|\zeta|=1$, because $\lim_{M=\infty} w_M(z)=0$. Hence such $U(z, p)$ does not depend on the subsequences. This $U(z, p)$ is the function stated in the theorem.

Next we shall show that $N(z, p) - U(z, p)$ is a generalized Green's function. We proved that $\int_{\partial G_L} N(z, p) ds = \lim_{n=\infty} \int_{\partial G_L} N_n(z, p) ds$[6] for almost all $L$ (i. e. the set of $L$ whose $\partial G_L$ does not satisfy the above condition is of measure zero), where $N_n(z, p)$ is a harmonic function in $R_n - R_0 - G_L$ such that $N_n(z, p)=0$ on $\partial R_0$, $N_n(z, p)=L$ on $\partial G_L \cap R_n$ and $\frac{\partial}{\partial n} N_n(z, p)=0$ on $\partial R_n - G_L$.

We call such $G_L$ a regular domain. Hence we can suppose without loss of generality that $G_L$ is regular. We see the following assertion from $\frac{\partial}{\partial n} N_n(z, p) > 0$ on $\partial G_L$, it is necessary and sufficient condition for

6) sec 3). p. 151.

the regularity of $G_L$ that *there exist* $n_0$ *and* $m_0$ *such that* $\int_{(R-R_m)\cap\partial G_L}\frac{\partial}{\partial n}N_n(z, p)ds$ $<\varepsilon$ *for* $n>n_0$ *and* $m>m_0$ *for any given positive number* $\varepsilon>0$.

Let $J_n(z)$ be a harmonic function in $R_n-R_0-(R_n\cap(G_M-G_L))$ such that $J_n(z)=0$ on $\partial G_M$, $J_n(z)=1$ on $\partial G_L$ and $\frac{\partial}{\partial n}J_n(z)=0$ on $\partial R_n\cap(G_M-G_L)$. Then $(M+(L-M)J_n(z))\to N(z, p)$ in mean, because $N(z, p)$ has M.D.I. over $G_M-G_L$. Hence $\lim_n\int_{\partial G_L}\frac{\partial}{\partial n}J_n(z)ds=\int_{\partial G_L}\lim_n\frac{\partial}{\partial n}J_n(z)ds$ and there exist $m_0$ and $n_0$ such that $\int_{(R-R_m)\cap\partial G_L}\frac{\partial}{\partial n}J_n(z)ds<\varepsilon$ for $n>n_0$ and $m>m_0$ for any given positive number $\varepsilon>0$. But $N_n'(z, p)>(L-M)J_n(z)$ in $G_M-G_L$ and $N_n'(z, p)=(M+(L-M)J_n(z))$ on $\partial G_L$ implies

$$(L-M)\frac{\partial}{\partial n}J_n(z)>\frac{\partial}{\partial n}N_n'(z, p)>0 \text{ on } \partial G_L.$$

Hence $\int_{\partial G_L}\lim_n\frac{\partial}{\partial n}N_n'(z, p)ds=\lim_n\int_{\partial G_L}\frac{\partial}{\partial n}N_n'(z, p)ds.$

Thus $\partial G_L$ *is also regular for* $N_n'(z, p)$.

Let $V_{M,n}(z, p)$ be a harmonic function $=N_n'(z, p)-U_{M,n}(z, p)$. Then $V_{M,n}(z, p)$ is harmonic in $R_n-R_0$, $V_{M,n}(z, p)=0$ on $\partial R_0+(\partial R_n-G_M)$ and $V_{M,n}(z, p)>L-M$ in $G_L$. By the regularity of $\partial G_L$, $\int_{\partial G_L\cap R_n}\frac{\partial}{\partial n}N_n'(z, p)ds$ $\to 2\pi$, as $n\to\infty$. Hence there exists a number $n_0$ for any given $\varepsilon$ such that $\int_{\partial G_L\cap R_n}\frac{\partial}{\partial n}N_n'(z, p)ds\leqq 2\pi+\varepsilon$ for $n>n_0$.

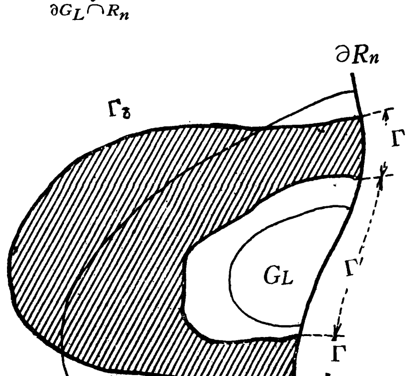


Fig. 1

Put $D=E[z\in R\colon \delta<V_{M,n}(z, p)<M'<M]$. $\Gamma_{M'}=E[z\in R: V_{M,n}(z, p)=M']$, $\Gamma_\delta=E[z\in R: V_{M,n}(z, p)=\delta]$, $\Gamma=\partial R_m\cap D$ and $\Gamma'=\partial R_n\cap E[z\in R: V_{M,n}(z, p)\geqq M']$. Then $D$ intersects only $\partial R_n\cap(G_M-G_L)$, because $N_n'(z, p)-U_{M,n}(z, p)=0$ on $\partial R_n-G_M$ and $N_n'(z, p)-U_{M,n}(z, p)>M'$ on $\partial G_L$ for $L>2M'$. Hence $\Gamma\subset\partial R_n\cap(G_M-G_L)$. Now $\frac{\partial}{\partial n}N_n'(z, p)=0$ on $\partial R_n-D_L$. Since $U_{M,n}(z, p)=\max U_{M,n}(z, p)=M$ on $\Gamma$, $\frac{\partial}{\partial n}U_{M,n}(z, p)>0$ on $\Gamma$ and

$$\int_{\Gamma} \frac{\partial}{\partial n} U_{M,n}(z, p) ds < \int_{\partial R_n \cap (G_N - G_L)} \frac{\partial}{\partial n} U_{M,n}(z, p) ds \leq \int_{\partial G_L \cap R_n} \frac{\partial}{\partial n} N_n'(z, p) ds \leq 2\pi + \varepsilon .$$

$$\int_{\Gamma_{M'}} \frac{\partial}{\partial n} N_n'(z, p) ds = \int_{\partial G_L \cap R_n} \frac{\partial}{\partial n} N_n'(z, p) ds < 2\pi + \varepsilon .$$

$$0 < \int_{\Gamma_{M'}} \frac{\partial}{\partial n} U_{M,n}(z, p) ds = \int_{\Gamma'} \frac{\partial}{\partial n} U_{M,n}(z, p) ds \leq \int_{\partial R_n \cap G_M} \frac{\partial}{\partial n} U_{M,n}(z, p) ds < 2\pi + \varepsilon .$$

$$\int_{\Gamma_\delta} \frac{\partial}{\partial n} N_n'(z, p) ds = - \int_{\partial G_L \cap R_n} \frac{\partial}{\partial n} N_n'(z, p) ds \geq -2\pi - \varepsilon .$$

$$\int_{\Gamma_\delta} \frac{\partial}{\partial n} U_{M,n}(z, p) ds = \int_{\Gamma + \Gamma'} \frac{\partial}{\partial n} U_{M,n}(z, p) ds$$

$$\geq - \int_{\partial R_n \cap G_M} \frac{\partial}{\partial n} U_{M,n}(z, p) ds > -2\pi - \varepsilon .$$

Hence $D(\min V_{M,n}(z, p), M') = D_D(V_{M,n}(z, p)) = \int_{\Gamma_\delta + \Gamma + \Gamma_{M'}} (N_n'(z, p) - U_{M,n}(z, p)) \frac{\partial}{\partial n} (N_n'(z, p) - U_{M,n}(z, p)) ds \leq M'(4\pi + 2\varepsilon) + \delta(2\pi + \varepsilon)$ and $D_{R_m - R_0}(\min(V_{M,n}(z, p), M') \leq M'(4\pi + 2\varepsilon) + \delta(2\pi + \varepsilon)$, for every $m$ (for every $n > 1$).

Let $n \to \infty$, then $N_n'(z, p) \to N(z, p)$ in $R - R_0 - G_L$, $U_{M,n}(z, p) \to U_M(z, p)$, $V_{M,n}(z, p) \to V(z, p)$ and derivatives of $V_{M,n}(z, p) \to$ derivatives of $V_M(z, p)$. By letting $n \to \infty$ and then $\delta \to 0$ and $\varepsilon \to 0$, $D_{R - R_m}(\min(V_M(z, p), M') \leq 4\pi M'$.

Let $L \to \infty$ and then $M \to \infty$. Then $U_M(z, p) \uparrow U(z, p)$ and $V_M(z, p) \downarrow V(z, p)$ and then by letting $m \to \infty$, we have

$$D_{R - R_0}(V(z, p), M')) \leq 4\pi M' .$$

On the other hand, clearly $V(z, p) = N(z, p) - U(z, p)$ has angular limits $= 0$ a.e. on $|\zeta| = 1$. Hence $V(z, p)$ *is a generalized Green's function.*

Since $U_M(z, p) = \lim_n U_{M,n}(z, p) = \lim_n U'_{M,n}(z, p)$, where $U'_{M,n}(z, p)$ is a harmonic function in $R_n - R_0$ such that $U'_{M,n}(z, p) = \min(M, N(z, p))$ on $\partial R_0 + \partial R_n$. Hence $U'_{M,n+i}(z, p) \leq U'_{M,n}(z, p)$ on $\partial R_n$, whence $U'_{M,n}(z, p) \downarrow U_M(z, p)$. Therefore there exists a number $n_0$ such that $U'_M(z, p) \leq U'_{M,n}(z, p) - \varepsilon$ for $n > n_0$ for any given positive number $\varepsilon$. Next since $N(z, p)$ is a continuous function of $p$ for any point $z \in R - R_0$, there exists a number $\delta_0$ such that

$$|N(z, p) - N(z, p_j)| < \varepsilon \text{ on } \partial R_n \text{ for } \delta(p, p_j) < \delta_0 .$$

Hence $U_M(z, p) \geqq U'_{M,n}(z, p) - \varepsilon \geqq U'_{M,n}(z, p_j) - 2\varepsilon \geqq U_M(z, p_j) - 2\varepsilon$.
*Thus $U_M(z, p)$ is an upper semicontinuous function of $p$*, whence $V_M(z, p)$ is a lower semicontinuous function of $p$ by the continuity of $N(z, p)$. $U_M(z, p) \uparrow U(z, p)$ and $V_M(z, p) \downarrow V(z, p)$ imply that $U(z, p)$ *and* $V(z, p)$ *are at most second class of Baire's functions.*

**Properties of generalized Green's functions.**

**Theorem 3.** *Let $V(z)$ be a generalized Green's function such that $D(\min(V(z), M)) \leqq \pi M$. Let $V'(z)$ be a non negative harmonic function such that $V'(z) \leqq V(z)$. Then $V'(z)$ is also a generalized Green's function such that $D(\min(V(z), M)) \leqq \pi M$.*

Put $D = E[z \in R: V'(z) < M$ and $V(z) > M]$. Let $V'_{n,n+i}(z)$ be a harmonic function in $R_{n+i} - R_0 - E[z \in R: V'(z) > M] - (D \cap (R_{n+i} - R_n))$ such that $V'_{n,n+i}(z) = V'(z)$ on $\partial R_0 + (E[z \in R: V'(z) \leqq M] \cap R_n)$, $\frac{\partial}{\partial n} V'_{n,n+i}(z) = 0$ on $\partial R_n \cap D$ and $V'_{n,n+i}(z) = V(z)$ on $\partial R_{n+i} - E[z \in R: V(z) > M]$. Then by the Dirichlet principle

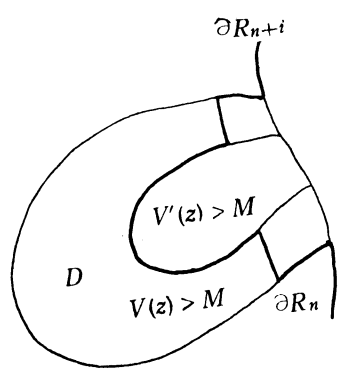

Fig. 2

$$D(\min M, V'_{n,n+i}(z)) \leqq D(\min(M, V(z)))$$

for every $i$ and $n$.

Next clearly $\lim_n \lim_i V'_{n,n+i}(z) = \tilde{V}(z)$ exists and $\tilde{V}(z)$ has angular limits $\leqq V(z)$ a.e. where $V(z)$ has angular limits $< M$. But $V(z)$ has angular limits $= 0$ a.e. on $|\zeta| = 1$, whence $\tilde{V}(z) = V'(z)$ and

$$D(\min(M, V'(z)) \leqq D(\min(M, V(z))) .$$

Hence $V'(z)$ is a generalized Green's function.

**Theorem 4.** Let $V(z)$ *be a generalized Green's function and put $R_\delta = E[z \in R: V(z) > \delta]$ and $D_M = E[z \in R: V(z) > M]$. Then $D_M$ determines a set of the ideal boundary of capacity zero.*

Let $V_{n,n+i}(z)$ be a harmonic function in $(R_\delta \cap R_{n+i}) - ((R_{n+i} - R_n) \cap D_M)$ such that $V_{n,n+i}(z) = 0$ on $\partial R_\delta \cap R_{n+i}$, $V_{n,n+i}(z) = 1$ on $\partial(D_M \cap (R_{n+i} - R_n))$ and $\frac{\partial}{\partial n} V_{n,n+i}(z) = 0$ and $\partial R_{n+i} \cap (R_\delta - D_M)$. Then by the Dirichlet principle

$$\int_{\partial(D_M \cap (R_{n+i} - R_n))} \frac{\partial}{\partial n} V_{n,n+i}(z) ds = D(V_{n,n+i}(z)) \leqq \frac{1}{(M-\delta)^2} D(V(z)) \leqq \frac{2\pi M}{(M-\delta)^2}$$

for every $i$ and $n$,

and clearly $V_{n,n+i}(z)$ converges to $V_n(z)$ in mean as $i\to\infty$.

$$\int\frac{\partial}{\partial n}(V_{n,n+i}(z)-V_{m,n+i}(z))\,V_{n,n+i}(z)=\int_{\partial(D_M\cap(R_{n+i}-R_n))}\frac{\partial}{\partial n}V_{n,n+i}(z)ds$$
$$-\frac{\partial}{\partial n}V_{m,n+i}(z)ds=D(V_{n,n+i}(z))-D(V_{m,n+i}(z)),\quad\text{for } n<m<n+i.$$

Since $V_{m,n+i}(z)\to V_n(z)$ in mean, we have

$D(V_n(z)-V_m(z),\ V_n(z))=D(V_n(z))-D(V_m(z))$ and $D(V_n(z)-V_m(z))=D(V_n(z))-D(V_m(z))$.

Hence $V_n(z)$ converges to a function $V^*(z)$ in mean as $n\to\infty$.

Map the universal couvring surface $R_\delta^\infty$ of $R_\delta$ onto $|\zeta|<1$. Then $V^*(z)$ has angular limits $=0$ a.e. on $|\xi|=1$ by that $V(z)$ has angular limits $=\delta$ a.e. on $|\zeta|=1$. Hence $V^*(z)=0$. Let $F$ be a closed arc on $\partial R_\delta$. Let $\omega_{n,n+i}(z)$ be a harmonic function in $R_\delta\cap R_{n+i}-((R_{n+i}-R_n)\cap D_M)$ such that $\omega_{n,n+i}(z)=0$ on $F$, $\omega_{n,n+i}(z)=1$ on $\partial(D_M\cap(R_{n+i}-R_n))$ and $\frac{\partial\omega_{n,n+i}}{\partial n}(z)=0$ on $\partial R_{n+i}-D_M$. Then by the Dirichlet principle

$$D(\omega_{n,n+i}(z))\leqq D(V_{n,n+i}(z))\,.$$

We see as above that $\omega_{n,n+i}(z)\to\omega_n(z)$ in mean and $\omega_n(z)\to\omega(z)$ in mean and by $V_n(z)\to V^*(z)$ in mean. We have $D(\omega(z))\leqq D(V^*(z))\leqq 0$.
Thus $D_M$ determines a set of the boundary of capacity zero.

**Theorem 5.** *Let* $V(z)$ *be a generalized Green's function. Then* $\int_c\frac{\partial}{\partial n}V(z)ds=k$ *on every niveau curve, where* $k$ *is a constant such that* $D(\min(M,\ V(z))=Mk$.

Let $\omega_n(z)$ and $D_M$ be in Theorem 4. Let $\omega_n'(z)$ be a harmonic function in $D_M\cap(R_n-R_{n_0})+(R_\delta\cap R_{n_0})$ such that $\omega_n'(z)=0$ on $F\cap R_{n_0}$, $\omega_n'(z)=1$ on $D_M\cap\partial R_n$ and $\frac{\partial\omega_n'}{\partial n}(z)=0$ on $(\partial R_\delta\cap R_{n_0})+(\partial R_{n_0}-D_M)+(\partial D_M\cap(R_n-R_{n_0}))-F$. Then clearly

$$D(\omega_n'(z))\leqq D(\omega_n(z))\,,$$

whence by Theorem 4 $\omega_n(z)\to 0$ as $n\to\infty$. Hence there exists for any given large number $T$, a number $n$ and a harmonic function $\omega_n^*(z)$ in $(R_\delta\cap R_{n_0})+(\partial R_{n_0}-D_M)+(\partial D_M\cap(R_n-R_{n_0}))$ such that $\omega_n^*(z)=0$ on $F$, $\frac{\partial\omega_n^*}{\partial n}(z)=0$ on $(\partial R_\delta\cap R_{n_0})-F+(\partial D_M\cap(R_n-R_{n_0}))+(\partial R_{n_0}-D_M)$, $\omega_n^*(z)=T$ on $\partial R_n\cap D_M$ and $\int_{F\cap R_{n_0}}\frac{\partial\omega_n^*}{\partial n}(z)=2\pi$.

Put $re^{i\theta}=\exp(\omega_n^*(z)+i\tilde{\omega}_n^*(z))$, where $\tilde{\omega}_n^*(z)$ is the conjugate function of $\omega_n^*(z)$. Put $L(r)=\int\left|\frac{\partial}{\partial n}V(z)\right|rd\theta$, where the integration is taken over $((R_\delta\cap R_{n_0})+(D_M-D_{M_2}))\cap(E[z\in R;\ \omega_n^*(z)=\log r])(M<M_2)$.

Suppose $L(r)>\varepsilon_0$ for every $r$. Then

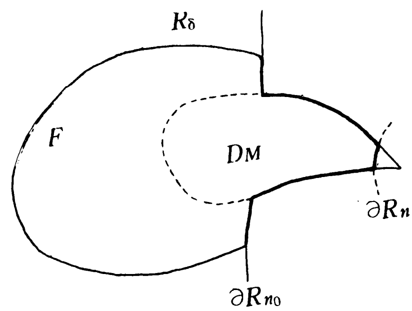


Fig. 3

$$\varepsilon_0^2\int_1^T\frac{1}{r}dr\leqq\int_1^T\frac{L^2(r)}{r}dr$$
$$\leqq\iint_{D_M-D_{M_2}}\left\{\left(\frac{\partial V(z)}{\partial r}\right)^2+r^2\left(\frac{\partial V(z)}{\partial\theta}\right)^2\right\}rdrd\theta$$
$$\leqq D_{R_\delta-D_{M_2}}(V(z))<\infty\,.$$

Let $T\to\infty$. Then $D(V(z))\to\infty$. This is a contradiction. Hence there exists a sequence $\{r_i\}$ such that $L(r_i)\to 0$. Since $\frac{\partial V}{\partial n}(z)<0$ on $\partial D_M$ and $\frac{\partial}{\partial n}V(z)>0$ on $\partial D_{M_2}$. Hence $k=\int_{\partial D_M}\frac{\partial}{\partial n}V(z)ds=\int_{\partial D_{M_2}}\frac{\partial}{\partial n}V(z)ds$ and $D_{D_M-D_{M_2}}(V(z))=k(M_2-M)$. Hence we have the theorem.

**Lemma 3.** *Let $V(z)$ be a positive harmonic function (not necessarily a generalized Green's function) in $R-R_0$. Let $G$ and $G'$ be non compact domains such that $R-R_0=\bar{G}+G'$.*[7] *Let ${}_nV_G^\alpha(z)\,({}_nV_G^\beta(z))$ be the lower (upper) envelope of super (sub) harmonic functions larger (smaller) than $V(z)$ in $G\cap(R-R_n)$. Put $V_G^\alpha(z)=\lim_n {}_nV_G^\alpha(z)$ and $V_G^\beta(z)=\lim_n {}_nV_G^\beta(z)$. Then*

$$\overset{\alpha}{G}(V_G^\alpha(z))=V_G^\alpha(z)\ \text{ and }\ \overset{\alpha}{G}(V_{G'}^\beta(z))=0\,.$$

Let $V_{n,n+i}(z)$ be a harmonic function in $R_n+((R_{n+i}-R_n)\cap G)-R_0$ such that $V_{n,n+i}(z)=0$ on $\partial R_0+(\partial R_{n+i}-G)$ and $V_{n,n+i}(z)=V(z)$ on $\partial G\cap(R_{n+i}-R_n)+G\cap(R-R_n)$. Then for every $G\cap(R-R_n)$ by $V_{n,n+i}(z)\uparrow V_n(z)$ and by $G_i(\zeta,z)\uparrow G(\zeta,z)$

$$\lim_i V_{n,n+i}(z)=V_n(z)=\int_{\partial(G\cap(R-R_n))+(G\cap\partial R_n)}V_n(\zeta)\frac{\partial}{\partial n}G(\zeta,z)ds,$$

where $G_i(\zeta,z)$ and $G(\zeta,z)$ are the Green's function of $R_{n+i}-R_0-(G\cap(R_{n+i}-R_n)$ and $R-R_0-(G\cap(R-R_n))$ respectively.

---

7) $\bar{G}$ means the closure of $G$.

Since $V_n(z) \downarrow V_G^\alpha(z)$, $V_G^\alpha(z) = \int_{\partial(G\cap(R-R_n))+(G\cap\partial R_n)} V_G^\alpha(\zeta)\frac{\partial}{\partial n}G(\zeta, z)ds$.

Next let $V'_{n,n+i}(z)$ be a harmonic function in $R_n+((R_{n+i}-R_n)\cap G)-R_0$ such that $V'_{n,n+i}(z)=0$ on $\partial R_0+(\partial R_{n+i}-G)$ and $V'_{n,n+i}(z)=V_G^\alpha(z)$ on $(\partial G\cap(R_{n+i}-R_n))+(G\cap\partial R_n)$. Then

$$\lim_i V'_{nn,+i}(z) = \int V_G^\alpha(\zeta)\frac{\partial}{\partial n}G(\zeta, z)ds ,$$

i. e. $\lim_i V'_{n,n+i}(z) = V_G^\alpha(z)$ for every $n$, hence

$${}_G^\alpha(V_G^\alpha(z)) = V_G^\alpha(z) . \tag{1}$$

Let $\tilde{V}_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-((R_{n+i}-R_n)\cap G)-R_0$ such that $\tilde{V}_{n,n+i}(z)=0$ on $\partial R_0+(\partial R_n\cap G)+(\partial G\cap(R_{n+i}-R_n))$ and $\tilde{V}_{n,n+i}(z)$ $=V(z)$ on $\partial R_{n+i}\subset G'$. Then

$$V(z) = V_{n,n+i}(z)+\tilde{V}_{n,n+i}(z), \quad \text{which implies}$$

$$V(z) = V_G^\alpha(z)+V_{G'}^\beta(z) . \tag{2}$$

From (1) we have
$$V(z) = {}_G^\alpha(V_G^\alpha(z)+V_{G'}^\beta(z))+V_{G'}^\beta(z)$$
$$= {}_G^\alpha(V_G^\alpha(z))+{}_G^\alpha(V_{G'}^\beta(z))+V_{G'}^\beta(z) ,$$

whence by (1) and (2) we have

$${}_G^\alpha(V_{G'}^\beta(z)) = 0 . \tag{3}$$

Let $V(z)$ be a generalized Green's function. Let $G(z, q)$ be the Green's function of $R-R_0$ with pole at $q$. Put $G=E[z\in R: G(z, q)>k]$ and $G'=E[z\ni R: G(z, q)<k]$. Then $V(z)=V_G^\alpha(z)+V_{G'}^\beta(z)$. We shall study the properties of $V_{G'}^\beta(z)$.

**Lemma 4.** *Let* $V(z)$ *be a generalized Green's function and put* $G=E[z\in R: G(z, q)>k]$ *and* $G'=E[z\in R: G(z, q)<k]$ *and* $D_M=E[z\in R: V_{G'}^\beta(z)>M]$. *Let* $H_{G'}^M(z)$ *be the lower envelope of superharmonic function larger than* $\min(M, V_{G'}^\beta(z))$ *on* $G'\cap D_M$. *Then*

$$\lim_{M=\infty} H_{G'}^M(z) = V_{G'}^\alpha(z) .$$

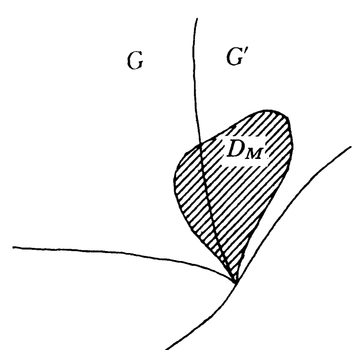

Fig. 4

For simplicity denote $V_{G'}^\beta(z)$ by $H(z)$. Let ${}_nH_{G'}^M(z)$ be a harmonic function in $R_n-R_0-(D_M\cap G')$ such that ${}_nH_{G'}^M(z)=0$ on

$\partial R_0+\partial R_n-(D_M\cap G')$ and ${}_nH_{G'}^M(z)=M$ on $\partial(D_M\cap G')$. Let ${}_n\check{H}_G^M(z)$ be a harmonic function in $R_n-R_0-(D\cap G)$ such that ${}_n\check{H}_G^M(z)=0$ on $\partial R_0+(\partial R_n-(D_M\cap G))$, ${}_n\check{H}_G^M(z)=M$ on $\partial D_M\cap G$ and ${}_n\check{H}_G^M(z)=H(z)-M$ on $\partial G\cap D_M$. Then cleary

$$\lim_n {}_nH_G^M(z)\leqq H(z)\leqq \lim_n {}_nH_{G'}^M(z)+\lim_n {}_n\check{H}_G^M(z)$$

and

$$\lim_{M\to\infty}(\lim_n {}_n\check{H}_G^M(z))\leqq {}_G^{\alpha}(H(z))={}_G^{\alpha}(V_{G'}^{\beta}(z))=0\,.$$

Hence

$$V_{G'}^{\beta}(z)=H(z)=\lim_{M=\infty}\lim_n {}_nH_{G'}^M(z)=\lim_{M\to\infty}H_{G'}^M(z)\,.$$

**Theorem 6.** *Let $V(z)$ be a generalized Green's function such that $D(\min M, V(z))\leqq M\pi$. Then by Lemma 3, $V(z)=V_G^{\alpha}(z)+V_{G'}^{\beta}(z)$, where $G'=E[z\in R:\ G(z,q)<k]$.*

$$\textit{Then}\ V_{G'}^{\beta}(q)\leqq\frac{k}{2}\,.$$

Clearly $V(z)\geqq V_G^{\alpha}(z)$ and $V(z)\geqq V_{G'}^{\beta}(z)\geqq 0$. If $V_{G'}^{\beta}(z)=0$, our assertion is trivial. Suppose $V_{G'}^{\beta}(z)>0$. Then by Theorem 3, $V_{G'}^{\beta}(z)$ is also a generalized Green's function such that $D(\min((M, V_{G'}^{\beta}(z))\leqq M\pi$.
Next by Lemma 3

$$V_{G'}^{\beta}(z)=H(z)=\lim_{M=\infty}\lim_n {}_nH_{G'}^M(z)\ \text{and}\ H(z)\geqq H_{G'}^M(z)=\lim_n {}_nH_{G'}^M(z)\,.$$

Hence by Theorem 5

$$\int_{\partial(D_M\cap G')}\frac{\partial}{\partial n}H_{G'}^M(z)ds\leqq\int_{\partial D_M}\frac{\partial}{\partial n}H(z)ds\leqq\pi \tag{4}$$

where $D_M=E[z\in R:\ H(z)>M]$.

Since $g_\delta=E[z\in R:\ H_{G'}^M(z)>\delta]\subset E[z\in R:\ H(z)>\delta]$, $(E[z\in R:\ H_{G'}^M(z)>L]=)D_L\cap G'$ determines a set of the boundary of capacity zero for $L>\delta$ by Theorem 4. Hence by $D({}_{G'}^{M}H(z))<\infty$ over $R-R_0-(D_M\cap G')$, we can prove as in Theorem 5

$$\int_{\Gamma_\delta}\frac{\partial}{\partial n}H_{G'}^M(z)ds=-\int_{\partial(D_M\cap G')}\frac{\partial}{\partial n}H_{G'}^M(z)ds\,,$$

where $\Gamma_\delta=E[z\in R:\ H_{G'}^M(z)=\delta]$.

Let $G_\delta(z,q)$ be the Green's function of $g_\delta\cap(R-R_0)$. Then $D(G_\delta(z,q))<\infty$ over a neighbourhood of the ideal boundary. Hence there exists a sequence of curves $\{\Gamma_i\}$ such that $\int_{\Gamma_i \cap D_M} \left| \frac{\partial}{\partial n} G_\delta(z, q) \right| ds \to 0$ as $i \to \infty$ and $\{\Gamma_i\}$ clusters at the ideal boundary as $i \to \infty$ and every $\Gamma_i$ separates the boundary determined by $D_M$ from $q$. Let $C = \partial(D_M \cap G')$ and $C_i$ be the part of $C_i$ contained in the domain $\ni q$ separated by $\Gamma_i$ and $C_i' = C - C_i$. Then

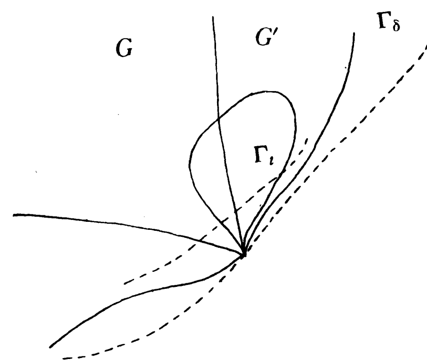


Fig. 5

$$\int_{C_i + C_i' + q + \Gamma_\delta} H_{G'}^M(z) \frac{\partial}{\partial n} G_\delta(z, q) ds = \int_{C + q + \Gamma_\delta} G_\delta(z, q) \frac{\partial}{\partial n} H_{G'}^M(z) ds,$$

$$M \int_{C_i + C_i'} \frac{\partial}{\partial n} G_\delta(z, q) ds + 2\pi H_{G'}^M(q) + \delta \int_{\Gamma_\delta} \frac{\partial}{\partial n} G_\delta(z, q) ds = \int_C G_\delta(z, q) \frac{\partial}{\partial n} H_{G'}^M(z) ds .$$

But the first term of the left hand side $\to 0$ as $i \to \infty$ and the remaining terms don't depend on $i$. Hence by letting $\delta \to 0$ and by $G_\delta(z, q) \uparrow G(z, q)$, we have

$$2\pi H_{G'}^M(q) = \int_C G(z, q) \frac{\partial}{\partial n} H_{G'}^M(z) ds \leqq k\pi ,$$

because $G(z, q) \leqq k$ in $G'$. Then by letting $M \to \infty$

$$H(q) = V_{G'}^\beta(q) \leqq \frac{k}{2} .$$

Put $V_G^\alpha(z) = V^{*k}(z)$ and $V_{G'}^\beta(z) = V'^k(z)$. Then by Theorem 6, $V'^k(z) \to 0$ as $k \to 0$. Then we have

**Theorem 7.** *Every generalized Green's function $V(z)$ is divided into two parts such that*

$$V(z) = V^{*k}(z) + V'^k(z) \text{ and } V(z) = \lim_{k \to 0} V^{*k}(z) .$$

Remark. $K(z, p_i) = \frac{G(z, p_i)}{G(p_0, p_i)}$ ($p_0$ is a fiexed point) is a positive harmonic function. Martin[8] defined *ideal boundary points* by using above functions and prove that every positive harmonic function is representable

8) R. S. Martin: Minimal positive harmonic functions. Trans. Amer. Math. Soc. 39, 1941.

by a unique mass distribution $\nu$ as follows: $\int_{B_1} K(z, p)d\nu(p)$, where $B_1$ is the set of minimal points. If $\overline{\lim_{i\to\infty}} G(p_i, q)>0$ as $p_i$ tends to a boundary point $p$ and $K(z, p_i) \to K(z, p)$, we call $p$ an irregular boundary point. In this case, $K(z, p)$ is a constant multiple of $G(z, p) = \lim_i G(z, p_i)$. We denote by $I_k$ the set of Martin's boundary point $p$ such that $\lim_{z\to p} G(z, q) \geqq k$. Then $V^{*k}(z)$ is represented by a mass distribution $\nu$ on $I_k$. Hence by Theorem 8 a generalized Green's function is represented by a mass distribution $\nu$ on $I=\bigcup_{k>0} I_k$.

**Theorem 8.** *Let $W(z)$ be a positive harmonic in $R-R_0$ and superharmonic function in $\overline{R-R_0}$ vanishing on $\partial R_0$. Then*

$$W(z) = \int N(z, p)d\mu(p) = \int U(z, p)d\mu(p) + \int V(z, p)d\mu(p) = U(z) + V(z),$$

*where $U(z) = \int U(z, p)d\mu(p)$ is a harmonic function representable by Poisson's integral and $V(z)= \int V(z, p)d\mu(p)$ is a generalized Green's function.*

Since $0< U(z, p) \leqq N(z, p)$, family $\{U(z, p)\}$ is uniformly bounded in every compact domain in $R-R_0$ and the partial derivatives of them are equicontinuous and $\Delta U(z, p)=0$, hence $U(z)$ and $V(z)$ are harmonic in $R-R_0$.

For a harmonic function $H(z)$ define $H^M(z) = \lim_n H_n^M(z)$, where $H_n^M(z)$ is a harmonic function in $R_n-R_0$ such that $H_n^M(z) = \min(M, H(z))$ on $\partial R_0+\partial R_n$. Then clealry ${}^M(H^M(z)) = H^M(z)$. Since $0< U(z, p) \leqq N(z, p)$ and $U^M(z) \uparrow U(z, p)$ as $M\uparrow\infty$, whe have

$$U(z) = \int U(z, p)d\mu(p) = \lim_{M=\infty} \int U^M(z, p)d\mu(p) \leqq \lim_{M=\infty} {}^M\Big[ \int N(z, p)d\mu(p) \Big]$$
$$= \lim_{M=\infty} \lim_n W_n^M(z),$$

where $W_n^M(z)$ is a harmonic function in $R-R_0$ such that $W_n^M(z) = \min(M, W(z))$ on $\partial R_0+\partial R_n$. Now $\lim_{M=\infty}\lim_n W_n^M(z) = W^p(z)$ *is representable by Poisson's integral.* $0< U(z) \leqq W^p(z)$ implies the *Poisson's integrability of* $U(z)$.

By the Remark $V(z, p) = \int_I K(z, q)d\nu(q)$, whence $V(z) = \int V(z, p)d\mu(p) = \int_I K(z, q)d\lambda(q)$. Hence there exist $n_0$ and $k_0$ such that

$$\int V(z, p)d\mu(p) < \int_{I_k} K(z, q)d\lambda(q) + \varepsilon \tag{5}$$

for $z \in R_n - R_0$, $n < n_0$ and $k < k_0$ for any given positive number $\varepsilon$, where $\lambda'$ is the restriction of $\lambda$ on $I_k$.

Denote by $(\int_{I_k} K(z, q)d\lambda'(q))^n_{I_k}$ the lower envelope of superharmonic functions larger than $\int_{I_k} K(z, q)d\lambda'(q)$ in $G \cap (R-R_0)$. Put $(\int_{I_k} K(z, q)d\lambda'(q))_{I_k} = \lim_n (\int_{I_k} K(z, q)d\lambda'(q))^n_{I_k}$. Then as in Lemma 3 and Theorem 2 it is proved that $(\int_{I_k} K(z, q)d\lambda'(q)) = (\int_{I_k} K(z, q)d\lambda'(q))_{I_k}$ and $(\int_{I_k} K(z, q)d\lambda'(q))$ has angular limits $=0$ a.e. on the ideal boundary[9]. In (5) let $\varepsilon \to 0$. Then $\int K(z, q)d\lambda(q) = \int V(z, p)d\mu(p)$ has angular limits $=0$ a.e. on the ideal bonndary. Hence $U(z) = \int U(z, p)d\mu(p)$ has the same angular limits as $\int N(z, p)d\mu(p)$ a.e. on the ideal bounary. Thus by Poisson's integrability of $U(z)$ and $W^p(z)$, we have $U(z) \equiv W^p(z)$ and $W(z) - W^p(z) \equiv \int V(z, p)d\mu(p)$. Now $W(z) - W^p(z) = \lim_{M'=\infty} \lim_n W'^{M'}_n(z)$, where $W'^{M'}_n(z)$ is a harmonic function in $R_n - R_0$ such that $W'^{M'}_n(z) = 0$ on $\partial R_0$ and $W'^{M'}_n(z) = W(z) - W^{M'}_n(z)$ on $\partial R_n$. Since $N(z, p)$ is a continuous function of $p$ for $z \in R$, there exists a sequence $\{W_m(z)\}$ $(m = 1, 2, \cdots)$ of the form $W_m(z) = \sum^m c_i N(z, p_i)$ $(c_i > 0$, $\sum c_i = \mu_0 = \int d\mu(p))$ such that $W_m(z) \to W(z)$ in $R - R_0$. On the other hand, let $V^{M'}_{n,m}(z)$ be a harmonic function in $R_n - R_0$ such that $V^{M'}_{n,m}(z) = 0$ on $\partial R_0$ and $V^{M'}_{n,m}(z) = \min(W^m(z) - M', 0)$ on $\partial R_n$. Then there exists a sequence $\{V^{M'}_{n,m}(z)\}$ which converges to $\lim_n W'^{M'}_n(z)$ as $n \to \infty$ and $m \to \infty$.

Since $V^{M'}_{n,m}(z)$ is constructed from $W_m(z) = \sum^m c_i N(z, p)$, we can prove by the method used for $V(z, p)$ and $N(z, p)$ that $D(\min(M, V^{M'}_{n,m}(z)) \leqq 4\pi(\sum c_i)M'$ for $M' < M$. Hence by letting $n \to \infty$, $m \to \infty$ and $M \to \infty$ we have

$$D(\min(M', V(z)) = D(\min(M', \lim_n \lim_m V^{M'}_{n,m}(z))$$
$$\leqq \lim_{M=\infty} \lim_{m,n} D(\min(M', V^{M}_{n,m}(z)) \leqq 4\pi(\sum c_i)M'.$$

Hence $\int V(z, p)d\mu(p)$ is a generalized Green's function. We have Theorem 8.

**Lemma 5.** *Let $V(z)$ be a generalized Green's function in $R - R_0$ such*

9) We map the universal covering surface of $(R - R_0)$ onto $|\zeta| < 1$. If the function $U(z)$ has angular limits$=0$ a.e. on the image of the ideal boundary on $|\zeta| = 1$. We say simply $U(z)$ has angular limits$=0$ a.e. on the ideal boundary.

*that* $D(\min(M, V(z)) \leqq M\pi$. *Then there exists a uniquely determined generalized Green's function* $V^*(z)$ *in* $R$ *such that* $D(\min(M, V^*(z)) \leqq M\pi$ *and sup* $(V^*(z)) - V(z)) < \infty$.

Since $\partial R_0$ is compact, there exists a contant $L$ such that $0 < \frac{\partial}{\partial n} V(z) \leqq L$ on $\partial R_0$. Let $\omega(z)$ be a positive bounded harmonic function in $R - R_0$ such that $\omega(z) = 1$ on $\partial R_0$ and $\omega(z)$ has angular limits $= 0$. a. e. on the ideal boundary of $R - R_0$. Put $\tilde{\omega}(z) \equiv 1$ in $R_0$ and $\tilde{\omega}(z) \equiv \omega(z)$ in $R - R_0$. Then $V(z) + K\tilde{\omega}(z) (K > L)$ is a superharmonic function in $R$. Let $V_n^*(z)$ be a harmonic function in $R_n$ such that $V_n^*(z) = V(z)$ on $\partial R_n$. Then $V(z) < V_n^*(z) \leqq V(z) + K\omega(z)$. Choose a subsequence $(n_1, n_2, , \cdots)$ so as $V_n^*(z)$ converges to $V^*(z)$. Then

$$V(z) \leqq V^*(z) \leqq V(z) + K\tilde{\omega}(z) .$$

Hence $V^*(z)$ has angular limits $= 0$ a. e. on the boundary of $R$ and by sup $(V^*(z) - V(z)) < \infty$, we see that such $V^*(z)$ does not depend on the above subsequence and $V^*(z)$ is uniquely determined.

Clearly $D(\min(M, V(z)) \leqq D(\min(M+K, V(z) + K\omega(z))$, hence
$D(\min(M, V(z)) \leqq 2D(\min(2M, V(z)) + 2D(\omega(z)) \leqq 10\pi M$, for large $M$.

But both $E[z \in R - R_0, V^*(z) > \delta]$ and $E[z \ni R - R_0, \omega(z) > \delta]$ determine sets of the boundary of capacity zero,[10] whence as in Theorem, we have

$$\int_C \frac{\partial}{\partial n} V^*(z) ds = k \leqq 10\pi ,$$

for every niveau curve $C$ of $V(z)$ and $D(\min(M, V^*(z)) \leqq 10\pi M$ for every $M$. Thus $V^*(z)$ is a generalized Green's function.

**Proof of Theorem 1.** Let $W^*(z)$ be a harmonic and superharmonic function in $\bar{R}$. Let $S(z)$ be a harmonic function in $R - R_0$ such that $S(z) = W^*(z)$ on $\partial R_0$ and $S(z)$ has M. D. I. over $R - R_0$. Then $S(z)$ is bounded and $W^*(z) - S(z) = W(z) = U(z) + V(z)$ in $R - R_0$ in Theorem 9. Let $U_n^*(z)$ be a harmonic function in $R_n$ such that $U_n^*(z) = U(z) + S(z)$ on $\partial R_n$. Let $V_n^*(z)$ be a harmonic function in Lemma 5. Then $W^*(z) = U_n^*(z) + V_n^*(z)$. Choose a subsequence $(n_1, n_2, \cdots)$ such that both $U_n^*(z)$ and $V_n^*(z)$ converge to $U^*(z)$ and $V^*(z)$ respectively. Then $U^*(z)$ is representable by Poisson's integral and $U^*(z)$ has angular limits as $U(z) + S(z)$ a. e. on the boundary of $R - R_0$, whence $U^*(z)$ does not depend on the above subseqence. Thus $W^*(z) = U^*(z) + V^*(z)$.

10) See 3) or Mass distributions. III (in this volume) (Properties of functiontheoretic equilibrium potential).

Apply our result to a unit-circle $|z|<1$. Then we have the following

**Proposition.** Let $U(z)$ be a logarithmic potential such that the total mass is bounded and whose mass does not exist in $|z|<1$. Then the potential $U(z)$ is representable by Poisson's integral in $|z|<1$, because in this case $|z|=1$ consists of only regular points of the Green's function and $V(z)=0$.

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