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On Harmonic Functions Representable by Poisson's Integral

By Zenjiro Kuramochi

Let $R$ be a Riemann surface with positive boundary and let $\{R_n\}$ ($n=0, 1, 2, \cdots$) be its exhaustion with compact relative boundaries $\partial R_n$. If an open set $G$ has relative boundary consisting of at most enumerably infinite number of analytic curves which cluster nowhere in $R$, we call $G$ a domain. Let $w_{n,n+i}(z)$ be a harmonic function in $R_{n+i} - (G \cap (R_{n+i} - R_n))$ such that $w_{n,n+i}(z) = 0$ on $\partial R_{n+i} - G$ and $w_{n,n+i}(z) = 1$ on $\partial (G \cap (R_{n+i} - R_n))$, and let $\omega_{n,n+i}(z)$ be a harmonic function in $R - R_0 - (G \cap (R_{n+i} - R_n))$ such that $\omega_{n,n+i}(z) = 0$ on $\partial R_0$, $\omega_{n,n+i}(z) = 1$ on $\partial (G \cap (R_{n+i} - R_n))$, and $\frac{\partial}{\partial n} \omega_{n,n+i}(z) = 0$ on $\partial R_{n+i} - G$. We call $\lim_{n \to \infty} \lim_{i \to \infty} w_{n,n+i}(z)$ and $\lim_{n \to \infty} \lim_{i \to \infty} \omega_{n,n+i}(z)$ the harmonic measure and the capacitary potential of the ideal boundary $(G \cap B)$ determined by $G$ respectively. We call a function $G(z)$ a generalized Green's function, if $G(z)$ is non negatively harmonic in $R$, the harmonic measure of $(B \cap E[z \in R : G(z) > \delta])$ is zero for $\delta > 0$ and the Dirichlet integral $D(\min(M, G(z))) \leq kM$ for $M < \infty$.

We map the universal covering surface $R^\mu$ of $R$ onto $|\xi| < 1$. Then

Theorem 1. Let $W(z)$ be a positive harmonic in $R$ and superharmonic in $\bar{R}^2$. Then $W(z) = U(z) + V(z)$, where $U(z)$ is a harmonic function in $R$ representable by Poisson's integral in $|\xi| < 1$ and $V(z)$ is a generalized Green's function. If furthermore $R$ has no irregular point of the Green's function, then $V(z) = 0$, therefore $W(z)$ is representable by Poisson's integral.

Let $W(z)$ be a function in Theorem 1. Then $W(z) - S(z)$ is also positively harmonic in $R - R_0$ and superharmonic in $R - R_0$ and $W(z) - S(z) = W'(z) = 0$ on $\partial R_0$, where $S(z)$ is harmonic in $R - R_0$ such that $S(z) = W(z)$ on $\partial R_0$ and $S(z)$ has M. D. I. (minimal Dirichlet integral).

2) Let $U(z)$ be a positively harmonic function which satisfies $D(\min(M, U(z))) < \infty$. If $U(z) > U_0(z)$ for every compact or noncompact domain $G$, we say $U(z)$ is superharmonic in $\bar{R}$, where $U_0(z) = \lim_{n \to \infty} U_{n,n+i}(z)$, $U_0^*(z) = \min(M, U(z))$ on $\partial G$ and $U_0^*(z)$ has minimal Dirichlet integral over $G.$
Then $W'(z)$ is representable by a positive mass distribution as follows: 3)

$$W'(z) = \int_{B_1} N(z, p) d\mu(p),$$

where $B_1$ is the set of minimal points and the total mass $\mu_0$ is given by

$$\text{ds} \int_{\partial R_0} \frac{\partial}{\partial n} W'(z)$$

and

$$D(\min(M, W(z)) \leq 2\pi M \mu_0.$$

First we shall prove for $N(z, p)$. Then

Theorem 2. Let $N(z, p)$ be a minimal function$^5$. Then $N(z, p) = U(z, p) + V(z, p)$, where $U(z, p)$ is a positive harmonic function representable by Poisson's integral and $V(z, p)$ is a generalized Green's function. $U(z, p)$ and $V(z, p)$ are functions of at most second class of Baire's function of $p$ for fixed $z \in \mathbb{R} - R_0$ with respect to Martin's topology.$^5$

If $\sup N(z, p) < \infty$, our assertion is trivial and in this case by the boundedness of $V(z, p)$, $V(z, p)$ reduces to constant zero. We shall suppose $\sup N(z, p) = \infty$. Put $G_M = E[z \in \mathbb{R} : N(z, p) > M]$. Then $G_M$ is a non compact domain. Consider a harmonic function $w_n(z)$ in $R_n - G_M - R_0$ such that $w_n(z) = 0$ on $\partial R_o + \partial R_n - G_M$ and $w_n(z) = 1$ on $\partial G_M$. Let $w_M(z) = \lim_n w_n(z)$. Since $N(z, p)$ has M. I. D. over $\mathbb{R} - R_0 - G_M$ among all functions with values 0 on $\partial R_0$ and $M$ on $\partial G_M$ respectively, $N(z, p) = \lim_n N_n(z, p)$, where $N_n(z, p)$ is harmonic in $R_n - R_0 - G_M$ such that $N_n(z, p) = M$ on $\partial G_M$, $N_n(z, p) = 0$ on $\partial R_0$ and $\frac{\partial}{\partial n} N_n(z, p) = 0$ on $\partial R_n - G_M$. Hence by the maximum principle $N(z, p) \geq M w_M(z)$, whence $\lim M w_M(z) = 0$. Map the universal covering surface $(\mathbb{R} - R_0)^\infty$ onto $|z| < 1$ and consider $w_M(z)$ in $|z| < 1$. Then $w_M(z)$ has angular limits = 0 a.e. (almost everywhere) on a set $E_M$ on $|z| = 1$ where $N(z, p)$ has angular limits $< M$. To the contrary, suppose that there exists a set $E$ of positive measure such that $w_M(z)$ has angular limits $> 0$ on $E$ and $N(z, p)$ has angular limits $< M$. Then there exists a closed set $E' \subset E$ such that $\text{mes}(E - E') < \varepsilon$, $N(z, p) < M - \varepsilon$ in angular domain $D_\varepsilon = [\arg |z - \xi_0| < \pi/2 - \varepsilon, \xi_0 \in E', |z| > 1 - \varepsilon]$ for any given positive number $\varepsilon$. Let $D'_\varepsilon$ be one of components of $D_\varepsilon$. Then the image of


$^4)$ See 3).

$^5)$ If $U(z)$ has no functions $V(z)$ such that both $V(z) > 0$ and $U(z) - V(z) > 0$ are harmonic and superharmonic in $R - R_0$ except its own multiples, we say that $U(z)$ is a minimal function.
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Let \( H(z) \) be a harmonic function in \( D' \) with values 1 on \( \partial D' \cap E [ |z| = 1] \) and 0 on \( \partial D' \cap E [ |z| = 1] \). Since \( \partial D' \) is rectifiable, \( H(z) = 0 \) a.e. on \( \partial D' \cap E [ |z| = 1] \). But \( w_M(z) \leq H(z) \), whence \( w_M(z) = 0 \) a.e. on \( E_M \).

Let \( N_n'(z, p) \) be a harmonic function in \( R_n - R_0 - G_L (= E \cap \{ N(z, p) > L \}) \) such that \( N_n'(z, p) = 0 \) on \( \partial R_0 \), \( N_n'(z, p) = L \) on \( \partial G_L \cap R_n \), \( N_n'(z, p) = N(z, p) \) on \( \partial R_n - G_M (M < L) \) and \( \frac{\partial}{\partial n} N_n'(z, p) = 0 \) on \( \partial R_n \cap (G_M - G_L) \). Then

\[
D_{R_n - G_L}(N_n'(z, p)) < D_{R_n - G_L}(N(z, p)).
\]

Since \( N(z, p) \) has M. D. I. over \( R - G_L \), \( N_n'(z, p) \rightarrow N(z, p) \) in mean.

Let \( U_{M_n}(z, p) \) be a harmonic function in \( R_n - R_0 - G_L \) such that \( U_{M_n}(z, p) = 0 \) on \( \partial R_0 \) and \( U_{M_n}(z, p) = N(z, p) \) on \( \partial R_n - G_L \). Then we have clearly \( \lim U_{M_n}(z, p) = \lim U_{M_n}(z, p) \) and \( U_{M_n}(z, p) > U_{M_n}(z, p) \) for \( M_2 > M_1 \).

Choose a subsequence \( (n_1', n_2', \ldots) \) from \( (n_1, n_2, \ldots) \) such that \( U_{M_n}(z, p) \) converges to \( U_{M_1}(z, p) \). Then \( U_{M_1}(z, p) \geq U_{M_1}(z, p) \). Let \( U(z, p) = \lim_{M \to \infty} U_{M_1}(z, p) \). Then \( U(z, p) \) is a function representable by Poisson's integral and \( U(z, p) \) has the same angular limits as \( N(z, p) \) a.e. on \( |z| = 1 \), because \( \lim_{M \to \infty} w_M(z) = 0 \). Hence such \( U(z, p) \) does not depend on the subsequences. This \( U(z, p) \) is the function stated in the theorem.

Next we shall show that \( N(z, p) - U(z, p) \) is a generalized Green's function. We proved that \( \int \limits_{\partial G_L} N(z, p) ds = \lim_{n \to \infty} \int \limits_{\partial G_L} N_n(z, p) ds \) for almost all \( \partial G_L \) (i.e. the set of \( L \) whose \( \partial G_L \) does not satisfy the above condition is of measure zero), where \( N_n(z, p) \) is a harmonic function in \( R_n - R_0 - G_L \) such that \( N_n(z, p) = 0 \) on \( \partial R_0 \), \( N_n(z, p) = L \) on \( \partial G_L \cap R_n \) and \( \frac{\partial}{\partial n} N_n(z, p) = 0 \) on \( \partial R_n - G_L \).

We call such \( G_L \) a regular domain. Hence we can suppose without loss of generality that \( G_L \) is regular. We see the following assertion from \( \frac{\partial}{\partial n} N_n(z, p) > 0 \) on \( \partial G_L \), it is necessary and sufficient condition for

\[\text{sec 3). p. 151.}\]
the regularity of \( G_L \) that there exist \( n_0 \) and \( m_0 \) such that
\[
\int_{\partial G_L} \frac{\partial}{\partial n} N_n(z, p) ds < \varepsilon \quad \text{for} \quad n > n_0 \quad \text{and} \quad m > m_0 \quad \text{for any given positive number} \quad \varepsilon > 0.
\]

Let \( J_n(z) \) be a harmonic function in \( R_n - R_0 - (R_n \cap (G_M - G_L)) \) such that \( J_n(z) = 0 \) on \( \partial G_M \), \( J_n(z) = 1 \) on \( \partial G_L \) and \( \frac{\partial}{\partial n} J_n(z) = 0 \) on \( \partial R_n \cap (G_M - G_L) \).

Then \( (M + (L - M) J_n(z)) \rightarrow N(z, p) \) in mean, because \( N(z, p) \) has M. D. I. over \( G_M - G_L \). Hence \( \lim_{n \to \infty} \int_{\partial G_L} \frac{\partial}{\partial n} J_n(z) ds = \lim_{n \to \infty} \int_{\partial G_L} \frac{\partial}{\partial n} J_n(z) ds \) and there exist \( m_0 \) and \( n_0 \) such that
\[
\int_{\partial G_L} \frac{\partial}{\partial n} J_n(z) ds < \varepsilon \quad \text{for} \quad n > n_0 \quad \text{and} \quad m > m_0 \quad \text{for any given positive number} \quad \varepsilon > 0.
\]

But \( N_n'(z, p) > (L - M) J_n(z) \) in \( G_M - G_L \) and
\[
N_n'(z, p) = (M + (L - M) J_n(z)) \quad \text{on} \quad \partial G_L \quad \text{implies}
\]
\[
(L - M) \frac{\partial}{\partial n} J_n(z) > \frac{\partial}{\partial n} N_n'(z, p) > 0 \quad \text{on} \quad \partial G_L.
\]

Hence
\[
\int_{\partial G_L} \frac{\partial}{\partial n} N_n'(z, p) ds = \lim_{n \to \infty} \int_{\partial G_L} \frac{\partial}{\partial n} N_n'(z, p) ds.
\]

Thus \( \partial G_L \) is also regular for \( N_n'(z, p) \).

Let \( V_{M,n}(z, p) \) be a harmonic function \( = N_n'(z, p) - U_{M,n}(z, p) \). Then \( V_{M,n}(z, p) \) is harmonic in \( R_n - R_0 \), \( V_{M,n}(z, p) = 0 \) on \( \partial R_0 + (\partial R_n - G_M) \) and \( V_{M,n}(z, p) \geq L - M \) in \( G_L \). By the regularity of \( \partial G_L \), \( \int_{\partial G_L \cap R_n} \frac{\partial}{\partial n} N_n'(z, p) ds \to 2\pi, \) as \( n \to \infty \). Hence there exists a number \( n_0 \) for any given \( \varepsilon \) such that
\[
\int_{\partial G_L \cap R_n} \frac{\partial}{\partial n} N_n'(z, p) ds \leq 2\pi + \varepsilon \quad \text{for} \quad n > n_0.
\]

Put \( D = E[z \in R : \delta < V_{M,n}(z, p) < M' < M'] \), \( \Gamma_{M'} = E[z \in R : V_{M,n}(z, p) = M'] \), \( \Gamma_{M} = E[z \in R : V_{M,n}(z, p) = \delta] \), \( \Gamma = \partial R_n \cap D \) and \( \Gamma' = \partial R_n \cap E[z \in R : V_{M,n}(z, p) \leq M'] \). Then \( D \) intersects only \( \partial R_n \cap (G_M - G_L) \), because \( N_n'(z, p) - U_{M,n}(z, p) = 0 \) on \( \partial R_n - G_M \) and \( N_n'(z, p) - U_{M,n}(z, p) > M' \) on \( \partial G_L \) for \( L > 2M' \). Hence \( \Gamma \subset \partial R_n \cap (G_M - G_L) \).

Now \( \frac{\partial}{\partial n} N_n'(z, p) = 0 \) on \( \partial R_n - D_L \).

Since \( U_{M,n}(z, p) = \max U_{M,n}(z, p) = M \) on \( \Gamma, \frac{\partial}{\partial n} U_{M,n}(z, p) > 0 \) on \( \Gamma' \) and
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\[ \int_{\partial R_n \cap G_n - G_L} \frac{\partial}{\partial n} U_{M_n}(z, p) \, ds < \int_{\partial R_n \cap G_n - G_L} \frac{\partial}{\partial n} U_{M_n}(z, p) \, ds \quad \text{for every } M > 0. \]

\[ \int_{\partial R_n \cap G_n - G_M} \frac{\partial}{\partial n} N_n'(z, p) \, ds = \int_{\partial R_n \cap G_M} \frac{\partial}{\partial n} N_n'(z, p) \, ds < 2\pi + \varepsilon. \]

\[ 0 < \int_{\partial R_n \cap G_M} \frac{\partial}{\partial n} U_{M_n}(z, p) \, ds = \int_{\partial R_n \cap G_M} \frac{\partial}{\partial n} U_{M_n}(z, p) \, ds \quad \text{for every } M > 0. \]

\[ \int_{\partial R_n \cap G_M} \frac{\partial}{\partial n} U_{M_n}(z, p) \, ds = -\int_{\partial R_n \cap G_M} \frac{\partial}{\partial n} N_n'(z, p) \, ds \quad \text{for every } M > 0. \]

Hence \( D(\min V_{M_n}(z, p), M') = D_D(V_{M_n}(z, p)) = \int_{\partial R_n \cap G_M} (N_n'(z, p) - U_{M_n}(z, p) \frac{\partial}{\partial n} (N_n'(z, p) - U_{M_n}(z, p))) \, ds \leq M'(4\pi + 2\varepsilon) + \delta(2\pi + \varepsilon) \quad \text{and} \]

\[ D_{R_n - R_0}(\min (V_{M_n}(z, p), M)) \leq M'(4\pi + 2\varepsilon) + \delta(2\pi + \varepsilon), \quad \text{for every } M > 0. \]

Let \( n \to \infty \), then \( N_n'(z, p) \to N(z, p) \) in \( R - R_0 - G_L \), \( U_{M_n}(z, p) \to U_M(z, p) \), \( V_{M_n}(z, p) \to V(z, p) \) and derivatives of \( V_{M_n}(z, p) \to \) derivatives of \( V_M(z, p) \).

By letting \( n \to \infty \) and then \( \delta \to 0 \) and \( \varepsilon \to 0 \), \( D_{R_n - R_0}(\min (V_M(z, p), M')) \leq 4\pi M' \).

Let \( L \to \infty \) and then \( M \to \infty \). Then \( U_M(z, p) \uparrow U(z, p) \) and \( V_M(z, p) \downarrow V(z, p) \) and then by letting \( m \to \infty \), we have

\[ D_{R_n - R_0}(V(z, p), M')) \leq 4\pi M'. \]

On the other hand, clearly \( V(z, p) = N(z, p) - U(z, p) \) has angular limits \( = 0 \) a.e. on \( |\xi| = 1 \). Hence \( V(z, p) \) is a generalized Green's function.

Since \( U_M(z, p) = \lim_{n} U_{M_n}(z, p) = \lim_{n} U'_{M_n}(z, p) \), where \( U_{M_n}(z, p) \) is a harmonic function in \( R_n - R_0 \) such that \( U_{M_n}(z, p) \leq \min (M, N(z, p)) \) on \( \partial R_n + \partial R_n \). Hence \( U'_{M_n}(z, p) \leq \min (M, N(z, p)) \) on \( \partial R_n \), whence \( U_{M_n}(z, p) \downarrow U_M(z, p) \). Therefore there exists a number \( n_0 \) such that \( U_M(z, p) \leq U'_{M_n}(z, p) - \varepsilon \) for \( n > n_0 \) for any given positive number \( \varepsilon \). Next since \( N(z, p) \) is a continuous function of \( p \) for any point \( z \in R - R_0 \), there exists a number \( \delta_0 \) such that

\[ |N(z, p) - N(z, p)| < \varepsilon \text{ on } \partial R_n \text{ for } \delta(p, p) < \delta_0. \]
Hence \( U_M(z, p) \geq U'_{M,n}(z, p) - \varepsilon \geq U_{M,n}(z, p) - 2\varepsilon \geq U_M(z, p) - 2\varepsilon \).

Thus \( U_M(z, p) \) is an upper semicontinuous function of \( p \), whence \( V_M(z, p) \) is a lower semicontinuous function of \( p \) by the continuity of \( N(z, p) \). \( U_M(z, p) \uparrow U(z, p) \) and \( V_M(z, p) \downarrow V(z, p) \) imply that \( U(z, p) \) and \( V(z, p) \) are at most second class of Baire's functions.

Properties of generalized Green's functions.

**Theorem 3.** Let \( V(z) \) be a generalized Green's function such that \( D(\min(V(z), M)) \leq \pi M \). Let \( V'(z) \) be a non-negative harmonic function such that \( V'(z) \leq V(z) \). Then \( V'(z) \) is also a generalized Green's function such that \( D(\min(V(z), M)) \leq \pi M \).

**Theorem 4.** Let \( V(z) \) be a generalized Green's function and put \( R_\delta = E[z \in R: V(z) > \delta] \) and \( D_M = E[z \in R: V(z) > M] \). Then \( D_M \) determines a set of the ideal boundary of capacity zero.

Let \( V_{n, n+i}(z) \) be a harmonic function in \( (R_\delta \cap R_{n+i}) - (R_{n+i} - R_n) \cap D_M \) such that \( V_{n, n+i}(z) = 0 \) on \( \partial R_\delta \cap R_{n+i} \), \( V_{n, n+i}(z) = 1 \) on \( \partial(D_M \cap (R_{n+i} - R_n)) \) and \( \partial V_{n, n+i}(z) = 0 \) on \( \partial D_M \cap (R_{n+i} - D_M) \). Then by the Dirichlet principle

\[
D(\min(M, V_{n, n+i}(z))) \leq D(\min(M, V(z)))
\]

for every \( i \) and \( n \).

Next clearly \( \lim_{n+i} V_{n, n+i}(z) = \tilde{V}(z) \) exists and \( \tilde{V}(z) \) has angular limits \( \leq V(z) \) a.e. where \( V(z) \) has angular limits \( < \pi M \). But \( V(z) \) has angular limits \( = 0 \) a.e. on \( |\xi| = 1 \), whence \( \tilde{V}(z) = V'(z) \) and

\[
D(\min(M, V'(z))) \leq D(\min(M, V(z)))
\]

Hence \( V'(z) \) is a generalized Green's function.
and clearly $V_{n,n+i}(z)$ converges to $V_n(z)$ in mean as $i \to \infty$.

$$
\int_{\partial R_n} (V_{n,n+i}(z) - V_{m,n+i}(z))V_{n,n+i}(z)ds = \int_{\partial(D_M \cap (R_{n+i} - R_n))} \partial_{\partial R_n} V_{n,n+i}(z)ds
$$

$$
- \partial_{\partial R_n} V_{n,n+i}(z)ds = D(V_{n,n+i}(z)) - D(V_{m,n+i}(z)), \text{ for } n < m < n + i.
$$

Since $V_{m,n+i}(z) \to V_n(z)$ in mean, we have

$$
D(V_n(z) - V_m(z), V_n(z)) = D(V_n(z)) - D(V_m(z)) \text{ and } D(V_n(z) - V_m(z)) = D(V_n(z)) - D(V_m(z)).
$$

Hence $V_n(z)$ converges to a function $V^*(z)$ in mean as $n \to \infty$.

Map the universal covering surface $R^*_{\delta}$ of $R_\delta$ onto $|f| = \delta$ a.e. on $|f| = \delta$. Hence $V^*(z) = 0$. Let $F$ be a closed arc on $\partial R_\delta$. Let $\omega_{n,n+i}(z)$ be a harmonic function in $R_\delta \cap R_{n+i} - ((R_{n+i} - R_n) \cap D_M)$ such that $\omega_{n,n+i}(z) = 0$ on $F$, $\omega_{n,n+i}(z) = 1$ on $\partial(D_M \cap (R_{n+i} - R_n))$ and $\partial_{\partial R_n} (z) = 0$ on $\partial R_{n+i} - D_M$. Then by the Dirichlet principle

$$
D(\omega_{n,n+i}(z)) \leq D(V_{n,n+i}(z)).
$$

We see as above that $\omega_{n,n+i}(z) \to \omega_n(z)$ in mean and $\omega_n(z) \to \omega(z)$ in mean and by $V_n(z) \to V^*(z)$ in mean. We have $D(\omega(z)) \leq D(V^*(z)) \leq 0$.

Thus $D_M$ determines a set of the boundary of capacity zero.

**Theorem 5.** Let $V(z)$ be a generalized Green's function. Then

$$
\int_{\partial R_n} V(z)ds = k \text{ on every niveau curve, where } k \text{ is a constant such that } D(\min(M, V(z))) = Mk.
$$

Let $\omega_n(z)$ and $D_M$ be in Theorem 4. Let $\omega'_n(z)$ be a harmonic function in $D_M \cap (R_n - R_0) + (R_\delta \cap R_n)$ such that $\omega'_n(z) = 0$ on $F \cap R_0$, $\omega'_n(z) = 1$ on $D_M \cap \partial R_n$ and $\partial_{\partial R_n} (z) = 0$ on $(\partial R_\delta \cap R_0) + (\partial R_0 - D_M) + (\partial D_M \cap (R_n - R_0)) - F$. Then clearly

$$
D(\omega'_n(z)) \leq D(\omega_n(z)),
$$

whence by Theorem 4 $\omega_n(z) \to 0$ as $n \to \infty$. Hence there exists for any given large number $T$, a number $n$ and a harmonic function $\omega^*_n(z)$ in $(R_\delta \cap R_n) + (\partial R_0 - D_M) + (\partial D_M \cap (R_n - R_0))$ such that $\omega^*_n(z) = 0$ on $F$, $\partial_{\partial R_n} (z) = 0$ on $(\partial R_\delta \cap R_n) - F + (\partial D_M \cap (R_n - R_0)) + (\partial R_0 - D_M)$, $\omega^*_n(z) = T$ on $\partial R_n \cap D_M$ and $\int_{F \cap R_0} \partial_{\partial R_n} (z) = 2\pi$. 
Put \( r e^{i\theta} = \exp(\omega_n^* + i\tilde{\omega}_n^*) \), where \( \tilde{\omega}_n^* \) is the conjugate function of \( \omega_n^* \). Put \( L(r) = \int \frac{1}{\partial_{n}} V(z) rd\theta \), where the integration is taken over \(((R_n \cap R_m) + (D_M - D_{M\bar{z}})) \cap (E[z \in R; \omega_n^*(z) = \log r]) \). Let \( \bar{M} = M \).

Suppose \( L(r) > \varepsilon_0 \) for every \( r \). Then

\[
\varepsilon_0^2 \int_1^r \frac{1}{r} dr \leq \int_1^r \frac{L(r)}{r} dr \leq \int \int_{D_M - D_{M\bar{z}}} \left[ \left( \frac{\partial V(z)}{\partial R} \right)^2 + r^2 \left( \frac{\partial V(z)}{\partial \theta} \right)^2 \right] r dr d\theta \leq D_{M - D_{M\bar{z}}}(V(z)) < \infty.
\]

Let \( T \to \infty \). Then \( D(V(z)) \to \infty \). This is a contradiction. Hence there exists a sequence \( \{ r_i \} \) such that \( L(r) \to 0 \). Since \( \frac{\partial V}{\partial\bar{n}}(z) < 0 \) on \( \partial D_M \) and \( \frac{\partial V}{\partial n}(z) > 0 \) on \( \partial D_{M\bar{z}} \). Hence \( k = \int_{\partial D_M} \frac{\partial}{\partial n} V(z) ds = \int_{\partial D_{M\bar{z}}} \frac{\partial}{\partial n} V(z) ds \) and \( D_{D_M - D_{M\bar{z}}}(V(z)) = k(M - M) \). Hence we have the theorem.

**Lemma 3.** Let \( V(z) \) be a positive harmonic function (not necessarily a generalized Green's function) in \( R - R_n \). Let \( G \) and \( G' \) be non compact domains such that \( R - R_n = \overline{G + G'} \). Let \( nV_G^a(z), nV_G^b(z) \) be the lower (upper) envelope of super (sub) harmonic functions larger (smaller) than \( V(z) \) in \( G \cap (R - R_n) \). Put \( V_G^a(z) = \lim nV_G^a(z) \) and \( V_G^b(z) = \lim nV_G^b(z) \). Then

\[
V_G^a(z) = V_G^b(z) = 0.
\]

Let \( V_{n.n+i}(z) \) be a harmonic function in \( R_n + ((R_{n+i} - R_n) \cap G) - R_n \) such that \( V_{n.n+i}(z) = 0 \) on \( \partial R_n + (\partial R_{n+i} - G) \) and \( V_{n.n+i}(z) = V(z) \) on \( \partial G \cap (R_{n+i} - R_n) \). Then for every \( G \cap (R - R_n) \) by \( V_{n.n+i}(z) \uparrow V_n(z) \) and by \( G_i(\xi, z) \uparrow G(\xi, z) \)

\[
\lim_{i} V_{n.n+i}(z) = V_n(z) = \int_{\partial(G \cap (R - R_n) + (G \cap \partial R_n)} V_n(\xi) \frac{\partial}{\partial n} G(\xi, z) ds,
\]

where \( G_i(\xi, z) \) and \( G(\xi, z) \) are the Green's function of \( R_{n+i} - R_n - (G \cap (R_{n+i} - R_n) \) and \( R - R_n - (G \cap (R - R_n) \) respectively.

7) \( \overline{G} \) means the closure of \( G \).
Since \( V_n(z) \downarrow V_C^n(z) \), \( V_C^n(z) = \int_{\partial G \cap (R-R_n^+ \cap (\partial R_n^+))} V_C^e(\zeta) \frac{\partial}{\partial n} G(\zeta, z) ds \).

Next let \( V'_{n, n+i}(z) \) be a harmonic function in \( R_n + ((R_{n+i} - R_n) \cap G) - R_0 \) such that \( V'_{n, n+i}(z) = 0 \) on \( \partial R_0 + (\partial R_{n+i} - G) \) and \( V'_{n, n+i}(z) = V_C^e(z) \) on \( (\partial G \cap (R_{n+i} - R_n)) + (G \cap \partial R_n) \). Then

\[
\lim_i V'_{n, n+i}(z) = \int V_C^e(\zeta) \frac{\partial}{\partial n} G(\zeta, z) ds,
\]

i.e. \( \lim_i V'_{n, n+i}(z) = V_C^n(z) \) for every \( n \), hence

\[
\mathcal{G}(V_C^n(z)) = V_C^n(z). 
\]

Let \( \tilde{V}_{n, n+i}(z) \) be a harmonic function in \( R_n + ((R_{n+i} - R_n) \cap G) - R_0 \) such that \( \tilde{V}_{n, n+i}(z) = 0 \) on \( \partial R_0 + (\partial R_{n+i} \cap G) + (\partial G \cap (R_{n+i} - R_n)) \) and \( \tilde{V}_{n, n+i}(z) = V(z) \) on \( \partial R_{n+i} \cap G' \). Then

\[
V(z) = V_{n, n+i}(z) + \tilde{V}_{n, n+i}(z), \text{ which implies } V(z) = V_C^n(z) + V_C^e(z). 
\]

From (1) we have

\[
V(z) = \mathcal{G}(V_C^n(z) + V_C^e(z)) + V_C^e(z) = \mathcal{G}(V_C^n(z)) + \mathcal{G}(V_C^e(z)) + V_C^e(z),
\]

whence by (1) and (2) we have

\[
\mathcal{G}(V_C^e(z)) = 0. 
\]

Let \( V(z) \) be a generalized Green's function. Let \( G(z, q) \) be the Green's function of \( R - R_0 \) with pole at \( q \). Put \( G = E[z \in R: G(z, q) > k] \) and \( G' = E[z \in R: G(z, q) < k] \). Then

\[
V(z) = V_C^n(z) + V_C^e(z). \]

We shall study the properties of \( V_C^e(z) \).

**Lemma 4.** Let \( V(z) \) be a generalized Green's function and put \( G = E[z \in R: G(z, q) > k] \) and \( G' = E[z \in R: G(z, q) < k] \) and \( D_M = E[z \in R: V_C^e(z) > M] \). Let \( H_{CM}^e(z) \) be the lower envelope of superharmonic function larger than \( \min(M, V_C^e(z)) \) on \( G' \cap D_M \). Then

\[
\lim_{M \to \infty} H_{CM}^e(z) = V_C^e(z). 
\]

For simplicity denote \( V_C^e(z) \) by \( H(z) \). Let \( H_{CM}^e(z) \) be a harmonic function in \( R_n - R_0 - (D_M \cap G') \) such that \( H_{CM}^e(z) = 0 \) on
∂R_0 + ∂R_m - (D_M ∩ G') and \( H_M^M(z) = M \) on \( \partial(D_M ∩ G') \). Let \( \tilde{H}_C^M(z) \) be a harmonic function in \( R_n - R_0 - (D ∩ G) \) such that \( \tilde{H}_C^M(z) = 0 \) on \( \partial R_0 + (\partial R_m - (D_M ∩ G)) \), \( \tilde{H}_C^M(z) = M \) on \( \partial D_M ∩ G \) and \( \tilde{H}_C^M(z) = H(z) - M \) on \( \partial G ∩ D_M \).

Then clearly
\[
\lim_{n} H_M^M(z) \leq H(z) \leq \lim_{n} H_C^M(z) + \lim_{n} \tilde{H}_C^M(z)
\]
and
\[
\lim_{M \to \infty} \lim_{n} H_M^M(z) \leq \varepsilon (H(z)) = \varepsilon (V_C^E(z)) = 0.
\]

Hence
\[
V_C^E(z) = H(z) = \lim_{M \to \infty} H_M^M(z) = \lim_{n} H_C^M(z).
\]

**Theorem 6.** Let \( V(z) \) be a generalized Green's function such that \( D(\min M, V(z)) \leq M\pi \). Then by Lemma 3, \( V(z) = V_C^E(z) + V_C^E(z) \), where \( G' = E[z \in R : G(z, q) < k] \).

Then \( V_C^E(q) \leq \frac{k}{2} \).

Clearly \( V(z) \geq V_C^E(z) \) and \( V(z) \geq V_C^E(z) \). If \( V_C^E(z) = 0 \), our assertion is trivial. Suppose \( V_C^E(z) > 0 \). Then by Theorem 3, \( V_C^E(z) \) is also a generalized Green's function such that \( D(\min (M, V_C^E(z)) \leq M\pi \).

Next by Lemma 3
\[
V_C^E(z) = H(z) = \lim_{M \to \infty} H_M^M(z) \quad \text{and} \quad H(z) \geq H_C^M(z) = \lim_{n} H_C^M(z).
\]

Therefore by Theorem 5
\[
\int_{\partial D_M ∩ G'} \frac{\partial}{\partial n} H_M^M(z)ds \leq \int_{\partial D_M} H(z)ds \leq \pi \tag{4}
\]

where \( D_M = E[z \in R : H(z) > M] \).

Since \( g_\delta = E[z \in R : H_M^M(z) > \delta] \subseteq E[z \in R : H(z) > \delta] \), \( E[z \in R : H_M^M(z) > L] = D_L ∩ G' \) determines a set of the boundary of capacity zero for \( L > \delta \) by Theorem 4. Hence by \( D(C_H^E(z)) \leq \infty \) over \( R - R_0 - (D_M ∩ G') \), we can prove as in Theorem 5
\[
\int_{\Gamma_\delta} \frac{\partial}{\partial n} H_M^M(z)ds = - \int_{\partial (D_M ∩ G')} \frac{\partial}{\partial n} H_M^M(z)ds,
\]
where \( \Gamma_\delta = E[z \in R : H_M^M(z) = \delta] \).

Let \( g_\delta(z, q) \) be the Green's function of \( g_\delta ∩ (R - R_0) \). Then \( D(G_\delta(z, q)) \leq \infty \) over a neighbourhood of the ideal boundary. Hence there exists
a sequence of curves \( \{\Gamma_i\} \) such that \( \left| \frac{\partial}{\partial n} G_\delta(z, q) \right| ds \to 0 \) as \( i \to \infty \) and \( \{\Gamma_i\} \) clusters at the ideal boundary as \( i \to \infty \) and every \( \Gamma_i \) separates the boundary determined by \( D_M \) from \( q \).

Let \( C = \partial(D_M \cap G') \) and \( C_i \) be the part of \( C \) contained in the domain \( \exists q \) separated by \( \Gamma_i \) and \( C_i' = C - C_i \).

Then

\[
G_\delta(2, q) \to 0 \quad \text{as} \quad i \to \infty \quad \text{and} \quad r_\delta \to \infty \]

But the first term of the left hand side \( \to 0 \) as \( i \to \infty \) and the remaining terms don't depend on \( i \). Hence by letting \( \delta \to 0 \) and by \( G_\delta(z, q) \uparrow G(z, q) \), we have

\[
2\pi H^M_\delta(q) = \int_G G(z, q) \frac{\partial}{\partial n} H^M_\delta(z) ds \leq k\pi,
\]

because \( G(z, q) \leq k \) in \( G' \). Then by letting \( M \to \infty \)

\[
H(q) = V^*_\delta(q) \leq \frac{k}{2}.
\]

Put \( V^*_\delta(z) = V^{*k}(z) \) and \( V^*_\delta(z) = V'^k(z) \). Then by Theorem 6, \( V'^k(z) \to 0 \) as \( k \to 0 \). Then we have

**Theorem 7.** Every generalized Green's function \( V(z) \) is divided into two parts such that

\[
V(z) = V^{*k}(z) + V'^k(z) \quad \text{and} \quad V(z) = \lim_{k \to 0} V^{*k}(z).
\]

**Remark.** \( K(z, p_i) = \frac{G(z, p_i)}{G(p_0, p_i)} \) (\( p_0 \) is a flexed point) is a positive harmonic function. Martin \( ^8 \) defined *ideal boundary points* by using above functions and prove that every positive harmonic function is representable.

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by a unique mass distribution \( \nu \) as follows: \( \int K(z, p)d\nu(p) \), where \( B \) is the set of minimal points. If \( \lim_{i \rightarrow \infty} G(p_i, q) > 0 \) as \( p_i \) tends to a boundary point \( p \) and \( K(z, p_i) \rightarrow K(z, p) \), we call \( p \) an irregular boundary point. In this case, \( K(z, p) \) is a constant multiple of \( G(z, p) = \lim_{i \rightarrow \infty} G(p_i, q) \). We denote by \( I_k \) the set of Martin's boundary point \( p \) such that \( \lim_{z \rightarrow p} G(z, q) \geq k \). Then \( V^{*k}(z) \) is represented by a mass distribution \( \nu \) on \( I_k \). Hence by Theorem 8 a generalized Green's function is represented by a mass distribution \( \nu \) on \( I = \bigcup I_k \).

**Theorem 8.** Let \( W(z) \) be a positive harmonic in \( R - R_0 \) and super-harmonic function in \( R - R_0 \) vanishing on \( \partial R_0 \). Then

\[
W(z) = \int N(z, p)d\mu(p) = \int U(z, p)d\mu(p) + \int V(z, p)d\mu(p) = U(z) + V(z),
\]

where \( U(z) = \int U(z, p)d\mu(p) \) is a harmonic function representable by Poisson's integral and \( V(z) = \int V(z, p)d\mu(p) \) is a generalized Green's function.

Since \( 0 < U(z, p) \leq N(z, p) \), family \( \{U(z, p)\} \) is uniformly bounded in every compact domain in \( R - R_0 \) and the partial derivatives of them are equicontinuous and \( \Delta U(z, p) = 0 \), hence \( U(z) \) and \( V(z) \) are harmonic in \( R - R_0 \).

For a harmonic function \( H(z) \) define \( H^M(z) = \lim_{n \rightarrow \infty} H^M_n(z) \), where \( H^M_n(z) \) is a harmonic function in \( R_n - R_0 \) such that \( H^M_n(z) = \min(M, H(z)) \) on \( \partial R_0 + \partial R_n \). Then clearly \( M(H^M(z)) = H^M(z) \). Since \( 0 < U(z, p) \leq N(z, p) \) and \( U^M(z) \uparrow U(z, p) \) as \( M \uparrow \infty \), we have

\[
U(z) = \int U(z, p)d\mu(p) = \lim_{M \rightarrow \infty} \int U^M(z, p)d\mu(p) \leq \lim_{M \rightarrow \infty} \left[ \int N(z, p)d\mu(p) \right]
\]

\[
= \lim_{M \rightarrow \infty} \lim_{n \rightarrow \infty} W^M_n(z),
\]

where \( W^M_n(z) \) is a harmonic function in \( R - R_0 \) such that \( W^M_n(z) = \min(M, W(z)) \) on \( \partial R_0 + \partial R_n \). Now \( \lim_{M \rightarrow \infty} W^M_n(z) = W^p(z) \) is representable by Poisson's integral. \( 0 < U(z) \leq W^p(z) \) implies the Poisson's integrability of \( U(z) \).

By the Remark \( V(z, p) = \int K(z, q)d\nu(q) \), whence \( V(z) = \int V(z, p)d\mu(p) = \int K(z, q)d\mu(q) \). Hence there exist \( n_0 \) and \( k_0 \) such that

\[
\int V(z, p)d\mu(p) \leq \int K(z, q)d\lambda(q) + \varepsilon
\]  
(5)
for \( z \in R_n - R_o, n < n_o \) and \( k < k_o \) for any given positive number \( \varepsilon \), where \( \lambda' \) is the restriction of \( \lambda \) on \( I_k \).

Denote by \(( \int K(z, q)d\lambda'(q))_{I_k} \) the lower envelope of superharmonic functions larger than \( \int K(z, q)d\lambda(q) \) in \( G \cap (R - R_o) \). Put \(( \int K(z, q)d\lambda'(q))_{I_k} = \lim ( \int K(z, q)d\lambda'(q))_{I_k} \). Then as in Lemma 3 and Theorem 2 it is proved that \(( \int K(z, q)d\lambda'(q))_{I_k} = ( \int K(z, q)d\lambda'(q))_{I_k} \) and \(( \int K(z, q)d\lambda'(q))_{I_k} \) has angular limits \( = 0 \) a.e. on the ideal boundary\(^9\). In (5) let \( \varepsilon \to 0 \). Then \( \int K(z, q)d\lambda(q) = \int V(z, p)d\mu(p) \) has angular limits \( = 0 \) a.e. on the ideal boundary. Hence \( U(z) = \int U(z, p)d\mu(p) \) has the same angular limits as \( \int N(z, p)d\mu(p) \) a.e. on the ideal boundary. Thus by Poisson's integrability of \( U(z) \) and \( W^p(z) \), we have \( U(z) = W^p(z) \) and \( W(z) - W^p(z) = \int V(z, p)d\mu(p) \).

Now \( W(z) - W^p(z) = \lim_{M' \to \infty} \lim_{m \to \infty} W^M(z) \), where \( W^M(z) \) is a harmonic function in \( R_n - R_o \) such that \( W^M_n(z) = 0 \) on \( \partial R_o \) and \( W^M_n(z) = W(z) - W^M_n(z) \) on \( \partial R_n \). Since \( N(z, p) \) is a continuous function of \( p \) for \( z \in R \), there exists a sequence \( \{ W_m(z) \} (m = 1, 2, \ldots) \) of the form \( W_m(z) = \sum c_i N(z, p_i) (c_i > 0, \sum c_i = \mu_0 = \int d\mu(p)) \) such that \( W_m(z) \to W(z) \) in \( R_n - R_o \). On the other hand, let \( V^M_n(z) \) be a harmonic function in \( R_n - R_o \) such that \( V^M_n(z) \to 0 \) on \( \partial R_o \) and \( V^M_n(z) = \min(\mathcal{W}^m(z) - M', 0) \) on \( \partial R_n \). Then there exists a sequence \( \{ V^M_n(z) \} \) which converges to \( W^M(z) \) as \( n \to \infty \) and \( m \to \infty \).

Since \( V^M_n(z) \) is constructed from \( W_m(z) = \sum c_i N(z, p_i) \), we can prove by the method used for \( V(z, p) \) and \( N(z, p) \) that \( D(\min(M', V^M_n(z))) \leq 4\pi(\sum c_i)M' \) for \( M' < M \). Hence by letting \( n \to \infty \), \( m \to \infty \) and \( M \to \infty \) we have

\[
D(\min(M', V(z))) = D(\min(M', \lim_{n \to \infty} \lim_{m \to \infty} V^M_n(z))) \\
\leq \lim_{M' \to \infty} \lim_{m \to \infty} D(\min(M', V^M_n(z))) \leq 4\pi(\sum c_i)M'.
\]

Hence \( \int V(z, p)d\mu(p) \) is a generalized Green's function. We have Theorem 8.

**Lemma 5.** Let \( V(z) \) be a generalized Green's function in \( R_n - R_o \) such

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\(^9\) We map the universal covering surface of \( R-R_o \) onto \( |z| < 1 \). If the function \( U(z) \) has angular limits \( = 0 \) a.e. on the image of the ideal boundary on \( |z| = 1 \). We say simply \( U(z) \) has angular limits \( = 0 \) a.e. on the ideal boundary.
that \( D(\min(M, V(z)) \leq M\pi \). Then there exists a uniquely determined generalized Green's function \( V^*(z) \) in \( R \) such that \( D(\min(M, V^*(z)) \leq M\pi \) and
\[
\sup (V^*(z) - V(z)) < \infty.
\]

Since \( \partial R_0 \) is compact, there exists a constant \( L \) such that \( 0 < \frac{\partial}{\partial n} V(z) \leq L \) on \( \partial R_0 \). Let \( \omega(z) \) be a positive bounded harmonic function in \( R - R_0 \) such that \( \omega(z) = 1 \) on \( \partial R_0 \) and \( \omega(z) \) has angular limits \( = 0 \) a.e. on the ideal boundary of \( R - R_0 \). Put \( \tilde{\omega}(z) \equiv 1 \) in \( R_0 \) and \( \tilde{\omega}(z) \equiv \omega(z) \) in \( R - R_0 \). Then \( V(z) + K\tilde{\omega}(z)(K > L) \) is a superharmonic function in \( R \). Let \( V_n^*(z) \) be a harmonic function in \( R_n \) such that \( V_n^*(z) = V(z) \) on \( \partial R_n \). Then \( V(z) < V_n^*(z) < V(z) + K\omega(z) \). Choose a subsequence \((n, n_2, \ldots)\) so as \( V_n^*(z) \) converges to \( V^*(z) \). Then
\[
V(z) \leq V^*(z) \leq V(z) + K\tilde{\omega}(z).
\]

Hence \( V^*(z) \) has angular limits \( = 0 \) a.e. on the boundary of \( R \) and \( \sup (V^*(z) - V(z)) < \infty \), we see that such \( V^*(z) \) does not depend on the above subsequence and \( V^*(z) \) is uniquely determined.

Clearly \( D(\min(M, V(z)) \leq D(\min(M + K, V(z) + K\omega(z))) \), hence
\[
D(\min(M, V(z)) \leq 2D(\min(M + K, V(z))) \leq 10\pi M, \text{ for large } M.
\]

But both \( E[z \in R - R_0, V^*(z) > \delta] \) and \( E[z \in R - R_0, \omega(z) > \delta] \) determine sets of the boundary of capacity zero,\(^\text{10}\) whence as in Theorem, we have
\[
\int_C \frac{\partial}{\partial n} V^*(z) ds = k \leq 10\pi,
\]
for every niveau curve \( C \) of \( V(z) \) and \( D(\min(M, V^*(z)) \leq 10\pi M \) for every \( M \). Thus \( V^*(z) \) is a generalized Green's function.

**Proof of Theorem 1.** Let \( W^*(z) \) be a harmonic and superharmonic function in \( \bar{R} \). Let \( S(z) \) be a harmonic function in \( R - R_0 \) such that \( S(z) = W^*(z) \) on \( \partial R_0 \) and \( S(z) \) has M.D.I. over \( R - R_0 \). Then \( S(z) \) is bounded and \( W^*(z) - S(z) = W(z) = U(z) + V(z) \) in \( R - R_0 \) in Theorem 9. Let \( U_n^*(z) \) be a harmonic function in \( R_n \) such that \( U_n^*(z) = U(z) + S(z) \) on \( \partial R_n \). Let \( V_n^*(z) \) be a harmonic function in Lemma 5. Then \( W^*(z) = U_n^*(z) + V_n^*(z) \). Choose a subsequence \((n_1, n_2, \ldots)\) such that both \( U_n^*(z) \) and \( V_n^*(z) \) converge to \( U^*(z) \) and \( V^*(z) \) respectively. Then \( U^*(z) \) is representable by Poisson's integral and \( U^*(z) \) has angular limits as \( U(z) + S(z) \) a.e. on the boundary of \( R - R_0 \), whence \( U^*(z) \) does not depend on the above subsequence. Thus \( W^*(z) = U^*(z) + V^*(z) \).

\(^{10}\) See 3 or Mass distributions. III (in this volume) (Properties of functiontheoretic equilibrium potential).
Apply our result to a unit-circle $|z|<1$. Then we have the following

**Proposition.** Let $U(z)$ be a logarithmic potential such that the total mass is bounded and whose mass does not exist in $|z|<1$. Then the potential $U(z)$ is representable by Poisson's integral in $|z|<1$, because in this case $|z|=1$ consists of only regular points of the Green's function and $V(z)=0$.

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