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INVARIANT COMPLEX STRUCTURES ON TANGENT AND COTANGENT LIE GROUPS OF DIMENSION SIX

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Abstract

This paper deals with left invariant complex structures on simply connected Lie groups, the Lie algebra of which is of the type $\mathrm{T}_\pi\mathfrak{h} = \mathfrak{h} \ltimes_\pi V$, where π is either the adjoint or the coadjoint representation. The main topic is the existence question of complex structures on $\mathrm{T}_\pi\mathfrak{h}$ for $\mathfrak{h}$ a three dimensional real Lie algebra. First it was proposed the study of complex structures J satisfying the constraint $J\mathfrak{h} = V$. Whenever π is the adjoint representation this kind of complex structures are associated to non-singular derivations of $\mathfrak{h}$. This fact allows different kinds of applications.

1. Introduction

It is well known that due to their particular properties as differential manifolds, the study of invariant complex structures and induced geometries on Lie groups can be realized at the Lie algebra level. While the existence of complex structures on reductive Lie algebras of even dimension has been solved in different steps (starting with [27] and [34]), the solvable case still remains an open problem. For dimensions up to four, complex structures were studied in [25, 29, 22], while the nilpotent case has been considered up to dimension six [1, 9, 10, 11, 14, 17, 21, 31, 33]. Since those works are mainly done on the basis of a case by case study, one of the principal obstructions in classifying complex (and more general) structures on solvable Lie groups of dimensions equal or greater than six relies in the high number of isomorphism classes. This implies that different criteria have to be developed in order to describe any kind of geometry on the corresponding Lie groups. An alternative and powerful tool that provides a new insight to the problem is provided by the notion of generalized complex structure, first introduced by Hitchin in [18], and treated by several authors (see for instance [2, 12, 16] and references therein). On the other hand, in order to study the complex geometry, special types of complex structures were considered, as the so called abelian [1, 5] and nilpotent [9, 10], specific for nilpotent Lie algebras, and which have been shown to be of considerable interest, in particular in combination with other compatible geometric structures.

The aim of this work is the study of complex structures on tangent and cotangent Lie algebras, that is Lie algebras which are semidirect products $\mathrm{T}_\pi\mathfrak{h} = \mathfrak{h} \ltimes_\pi V$, where

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$\dim V = \mathfrak{h}$ and π denotes either the adjoint or coadjoint representation. We specifically focus on the following questions:

1) Complex structures satisfying the condition $J\mathfrak{h} = V$.
2) Complex structures on $\mathrm{T}\mathfrak{h}$ and $\mathrm{T}^*\mathfrak{h}$ where $\mathfrak{h}$ is a three dimensional real Lie algebra.
3) Symplectic structures which are compatible for a complex structure in 2), therefore inducing pseudo-Kähler geometries.

Complex structures appearing in 1) are called *totally real*. They have become objects of importance in the construction of weak mirror pairs (see for instance [7] and references therein).

Complex and symplectic geometry constitute extreme special cases of generalized complex geometry. Once a Lie algebra $\mathfrak{h}$ has been fixed, the corresponding underlying geometric structure arises either as a complex structure on $\mathfrak{h}$ or as a totally real complex structure on $\mathrm{T}^*\mathfrak{h}$, which is Hermitian for the canonical metric on $\mathrm{T}^*\mathfrak{h}$.

For the adjoint representation we prove that a totally real complex structure corresponds to a non-singular derivation of $\mathfrak{h}$. Therefore the existence of such totally real complex structures on $\mathrm{T}\mathfrak{h}$ imposes on $\mathfrak{h}$ the nilpotency constraint (Theorem 3.5). Hence, in dimension three one gets $\mathrm{T}\mathfrak{h}_1$, where $\mathfrak{h}_1$ denotes the Heisenberg Lie algebra of dimension three. As application, we prove the existence of a generalized complex structure of symplectic type on some types of nilpotent Lie algebras $\mathfrak{h}$ and the existence of Lagrangian symplectic structures on $\mathrm{T}^*\mathfrak{h}$.

For the coadjoint representation, we give the general form of totally real complex structures J on $\mathrm{T}^*\mathfrak{h}$, proving the existence whenever $\mathfrak{h}$ is one of the following three dimensional Lie algebras: the Heisenberg Lie algebra, the Lie algebra of the group of rigid motions of the Minkowski 2-space $\mathfrak{r}_{3,-1}$, the Lie algebra of the group of rigid motions of the Euclidean 2-space $\mathfrak{r}'_{3,0}$ and the one dimensional trivial central extension of the Lie algebra of the group of affine motions, usually denoted by $\mathfrak{aff}(\mathbb{R})$.

In addition to the Lie algebras obtained in 1), the six dimensional tangent Lie algebras admitting complex structures correspond to a Lie algebra $\mathfrak{h}$ which is either simple $\mathfrak{sl}(2)$, $\mathfrak{so}(3)$ or solvable and isomorphic to $\mathbb{R} \times \mathfrak{aff}(\mathbb{R})$. In the cotangent case we add $\mathfrak{sl}(2)$, $\mathfrak{so}(3)$, $\mathfrak{r}_{3,1}$ and $\mathfrak{r}'_{3,\eta}$ for $\eta > 0$.

Concerning 3), the only Lie algebras carrying a pseudo-Kähler structure are: the tangent and the cotangent of the Heisenberg Lie algebra and the tangent of $\mathbb{R} \times \mathfrak{aff}(\mathbb{R})$, a case investigated in more detail. In the nilpotent case one can see that there are flat and non-flat pseudo-Kähler metrics [11, 14]. In $\mathrm{T}(\mathbb{R} \times \mathfrak{aff}(\mathbb{R}))$ the resulting metric is non-flat. However one gets flat distributions. Again in this situation, totally real complex structures provide examples for pseudo-Kähler pairs.

2. Generalities on complex structures

An *almost complex* structure on a Lie algebra $\mathfrak{g}$ is an endomorphism $J\colon \mathfrak{g} \to \mathfrak{g}$ satisfying $J^2 = -\mathrm{I}$, where I is the identity map.

Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ denote the complexification of $\mathfrak{g}$ whose elements have the form $v \otimes c$, with $v \in \mathfrak{g}$, $c \in \mathbb{C}$. An almost complex structure J on $\mathfrak{g}$ can be extended to a complex linear endomorphism of $\mathfrak{g}^{\mathbb{C}}$ that we also denote by J, by setting $J(v \otimes c) = Jv \otimes c$.

As usual, we identify $v \in \mathfrak{g}$, with $v \otimes 1 \in \mathfrak{g}^{\mathbb{C}}$, and hence any element in $\mathfrak{g}^{\mathbb{C}}$ can be written as $x + iy$ where $x, y \in \mathfrak{g}$. With this identification, the eigenspace corresponding to the imaginary eigenvalue i of J is the subspace $\mathfrak{m}$ of $\mathfrak{g}^{\mathbb{C}}$ given by

$$\mathfrak{m} = \{x - iJx \colon x \in \mathfrak{g}\}.$$

If we denote by σ the conjugation map on $\mathfrak{g}^{\mathbb{C}}$, that is, $\sigma(x + iy) = x - iy$, the eigenspace corresponding to $-i$ is $\sigma\mathfrak{m}$, and we obtain the direct sum of vector spaces

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{m} \oplus \sigma\mathfrak{m}. \tag{1}$$

Conversely any decomposition of type (1) induces an almost complex structure on $\mathfrak{g}$. In fact let $J \colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ be the linear map given by $J(x + \sigma y) = ix - i\sigma y$ for all $x, y \in \mathfrak{m}$. Clearly $J^2 = -1$ and since $J \circ \sigma = \sigma \circ J$ the map J gives rise to an almost complex structure on $\mathfrak{g}$.

Notice that any J-invariant subspace must be even dimensional.

The integrability condition of an almost complex structure J is expressed in terms of the Nijenhuis tensor N_J

$$N_J(x, y) = [Jx, Jy] - [x, y] - J[Jx, y] - J[x, Jy], \quad \text{for all} \quad x, y \in \mathfrak{g}. \tag{2}$$

It is straightforward to verify that N_J is bilinear, skew-symmetric and it satisfies $N_J(Jx, Jy) = -N_J(x, y)$ and $N_J(Jx, y) = -JN_J(x, y)$ for any $x, y \in \mathfrak{g}$. Hence, if $\mathfrak{g} = \mathfrak{u} \oplus J\mathfrak{u}$ is a direct sum as vector subspaces, then $N_J \equiv 0$ if and only if $N_J(u, v) = 0$ for all $u, v \in \mathfrak{u}$.

An almost complex structure J on $\mathfrak{g}$ is called *integrable* if $N_J \equiv 0$. In this case J is called a *complex structure* on $\mathfrak{g}$. Equivalently, J is integrable if and only if $\mathfrak{m}$ (and hence $\sigma\mathfrak{m}$) satisfying (1) is a complex subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

Special types of almost complex structures are determined by those endomorphisms $J \colon \mathfrak{g} \to \mathfrak{g}$ satisfying $J^2 = -\mathrm{I}$ and one of the following conditions for any $x, y \in \mathfrak{g}$:

$$\text{c1)} \quad J[x, y] = [x, Jy], \qquad \text{c2)} \quad [Jx, Jy] = [x, y].$$

In any case they are integrable. Complex structures of type c1) determine a structure of complex Lie algebra on $\mathfrak{g}$, they are sometimes called *bi-invariant*. The subalgebra corresponding to the eigenvalue $\pm i$ is actually an ideal of $\mathfrak{g}^{\mathbb{C}}$. Structures of type c2) are called *abelian* [5], and the corresponding eigenspaces for the eigenvalues $\pm i$ are complex abelian subalgebras of $\mathfrak{g}^{\mathbb{C}}$.

Note that if $\mathfrak{g}$ carries an abelian complex structure, then the center of $\mathfrak{g}$ must be J-invariant and therefore even dimensional. Another necessary condition to have abelian complex structures is that $\mathfrak{g}$ is 2-step solvable, which means that the commutator subalgebra $C(\mathfrak{g})$ is abelian (see [24] for instance).

Let $\mathfrak{g}$ be a Lie algebra and let J be a fixed almost complex structure on $\mathfrak{g}$. For any $l \geq 0$ we define the set $\mathfrak{a}_l(J)$ inductively as:

$$\mathfrak{a}_0(J) = \{0\}, \quad \mathfrak{a}_l(J) = \{X \in \mathfrak{g};\ [X, \mathfrak{g}] \subset \mathfrak{a}_{l-1}(J) \text{ and } [JX, \mathfrak{g}] \subset \mathfrak{a}_{l-1}(J)\}, \quad l \geq 1.$$

It is easy to verify that

$$\mathfrak{a}_0(J) \subseteq \mathfrak{a}_1(J) \subseteq \mathfrak{a}_2(J) \subseteq \cdots$$

For a fixed $X \in \mathfrak{a}_{i+1}(J)$ we have that $[X, Y] \in \mathfrak{a}_i(J) \subseteq \mathfrak{a}_{i+1}(J)$ for all $Y \in \mathfrak{g}$, and clearly $[J[X, Y], Z] \in \mathfrak{a}_i(J) \subset \mathfrak{a}_{i+1}(J)$ for all $Y, Z \in \mathfrak{g}$. Therefore $\mathfrak{a}_i(J)$ is a J-invariant ideal of $\mathfrak{g}$ for any $i \geq 0$.

The almost complex structure J is called *nilpotent* if there exists a t such that $\mathfrak{a}_t(J) = \mathfrak{g}$. This implies that $\mathfrak{g}$ must be nilpotent. For a nilpotent almost complex structure J on an s-step nilpotent Lie algebra of dimension $2n$ we shall say that it is r-step nilpotent if r is the first nonnegative integer such that $\mathfrak{a}_r(J) = \mathfrak{g}$; this satisfies the inequality $s \leq r \leq n$ and these bounds are actually reached ([10]). Notice that if J is a nilpotent almost complex structure on a nilpotent Lie algebra $\mathfrak{g}$, then any term of the ascending series of $\mathfrak{g}$ admits a two dimensional J-invariant subspace. Clearly, if J is integrable, the condition of being nilpotent is stronger than asking the corresponding $\mathfrak{m}$ for J to be nilpotent.

EXAMPLE 2.1. The canonical complex structure of a nilpotent complex Lie algebra is nilpotent (see Example 4.1).

An equivalence relation is defined among Lie algebras with complex structures. Lie algebras with complex structures $(\mathfrak{g}_1, J_1)$ and $(\mathfrak{g}_2, J_2)$ are called *holomorphically equivalent* if there exists an isomorphism of Lie algebras $\alpha\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $J_2 \circ \alpha = \alpha \circ J_1$. In particular when $\mathfrak{g}_1 = \mathfrak{g}_2$ we simply say that J_1 and J_2 are *equivalent* and a classification of complex structures can be done.

Lemma 2.2. *Let $\mathfrak{g}$ be an even dimensional real Lie algebra.*
i) *The class of an abelian complex structure, if non-empty, consists only of abelian complex structures.*
ii) *Let J, J' be complex structures on $\mathfrak{g}$ such that $J' = \alpha J \alpha^{-1}$ for $\alpha \in \mathrm{Aut}(\mathfrak{g})$. Then $\alpha \mathfrak{a}_l(J) = \mathfrak{a}_l(J')$ for any $l \geq 0$.*

In particular the class of a nilpotent complex structure on a given nilpotent Lie algebra consists only of nilpotent complex structures, all of them being nilpotent of the same type.
iii) *The class of a bi-invariant complex structure has only bi-invariant complex structures.*

REMARK. From the definitions above it is not immediately clear which is the relationship between nilpotent complex structures and complex structures whose corresponding i-eigenspace $\mathfrak{m}$ is nilpotent. In Proposition 4.6 we see that the tangent Lie algebra of the Heisenberg Lie algebra $\mathfrak{h}_1$ carries only 2-step nilpotent complex structures, some of them being abelian, and others having $\pm i$-eigenspaces which are 2-step nilpotent subalgebras.

3. Totally real complex structures on tangent and cotangent Lie algebras

The aim of this section is the study of totally real complex structures on tangent and cotangent Lie algebras, that is complex structures J on $\mathrm{T}_\pi\mathfrak{h}$ such that $J\mathfrak{h} = V$.

We briefly recall the construction. Let $\mathfrak{h}$ denote a real Lie algebra and let (π, V) be a finite dimensional representation of $\mathfrak{h}$. By endowing V with the trivial Lie bracket, consider the semidirect product of $\mathfrak{h}$ and V relative to π, $\mathrm{T}_\pi\mathfrak{h} := \mathfrak{h} \ltimes_\pi V$, where the Lie bracket is:

$$[(x, v), (x', v')] = ([x, x'], \pi(x)v' - \pi(x')v), \quad x, x' \in \mathfrak{h},\ v, v' \in V.$$

We mainly concentrate on the adjoint and coadjoint representations. In both cases, V is a real vector space with the same dimension as that of $\mathfrak{h}$. The adjoint representation $\mathrm{ad}\colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h})$ is given by $\mathrm{ad}(x)y = [x, y]$, and it defines the *tangent* Lie algebra that we denote with $\mathrm{T}\mathfrak{h}$. For the coadjoint representation $\mathrm{ad}^*\colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h}^*)$, that is $V = \mathfrak{h}^*$, take

$$\mathrm{ad}(x)^*\varphi(y) = -\varphi \circ \mathrm{ad}(x)y, \quad x, y \in \mathfrak{h},\ \varphi \in \mathfrak{h}^*; \tag{3}$$

we call the resulting Lie algebra *cotangent Lie algebra* and we denote it as $\mathrm{T}^*\mathfrak{h}$.

A question concerning complex structures when we look at the algebraic structure of the Lie algebra $\mathrm{T}_\pi\mathfrak{h} = \mathfrak{h} \ltimes_\pi V$ is whether there exists an almost complex structure J such that $J\mathfrak{h} = V$. Such a J induces a linear isomorphism $j\colon \mathfrak{h} \to V$, and conversely any such $j\colon \mathfrak{h} \to V$ determines an almost complex structure on $\mathrm{T}_\pi\mathfrak{h}$ such that $J\mathfrak{h} = V$, by means of

$$J(x, v) = (-j^{-1}v, jx), \quad x \in \mathfrak{h},\ v \in V,\ j\colon \mathfrak{h} \to V. \tag{4}$$

It follows that both $\mathfrak{h}$ and V are totally real with respect to J. We further adopt the following terminology [7]:

DEFINITION 3.1. Let $\mathrm{T}_\pi\mathfrak{h} := \mathfrak{h} \ltimes_\pi V$ be the semidirect product of a Lie algebra $\mathfrak{h}$ with the real vector space V such that $\dim V = \dim \mathfrak{h}$ and let J denote an (almost) complex structure on $\mathrm{T}_\pi\mathfrak{h}$. If $J\mathfrak{h} = V$ we say that J is a *totally real* (almost) complex structure on $\mathrm{T}_\pi\mathfrak{h}$.

Suppose that (π, V) denotes a finite dimensional representation of $\mathfrak{h}$ and let J be a totally real almost complex structure on $\mathrm{T}_\pi\mathfrak{h} = \mathfrak{h} \ltimes_\pi V$ (like in (4)). In this case, the integrability condition for J reduces to

$$0 = [x, y] - j^{-1}\pi(x)jy + j^{-1}\pi(y)jx \quad \text{for all} \quad x, y \in \mathfrak{h}. \tag{5}$$

For a fixed Lie algebra $\mathfrak{h}$, recall that the representations (V, π) and (V', π') are called *equivalent* if there is a linear isomorphism $T\colon V \to V'$ such that $T^{-1}\pi'(x)T = \pi(x)$ for all $x \in \mathfrak{h}$.

Actually, for any $\psi \in \mathrm{Aut}(\mathfrak{h})$, the map $\varphi\colon \mathrm{T}_\pi\mathfrak{h} \to \mathrm{T}_{\pi'}\mathfrak{h}$ given by $\varphi = \psi + T$ is a Lie algebra isomorphism. In fact, for arbitrary elements $x, y \in \mathfrak{h}$, $u, v \in V$ we get the condition

$$\begin{aligned}
\varphi[x+u, y+v] &= \varphi([x, y] + \pi(x)v - \pi(y)u) \\
&= \psi[x, y] + T\pi(x)v - T\pi(y)u \\
&= [\psi x, \psi y] + \pi'(x)Tv - \pi'(y)Tu \\
&= [\varphi(x+u), \varphi(y+v)].
\end{aligned}$$

Thus, if J denotes a complex structure on $\mathrm{T}_\pi\mathfrak{h}$, then $J' := \varphi \circ J \circ \varphi^{-1}$ constitutes a complex structure on $\mathrm{T}_{\pi'}\mathfrak{h}$ making of $(\mathrm{T}_\pi\mathfrak{h}, J)$ and $(\mathrm{T}_{\pi'}\mathfrak{h}, J')$ a pair of holomorphically equivalent Lie algebras.

In particular, if J is a totally real complex structure on $\mathrm{T}_\pi\mathfrak{h}$, then $(\mathrm{T}_\pi\mathfrak{h}, J)$ is holomorphically equivalent to $(\mathrm{T}_{\pi'}\mathfrak{h}, \tilde{J})$, where $\tilde{J}_{|_\mathfrak{h}}\colon \mathfrak{h} \to V'$ is $\tilde{J} = T \circ J_{|_\mathfrak{h}}$ and extended as in (4). The proof of the following result follows by using these relations and the integrability condition (5).

Proposition 3.2. *Let (V, π) and (V', π') be equivalent representations of a Lie algebra $\mathfrak{h}$ such that* $\dim V = \dim V' = \dim \mathfrak{h}$. *Complex structures on $\mathrm{T}_\pi\mathfrak{h}$ are in one to one correspondence with complex structures on $\mathrm{T}_{\pi'}\mathfrak{h}$. In particular, totally real complex structures on $\mathrm{T}_\pi\mathfrak{h}$ are holomorphically equivalent to totally real complex structures on $\mathrm{T}_{\pi'}\mathfrak{h}$.*

A first consequence of (5) concerns abelian complex structures.

Corollary 3.3. *Let $\mathfrak{h}$ be a Lie algebra and let V denote the underlying vector space of $\mathfrak{h}$. Let $\mathrm{T}_\pi\mathfrak{h} := \mathfrak{h} \ltimes_\pi V$ denote the semidirect product and let J be an abelian totally real complex structure on $\mathrm{T}_\pi\mathfrak{h}$. Then $\mathfrak{h}$ is abelian and π and J are related by the condition $\pi(x)Jy = \pi(y)Jx$, where $x, y \in \mathfrak{h}$.*

REMARK. The converse of the previous corollary is also true. Let $\mathfrak{h}$ denote an abelian Lie algebra and let π be a representation of $\mathfrak{h}$ into $\mathfrak{h}$. If $j\colon \mathfrak{h} \to \mathfrak{h}$ is a non singular map such that $\pi(x)jy = \pi(y)jx$ for all $x, y \in \mathfrak{h}$, then the almost complex structure on $\mathfrak{h} \ltimes_\pi \mathfrak{h}$ given as in (4) is integrable and totally real with respect to $\mathfrak{h}$.

EXAMPLE 3.4. Consider $\mathbb{R}^n$ with the canonical basis $\{e_1, e_2, \ldots, e_n\}$ and let A be a non-singular $n \times n$ real matrix. Let C_A denote the centralizer of A in $gl(n, \mathbb{R})$ that is, the set of $n \times n$ matrices B such that $BA = AB$. Let $B_1, B_2, \ldots, B_n$ be n matrices in C_A such that they are pairwise in involution, $B_i B_j = B_j B_i$ for all i, j. Take π the representation of $\mathbb{R}^n$ which extends linearly the mapping $e_i \to B_i$ (notice that this could be trivial depending on A). The map j represented by A amounts to a totally real abelian complex structure on $\mathrm{T}_\pi \mathbb{R}^n$.

Recall that a *derivation* of a Lie algebra $\mathfrak{h}$ is a linear map $d \colon \mathfrak{h} \to \mathfrak{h}$ such that

$$d[x, y] = [dx, y] + [x, dy], \quad \text{for all} \quad x, y \in \mathfrak{h}.$$

Jacobson proved that if a Lie algebra $\mathfrak{h}$ admits a non-singular derivation then it must be nilpotent [19].

Theorem 3.5. *Let* $\mathrm{T}\mathfrak{h}$ *denote the tangent Lie algebra of* $\mathfrak{h}$. *The set of totally real complex structures on* $\mathrm{T}\mathfrak{h}$ *is in one to one correspondence with the set of non-singular derivations of* $\mathfrak{h}$. *If one* (*and therefore both*) *of these sets is non-empty, then* $\mathfrak{h}$ *is nilpotent.*

Proof. Let ad denote the adjoint representation of $\mathfrak{h}$ and take $\pi = \mathrm{ad}$ in (5), so one becomes

$$0 = j[x, y] - \mathrm{ad}(x)jy + \mathrm{ad}(y)jx, \quad \text{for all} \quad x, y \in \mathfrak{h}.$$

By identifying j with a linear map on $\mathfrak{h}$ the previous equality shows that j corresponds to a derivation of $\mathfrak{h}$. Since j is non-singular, the proof is completed applying the result of Jacobson. □

EXAMPLE 3.6. Let $\mathfrak{h}_n$ denote the Heisenberg Lie algebra of dimension $2n + 1$, that is $\mathfrak{h}_n = span\{x_1, y_1, \ldots, x_n, y_n, z\}$ with the Lie bracket $[x_i, y_j] = \delta_{ij} z$. Any non-singular derivation d has a matrix representation given by:

$$\begin{pmatrix} A & 0 \\ * & tr(A) \end{pmatrix}, \quad \text{with} \quad A \in GL(2n, \mathbb{R}) \quad \text{and} \quad tr(A) \neq 0$$

where tr denotes the trace of the matrix A. Hence $T\mathfrak{h}_n$ has several totally real complex structures.

Let $\mathfrak{s}$ denote a semisimple Lie algebra. Since the Killing form is non-degenerate this induces an ad-invariant metric on $\mathfrak{s}$ and therefore the adjoint and coadjoint representation are equivalent. A consequence of Proposition 3.2 and Theorem 3.5 is the next corollary.

Corollary 3.7. *There is no totally real complex structure on* $\mathrm{T}^*\mathfrak{s}$ *for any semisimple Lie algebra* $\mathfrak{s}$.

We now proceed to analyze the existence of totally real complex structures on six dimensional cotangent Lie algebras $\mathrm{T}^*\mathfrak{h}$ for $\mathfrak{h}$ (for the list of Lie algebras $\mathfrak{h}$ of dimension three see Theorem 4.2).

Proposition 3.8. *Let* $\mathrm{T}^*\mathfrak{h} = \mathfrak{h} \ltimes \mathfrak{h}^*$ *be a cotangent Lie algebra of a three dimensional Lie algebra* $\mathfrak{h}$. *Then totally real complex structures on* $\mathrm{T}^*\mathfrak{h}$ *exist whenever* $\mathfrak{h}$ *is either unimodular or isomorphic to* $\mathbb{R} \times \mathfrak{aff}(\mathbb{R})$. *In those cases the map* $j : \mathfrak{h} \to \mathfrak{h}^*$ *admits a matrix representation as follows*

$$\mathrm{T}^*\mathfrak{h}_1 \begin{pmatrix} a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \\ -a_{43} & -a_{53} & 0 \end{pmatrix}; \quad \mathrm{T}^*\mathfrak{r}_{3,-1} \begin{pmatrix} a_{41} & a_{42} & a_{43} \\ -a_{42} & 0 & a_{53} \\ -a_{43} & -a_{53} & 0 \end{pmatrix};$$

$$\mathrm{T}^*\mathfrak{r}_{3,0} \begin{pmatrix} a_{41} & a_{42} & a_{43} \\ -a_{42} & 0 & 0 \\ a_{61} & 0 & a_{63} \end{pmatrix}; \qquad \mathrm{T}^*\mathfrak{r}'_{3,0} \begin{pmatrix} a_{41} & a_{42} & a_{43} \\ -a_{42} & 0 & a_{53} \\ -a_{43} & -a_{53} & 0 \end{pmatrix};$$

where the matrix should be non-singular.

Proof. The proof follows by direct computation of (5) taking π as the coadjoint representation. In the cases not listed above, the maps j solving (5) are singular, hence they cannot induce a complex structure on $\mathrm{T}^*\mathfrak{h}$. □

4. Complex structures on tangent and cotangent Lie algebras of dimension six

Examples of six dimensional real Lie algebras with complex structures arise from three dimensional complex Lie algebras. In fact let $\tilde{\mathfrak{g}}$ denote a three dimensional complex Lie algebra, then the underlying real Lie algebra $\mathfrak{g} := \tilde{\mathfrak{g}}_{\mathbb{R}}$, is naturally equipped with a bi-invariant complex structure induced by the multiplication by i on $\tilde{\mathfrak{g}}$. In this way this complex structure on $\mathfrak{g}$ is bi-invariant.

EXAMPLE 4.1. Let $\mathfrak{g}$ denote a six dimensional two-step nilpotent Lie algebra equipped with bi-invariant complex structure J. Then $\mathfrak{g}$ is isomorphic to the real Lie algebra underlying $\mathfrak{h}_1 \otimes \mathbb{C}$, the complexification of the Heisenberg Lie algebra of dimension three.

Now we shall study the existence problem of complex structures on any tangent or cotangent Lie algebra corresponding to a three dimensional real Lie algebra.

Recall the classification of three dimensional Lie algebras as given e.g. in [15] or [20].

Theorem 4.2. *Let $\mathfrak{h}$ be a real Lie algebra of dimension three spanned by e_1, e_2, e_3. Then it is isomorphic to one in the following list*:

$$
(6)\qquad
\begin{array}{ll}
\mathfrak{h}_1 & [e_1, e_2] = e_3,\\
\mathfrak{r}_3 & [e_1, e_2] = e_2,\ [e_1, e_3] = e_2 + e_3,\\
\mathfrak{r}_{3,\lambda} & [e_1, e_2] = e_2,\ [e_1, e_3] = \lambda e_3, \quad |\lambda| \le 1,\\
\mathfrak{r}'_{3,\eta} & [e_1, e_2] = \eta e_2 - e_3,\ [e_1, e_3] = e_2 + \eta e_3, \quad \eta \ge 0,\\
\mathfrak{sl}(2) & [e_1, e_2] = e_3,\ [e_3, e_1] = 2e_1,\ [e_3, e_2] = -2e_2,\\
\mathfrak{so}(3) & [e_1, e_2] = e_3,\ [e_3, e_1] = e_2,\ [e_3, e_2] = -e_1.
\end{array}
$$

A Lie algebra $\mathfrak{g}$ which satisfies $tr(\mathrm{ad}(x)) = 0$ for all $x \in \mathfrak{g}$ is called unimodular. Among the Lie algebras above, the unimodular solvable ones are: $\mathfrak{h}_1$, $\mathfrak{r}_{3,-1}$ and $\mathfrak{r}'_{3,0}$.

The simple case. Among the Lie algebras listed in Theorem 4.2 the simple ones are $\mathfrak{sl}(2)$ and $\mathfrak{so}(3)$. Since the adjoint and the coadjoint representations are equivalent, after Proposition 3.2 for a semisimple Lie algebra $\mathfrak{s}$, the existence of a complex structure on $\mathrm{T}\mathfrak{s}$ determines it on $\mathrm{T}^*\mathfrak{s}$ and vice versa. Recall that complex structures on compact semisimple and more generally on reductive Lie algebras were extensively studied (see for instance [25, 26, 28, 27, 34]).

Let J be the almost complex structure on $\mathrm{T}\mathfrak{sl}(2)$ and $\mathrm{T}\mathfrak{so}(3)$ defined by

$$Je_3 = e_6, \quad Je_2 = e_1, \quad Je_4 = e_5.$$

By calculating N_J one verifies that J is integrable (see the Lie brackets in Proposition 4.3). Hence *the tangent Lie algebras* $\mathrm{T}\mathfrak{so}(3)$ *and* $\mathrm{T}\mathfrak{sl}(2)$ (*and therefore* $\mathrm{T}^*\mathfrak{so}(3)$ *and* $\mathrm{T}^*\mathfrak{sl}(2)$) *carry complex structures*.

The solvable case. Suppose that $\mathfrak{g}$ is a six dimensional tangent $\mathrm{T}\mathfrak{h}$ or cotangent Lie algebra $\mathrm{T}^*\mathfrak{h}$ being $\mathfrak{h}$ a solvable real Lie algebra of dimension three. It admits a complex structure if and only if $\mathfrak{g}^{\mathbb{C}}$ decomposes as a direct sum of vector subspaces $\mathfrak{g}^{\mathbb{C}} = \mathfrak{m} \oplus \sigma\mathfrak{m}$, where $\mathfrak{m}$ (resp. $\sigma\mathfrak{m}$) is a complex subalgebra. Without loss of generality assume that $\mathfrak{m}$ is spanned by vectors U, V, W as follows:

$$
(7)\qquad
\begin{aligned}
U &= e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6,\\
V &= b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 + b_6e_6,\\
W &= c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6, \quad a_i, b_j, c_k \in \mathbb{C},\ \forall i, j, k = 2, \ldots 6.
\end{aligned}
$$

Let $\mathfrak{a} := span\{V, W\}$. We claim that $\mathfrak{a}$ is an ideal in $\mathfrak{m}$. In fact, according to the Lie brackets in $\mathfrak{g}$ (see Proposition 4.3 and Proposition 4.8 below), one verifies that $U \notin$

$C(\mathfrak{g}^{\mathbb{C}})$, hence for $x, y \in \mathfrak{m}$, one has $[x, y] \in C(\mathfrak{m}) \subseteq \mathfrak{a}$. Thus $\mathfrak{m} = \mathbb{C}U \ltimes \mathfrak{a}$, where $\mathfrak{a}$ an ideal of $\mathfrak{m}$ of dimension two and therefore isomorphic either to i) $\mathbb{C}^2$ or to ii) $\mathfrak{aff}(\mathbb{R})$, the two dimensional complex Lie algebra spanned by vector X, Y with $[X, Y] = Y$. We may assume in the last situation that V, W satisfy the Lie bracket relation $[V, W] = W$.

In case $\mathfrak{m} = \mathbb{C}U \ltimes \mathbb{C}^2$, the action of U on $\mathfrak{a}$ admits a basis whose matrix is one of the following ones

$$\text{(1)} \quad \begin{pmatrix} \nu & 0 \\ 0 & \mu \end{pmatrix}, \ \nu, \mu \in \mathbb{C}; \quad \text{(2)} \quad \begin{pmatrix} \nu & 1 \\ 0 & \nu \end{pmatrix}, \ \nu \in \mathbb{C}. \tag{8}$$

In case $\mathfrak{m} = \mathbb{C}U \ltimes \mathfrak{aff}(\mathbb{R})$ the action of U on $\mathfrak{a}$ is a derivation of $\mathfrak{aff}(\mathbb{R})$ thus over the basis $\{V, W\}$ we have a matrix

$$\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}, \quad a, b \in \mathbb{C}. \tag{9}$$

By making use of this we shall deduce the existence or non-existence of complex structures on any tangent or cotangent Lie algebra corresponding to a three dimensional solvable real Lie algebra.

4.1. Complex structures on six dimensional tangent Lie algebras. If H denotes a Lie group, its tangent bundle TH is identified with $H \times \mathfrak{h}$, which inherits a natural Lie group structure as the semidirect product under the adjoint representation. Its Lie algebra, the tangent Lie algebra $\mathrm{T}\mathfrak{h}$, is the semidirect product via the adjoint representation $\mathfrak{h} \ltimes_{\mathrm{ad}} V$, where V is the underlying vector space to $\mathfrak{h}$ equipped with the trivial Lie bracket.

Proposition 4.3. *Let $\mathfrak{h}$ be a real Lie algebra of dimension three and let $\mathrm{T}\mathfrak{h}$ denote the tangent Lie algebra spanned by $e_1, e_2, e_3, e_4, e_5, e_6$. Then the non-zero Lie brackets are presented in the following list*:

$\mathrm{T}\mathfrak{h}_1$:	$[e_1, e_2] = e_3,\ [e_1, e_5] = e_6,\ [e_2, e_4] = -e_6,$
$\mathrm{T}\mathfrak{r}_3$:	$[e_1, e_2] = e_2,\ [e_1, e_3] = e_2 + e_3,$
	$[e_1, e_5] = e_5,\ [e_1, e_6] = e_5 + e_6,\ [e_2, e_4] = -e_5,\ [e_3, e_4] = -e_5 - e_6,$
$\mathrm{T}\mathfrak{r}_{3,\lambda}$:	$[e_1, e_2] = e_2,\ [e_1, e_3] = \lambda e_3,$
$\lvert\lambda\rvert \leq 1$	$[e_1, e_5] = e_5,\ [e_1, e_6] = \lambda e_6,\ [e_2, e_4] = -e_5,\ [e_3, e_4] = -\lambda e_6,$
$\mathrm{T}\mathfrak{r}'_{3,\eta}$:	$[e_1, e_2] = \eta e_2 - e_3,\ [e_1, e_3] = e_2 + \eta e_3,\ [e_1, e_5] = \eta e_5 - e_6,$
$\eta \geq 0$	$[e_1, e_6] = e_5 + \eta e_6,\ [e_2, e_4] = -\eta e_5 + e_6,\ [e_3, e_4] = -e_5 - \eta e_6,$
$\mathrm{T}\mathfrak{sl}(2)$	$[e_1, e_2] = e_3[e_3, e_1] = 2e_1[e_3, e_2] = -2e_2,$
	$[e_1, e_5] = e_6[e_1, e_6] = -2e_4[e_2, e_4] = -e_6,$
	$[e_2, e_6] = 2e_5[e_3, e_4] = 2e_4[e_3, e_5] = -2e_5,$
$\mathrm{T}\mathfrak{so}(3)$	$[e_1, e_2] = e_3[e_3, e_1] = e_2[e_3, e_2] = -e_1,$
	$[e_1, e_5] = -e_6[e_1, e_6] = e_5[e_2, e_4] = -e_6,$
	$[e_2, e_6] = e_4[e_3, e_4] = -e_5[e_3, e_5] = e_4.$

Theorem 4.4. *Let $\mathfrak{h}$ denote a three dimensional Lie algebra, then* $T\mathfrak{h}$ *admits a complex structure if and only if* $\mathfrak{h}$ *is either isomorphic to* $\mathfrak{h}_1$ *or* $\mathbb{R} \times \mathfrak{aff}(\mathbb{R})$.

The proof can be derived from the next paragraphs.

Lemma 4.5. *If* $\mathfrak{m}$ *is a complex subalgebra of* $T\mathfrak{h}$ *being* $\mathfrak{h}$ *a three dimensional solvable real Lie algebra such that* $T\mathfrak{h}^{\mathbb{C}} = \mathfrak{m} \oplus \sigma\mathfrak{m}$ *then* $\mathfrak{m} \simeq \mathbb{C} \ltimes \mathbb{C}^2$.

Proof. According to the previous paragraphs it should hold $\mathfrak{m} \simeq \mathbb{C} \ltimes \mathbb{C}^2$ or $\mathfrak{m} \simeq \mathbb{C} \ltimes \mathfrak{aff}(\mathbb{R})$. We shall prove that the last situation is not possible. In fact, from the Lie brackets in Proposition 4.3 we see that $[V, W] \in span\{e_5, e_6\}$ so that $c_2 = 0 = c_3 = c_4$. But by computing one has $[V, W] = 0$ implying $W = 0$ and therefore no complex structure can be derived from this situation. $\square$

With the previous Lemma it follows to analyze next the existence of complex structures attached to complex Lie subalgebras $\mathfrak{m}$ such that $\mathfrak{m} \simeq \mathbb{C} \ltimes \mathbb{C}^2$.

Recall that any totally real complex structure on $T\mathfrak{h}_1$ corresponds to a non-singular derivation of $\mathfrak{h}_1$ (Example 3.6). No one of these complex structures is abelian. However $T\mathfrak{h}_1$ can be equipped with abelian complex structures as we show below.

Let $\mathfrak{m}$ be a complex subalgebra of $T\mathfrak{h}_1$ spanned by vectors U, V, W as in (7). The subspace $\mathfrak{a} = span\{V, W\}$ is an ideal of $\mathfrak{m}$ and $\mathfrak{m} = \mathbb{C}U \ltimes \mathfrak{a}$. Since $T\mathfrak{h}_1$ is nilpotent, $\mathfrak{a}$ is abelian and the action of U on $\mathfrak{a}$ is of type (8) and moreover case 1) holds for $\mu = \nu = 0$ while case 2) holds for $\nu = 0$. Case 1) gives rise to abelian complex structures, while case 2) corresponds to non abelian ones.

Computing the Lie brackets $[V, W]$, $[U, V]$ and $[U, W]$, and imposing these brackets to be zero, we get

$$U = e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6,$$
$$V = b_3e_3 + b_4e_4 - a_2b_4e_5 + b_6e_6, \quad W = c_3e_3 + c_4e_4 - a_2c_4e_5 + c_6e_6.$$

If the set $\{U, V, W, \sigma U, \sigma V, \sigma W\}$ spans a basis of $(T\mathfrak{h}_1)^{\mathbb{C}}$, the tangent algebra $T\mathfrak{h}_1$ carries an abelian complex structure J. For instance the linear homomorphism J given by

$$Je_1 = e_2, \quad Je_6 = e_3, \quad Je_4 = e_5, \tag{10}$$

and such that $J^2 = -\mathrm{I}$ defines an abelian complex structure on $T\mathfrak{h}_1$. Following [21], there is only one class among abelian complex structures (see [1]).

Any abelian complex structure is 2-step nilpotent. In fact, since J is abelian $\mathfrak{a}_1(J) = \mathfrak{z}(T\mathfrak{h}_1)$ and clearly the condition $C(T\mathfrak{h}_1) = \mathfrak{z}(T\mathfrak{h}_1)$ shows that $\mathfrak{a}_2(J) = T\mathfrak{h}_1$. On the other hand, the following set of vectors on $T\mathfrak{h}_1^{\mathbb{C}}$ is a basis of the complex subalgebra $\mathfrak{m}$ corresponding to a totally real complex structure on $T\mathfrak{h}_1$:

$$e_1 - i(ae_4 + be_5 + ee_6), \quad e_2 - i(ce_4 + de_5 + fe_6), \quad e_3 - i(a + d)e_6$$

with $a, b, c, d, e, f \in \mathbb{R}$, $a + d \neq 0$ and $ad - bc \neq 0$. They induce non abelian complex structures, and furthermore there are more non-abelian complex structures than the totally real ones. Let $\mathfrak{m}$ be a complex subalgebra of $(\mathrm{T}\mathfrak{h}_1)^{\mathbb{C}}$ spanned by U, V, W as in (7). Requiring that $[U, V] = 0 = [V, W]$ and $[U, W] = V$ we deduce that any complex subalgebra $\mathfrak{m}$ of $(\mathrm{T}\mathfrak{h}_1)^{\mathbb{C}}$ spanned by

$$U = e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6,$$
$$V = c_2e_3 + (c_5 - a_2c_4 + a_4c_2)e_6, \quad W = c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6,$$

and such that $U, V, W, \sigma U, \sigma V, \sigma W$ is a basis of $(\mathrm{T}\mathfrak{h}_1)^{\mathbb{C}}$, induces a non-abelian complex structure on $\mathrm{T}\mathfrak{h}_1$.

The class of non-abelian complex structures J is 2-step nilpotent. Actually the vector $X := W + \sigma W$ belongs to the center of $\mathrm{T}\mathfrak{h}_1$ and also $JX \in \mathfrak{z}(\mathrm{T}\mathfrak{h}_1)$. Since $\mathfrak{a}_1(J) = span\{X, JX\} = \mathfrak{z}(\mathrm{T}\mathfrak{h}_1)$ and $C(\mathrm{T}\mathfrak{h}_1) = \mathfrak{z}(\mathrm{T}\mathfrak{h}_1)$, we conclude that $\mathfrak{a}_2(J) = \mathrm{T}\mathfrak{h}_1$.

After [21] in the set of non-abelian complex structures, one has the following non equivalent complex structures (the extension is such that $J^2 = -\mathrm{I}$):

$$J_se_1 = e_4, \quad J_se_2 = -se_4 + e_5, \quad J_se_3 = 2e_6, \quad s = 0, 1, \tag{11}$$

which are totally real, and next

$$J_\nu e_1 = e_2 + (1-\nu)e_4 + \frac{1-\nu}{\nu}e_5, \quad Je_2 = -\nu e_1 + (1-\nu)e_4, \quad J_\nu e_3 = e_6, \quad \nu \in \mathbb{R} - \{0\} \tag{12}$$

which are neither abelian nor totally real.

By following a similar approach as that already done, that is by scanning the equations that make of the set U, V, W a subalgebra $\mathfrak{m}$ such that $\mathfrak{m} \oplus \sigma\mathfrak{m} = \mathfrak{g}^{\mathbb{C}}$, one can verify the following result.

Proposition 4.6. *Let $\mathfrak{h}$ denote a Lie algebra of dimension three* (4.2).

i) *The tangent Lie algebra $\mathrm{T}\mathfrak{h}_1$ admits abelian and non-abelian complex structures, which are in every case 2-step nilpotent.*

ii) $\mathrm{T}\mathfrak{r}_3$ *and* $\mathrm{T}\mathfrak{r}'_{3,\eta}$ $(\eta \geq 0)$ *do not admit complex structures.*

REMARK. The Lie algebra $\mathrm{T}\mathfrak{h}_1$ is isomorphic to $\mathcal{G}_{6,1}$ in [21] and to $\mathfrak{h}_4$ in [11].

Proposition 4.7. *The following statements are equivalent*:

i) $\mathrm{T}\mathfrak{r}_{3,\lambda}$ *can be endowed with a complex structure*;

ii) $\mathrm{T}\mathfrak{r}_{3,\lambda}$ *carries an abelian complex structure*;

iii) $\lambda = 0$.

Proof. We proceed proving the implications i) $\Rightarrow$ iii) $\Rightarrow$ ii) $\Rightarrow$ i). It is easy to see ii) $\Rightarrow$ iii).

Let $\mathfrak{m}$ denote a complex subalgebra of $\mathfrak{Tr}^{\mathbb{C}}_{3,\lambda}$ spanned by vectors U, V, W as in (7), with $[V, W] = 0$. Evaluating the Lie brackets we obtain the expressions

$$[V, W] = (b_4c_2 - b_2c_4)e_5 + \lambda(b_4c_3 - b_3c_4)e_6,$$

$$[U, V] = b_2e_2 + \lambda b_3e_3 + b_5e_5 + \lambda b_6e_6 + (a_4b_2 - a_2b_4)e_5 + \lambda(a_4b_3 - a_3b_4)e_6,$$

$$[U, W] = c_2e_2 + \lambda c_3e_3 + c_5e_5 + \lambda c_6e_6 + (a_4c_2 - a_2c_4)e_5 + \lambda(a_4c - 3 - a - 3c_4)e_6.$$

If the action of U on $span\{V, W\}$ is of type (1) in (8), by solving the corresponding system, one gets a basis of $(\mathfrak{Tr}_{3,\lambda})^{\mathbb{C}}$ only if $\lambda = 0$, with the additional constraints $\nu = 0 = \mu$. Explicitly, the vectors adopt the form

$$\begin{aligned} &U = e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6, \\ &V = b_3e_3 + b_4e_4 - a_2b_4e_5 + b_6e_6, \quad W = c_3e_3 + c_4e_4 - a_2c_4e_5 + c_6e_6, \end{aligned} \tag{13}$$

whenever $U, V, W, \sigma U, \sigma V, \sigma W$ is a basis of $(\mathfrak{Tr}_{3,0})^{\mathbb{C}}$. It follows at once that the induced complex structure on $\mathfrak{Tr}_{3,0}$ is abelian.

If the action of U on $span\{V, W\} \simeq \mathbb{C}^2$ is of type (2) in (8), then we cannot find a complex structure, regardless of the value of λ. This argument shows i) $\Rightarrow$ iii).

For ii) $\Rightarrow$ iii) one works out the equations deriving from $[V, W] = 0 = [U, V] = [U, W]$ to obtain that a solution exists only for $\lambda = 0$. In this case, one gets the vectors U, V, W above (13). For instance, the following J gives rise to an abelian complex structure on $\mathfrak{Tr}_{3,0}$:

$$Je_1 = e_2, \quad Je_3 = -e_6, \quad Je_4 = e_5. \tag{14}$$

To prove iii) $\Rightarrow$ ii), we have to solve the equation $[V, W] = 0$, $[U, V] = \nu V$ and $[U, W] = \mu W$ for $\lambda = 0$. It is possible to see that the only way to get solutions is imposing that $\nu = \mu = 0$, finishing the proof. □

REMARK. Notice that any complex structure on $\mathfrak{Tr}_{3,0}$ is abelian. See [1] for a classification of abelian complex structures in dimension six.

4.2. Complex structures on cotangent Lie algebras of dimension six. Recall the Lie group counterpart of the cotangent Lie algebra. The zero section in the cotangent bundle T^*H of a Lie group H can be identified with H, as well as the fiber over $(e, 0)$ with $\mathfrak{h}^*$. As a Lie group, the cotangent bundle of H is the semidirect product of H with $\mathfrak{h}^*$ via the coadjoint representation. The tangent space of T^*H at the identity is naturally identified with the *cotangent* Lie algebra $\mathrm{T}^*\mathfrak{h} := \mathfrak{h} \ltimes_{\mathrm{coad}} \mathfrak{h}^*$, the semidirect product of $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ via the coadjoint action.

Proposition 4.8. *Let $\mathfrak{h}$ be a solvable real Lie algebra of dimension three and let $\mathrm{T}^*\mathfrak{h}$ denote the cotangent Lie algebra spanned by $e_1, e_2, e_3, e_4, e_5, e_6$. The non-zero Lie brackets are listed below*:

$$\begin{array}{ll}
\mathrm{T}^*\mathfrak{h}_1: & [e_1,e_2]=e_3,\ [e_1,e_6]=-e_5,\ [e_2,e_6]=e_4,\\
\mathrm{T}^*\mathfrak{r}_3: & [e_1,e_2]=e_2,\ [e_1,e_3]=e_2+e_3,\\
& [e_1,e_5]=-e_5-e_6,\ [e_1,e_6]=-e_6,\ [e_2,e_5]=e_4,\ [e_3,e_5]=e_4,\ [e_3,e_6]=e_4,\\
\mathrm{T}^*\mathfrak{r}_{3,\lambda}: & [e_1,e_2]=e_2,\ [e_1,e_3]=\lambda e_3,\\
|\lambda|\le 1 & [e_1,e_5]=-e_5,\ [e_1,e_6]=-\lambda e_6,\ [e_2,e_5]=e_4,\ [e_3,e_6]=\lambda e_4,\\
\mathrm{T}^*\mathfrak{r}'_{3,\eta}: & [e_1,e_2]=\eta e_2-e_3,\ [e_1,e_3]=e_2+\eta e_3,\ [e_1,e_5]=-\eta e_5-e_6,\\
\eta\ge 0 & [e_1,e_6]=e_5-\eta e_6,\ [e_2,e_5]=\eta e_4,\ [e_2,e_6]=-e_4,\ [e_3,e_5]=e_4,\ [e_3,e_6]=\eta e_4.
\end{array}$$

Theorem 4.9. *Let $\mathfrak{h}$ denote a three dimensional solvable real Lie algebra. If $\mathrm{T}^*\mathfrak{h}$ admits a complex structure, then $\mathfrak{h}$ is isomorphic to one of the following Lie algebras*: $\mathfrak{h}_1$, $\mathbb{R}\times\mathfrak{aff}(\mathbb{R})$, $\mathfrak{r}_{3,1}$, $\mathfrak{r}_{3,-1}$, $\mathfrak{r}'_{3,\eta}$ *for any* $\eta\ge 0$.

The proof of this theorem is a straightforward consequence of the following results.

Suppose that $\mathfrak{m}$ is a complex subalgebra of $\mathrm{T}^*\mathfrak{h}$ with $V, W\in\mathfrak{m}$ satisfying $[V,W]=W$. The Lie bracket relations in Proposition 4.8 immediately imply that $W=c_4e_4$ and thus $W=0$.

Corollary 4.10. *Let $\mathfrak{h}$ denote a solvable real Lie algebra of dimension three. If $(\mathrm{T}^*\mathfrak{h})^{\mathbb{C}}$ splits as a direct sum as vector spaces $(\mathrm{T}^*\mathfrak{h})^{\mathbb{C}}=\mathfrak{m}\oplus\sigma\mathfrak{m}$, where $\mathfrak{m}$ is a complex subalgebra and σ is the conjugation map with respect to $\mathrm{T}^*\mathfrak{h}$, then $\mathfrak{m}\simeq\mathbb{C}\ltimes\mathbb{C}^2$.*

Proposition 4.11. *Let $\mathfrak{h}$ denote a Lie algebra of dimension three. Then*

i) *Every complex structure on the Lie algebra $\mathrm{T}^*\mathfrak{h}_1$ is 3-step nilpotent.*

ii) *The Lie algebra $\mathrm{T}^*\mathfrak{r}_3$ cannot be endowed with a complex structure.*

iii) *If the Lie algebra $\mathrm{T}^*\mathfrak{r}_{3,\lambda}$ admits a complex structure then $\lambda=0, 1, -1$.*

iv) *The Lie algebra $\mathrm{T}^*\mathfrak{r}'_{3,\eta}$ carries a complex structure for any $\eta\ge 0$.*

Proof. i) Since the center of $\mathrm{T}^*\mathfrak{h}_1$ is odd-dimensional, $\mathrm{T}^*\mathfrak{h}_1$ cannot admit an abelian complex structure. As proved in [9, 31] this Lie algebra has a complex structure, such that $\mathfrak{m}=span\{U, V, W\}$ with $[V, W]=0$, and

$$U=e_1+a_2e_2+a_3e_3+a_4e_4+a_5e_5+a_6e_6,$$

$$V=c_2e_3+(a_2c_6-a_6c_2)e_4-c_6e_5,\quad W=c_2e_2+c_3e_3+c_4e_4+c_5e_5+c_6e_6$$

induce complex structures if and only if the vectors $U, V, W, \sigma U, \sigma V, \sigma W$ span a basis of $(\mathrm{T}^*\mathfrak{h}_1)^{\mathbb{C}}$. Following [21], there is only one class of complex structures, thus they are equivalent to the complex structure J given by

$$Je_1=e_4,\quad Je_2=e_6,\quad Je_5=e_3. \tag{15}$$

For this complex structure, note that $\mathfrak{a}_0(J) = \{0\}$, $\mathfrak{a}_1(J) = span\{e_3, e_5\}$, $\mathfrak{a}_2(J) = span\{e_1, e_3, e_4, e_5\}$, $\mathfrak{a}_3(J) = \mathfrak{g}$. Hence from Lemma 2.2 we conclude that any complex structure on $\mathrm{T}^*\mathfrak{h}_1$ is nilpotent.

Observe however that there are complex structures which are not totally real, as for example the following

$$Je_1 = e_2 - e_4, \quad Je_2 = e_6, \quad Je_5 = e_3 + e_4. \tag{16}$$

For the rest of the proof we shall sketch the basic ideas and results (for more details see [13]).

ii) This follows from the following steps: first find conditions for $\mathfrak{m} = span\{U, V, W\}$ to build a subalgebra of $(\mathrm{T}^*\mathfrak{r}_3)^{\mathbb{C}}$; second impose the condition $\mathfrak{m} \oplus \sigma\mathfrak{m} = (\mathrm{T}^*\mathfrak{r}_3)^{\mathbb{C}}$ to see that this is not possible.

iii) Write out the corresponding equations for $[U, V]$, $[V, W]$ and $[U, W]$. Assume the action of U on $\mathfrak{a} = span\{V, W\}$ is of type (1) in (8). The conditions $[U, V] = \nu V$ and $[U, W] = \mu W$ show that a subalgebra $\mathfrak{m}$ exists if $\lambda \in \{0, 1, -1\}$. Moreover, such an $\mathfrak{m}$ is spanned by U, V, W as given in the following table:

$\lambda = 0$	$U = e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6$, $V = b_3e_3 + b_4e_4 + b_6e_6$, $W = -a_2c_5e_4 + c_5e_5$ or U, V as above and $W = c_2e_2 - a_5c_2e_4$
$\lambda = 1$	$U = e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6$, $V = b_2e_2 + b_3e_3 - (a_5b_2 + a_6b_3)e_4$, $W = -(c_5/b_3)(a_2b_3 - a_3b_2) + c_5e_5 - (b_2c_5/b_3)e_6$ with $b_3 \neq 0$ or U as above and $V = -(a_2b_5 + a_3b_6)e_4 + b_5e_5 + b_6e_6$, $W = c_2e_2 - (b_5c_2/b_6)e_3 - (c_2/b_6)(a_5b_6 - a_6b_5)$ with $b_6 \neq 0$
$\lambda = -1$	$U = e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6$, $V = -(c_3b_6/c_5)e_2 - (b_6/c_5)(a_5c_3 - a_6c_5)e_4 + b_6e_6$, $W = c_3e_3 - (a_2c_5 + a_6c_3)$ with $c_5 \neq 0$ or U as above and $V = b_3e_3 - (a_2b_5 + a_6b_3)e_4 + b_5e_5$, $W = c_2e_2 - (c_2/b_3)(a_3b_5 - a_5b_3)e_4 - (b_5c_2/b_3)e_6$ with $b_3 \neq 0$

In all cases, U, V, W, σU, σV, σW turn out to be a basis of $(\mathrm{T}^*\mathfrak{r}_{3,\lambda})^{\mathbb{C}}$. We also observe that none of these complex structures is abelian. For instance, the linear map on $\mathrm{T}^*\mathfrak{r}_{3,0}$ given by

$$Je_1 = e_5, \quad Je_2 = -e_4, \quad Je_3 = e_6 \tag{17}$$

and such that $J^2 = -\mathrm{I}$ defines a totally real complex structure on the cotangent Lie algebra $\mathrm{T}^*\mathfrak{r}_{3,0}$, while the J taken as

$$Je_1 = e_2, \quad Je_4 = e_5, \quad Je_3 = e_6 \tag{18}$$

gives rise to a complex structure which is not totally real.

For $\lambda = -1$ the linear homomorphism such that $J^2 = -\mathrm{I}$ given by

$$Je_1 = e_4, \quad Je_2 = e_6, \quad Je_3 = -e_5 \tag{19}$$

gives a totally real complex structure on $\mathrm{T}^*\mathfrak{r}_{3,-1}$. On the other hand, the J satisfying $J^2 = -\mathrm{I}$ and given by

$$Je_3 = -(e_1 + e_6), \quad Je_5 = e_3 - e_4, \quad Je_6 = -(e_2 + e_4) \tag{20}$$

induces a non totally real complex structure on $\mathrm{T}^*\mathfrak{r}_{3,-1}$. Finally, for $\lambda = 1$ no complex structure on $\mathrm{T}^*\mathfrak{r}_{3,1}$ is totally real. Actually

$$Je_1 = e_4, \quad Je_2 = e_3, \quad Je_5 = e_6 \tag{21}$$

give rise to a complex structure on $\mathrm{T}^*\mathfrak{r}_{3,1}$.

By proceeding in a similar way whenever the action of U on $\mathfrak{a} = span\{V, W\}$ is of type (2) in (8) one obtains that there is no Lie subalgebra $\mathfrak{m} = span\{U, V, W\}$ satisfying the requirements to induce a complex structure.

iv) The linear isomorphisms J such that $J^2 = -\mathrm{I}$ given by

$$Je_1 = \pm e_4, \quad Je_2 = e_3, \quad Je_5 = e_6 \tag{22}$$

define complex structures on $\mathrm{T}^*\mathfrak{r}'_{3,\eta}$ for any $\eta \geq 0$. Note that on $\mathrm{T}^*\mathfrak{r}'_{3,0}$ one has totally real complex structures Proposition 3.8, for instance

$$Je_1 = \pm e_4, \quad Je_2 = e_6, \quad Je_3 = -e_5. \tag{23}$$

□

REMARK. The Lie algebra $\mathrm{T}^*\mathfrak{h}_1$ is isomorphic to $\mathcal{G}_{6,3}$ in [21] and to $\mathfrak{h}_7$ in [11].

5. Complex structures and related geometric structures

In this section we relate complex structures to some geometric structures. In this analysis, we are mainly interested on Hermitian, symplectic and pseudo-Kähler structures.

5.1. On Hermitian complex structures. A *metric* on a Lie algebra $\mathfrak{g}$ is a non-degenerate symmetric bilinear map, $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. It is called *ad-invariant* if the constraints

$$\langle [x, y], z\rangle + \langle y, [x, z]\rangle = 0, \quad \forall x, y \in \mathfrak{g}$$

holds.

EXAMPLE 5.1. The *canonical metric* on a cotangent Lie algebra $T^*\mathfrak{h}$ is defined by

$$\langle (x, \varphi), (x', \varphi') \rangle = \varphi'(x) + \varphi(x'), \quad \forall x, x' \in \mathfrak{h},\ \varphi, \varphi' \in \mathfrak{h}^*.$$

It is neutral and ad-invariant.

A subspace $\mathfrak{w} \subseteq (\mathfrak{g}, \langle\ ,\ \rangle)$ is called *isotropic* if $\langle x, y\rangle = 0$ for all $x, y \in \mathfrak{w}$, that is, if $\mathfrak{w} \subseteq \mathfrak{w}^\perp$, where

$$\mathfrak{w}^\perp = \{y \in \mathfrak{g} \text{ such that } \langle x, y\rangle = 0 \text{ for all } x \in \mathfrak{w}\}.$$

Further, $\mathfrak{w}$ is called *totally isotropic* whenever $\mathfrak{w} = \mathfrak{w}^\perp$.

EXAMPLE 5.2. On $T^*\mathfrak{h}$ equipped with its canonical metric, both subspaces $\mathfrak{h}$ and $\mathfrak{h}^*$ are totally isotropic.

Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ denote a real Lie algebra equipped with a metric. An (almost) complex structure J on $\mathfrak{g}$ is called *Hermitian* if

$$\langle Jx, Jy\rangle = \langle x, y\rangle, \quad \forall x, y \in \mathfrak{g}. \tag{24}$$

If the metric is positive definite the Hermitian complex structure J is also called orthogonal.

Observe that if J is Hermitian, then $\langle x, Jx\rangle = 0$ for all $x \in \mathfrak{g}$. The non-degeneracy property of $\langle\ ,\ \rangle$ says that there is $y \in \mathfrak{g}$ such that $\langle x, y\rangle \neq 0$. Therefore the subspace $\mathfrak{w} = span\{x, Jx, y, Jy\} \subseteq \mathfrak{g}$ is non-degenerate and J-invariant. Furthermore

$$\mathfrak{g} = \mathfrak{w} \oplus \mathfrak{w}^\perp$$

is a orthogonal direct sum of J-invariant non degenerate subspaces of $\mathfrak{g}$. A similar argument can be done in the proof of the following lemma.

Lemma 5.3. *Let $\mathfrak{g}$ denote a real Lie algebra endowed with a metric $\langle\ ,\ \rangle$ and let J be an almost complex structure on $\mathfrak{g}$. Assume $\mathfrak{v}$ is a totally real and totally isotropic subspace on $\mathfrak{g}$, then*

i) *$\mathfrak{g}$ admits a decomposition into a direct sum of totally real and totally isotropic vector subspaces*

$$\mathfrak{g} = \mathfrak{v} \oplus J\mathfrak{v};$$

ii) *$\mathfrak{g}$ splits into an orthogonal direct sum*

$$\mathfrak{g} = \mathfrak{w}_1 \oplus \mathfrak{w}_2 \oplus \cdots \oplus \mathfrak{w}_n$$

of J-invariant non-degenerate subspaces $\mathfrak{w}_1, \ldots, \mathfrak{w}_n$, where $\dim \mathfrak{w}_i \equiv 0 \pmod 4$.

DEFINITION 5.4. A generalized complex structure on a Lie algebra $\mathfrak{h}$ is a Hermitian complex structure on $(\mathrm{T}^*\mathfrak{h}, \langle\ ,\ \rangle)$ where $\langle\ ,\ \rangle$ denotes the canonical metric on $\mathrm{T}^*\mathfrak{h}$.

A Hermitian complex structure J on $\mathrm{T}^*\mathfrak{h}$ which leaves $\mathfrak{h}$ invariant is called a generalized complex structure of *complex type* and it corresponds to a complex structure on $\mathfrak{h}$. A Hermitian complex structure J on $\mathrm{T}^*\mathfrak{h}$ which is totally real, that is $J\mathfrak{h} = \mathfrak{h}^*$, is said a generalized complex structure of *symplectic type*. It corresponds to a symplectic structure on $\mathfrak{h}$.

Proposition 3.2 and Theorem 3.5 of Section 3 imply that a totally real complex structure J on $\mathrm{T}^*\mathfrak{h}$, with $J\mathfrak{h} = \mathfrak{h}^*$, corresponds to a non-singular derivation d of $\mathfrak{h}$. If $\mathfrak{h}$ is equipped with an ad-invariant metric $(\ ,\ )$ and one asks J to be Hermitian, then d must be skew-symmetric with respect to $(\ ,\ )$. Explicitly, a non-singular derivation d on $\mathfrak{h}$ induces the map $l \circ d : \mathfrak{h} \to \mathfrak{h}^*$ giving rise to a complex structure on $\mathrm{T}^*\mathfrak{h}$, where $l : x \to (x,\ \cdot)$.

Corollary 5.5. *Let* $\mathfrak{h}$ *denote an even dimensional Lie algebra endowed with an ad-invariant metric* $(\ ,\ )$. *The following statements are equivalent*

i) $\mathfrak{h}$ *admits a generalized complex structure of symplectic type*;

ii) $\mathfrak{h}$ *admits a symplectic structure*;

iii) $\mathfrak{h}$ *admits a non-singular derivation which is skew symmetric for* $(\ ,\ )$.

In addition, if any of these structures exists, then $\mathfrak{h}$ *is nilpotent.*

5.2. Complex structures and symplectic structures. A *symplectic structure* on an even dimensional Lie algebra $\mathfrak{g}$ is a closed 2-form $\omega \in \Lambda^2(\mathfrak{g}^*)$ of maximal rank, i.e., it satisfies the conditions $\bigwedge^{(1/2)\dim\mathfrak{g}} \omega \neq 0$ and

$$\omega([x, y], z) + \omega([y, z], x) + \omega(z, [x, y]) = 0, \quad \text{for all} \quad x, y, z \in \mathfrak{g}. \tag{25}$$

Let $(\mathrm{T}_\pi\mathfrak{h} = \mathfrak{h} \ltimes V, \omega)$ denote a semidirect product equipped with a symplectic structure. Following [7], we say that $\mathrm{T}_\pi\mathfrak{h}$ is *Lagrangian* if both $\mathfrak{h}$ and V are Lagrangian subspaces relative to ω. We also say that ω is *Lagrangian symplectic*.

Let $\mathrm{T}_\pi\mathfrak{h}$ denote a generalized tangent Lie algebra, then its dual Lie algebra is the semidirect product $\mathrm{T}_{\pi^*}\mathfrak{h} := \mathfrak{h} \ltimes_{\pi^*} V^*$, where π^* is the dual representation

$$(\pi^*(x)\varphi)(u) := -\varphi(\pi(x)(u)), \quad x \in \mathfrak{h},\ \varphi \in V^*,\ u \in V.$$

Note that the cotangent Lie algebra $\mathrm{T}^*\mathfrak{h}$ is the dual of the tangent Lie algebra $\mathrm{T}\mathfrak{h}$.

Suppose $\mathrm{T}_\pi\mathfrak{h} = \mathfrak{h} \ltimes_\pi V$ is a Lie algebra equipped with a totally real complex structure J (that is, $J\mathfrak{h} = V$). This enables us to define on $\mathrm{T}_{\pi^*}\mathfrak{h} := \mathfrak{h} \ltimes_{\pi^*} V^*$ a two-form ω_J by

$$\omega_J(x + u, y + v) := v(Jx) - u(Jy),$$

where x, y are in $\mathfrak{h}$ and u, v are in $(J\mathfrak{h} = V)^*$.

Then ω_J is non-degenerate and symplectic since J is integrable (see [4] or [7] for instance). We remark that the converse is also true, that is, Lagrangian symplectic structures on $\mathrm{T}_\pi\mathfrak{h}$ give rise to totally real complex structures on $\mathrm{T}_{\pi^*}\mathfrak{h}$. Thus totally real complex structures on $\mathrm{T}_\pi\mathfrak{h} = \mathfrak{h} \ltimes_\pi V$ are in correspondence to Lagrangian symplectic structures on $\mathrm{T}_{\pi^*}\mathfrak{h}$.

Corollary 5.6. i) *Let* $\mathrm{T}^*\mathfrak{h}$ *denote a cotangent Lie algebra. If it admits a Lagrangian symplectic structure, then* $\mathfrak{h}$ *is nilpotent.*
ii) *The tangent Lie algebra* $\mathrm{T}\mathfrak{h}$ *admits a Lagrangian symplectic structure for any* $\mathfrak{h}$ *isomorphic to* $\mathfrak{h}_1$, $\mathfrak{r}_{3,-1}$, $\mathfrak{r}'_{3,0}$ *or* $\mathbb{R} \times \mathfrak{aff}(\mathbb{R})$.

We denote by $e^{ij\cdots}$ the wedge product $e^i \wedge e^j \wedge \cdots$, $e^1, \ldots, e^6$ being the dual basis of $e_1, \ldots, e_6$.

To determine the existence of symplectic structures one can proceed as follows. Let $\alpha_{ij} \in \mathbb{R}$ be arbitrary constants and define the generic 2-form on $\mathrm{T}^*\mathfrak{h}$

$$\theta = \sum_{i<j} \alpha_{ij} e^{ij}, \quad i = 1, \ldots, 5. \tag{26}$$

If one requires θ to be closed, the condition $d\theta = 0$ generates a system depending on the parameters α_{ij}.

We exemplify here one case. The Maurer–Cartan equations on $\mathrm{T}^*\mathfrak{r}_{3,\lambda}$ are given by

$$\begin{aligned} &de^1 = 0, \quad de^2 = e^{12}, \quad de^3 = \lambda e^{13}, \\ &de^4 = e^{25} + \lambda e^{36}, \quad de^5 = -e^{15}, \quad de^6 = -\lambda e^{16}. \end{aligned}$$

By the expansion of this expression making use of $de^{ij} = de^i \wedge e^j - e^i \wedge de^j$, we subsequently obtain conditions on the parameters α_{ij}. In the case of $\mathrm{T}^*\mathfrak{r}_{3,\lambda}$, one obtains that $\alpha_{1j} = 0$ for all $j = 1, \ldots, 6$, therefore a closed 2-form θ belongs to $\Lambda^2\mathfrak{v}^*$ being $\mathfrak{v} = span\{e_1, \ldots, e_5\}$, which implies that θ cannot be symplectic.

A similar reasoning applies on $\mathrm{T}^*\mathfrak{r}'_{3,\eta}$ and so one proves the next result.

Lemma 5.7. *The following Lie algebras do not carry a symplectic structure*:
i) $\mathrm{T}^*\mathfrak{r}_{3,\lambda}$ *for any* λ.
ii) $\mathrm{T}^*\mathfrak{r}'_{3,\eta}$ *for any* $\eta \geq 0$.

The next natural step to be analyzed in Lie algebras $\mathfrak{g}$ carrying both a symplectic structure ω and a complex structure J, is the compatibility of these structure. A pair (ω, J) is called a pseudo-Kähler structure whenever the following condition

$$\omega(Jx, Jy) = \omega(x, y), \quad \forall x, y \in \mathfrak{g} \tag{27}$$

is satisfied.

Let $\psi \in \mathrm{Aut}(\mathfrak{g})$ denote a automorphism of $\mathfrak{g}$. Since $\wedge^n \psi^* \omega = \psi^* \wedge^n \omega$ and $\psi^* d\omega = d(\psi^*\omega)$, one has that the existence of a compatible symplectic form for a fixed complex structure J is equivalent to the existence of a compatible symplectic structure for every complex structure in the orbit of J under the action of the group $\mathrm{Aut}(\mathfrak{g})$. In fact, if J is compatible with ω and $J' = \psi^{-1} J \psi$, then $\omega' = \psi^*\omega$ is compatible with J':

$$\omega'(J'x, J'y) = \omega'(\psi^{-1}J\psi x, \psi^{-1}J\psi y) = \omega(J\psi x, J\psi y) = \omega(\psi x, \psi y) = \omega'(x, y).$$

Lemma 5.8. *Let ω denote a two form on $\mathfrak{g}$ which is compatible with the complex structure J. Let $\psi \in \mathrm{Aut}(\mathfrak{g})$ be an automorphism such that $J' = \psi^{-1} J \psi$. Then $\psi^*\omega$ is compatible with J'.*

A *pseudo-Kähler Lie algebra* is a triple $(\mathfrak{g}, J, \omega)$ consisting of a Lie algebra equipped with a pseudo-Kähler structure. The pseudo-Kähler pair (J, ω) originates an Hermitian structure on $\mathfrak{g}$ by means of defining a metric g as

$$g(x, y) = \omega(Jx, y), \quad \text{for all} \quad x, y \in \mathfrak{g}. \tag{28}$$

This kind of Hermitian structures satisfies the parallel condition

$$\nabla J \equiv 0,$$

where ∇ denotes the Levi-Civita connection for g. The pair (J, g) is called a *pseudo-Kähler metric* on $\mathfrak{g}$.

REMARK. A Lie algebra $\mathfrak{h}$ equipped with an ad-invariant metric $(\, , \,)$ cannot carry a complex structure J which is Hermitian and parallel with respect to the Levi-Civita connection of $(\, , \,)$ (see [3]).

It is our aim to investigate the existence of pseudo-Kähler metrics on the Lie algebras $\mathrm{T}\mathfrak{h}$ and $\mathrm{T}^*\mathfrak{h}$ treated previously.

For the complex structure on $\mathrm{T}\mathfrak{h}_1$ given by

$$Je_1 = 2e_4, \quad Je_2 = -e_5, \quad Je_3 = e_6. \tag{29}$$

one can find several compatible closed two forms:

$$\theta = a(e^{45} - 2e^{12}) + be^{14} + c(e^{24} - 2e^{15}) + de^{25} + e(e^{26} + e^{35}) + fe^{36}.$$

For instance, the following two forms give rise to pseudo-Kähler pairs

$$\omega = e^{45} - 2e^{12} + \mu e^{36}, \quad \mu \neq 0, \tag{30}$$

$$\omega = e^{14} + \nu(e^{26} - e^{35}), \quad \nu \neq 0. \tag{31}$$

On $\mathrm{T}^*\mathfrak{h}_1$ consider the complex structure given by (15):

$$Je_1 = e_4, \quad Je_2 = e_6, \quad Je_3 = -e_5.$$

It follows by canonical computations that any two form on $\mathfrak{g}$ which is compatible with J has the form

$$\omega = a(e^{12} + e^{46}) + b(2e^{14} - e^{25} + e^{36}) + c(e^{16} + e^{24}) + d(e^{23} + e^{56}) + ee^{26} + fe^{35}.$$

On $\mathrm{T}\mathfrak{r}_{3,0}$ let J denote the complex structure given by (14):

$$Je_1 = e_2, \quad Je_3 = -e_6, \quad Je_4 = e_5.$$

Canonical computations show that this complex structure is compatible with the symplectic forms

$$\omega_{\alpha,\beta,\gamma} = \alpha e^{12} + \beta(e^{15} - e^{24}) + \gamma e^{36}, \quad \beta\gamma \neq 0, \tag{32}$$

thus the pairs $(J, \omega_{\alpha,\beta,\gamma})$ amount to pseudo-Kähler pairs on $\mathrm{T}\mathfrak{r}_{3,0}$.

Lemma 5.9. *The Lie algebras* $\mathrm{T}\mathfrak{h}_1$, $\mathrm{T}^*\mathfrak{h}_1$ *and* $\mathrm{T}(\mathbb{R}\times\mathfrak{aff}(\mathbb{R}))$ *carry several pseudo-Kähler structures.*

However no pseudo-Kähler pair (J, ω) *on* $\mathrm{T}\mathfrak{h}_1$ *corresponds to an abelian Lie algebra* ([11]).

In view of explanations before, the proof of the theorem below is straightforward.

Theorem 5.10. *Let* $\mathfrak{h}$ *denote a real Lie algebra of dimension three.*
i) *If* $\mathfrak{h}$ *is solvable and* $\mathrm{T}\mathfrak{h}$ *admits a complex structure then it carries a pseudo-Kähler structure.*
ii) *If* $\mathrm{T}^*\mathfrak{h}$ *carries a pseudo-Kähler structure then* $\mathfrak{h}$ *is nilpotent.*

5.3. On the geometry of some pseudo-Kähler homogeneous manifolds. The goal of this paragraph is to point out some geometric features on the homogeneous manifolds arising in the previous paragraphs in Lemma 5.9.

In the nilpotent case, the corresponding computations on the simply connected Lie groups give rise to the results we summarize below.[1]

Proposition 5.11. *The Lie algebra* $\mathrm{T}\mathfrak{h}_1$ *carries flat and non-flat pseudo-Kähler metrics.*

The Lie algebra $\mathrm{T}^*\mathfrak{h}_1$ *admits non-flat but Ricci flat pseudo-Kähler metrics.*

[1]The Lie algebra $\mathrm{T}^*\mathfrak{h}_1$ is the free 2-step nilpotent Lie algebra in three generators.

The first statement was proved in [14] and the second one in [11].

Now we proceed to the study in the solvable case. The simply connected Lie group G with Lie algebra $\mathfrak{Tr}_{3,0}$ is, as a manifold, diffeomorphic to $\mathbb{R}^6$. Let $(r_1,r_2,\ldots,r_6)$ denote an arbitrary element in $\mathbb{R}^6$, then the rule multiplication is given by

$$\begin{aligned}&(r_1, r_2, r_3, r_4, r_5, r_6)\cdot(s_1, s_2, s_3, s_4, s_5, s_6)\\ &= \left(r_1+s_1, r_2+e^{r_6}s_2, r_3+e^{r_6}s_3+\frac{e^{r_6}}{2}(r_1s_2-r_2s_1), r_4+s_4, r_5+s_5, r_6+s_6\right).\end{aligned}$$

The left invariant vector fields at $Y=(s_1, s_2, s_3, s_4, s_5, s_6)\in G$ are

$$\begin{aligned}&e_1(Y)=\frac{\partial}{\partial r_1}-e^{s_6}s_2\frac{\partial}{\partial r_3},\quad e_2(Y)=e^{s_6}\left(\frac{\partial}{\partial r_2}+s_1\frac{\partial}{\partial r_3}\right),\\ &e_3(Y)=e^{s_6}\frac{\partial}{\partial r_3},\quad e_4(Y)=\frac{\partial}{\partial r_4},\quad e_5(Y)=\frac{\partial}{\partial r_5},\quad e_6(Y)=\frac{\partial}{\partial r_6},\end{aligned}$$

and let e^i denote the dual 1-forms for $i=1, 2, 3, 4, 5, 6$.

Consider the metric $\langle\ ,\ \rangle$ on G for which the vector fields above satisfy the non-zero relations

$$g=\alpha(e^1\cdot e^1+e^2\cdot e^2)+\beta(e^1\cdot e^4+e^2\cdot e^5)+\gamma(e^3\cdot e^3+e^6\cdot e^6),\quad \beta\gamma\neq 0.$$

This metric is clearly indefinite.

The complex structure on G is defined as the linear map $J\colon T_YG\to T_YG$ such that $J^2=-1$ and

$$Je_1=e_2,\quad Je_3=e_6,\quad Je_4=e_5.$$

This gives a complex structure on $\mathbb{R}^6$ which is invariant under the action of the Lie group G, the action is induced from the multiplication on G. Moreover, the complex structure is Hermitian for the metric above and it is parallel for the corresponding Levi-Civita connection ∇, which on the basis of left invariant vector fields is given by

$$\nabla_X=\begin{pmatrix}0 & x_2 & 0 & 0 & 0 & 0\\ -x_2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & x_5 & 0 & 0 & x_2 & 0\\ -x_5 & 0 & 0 & -x_2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}\quad\text{for}\quad X=\sum_{i=1}^{6}x_ie_i.$$

If $\mathfrak{h}$ denotes the involutive distribution spanned by e_2, e_5, e_6, then it admits a complementary orthogonal distribution $J\mathfrak{h}$, therefore $T_YG=\mathfrak{h}\oplus J\mathfrak{h}$ is an orthogonal direct sum. At the Lie algebra level, one has the following short exact sequence

$$0\to\mathfrak{h}\to\mathfrak{g}\to J\mathfrak{h}\to 0.$$

Notice that $\mathfrak{h}$ is an abelian ideal, while $J\mathfrak{h}$ is an abelian subalgebra. Moreover, *the complex structure J is totally real* with respect to this decomposition and the representation $\pi : J\mathfrak{h} \to \mathfrak{h}$ is given in the basis e_2, e_5, e_6 by

$$\pi(e_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \pi(e_3) \equiv 0, \quad \pi(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Taking the Lie subgroup H corresponding to the distribution $\mathfrak{h}$ and JH the Lie subgroup corresponding to $J\mathfrak{h}$, the latter is totally geodesic. In fact, making use of the formula for ∇, one verifies that

$$\nabla_X Y \subseteq J\mathfrak{h}, \quad \nabla_{JX} JY \subseteq J\mathfrak{h}, \quad X, Y \in \mathfrak{h}$$

and since (J, g) is pseudo-Kähler, $\nabla_X JY = J\nabla_X Y$ and $\nabla_{JX} Y = -J\nabla_{JX} JY$ for $X, Y \in \mathfrak{h}$.

The curvature tensor R is given by

$$R(x, y) = -\nabla_{[x,y]}$$

which implies that JH is flat. The Ricci tensor r follows $r(X, Y) = 2(x_1 y_1 + x_2 y_2)$ for $X = \sum x_i e_1$, $Y = \sum y_i e_i$, therefore G is neither flat nor Ricci flat.

Notice that since $\mathrm{Tr}_{3,0} = \mathfrak{h} \oplus J\mathfrak{h}$, where $\mathfrak{h}$ is an ideal and $J\mathfrak{h}$ a Lie subalgebra, then J is totally real.

The pseudo-Kähler metric for J is non-flat, however the Lie subgroup for $J\mathfrak{h}$ which is abelian, is totally geodesic and flat.

REMARK. The results above and those in [8] suggest that totally real complex structures are interesting objects to be consider in presence of compatible structures.

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