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ON WEAK POTENTIAL OPERATORS FOR RECURRENT MARKOV CHAINS WITH CONTINUOUS PARAMETERS

RYŌJI KONDŌ

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Let S be a denumerable (possibly finite) state space and $(P_t)_{t\geqq 0}$ a recurrent semi-group of Markov kernels on S with an invariant measure μ. We shall say that a real valued function f defined on S is a *null charge* for $(P_t)_{t\geqq 0}$ if it has finite support and $\sum_{x\in S}\mu(x)f(x)=0$. Throughout this work we shall denote by $\boldsymbol{B}$ the space of all real valued and bounded functions on S and by $\boldsymbol{N}$ the space of all null charges for $(P_t)_{t\geqq 0}$. A linear operator R from $\boldsymbol{N}$ to $\boldsymbol{B}$ will be called a *weak potential operator* for $(P_t)_{t\geqq 0}$ if it satisfies the following condition:

$$\text{(W. P)} \qquad (I-P_t)Rf(x)=\int_0^t P_s f(x)ds \quad \text{for any } f\in\boldsymbol{N},\ t\geqq 0 \text{ and } x\in S,$$

where I denotes the identity operator. Our definition of the weak potential operator is a version for continuous parameter of the weak inverse which was introduced by Orey [18] for discrete parameter Markov chains. Orey has shown that, for any recurrent Markov chain, there is always a weak inverse unique up to a linear functional on the space of null charges.

In the present paper we shall prove that any recurrent Markov chain with continuous parameter has a weak potential operator by systematically studying those Markov chains admitting instantaneous states. Moreover we shall show that a recurrent semi-group is determined uniquely from the pair of its own invariant measure and weak potential operator.

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1. Preliminaries on Markov chains

Throughout this work matrix notation is adopted. A kernel on S is a matrix, that is, a real valued function defined on $S\times S$ and a function (measure) is a column (row) vector. Let us denote the functions on S, by $f, g, \cdots$ the measures on S by $\mu, \nu, \cdots$ and the kernels on S respectively by $K, H, \cdots$. Then the function Kf, measure μK and kernel KH are, respectively, defined by

$$Kf(x) = \sum_{y \in S} K(x, y) f(y) \qquad (x \in S),$$
$$\mu K(y) = \sum_{x \in S} \mu(x) K(x, y) \qquad (y \in S),$$
$$KH(x, y) = \sum_{z \in S} K(x, z) H(z, y) \qquad ((x, y) \in S \times S).$$

To avoid confusion, however, we shall denote $\langle \mu, f \rangle$ and $f \otimes \mu$ instead of μf and $f \mu$ respectively, that is,

$$\langle \mu, f \rangle = \sum_{x \in S} \mu(x) f(x),$$
$$f \otimes \mu(x, y) = f(x) \mu(y) \qquad ((x, y) \in S \times S).$$

We shall also list some trivial convention for clarity. A function or a measure on S is *non-negative* (*strictly positive*) if it is non-negative (positive) for every state in S. That a kernel on S is *non-negative* is understood in the same way. Convergence is always pointwise convergence. The indicator function of a set Γ is denoted by χ_Γ. χ_S is written by 1. A kernel K is said to be a *Markov kernel* if $K \geqq 0$ and $K1 = 1$.

A family of Markov kernels $(P_t)_{t \geqq 0}$ on S will be called a *standard semi-group of Markov kernels* or simply the *semi-group* if it satisfies the following conditions:

(P. 1) $\qquad P_{s+t} = P_s P_t \qquad$ for all $s, t \geqq 0$,

(P. 2) $\qquad \lim_{t \to 0} P_t = I$.

From these properties it follows that, for any $(x, y) \in S \times S$, the mapping $t \to P_t(x, y)$ is uniformly continuous on $[0, \infty)$ ([2, p. 124]). If we introduce the family of kernels $(R_\alpha)_{\alpha > 0}$ by

$$R_\alpha(x, y) = \int_0^\infty e^{-\alpha t} P_t(x, y) dt \qquad \text{for } (x, y) \in S \times S \text{ and } \alpha > 0,$$

it satisfies the following conditions:

(R. 1) $\qquad R_\alpha \geqq 0$ and $\alpha R_\alpha 1 = 1 \qquad$ for all $\alpha > 0$,

(R. 2) $\qquad R_\alpha - R_\beta + (\alpha - \beta) R_\alpha R_\beta = 0 \qquad$ for all $\alpha, \beta > 0$,

(R. 3) $\qquad \lim_{\alpha \to \infty} \alpha R_\alpha = I$.

We shall call $(R_\alpha)_{\alpha > 0}$ the *resolvent* of the semi-group $(P_t)_{t \geqq 0}$. The equation (R. 2) is called the *resolvent equation*. Using the uniqueness of the inverse Laplace transform and continuity of the semi-group we can see that the semi-group is uniquely determined from its resolvent.

In the rest of this section we will give a definition of Markov process which may have branching points and introduce a natural Markov process associated with a semi-group $(P_t)_{t \geqq 0}$ given on S, which will be called a Ray process.

The definition and terminology of Markov process are taken from Dynkin [5] with a slight change. Let $(E, \mathfrak{B})$ be a measurable space for which each one-point set is measurable and Δ a point adjoined to E as the extra point. We write $E_\Delta = E \cup \{\Delta\}$, and let $\mathfrak{B}_\Delta$ is the σ-field of subsets of E_Δ generated by the sets in $\mathfrak{B}$ and $\{\Delta\}$. We consider the collection $X = (\Omega, \mathfrak{M}, (X_t)_{t \geqq 0}, (\theta_t)_{t \geqq 0}, (P_x)_{x \in E_\Delta})$, where Ω the sample space is a set with a distinguished element ω_Δ, $\mathfrak{M}$ is a σ-field of subsets of Ω, $(X_t)_{t \geqq 0}$ is a family of mappings from Ω to E_Δ, $(\theta_t)_{t \geqq 0}$ is a family of mappings (shift operators) from Ω to Ω and finally, $(P_x)_{x \in E_\Delta}$ is a family of probability measures on $(\Omega, \mathfrak{M})$. We shall say that X is a *Markov process* with state space E if the following conditions are satisfied:

(M. 1) For each $\omega \in \Omega$, if $X_t(\omega) = \Delta$ for some $t \geqq 0$, then $X_s(\omega) = \Delta$ for all $s \geqq t$ and $X_s(\omega_\Delta) = \Delta$ for all $s \geqq 0$.

(M. 2) For each $t \geqq 0$, the mapping $X_t \colon \Omega \to E_\Delta$ is $\mathfrak{M}$–$\mathfrak{B}_\Delta$ measurable, that is, $X_t^{-1}(\mathfrak{B}_\Delta) \subseteqq \mathfrak{M}$. We shall denote by $\mathfrak{B}_t$ the σ-field of subsets of Ω generated by $X_s^{-1}(\Gamma)$, where $s \leqq t$ and $\Gamma \in \mathfrak{B}_\Delta$, and by $\mathfrak{B}_\infty$ the σ-field generated by $\cup_{t \geqq 0} \mathfrak{B}_t$.

(M. 3) For each $\Lambda \in \mathfrak{B}_\infty$, the mapping $x \to P_x(\Lambda)$ is $\mathfrak{B}_\Delta$ measurable and $P_\Delta(X_0 = \Delta) = 1$.

(M. 4) $X_{s+t}(\omega) = X_s(\theta_t \omega)$ for all $\omega \in \Omega$, $s, t \geqq 0$. From this it follows $\theta_t^{-1}(\mathfrak{B}_\infty) \subseteqq \mathfrak{B}_\infty$ for all $t \geqq 0$.

(M. 5) For each bounded, $\mathfrak{B}_\infty$ measurable function F, $t \geqq 0$, $\Lambda \in \mathfrak{B}_t$ and $x \in E_\Delta$,

$$E_x(F \circ \theta_t \colon \Lambda) = E_x(E_{X_t}(F) \colon \Lambda),$$

where $E_x(F \colon \Lambda)$ denotes the integral $\int_\Lambda F(\omega) P_x(d\omega)$.

We now extend the definitions by:

$$X_\infty(\omega) = \Delta \quad \text{and} \quad \theta_\infty(\omega) = \omega_\Delta \qquad \text{for each } \omega \in \Omega$$

and if τ is a function defined on Ω with values in $[0, \infty]$, then

$$X_\tau(\omega) = X_{\tau(\omega)}(\omega) \quad \text{rand} \quad \theta_\tau(\omega) = \theta_{\tau(\omega)}(\omega).$$

The function $\zeta(\omega) = \inf\{t \colon X_t(\omega) = \Delta\}$ is called the life time of X. The functions $t \to X_t(\omega)$ are called the sample functions of X.

For each bounded measure μ on $(E_\Delta, \mathfrak{B}_\Delta)$ we may define a measure P_μ on $(\Omega, \mathfrak{B}_\infty)$ by $P_\mu(\Lambda) = \int \mu(dx) P_x(\Lambda)$. We use E_μ to denote integrals with respect to P_μ. We now define $\mathfrak{F}_\infty$ to be the intersection over all μ of the P_μ-completions of $\mathfrak{B}_\infty$. Each of the measure P_μ extends unipuely to $\mathfrak{F}_\infty$. We define the σ-field $\mathfrak{F}_t$ as follows: $\Lambda \in \mathfrak{F}_t$ if for each μ there exists $\Lambda_\mu \in \mathfrak{B}_t$ such that $\Lambda \backslash \Lambda_\mu$ and $\Lambda_\mu \backslash \Lambda$ are in $\mathfrak{F}_\infty$ and $P_\mu(\Lambda \backslash \Lambda_\mu) = P_\mu(\Lambda_\mu \backslash \Lambda) = 0$. A mapping $\tau \colon \Omega \to [0, \infty]$ is called a *Markov time* provided that $\{\tau < t\} \in \mathfrak{F}_t$ for all $t > 0$. The σ-field $\mathfrak{F}_{\tau+}$ of Markov time τ consists of all $\Lambda \in \mathfrak{F}_\infty$ such that $\Lambda \cap \{\tau < t\} \in \mathfrak{F}_t$ for all $t > 0$.

Let E be a locally compact, separable Hausdorff space and $\mathfrak{B}$ the σ-field of Borel subsets of E. In this case we adjoin Δ to E as an isolated point. We shall say that a Markov process with state space E is *right-continuous* if all sample functions are right-continuous on $[0, \infty)$. If X is right-continuous and τ is a Markov time, then $X_\tau^{-1}(\Gamma)\in\mathfrak{F}_\infty$ for all universally measurable subsets Γ of E_Δ and $\theta_\tau^{-1}(\Lambda)\in\mathfrak{F}_\infty$ for all $\Lambda\in\mathfrak{F}_\infty$. We shall say that a right-continuous Markov process X has the *strong Markov property* if it satisfies:

(M. 6) For each bounded, $\mathfrak{F}_\infty$ measurable function F and Markov time τ one has

$$E_\mu(F\circ\theta_\tau\colon \Lambda)=E_\mu(E_{X_\tau}(F)\colon \Lambda)$$

for all $\Lambda\in\mathfrak{F}_{\tau+}$ and μ.

Let us now return to the case of denumerable state space. As before let S be a denumerable state space and $(P_t)_{t\geqq 0}$ a semi-group on S with resolvent $(R_\alpha)_{\alpha>0}$. If we consider S as a topological space with discrete topology, it is a locally compact, separable Hausdorff space and $\boldsymbol{B}$ coincides with the space of bounded and continuous functions defined on S. Now let f be a function in $\boldsymbol{B}$ with $0\leqq f\leqq 1$. Since, for each x in S, the functions $t\to P_tf(x)$ and $t\to P_t(1-f)(x)$ are lower semi-continuous on $[0, \infty)$, then, noting the relation $P_t1=1$ for all $t\geqq 0$, we see that the function $t\to P_tf(x)$ is continuous on $[0, \infty)$. Thus, for each f in $\boldsymbol{B}$ and for each x in S, the function $t\to P_tf(x)$ is continuous on $[0, \infty)$. Further if we denote by $\boldsymbol{B}_1$ the set of the functions of the form: $R_1(\ , y)$, $y\in S$, then $\boldsymbol{B}_1$ is a countable subset of $\boldsymbol{B}^+$, the cone of non-negative functions in $\boldsymbol{B}$, separating two points in S and satisfying the following condition:

$$\alpha R_{\alpha+1}f\leqq f \qquad \text{for all } f\in\boldsymbol{B}_1,\ \alpha>0\,.$$

Therefore, according to Kunita-Watanabe [14, Theorem 1] and Ray [19], if we take an appropriate, compact metric space $\bar{S}$ containing S as a dense subset, we can find a right-continuous, strong Markov process X with state space $\bar{S}$ which has the following properties:

(i) For each $x\in\bar{S}$, with P_x-measure one, Lebesgue measure of the set $\{t\colon X_t\in\bar{S}\backslash S\}$ is equal to zero.

(ii) For each bounded, continuous function $\bar{f}$ on $\bar{S}$, $\alpha>0$, the function $\bar{R}_\alpha\bar{f}$ defined by

$$\bar{R}_\alpha\bar{f}(x)=E_x\left(\int_0^\zeta e^{-\alpha t}\bar{f}(X_t)dt\right) \qquad \text{for } x\in\bar{S}$$

is continuous on $\bar{S}$.

(iii) For any $(x, y)\in S\times S$, $t\geqq 0$, $P_x(X_t=y)=P_t(x, y)$. We shall call such a Markov process a *Ray process* associated with the semi-group $(P_t)_{t\geqq 0}$. In the following, when we consider a Ray process, we always extend the function defined on S to the function on $\bar{S}$ by putting the values on $\bar{S}\backslash S$ equal to zero.

Let X be a Ray process associated with the semi-group $(P_t)_{t\geqq 0}$ and V a subset of $\bar{S}_\Delta$. We shall denote by σ^V the first hitting time of V, that is,

$$\sigma^V = \begin{cases} \inf\{t \geqq 0\colon X_t \in V\} \\ \infty \quad \text{if the set in braces is empty.} \end{cases}$$

If V is a Borel subset of $\bar{S}$, σ^V is a Markov time. If V is a subset of $\bar{S}$, then $\tau^V = \sigma^{\bar{S}_\Delta \setminus V}$ is called the first exist time from V. Further we introduce σ_U^V by: $\sigma_U^V = \tau^U + \sigma^V \circ \theta_{\tau^U}$, which shows the first hitting time of the set V after the first exist from the set U. If both U and V are Borel subsets of $\bar{S}$, then τ^V and σ_U^V are Markov times. When V has a form $\{a\}$ with a single element a in S, we shall use σ^a, τ^a and σ_U^a to denote $\sigma^{\{a\}}$, $\tau^{\{a\}}$ and $\sigma_U^{\{a\}}$ respectively. Note that $\zeta = \sigma^\Delta$ and therefore ζ is a Markov time. For each x in S, since $P_x(\zeta > t) \geqq P_x(X_t \in S) = 1$ for all $t \geqq 0$, we have $P_x(\zeta = \infty) = 1$.

For later use we prove here the next property of Ray process.

(iv) If a state a in S is not a trap (a is a *trap* if $P_t(a, a) = 1$ for all $t \geqq 0$), then we can find a neighborhood U (in $\bar{S}$) of a such as $E_a(\tau^U) < \infty$, which will be called an exit neighborhood of a.

Let $\boldsymbol{C}$ be the space of continuous functions on $\bar{S}$ and $(\bar{R}_\alpha)_{\alpha>0}$ the resolvent of Ray process X. $\bar{R}_\alpha f = 0$ implies $\bar{R}_\beta f = 0$ for all $\beta > 0$. Since $\lim_{\beta\to 0} \beta \bar{R}_\beta f(a) = f(a)$ for any state a of S by (iii) and (P. 2), $f = 0$ on S. However, since f is uniformly continuous on $\bar{S}$ and S is a dense subset of $\bar{S}$, we have $f = 0$ on $\bar{S}$. Therefore $\bar{R}_\alpha$ is invertible. It is easily verified that $\bar{R}_\alpha(\boldsymbol{C})$ is independent of α and that $\bar{G} = \alpha - \bar{R}_\alpha^{-1}\colon \bar{R}_\alpha(\boldsymbol{C}) \to \boldsymbol{C}$ is independent of α. If $a \in S$ and $\bar{G}f(a) = 0$ for all $f \in \bar{R}_\alpha(\boldsymbol{C})$, so we have $\alpha \bar{R}_\alpha g(a) = g(a)$ for all $g \in \boldsymbol{C}$. Consequently $\alpha R_\alpha(a, a) = 1$ for all $\alpha > 0$, which implies $P_t(a, a) = 1$ for all $t \geqq 0$. Therefore if a is not a trap, there is a function f in $\bar{R}_\alpha(\boldsymbol{C})$ with $\bar{G}f(a) > 1$. In the sameway as in [11; p. 99], we can prove $E_a(\tau^U) \leqq 2 \sup |f| < \infty$ for a small neighborhood U of a.

2. Recurrent semi-groups

A semi-group $(P_t)_{t\geqq 0}$ is said to be *irreducible recurrent* or simply *recurrent* if the following condition is satisfied:

$$\text{(P. 3)} \qquad \int_0^\infty P_t(x, y)dt = \infty \qquad \text{for all } (x, y) \in S \times S\,.$$

In this section we shall study some prorerties of recurrent semi-groups and give a formula of the invariant measure.

Let X be a Ray process associated with a recurrent semi-group $(P_t)_{t\geqq 0}$. Using the assumption (P. 3), we can easily verify that any state in S is not a trap and therefore has an exit neighborhood.

Lemma 1. *Let a be a state in S and U an exit neighborhood of a, then*

$$P_a(0<\sigma_U^a<\infty)=1\,.$$

Furthermore, if we introduce a sequence of Markov times $(\sigma_n)_{n\geqq 0}$ *by*

$$\sigma_0=0 \quad and \quad \sigma_n=\sigma_{n-1}+\sigma_U^a\circ\theta_{\sigma_{n-1}} \qquad for\ n\geqq 1\,, \tag{2. 1}$$

then with P_a*-measure one, we have*

$$0=\sigma_0<\sigma_1<\sigma_2\cdots<\infty \quad and \quad \lim_{n\to\infty}\sigma_n=\infty\,.$$

Proof. The right-continuity of sample functions implies $P_a(\sigma_U^a>0)=1$, so we have only to prove $P_a(\sigma_U^a<\infty)=1$. Let $(R_\alpha)_{\alpha>0}$ be the resolvent of $(P_t)_{t\geqq 0}$, then, using the strong Markov property, we have

$$R_\alpha(a,a)=E_a\Big(\int_0^{\sigma_U^a}e^{-\alpha t}\chi_{\{a\}}(X_t)\,dt\Big)+E_a(e^{-\alpha\sigma_U^a}:\sigma_U^a<\infty)R_\alpha(a,a)$$

for all $\alpha>0$. Consequently

$$(1-E_a(e^{-\alpha\sigma_U^a}:\sigma_U^a<\infty))\leqq E_a\Big(\int_0^{\sigma_U^a}\chi_{\{a\}}(X_t)dt\Big)\Big/R_\alpha(a,a)$$
$$\leqq E_a(\tau^U)/R_\alpha(a,a)\,.$$

However, since $\lim_{\alpha\to 0}E_a(e^{-\alpha\sigma_U^a}:\sigma_U^a<\infty)=P_a(\sigma_U^a<\infty)$, $\lim_{\alpha\to 0}R_\alpha(a,a)=\infty$ and $E_a(\tau^U)<\infty$, we have $P_a(\sigma_U^a<\infty)=1$. Thus the first assertion of the lemma was proved. Next let $(\sigma_n)_{n\geqq 0}$ be the sequence defined by (2. 1). Using the strong Markov property, we can easily verify that, for any $n\geqq 1$ and $\alpha_0,\alpha_1,\cdots,\alpha_n>0$,

$$E_a[\exp(-\sum_{k=0}^n\alpha_k\sigma_U^a\circ\theta_{\sigma_k})]=\prod_{k=0}^n E_a[\exp(-\alpha_k\sigma_U^a)]\,,$$

which implies, as random variables on probability space $(\Omega,\mathfrak{M},P_a)$, the sequence $(\sigma_U^a\circ\theta_{\sigma_n})_{n\geqq 0}$ is independent and that each $\sigma_U^a\circ\theta_{\sigma_n}$ has the same distribution with that of σ_U^a. Since

$$\sigma_n=\sum_{k=0}^{n-1}\sigma_U^a\circ\theta_{\sigma_k} \qquad \text{for all } n\geqq 1\,,$$

the second assertion of the lemma is followed from Levy's theorem.

Lemma 2. *Let a be a state in S and U an exit neighborhood of a, then*

$$P_a(\sigma_U^a>\sigma^b)>0 \qquad for\ all\ b\in S\,. \tag{2. 2}$$

Proof. If there were some $b\in S$ with $P_a(\sigma_U^a>\sigma^b)=0$, then we should have $P_a(\sigma_U^a<\sigma^b)=1$ since $a\neq b$. Let $(\sigma_n)_{n\geqq 0}$ be the sequence introduced in Lemma 1, then, using the strong Markov properties, we should have $P_a(\sigma^b>\sigma_n)=(P_a(\sigma^b>\sigma_U^a))^n=1$ for all $n\geqq 1$. Therefore $P_a(\sigma^b=\infty)=1$ by Lemma 1. Hence we should have $P_t(a,b)=0$ for all $t\geqq 0$, which contradicts the assumption (P. 3).

Lemma 3. *For any a, $b \in S$, $P_a(\sigma^b < \infty) = 1$.*

Proof. We may assume $a \neq b$ since the other case is trivial. Let U be an exit neighborhood of a and $(\sigma_n)_{n \geqq 0}$ the sequence defined by (2. 1). Then, using Lemma 1 and (2. 2), we have

$$\begin{aligned} P_a(\sigma^b < \infty) &= \sum_{n=0}^{\infty} P_a(\sigma_n < \sigma^b < \sigma_{n+1}) \\ &= P_a(\sigma^b < \sigma_U^a) \sum_{n=0}^{\infty} (P_a(\sigma_U^a < \sigma^b))^n \\ &= P_a(\sigma^b < \sigma_U^a)/(1 - P_a(\sigma_U^a < \sigma^b)) = 1 . \end{aligned}$$

Lemma 4. *Let a be a state in S, then*

$$E_x\Big(\int_0^{\sigma^a} \chi_{\{y\}}(X_t)dt\Big) < \infty \qquad \textit{for all } (x, y) \in S \times S .$$

Proof. For any $(x, y) \in S \times S$, since we have

$$\begin{aligned} E_x\Big(\int_0^{\sigma^a} \chi_{\{y\}}(X_t)dt\Big) &= E_x\Big(\int_{\sigma^y}^{\sigma^a} \chi_{\{y\}}(X_t)dt \colon \sigma^y < \sigma^a\Big) \\ &= P_x(\sigma^y < \sigma^a) E_y\Big(\int_0^{\sigma^a} \chi_{\{y\}}(X_t)dt\Big) \\ &\leqq E_y\Big(\int_0^{\sigma^a} \chi_{\{y\}}(X_t)dt\Big) , \end{aligned}$$

we have only to prove

$$E_y\Big(\int_0^{\sigma^a} \chi_{\{y\}}(X_t)\, dt\Big) < \infty \qquad \text{for any } y \in S .$$

We may assume $y \neq a$ since the other case is trivial. Let V an exit neighborhood of y not containing a and $(\tau_n)_{n \geqq 0}$ the sequence of Markov times defined by

$$\tau_0 = 0 \quad \text{and} \quad \tau_n = \tau_{n-1} + \sigma_V^y \circ \theta_{\tau_{n-1}} \qquad \text{for } n \geqq 1 .$$

Then, from the preceeding lemmas it follows that

$$\begin{aligned} E_y\Big(\int_0^{\sigma^a} \chi_{\{y\}}(X_t)dt\Big) &= \sum_{n=0}^{\infty} E_y\Big(\int_0^{\sigma^a} \chi_{\{y\}}(X_t)dt \colon \tau_n < \sigma^a < \tau_{n+1}\Big) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} E_y\Big(\int_{\tau_k}^{\tau_{k+1}} \chi_{\{y\}}(X_t)dt \colon \tau_n < \sigma^a < \tau_{n+1}\Big) \\ &= \sum_{k=0}^{\infty} E_y\Big(\int_{\tau_k}^{\tau_{k+1}} \chi_{\{y\}}(X_t)\, dt \colon \tau_k < \sigma^a\Big) \\ &= E_y\Big(\int_0^{\sigma_V^y} \chi_{\{y\}}(X_t)\, dt\Big) \sum_{k=0}^{\infty} (P_y(\sigma_V^y < \sigma^a))^n \\ &\leqq E_y(\tau^V)/P_y(\sigma^a < \sigma_V^y) < \infty . \end{aligned}$$

Thus the Lemma was proved.

From now on we use ${}^aR$ to denote the kernel defined by

$$ {}^aR(x, y) = E_x\left(\int_0^{\sigma^a} \chi_{\{y\}}(X_t)dt\right) \qquad ((x, y)\in S\times S). \tag{2.3} $$

As we have seen in the proof of Lemma 4, it satisfies:

$$ {}^aR(x, y) \leqq {}^aR(y, y) \qquad \text{for all } (x, y)\in S\times S. \tag{2.4} $$

A non-negative function f defined on S is said to be *excessive* if $P_t f \leqq f$ for all $t\geqq 0$. Although the next lemma is an easy consequence of the general theory of excessive functions, we will give here a simple direct proof.

Lemma 5. *Any excessive function is constant.*

Proof. Let f be an excessive function and $(R_\alpha)_{\alpha>0}$ the resolvent of $(P_t)_{t\geqq 0}$. From the definition of excessive function it follows that $\alpha R_\alpha f \leqq f$ for all $\alpha > 0$. However we may assume $\alpha R_\alpha f = f$ for all $\alpha > 0$. For, if the contrary were true, there would be some $\beta > 0$ and some $a \in S$ with $\beta R_\beta f(a) < f(a)$. Put $g = f - \beta R_\beta f$, then, using the resolvent equations, we should have

$$ \begin{aligned} R_\alpha(a, a)g(a) &\leqq R_\alpha g(a) \\ &= [\beta R_\beta f(a) - \alpha R_\alpha f(a)]/(\alpha-\beta) \\ &\leqq f(a)/(\beta-\alpha) \end{aligned} $$

for all α smaller than β. Thus, letting $\alpha \to 0$, we should have $f(a) = \infty$, which contradicts the finiteness of the values of f. Now let a and b be any two states in S, then, using the strong Markov property, we have

$$ f(a) = \alpha R_\alpha f(a) \geqq E_a(e^{-\alpha\sigma^a})\alpha R_\alpha f(b) = E_a(e^{-\alpha\sigma^a}) f(b). $$

Then, letting $\alpha \to 0$, we have $f(a) \geqq f(b)$. By the exactly same reason we have $f(b) \geqq f(a)$. Thus f must be constant.

A strictly positive measure μ on S is called an *invariant measure* of the semi-group $(P_t)_{t\geqq 0}$ if $\mu P_t = \mu$ for all $t \geqq 0$. For discussions on invariant measure of Markov process, see, for example, [4], [7] and [12] in the time discrete case, [1], [8], [15] and [20] in the time continuous case. We give here a formula of invariant measure which is used in the next section.

Theorem 1. *For any recurrent semi-group $(P_t)_{t\geqq 0}$, there is an invariant measure, unique except for a constant multiplier, and this is the only invariant measure.*

Proof. First we show the uniqueness of the invariant measure by using the same idea as Kemeny-Snell [12]. Let μ and ν be any two invariant measures of recurrent semi-group $(P_t)_{t\geqq 0}$. If we introduce the family of kernels $(\hat{P}_t)_{t\geqq 0}$ by

$$\hat{P}_t(x, y) = \mu(y)P_t(y, x)/\mu(x) \qquad \text{for } (x, y)\in S\times S,$$

then it is easily verified that $(\hat{P}_t)_{t\geqq 0}$ is a recurrent semi-group on S and that the function $\hat{f}$ defined by $\hat{f}(x)=\nu(x)/\mu(x)$, $x\in S$, is an excessive function for $(\hat{P}_t)_{t\geqq 0}$. Thus, $\hat{f}$, that is, ν/μ is constant by Lemma 5.

We now show the existence of an invariant measure. Let X be a Ray process associated with $(P_t)_{t\geqq 0}$ and let T be any Markov time such that $P_a(T>0)=1$ and $E_a(T)<\infty$. We shall prove that the measure μ defined by

$$\mu(y) = E_a\left(\int_0^{T+\sigma^a\circ\theta_T} \chi_{\{y\}}(X_s)ds\right)$$

is an invariant measure of $(P_t)_{t\geqq 0}$.[1] $\mu(y)$ is finite, for

$$\mu(y) \leqq E_a(T)+{}^aR(y, y)<\infty .$$

Next we prove that μ is invariant under P_t. For short, set $T^a=T+\sigma^a\circ\theta_T$. Noting that $P_a(X_{T^a}=a)=1$, we have

$$\begin{aligned}\mu P_t(y) &= E_a\left(\int_0^{T^a} E_{X_s}(\chi_{\{y\}}(X_t)ds\right) = E_a\left(\int_t^{t+T^a} \chi_{\{y\}}(X_s)ds\right)\\ &= E_a\left(\int_0^{T^a} \chi_{\{y\}}(X_s)ds\right)+E_a\left(\int_{T^a}^{T^a+t} \chi_{\{y\}}(X_s)ds\right)-E_a\left(\int_0^t \chi_{\{y\}}(X_s)ds\right)\\ &= E_a\left(\int_0^{T^a} \chi_{\{y\}}(X_s)ds\right) = \mu(y) .\end{aligned}$$

It remains to prove $\mu(y)>0$ for any $y\in S$. Since

$$\sum_{y\in S} \mu(y) \geqq E_a(T)>0 ,$$

there exists some x_0 such that $\mu(x_0)>0$. But we have for any y

$$\begin{aligned}\mu(y) &= \mu P_t(y)\\ &\geqq \mu(x_0)P_t(x_0, y)>0 ,\end{aligned}$$

for $P_t(x_0, y)>0$ for some $t>0$ by (P. 3). Thus Theorem 1 was proved.

Let T be a Markov time which is independent of $(X_t)_{t\geqq 0}$ under P_a and has exponential distribution with expectation $1/\alpha$. In this case, we have

$$\begin{aligned}\mu(y) &= E_a\left(\int_0^T \chi_{\{y\}}(X_s)ds\right)+E_a\left[E_{X_T}\left(\int_0^{\sigma^a} \chi_{\{y\}}(X_s)ds\right)\right]\\ &= E_a\left[\int_0^\infty e^{-\alpha s}\chi_{\{y\}}(X_s)\, ds\right]+\alpha E_a\left[\int_0^\infty e^{-\alpha s} E_{X_s}\left(\int_0^{\sigma^a} \chi_{\{y\}}(X_u)\, du\right) ds\right]\\ &= R_\alpha(a, y)+\alpha R_\alpha{}^aR(a, y) .\end{aligned}$$

---

1) The following proof is indepted to H. Tanaka and T. Watanabe.

Corollary. *The measure* μ *defined by*

$$\mu(y) = R_\alpha(a, y) + \alpha R_\alpha{}^a R(a, y) \tag{2.5}$$

is an invariant measure for $(P_t)_{t\geqq 0}$.

3. Weak potential operators

Let $(P_t)_{t\geqq 0}$ be a recurrent semi-group on S with an invariant measure μ. Further let $\boldsymbol{N}$ be the space of null charges for $(P_t)_{t\geqq 0}$ and $(R_\alpha)_{\alpha>0}$ the resolvent of $(P_t)_{t\geqq 0}$. We can easily show that the condition (W.P) of weak potential operator is equivalent to the condition:

(W.P′) $\quad (I-\alpha R_\alpha)Rf = R_\alpha f \quad$ for all $\quad f \in \boldsymbol{N}$ and $\alpha>0$.

In the first place we will prove the Dynkin formula for weak potential operator provided that it exists.

Lemma 6. *Let R be a weak potential operator for $(P_t)_{t\geqq 0}$ and X a Ray process associated with $(P_t)_{t\geqq 0}$. If τ is a Markov time such that $P_x(\tau<\infty)=1$ and $E_x\left(\int_0^\tau \chi_{\{y\}}(X_t)dt\right)<\infty$ for any $x, y \in S$, then, for each $f \in \boldsymbol{N}$ and for each $x \in S$, we have*

$$Rf(x) - E_x(Rf(X_\tau)) = E_x\left(\int_0^\tau (f(X_t)dt\right). \tag{3.1}$$

Proof. Let $f \in \boldsymbol{N}$ and $g = Rf$, then $g \in \boldsymbol{B}$ and $g = R_\alpha f + \alpha R_\alpha g$. Using this and the strong Markov property, we have

$$\begin{aligned} g(x) = E_x&\left(\int_0^\tau e^{-\alpha t} f(X_t)dt\right) + E_x(e^{-\alpha\tau} g(X_\tau)) \\ &+ E_x\left(\int_0^\tau \alpha e^{-\alpha t} g(X_t)dt\right) \end{aligned}$$

for x in S. Since f has finite support, we have

$$\lim_{\alpha\to 0} E_x\left(\int_0^\tau e^{-\alpha t} f(X_t)dt\right) = E_x\left(\int_0^\tau f(X_t)dt\right).$$

Further we obtain easily

$$|E_x(e^{-\alpha\tau} g(X_\tau) - E_x(g(X_\tau))| \leqq ||g||(1-E_x(e^{-\alpha\tau})),$$

$$\left|E_x\left(\int_0^\tau \alpha e^{-\alpha t} g(X_t)dt\right)\right| \leqq ||g||(1-E_x(e^{-\alpha\tau})),$$

where $||g|| = \sup_{x\in S} |g(x)|$. Therefore, letting $\alpha\to 0$, we have

$$g(x) = E_x\left(\int_0^\tau f(X_t)dt\right) + E_x(g(X_\tau)),$$

which implies (3.1).

Giving a few particular Markov times as τ in (3.1), we have some information about the weak potential operator.

EXAMPLE 1. Assume that all states in S are stable and conservative in the sense: $P_x(0<\tau^x<\infty)=1$ and $P_x(X_{\tau^x}\in S)=1$ for all $x\in S$. Let us introduce the function q and the kernel Π on S by

$$q(x) = (E_x(\tau^x))^{-1} \quad \text{and} \quad \Pi(x, y) = P_x(X_{\tau^x} = y)$$

respectively, then q is strictly positive and Π is a Markov kernel. It is familiar to us that the kernel D defined by

$$D(x, y) = q(x)(\Pi(x, y)-I(x, y)) \qquad \text{for} \quad (x, y)\in S\times S$$

plays the same role as Laplacian does in the classicial potential theorey. In this case if we set $\tau=\tau^x$ in (3.1), we have $DRf=-f$, which implies that for each $f\in \boldsymbol{N}$ the function $g=Rf$ is a bounded solution of the "Poisson equation" $Dg=-f$.

EXAMPLE 2. For some $a\in S$, if we set $\tau=\sigma^a$ in (3.1), then

$$Rf(x) = E_x\Big(\int_0^{\sigma^a} f(X_t)dt\Big)+Rf(a)$$

which implies that a weak potential operator R, if it exists, should have the form:

$$Rf = {}^aRf+l(f)\,,$$

where ${}^aR$ is a kernel on S defined by (2.3) and l is a linear functional (in the algebraic sense) on $\boldsymbol{N}$.

EXAMPLE 3. Let E be the set $\{f>0\}$ and set $\tau=\sigma^E$ in (3.1), then

$$\begin{aligned} Rf(x) &= E_x(Rf(X_{\sigma E}))+ E_x\Big(\int_0^{\sigma^E} f(X_t)\,dt\Big) \\ &\leqq E_x(Rf(X_{\sigma E}))\,. \end{aligned}$$

Consequently the weak potential operator statisfies a sort of maximum principle as follows: For any function f in $\boldsymbol{N}$ and any real number m, if $Rf\leqq m$ on the set $\{f>0\}$, then $Rf\leqq m$ on S.

We now prove a time continuous version of Orey's result.

Theorem 2. *Let $(P_t)_{t\geqq 0}$ be a recurrent semi-group with the space $\boldsymbol{N}$ of null charges and R a linear operator from $\boldsymbol{N}$ to $\boldsymbol{B}$, then R is a weak potential operator for $(P_t)_{t\geqq 0}$ if and only if it has the form*:

$$Rf = {}^aRf+l(f) \tag{3.2}$$

with some linear functional l on $\boldsymbol{N}$.

Proof. We have already seen in EXAMPLE 2 that a weak potential operator has the form (3.2), so we have only to prove that a linear operator defined by (3.2) is a weak potential operator for $(P_t)_{t\geqq 0}$. Let μ be an invariant measure of $(P_t)_{t\geqq 0}$ and X a Ray process associated with $(P_t)_{t\geqq 0}$. Let us introduce a family of kernels $({}^aR_\alpha)_{\alpha>0}$ by

$${}^aR_\alpha(x, y) = E_x\left(\int_0^{\sigma^a} e^{-\alpha t}\chi_{\{y\}}(X_t)dt\right) \qquad \text{for all} \quad (x, y)\in S\times S\,.$$

Since $\{t<\sigma^a\}\subseteqq\{\sigma^a=t+\sigma^a\circ\theta_t\}$ for all $t\geqq 0$, $({}^aR_\alpha)_{\alpha>0}$ satisfies the resolvent equations and

$$(3.3) \qquad \alpha\,{}^aR_\alpha{}^aR = {}^aR-{}^aR_\alpha \qquad \text{for all} \quad \alpha>0\,.$$

We obtain easily

$$R_\alpha(x, y) = {}^aR_\alpha(x, y)+E_x(e^{-\alpha\sigma^a})R_\alpha(a, y)$$

and in particular

$$R_\alpha(x, a) = E_x(e^{-\alpha\sigma^a})R_\alpha(a, a)\,,$$

therefore we have

$$(3.4) \qquad R_\alpha(x, y) = {}^aR_\alpha(x, y)+R_\alpha(x, a)R_\alpha(a, y)/R_\alpha(a, a)\,.$$

Combining (3.3) with (3.4), we obtain

$$(3.5) \qquad \begin{aligned}&(I-\alpha R_\alpha){}^aR(x, y)\\ &\quad = R_\alpha(x, y)-R_\alpha(x, a)[R_\alpha(a, y)+\alpha R_\alpha{}^aR(a, y)]/R_\alpha(a, a)\,,\end{aligned}$$

However, as we have seen in Theorem 1, $R_\alpha(a,\ )+\alpha R_\alpha{}^aR(a,\ )$ is an invariant measure of $(P_t)_{t\geqq 0}$, then, from the uniqueness of invariant measure we have

$$(3.6) \qquad (I-\alpha R_\alpha){}^aR(x, y) = R_\alpha(x, y)-R_\alpha(x, a)\mu(y)/\mu(a)\,.$$

Thus, if f is a null charge for $(P_t)_{t\geqq 0}$, we have

$$(I-\alpha R_\alpha){}^aRf = R_\alpha f\,,$$

which implies ${}^aR$ is a weak potential operator. Since $(I-\alpha R_\alpha)l(f)=0$, we have proved the theorem.

The next theorem shows that a recurrent semi-group is uniquely determined from the pair of its own invariant measure and weak potential operator, or, roughly speaking, that a weak potential operator contains a complete information for its recurrent semi-group.

Theorem 3. *Let $(P_t)_{t\geqq 0}$, $(\tilde{P}_t)_{t\geqq 0}$ be recurrent semi-groups on S with the invariant measures μ, $\tilde{\mu}$, the space of the null charges $\boldsymbol{N}$, $\tilde{\boldsymbol{N}}$ and the weak potential* *operators R, $\tilde{R}$ respectively. If $\tilde{\mu}=c\mu$ with some positive constant c (then $N=\tilde{N}$) and if, for all $f\in N$, $\tilde{R}f=Rf+l(f)$ with some linear functional l on N, then we have $\tilde{P}_t=P_t$ for all $t\geqq 0$.*

Proof. Let X and $\tilde{X}$ be Ray processes associated with $(P_t)_{t\geqq 0}$ and $(\tilde{P}_t)_{t\geqq 0}$ respectively. In the course of this proof, we shall denote the quantities related with $\tilde{X}$ by putting the sign "$\sim$" over the corresponding quantities related with X, for example

$$^a\tilde{R}_\alpha(x, y) = \tilde{E}_x\left(\int_0^{\tilde{\sigma}^a} e^{-\alpha t}\chi_{\{y\}}(\tilde{X}_t)dt\right),$$

where $\tilde{E}_x$ denotes the expectation with respect to $\tilde{X}$ and $\tilde{\sigma}^a$ is the first hitting time of $\{a\}$ with respect to $\tilde{X}$. Let us now introduce the function f_y for each $y\in S$, $y\neq a$, by

$$f_y(x) = \begin{cases} 1 & (x=y) \\ -\mu(y)/\mu(a) & (x=a) \\ 0 & (\text{otherwise}), \end{cases}$$

then $f_x\in N$. Therefore, using (3.2) and the assumption of the theorem, we obtain easily

$$^a\tilde{R}(x, y) = \tilde{R}f_y(x)-\tilde{R}f_y(a) = Rf_y(x)-Rf_y(a) = {}^aR(x, y)$$

for all $x\in S$. Evidently $^a\tilde{R}(x, a)={}^aR(x, a)=0$ for all $x\in S$, then we have $^a\tilde{R}={}^aR$. We remark here that the operator ^aR satirfies the complete maximam principle on $^aS=S\backslash\{a\}$, that is, if, for any function f with finite support in ^aS, we have $^aRf\leqq m$ on the set $\{f>0\}$ with some $m\geqq 0$, then we have $^aRf\leqq m$ on ^aS. Then, according to Deny [3] or Meyer [16, p. 205], the sub-Markov resolvent[2] $(^aR_\alpha)_{\alpha>0}$ satisfying the relation (3.3) is unique. Consequently we have $^a\tilde{R}_\alpha={}^aR_\alpha$ for all $\alpha>0$.

Let us now introduce the quantities e_α, λ_α by

$$e_\alpha(x) = 1-\alpha{}^aR_\alpha 1(x) \qquad (x\in S),$$
$$\lambda_\alpha(y) = \mu(y)-\alpha\mu{}^aR_\alpha(y) \qquad (y\in S).$$

Since μ is an invariant measure of $(P_t)_{t\geqq 0}$, we have $\alpha\mu R_\alpha=\mu$ for all $\alpha>0$. Then, multiplying $\alpha\mu(x)$ to the both side of (3.4) and summing up with respect to x over S, we have

$$\mu(y) = \alpha\mu{}^aR_\alpha(y)+\mu(a)R_\alpha(a, y)/R_\alpha(a, a) \tag{3.7}$$

for all $y\in S$. Therefore λ_α is a non-negative measure:

2) A family of kernels $(R_\alpha)_{\alpha>0}$ on S is called a sub-Markov resolvent if it satisfies: (R. 1) $R_\alpha\geqq 0$ and $\alpha R_\alpha 1\leqq 1$ for all $\alpha>0$, and (R. 2).

(3. 8) $$\lambda_\alpha(y) = \mu(a)R_\alpha(a, y)/R_\alpha(a, a) \qquad (y\in S)$$

with the total mass:

(3. 9) $$\langle\lambda_\alpha, 1\rangle = \mu(a)/\alpha R_\alpha(a, a)\,.$$

On the other hand, summing up the both side of (3.3) with respect to y over S, we have

$$1 = \alpha^a R_\alpha 1(x) + R_\alpha(x, a)/R_\alpha(a, a) \qquad (x\in S)\,,$$

consequently

(3. 10) $$e_\alpha(x) = R_\alpha(x, a)/R_\alpha(a, a) \qquad (x\in S)\,.$$

Combining (3.8), (3.9) and (3.10) with (3.3), we have

(3. 11) $$R_\alpha = {}^aR_\alpha + e_\alpha\otimes\lambda_\alpha/\alpha\langle\lambda_\alpha, 1\rangle$$

for all $\alpha>0$. It is easily verified that $\tilde{\lambda}_\alpha=c\lambda_\alpha$, $\tilde{e}_\alpha=e_\alpha$, then we have for all $\alpha>0$

$$\tilde{R}_\alpha = {}^a\tilde{R}_\alpha + \tilde{e}_\alpha\otimes\tilde{\lambda}_\alpha/\alpha\langle\tilde{\lambda}_\alpha, 1\rangle = {}^aR_\alpha + e_\alpha\otimes\lambda_\alpha/\alpha\langle\lambda_\alpha, 1\rangle = R_\alpha\,,$$

which implies $\tilde{P}_t=P_t$ for all $t\geqq 0$. Thus the theorem was proved.

4. Additional remarks

In the rest of this work we shall study some properties of the weak potential and apply them to the operator of the form:

(4. 1) $$R_0 f(x) = \lim_{t\to\infty}\int_0^t P_s f(x)ds \qquad (f\in \boldsymbol{N},\ x\in S)\,,$$

which is defined for some recurrent semi-group. The results in this section are counterparts in the continuous parameter case of Orey's results in [18, Section 1.].

Let $(P_t)_{t\geqq 0}$ be a recurrent semi-group with an invariant measure μ, the space of null charges $\boldsymbol{N}$ and a weak potential operator R.

Lemma 7. *R is non-singular in the sense as follows: For each null charge f, if Rf is equal to a constant on the support of f, then f is equal to zero on S.*

Proof. If $f\in\boldsymbol{N}$ and $Rf=c$ on the support of f, then, according to the maximum principle on R (see EXAMPLE 3 in the section 3), we have $Rf=c$ on S. Therefore from (W.P′) we have $R_\alpha f=0$ for all $\alpha>0$. Since $\lim_{\alpha\to\infty}\alpha R_\alpha f=f$, we have $f=0$.

In the following we shall denote by $\mathfrak{K}$ the set of all non-empty, finite subsets of S. For each $E\in\mathfrak{K}$, we shall use the following notations;

f_E The function restricted to E.
ν_E The measure restricted to E.
K_E The kernel restricted to $E\times E$.
$\boldsymbol{B}^E$ The space of functions with supports in E.
$\boldsymbol{B}_E$ The space of functions f_E.
$\boldsymbol{N}^E$ The space $\boldsymbol{N}\cap\boldsymbol{B}^E$.

Lemma 8. *For each weak potential operator R, we can find a family of (signed) measures $(\lambda^E)_{E\in\mathfrak{K}}$ with the following properties:* (λ. 1) *Each measure λ^E has the support in E.* (λ. 2) $\langle\lambda^E, 1\rangle=1$. ($\lambda$. 3) $\langle\lambda^E, Rf\rangle=0$ *for all $f\in\boldsymbol{N}^E$. And such a family is uniquely determined from R.*

Proof. If $E\in\mathfrak{K}$ and E contains exactly n elements, then the linear dimensions of $\boldsymbol{B}_E$ and $\boldsymbol{N}^E$ are n and $n-1$ respectively. Let us introduce a linear operator R_E from $\boldsymbol{N}^E$ to $\boldsymbol{B}_E$ by

$$R_Ef=(Rf)_E \qquad \text{for} \quad f\in\boldsymbol{N}^E. \tag{4.2}$$

If $f\in\boldsymbol{N}^E$ and $R_Ef=0$, then, according to Lemma 7, we have $f=0$, which implies the linear dimension of $R_E(\boldsymbol{N}^E)$ is equal to $\boldsymbol{N}^E$, that is, the linear dimension of the factor space $\boldsymbol{B}_E/R_E(\boldsymbol{N}^E)$ is equal to one. On the other hand, using again Lemma 7, we can easily show that 1_E does not belong to $R_E(\boldsymbol{N}^E)$. Therefore we can find exactly one linear functional l_E on $\boldsymbol{B}_E$ such that $\langle l_E, g_E\rangle=0$ for all $g_E\in R_E(\boldsymbol{N}^E)$ and $\langle l_E, 1_E\rangle=1$. If we define the measure λ^E by: $\lambda^E(y)=\langle l_E, (\chi_{\{y\}})_E\rangle$ for $y\in E$ and $\lambda^E(y)=0$ for $y\in S\backslash E$, then the family $(\lambda^E)_{E\in\mathfrak{K}}$ is the desired one.

The family $(\lambda^E)_{E\in\mathfrak{K}}$ was first introduced by Kemeny-Snell [12] to investigate normal chains and studied by Orey [18] in a more abstract way in the time discrete case.

Let X be a Ray process associated with $(P_t)_{t\geqq 0}$. For each $E\in\mathfrak{K}$, let us define the kernel H^E on S by

$$H^E(x, y)=P_x(X_{\sigma E}=y) \qquad ((x, y)\in S\times S),$$

then $H^E\geqq 0$ and $H^E1=1$, each measure $H^E(x, \ )$ has support in E and $H^EH^E=H^E$. Using $(\lambda^E)_{E\in\mathfrak{K}}$ and $(H^E)_{E\in\mathfrak{K}}$, we can characterize a weak potential in the next form:

Lemma 9. *A function g of $\boldsymbol{B}$ is a weak potential of null charge of $\boldsymbol{N}^E$ if and only if $\langle\lambda^E, g\rangle=0$ and $H^Eg=g$.*

Proof. Let $g=Rf$ with some $f\in\boldsymbol{N}^E$. Then from the definition of λ^E we have $\langle\lambda^E, g\rangle=\langle\lambda^E_E, R_Ef\rangle=0$ and, from Dynkin formula (3.1) for weak potential operator, we have easily $H^Eg=g$. Conversely if $\langle\lambda^E, g\rangle=0$ and $H^Eg=g$, we can find exactly one $f \in \boldsymbol{N}^E$ such that $g_E = R_E f$, since dim $(B_E/R_E(N^E))=1$. Therefore

$$g = H^E g = H^E R_E f = H^E R f = Rf .$$

REMARK. If $(P_t)_{t\geqq 0}$ is conservative, stable (see EXAMPLE 1 in the section 3) and minimal in the sense of Feller [6], then g is a weak potential of null charge of $\boldsymbol{N}^E$ if and only if $\langle \lambda^E, g \rangle = 0$ and $Dg=0$ in $S \backslash E$.

The next theorem corresponds to Theorem 1.2.5 of Orey[18].

Theorem 4. *Let $(P_t)_{t\geqq 0}$ be a recurrent semi-group with a weak potential operator R. Then $P_t Rf$ converges as $t\to\infty$ for every $f \in \boldsymbol{N}$ if and only if $P_t H^E g$ converges as $t\to\infty$ for every $E \in \mathfrak{K}$ and $g \in \boldsymbol{B}$. $P_t Rf$ will converges to 0 for all $f \in \boldsymbol{N}$ if and only if $\langle \lambda^E, g \rangle = \lim_{t\to\infty} P_t H^E g$ for all $E \in \mathfrak{K}$ and $g \in \boldsymbol{B}$, where $(\lambda^E)_{E\in\mathfrak{K}}$ is the family of measures introduced in Lemma* 8.

Proof. Let $g \in \boldsymbol{B}$ and $E \in \mathfrak{K}$. If we put $h = H^E g - \langle \lambda^E, g \rangle$, then $\langle \lambda^E, h \rangle = 0$ and $H^E h = h$, then, according to Lemma 9, we can find exactly one $f \in \boldsymbol{N}^E$ such that

$$H^E g - \langle \lambda^E, g \rangle = Rf . \tag{4.3}$$

Conversely, for aech $f \in \boldsymbol{N}^E$, if we put $g = Rf$, then $g \in \boldsymbol{B}$ and satisfies the relation (4.3). Since

$$P_t H^E g - \langle \lambda^E, g \rangle = P_t Rf \qquad \text{for all} \quad t \geqq 0 ,$$

the proof of the theorem is easily obtained.

The next theorem gives some information about the operator R_0 defined by (4.1).

Theorem 5. *For any recurrent semi-group $(P_t)_{t\geqq 0}$, the following two conditions are equivalent*:

(a) $\int_0^t P_s f(x)\, ds$ *converges as $t\to\infty$ for every $f \in \boldsymbol{N}$ and $x \in S$.*

(b) *$P_t H^E g$ converges as $t\to\infty$ for every $E \in \mathfrak{K}$ and $g \in \boldsymbol{B}$.*

If $(P_t)_{t\geqq 0}$ satisfies one of these conditions, then the operator R_0 defined by (4.1) *is a weak potential operator for $(P_t)_{t\geqq 0}$ and the family $(\lambda^E)_{E\in\mathfrak{K}}$ associated with R_0 is given by*

$$\langle \lambda^E, g \rangle = \lim_{t\to\infty} P_t H^E g \tag{4.4}$$

for all $E \in \mathfrak{K}$ and $g \in \boldsymbol{B}$.

Proof. We have seen in Theorem 2 that ${}^a R$ is a weak potential operator for $(P_t)_{t\geqq 0}$, then

$$(4.5) \qquad (I-P_t)^a Rf(x) = \int_0^t P_s f(x) ds$$

for every $f \in \boldsymbol{N}$, $x \in S$ and $t \geqq 0$. Therefore $\int_0^t P_s f(x) ds$ converges as $t \to \infty$ for every $f \in \boldsymbol{N}$ and $x \in S$ if and only if $P_t{}^a Rf$ converges as $t \to \infty$ for every $f \in \boldsymbol{N}$, which is equivalent to that $P_t H^E g$ converges as $t \to \infty$ for every $E \in \mathfrak{K}$ and $g \in B$ by Theorem 4. Consequently (a) and (b) are equivalent. Next let $(P_t)_{t \geqq 0}$ be a recurrent semi-group satisfying (a) or (b). Then, according to (4.5), the limit ${}^a g = \lim_{t \to \infty} P_t{}^a Rf$ exists for each $f \in \boldsymbol{N}$ and the function ${}^a g$ is bounded. However, since $P_t{}^a g = {}^a g$ for all $t \geqq 0$, it must be constant on S by Lemma 5, that is, the limit of $P_t{}^a Rf$ defines a linear functional l on $\boldsymbol{N}$. Therefore we have

$$R_0 f = {}^a Rf + l(f)$$

for all $f \in \boldsymbol{N}$, which shows that R_0 is a weak potential operator for $(P_t)_{t \geqq 0}$. Finally, using the relation:

$$(I-P_t) R_0 f(x) = \int_0^t P_s f(x) ds \qquad \text{for} \quad f \in \boldsymbol{N}, \quad x \in S,$$

we have $\lim_{t \to \infty} P_t R_0 f = 0$ for all $f \in \boldsymbol{N}$, which implies $\langle \lambda^E, g \rangle = \lim_{t \to \infty} P_t H^E g$ for all $E \in \mathfrak{K}$ and $g \in \boldsymbol{B}$. Thus the theorem was proved.

An irreducible recurrent semi-group $(P_t)_{t \geqq 0}$ is said to be positive or ergodic if it has a bounded invariant measure. We know that, for any ergodic semi-group $(P_t)_{t \geqq 0}$, the measure ν defined by:

$$\nu(y) = \lim_{t \to \infty} P_t(x, y) \qquad \text{for} \quad x, y \in S,$$

is an invariant probability measure (see [2, p. 178]). In this case we can easily prove that, for each $E \in \mathfrak{K}$ and $g \in \boldsymbol{B}$, $P_t H^E g$ converges to $\langle \nu H^E, g \rangle$ as $t \to \infty$, so we can define a weak potential operator R_0 by (4.1) and the family $(\lambda^E)_{E \in \mathfrak{K}}$ associated with R_0 is given by $\lambda^E = \nu H^E$.

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