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Constitutive Equation for Thermal Elastic-Plastic Creep State†

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Abstract

High-temperature structural components such as pressure vessels and pipes in nuclear reactors and chemical plants, are subjected to repeated loading due to the changes of internal pressure and temperature etc.. In recent years, it has become more important to perform more accurate theoretical analysis of mechanical behaviors of the above structures for rational design and for critical evaluation of safety. In this study, the thermal elastic-plastic creep theory represented by the authors will be further developed by introducing more general forms of workhardening rule and creep law respectively.

KEY WORDS: (Thermal Elastic-Plastic Creep) (Constitutive Equation) (Creep Constitutive Equation) (Workhardening Rule) (Finite Element Method)

1. Introduction

Pressure vessels, pipes and so forth which are the main structural components in nuclear reactors and chemical plants, are operated at high temperature and loaded cyclically due to the changes of internal pressure and temperature, etc.. For some components of these structures, stress relief annealing is applied in the process of construction. In recent years, it has become more important to perform more accurate theoretical analysis of the mechanical behaviors, including creep, of structural components and their materials to be used at high temperature for rational design and for critical evaluation of safety.

The authors have already presented the theory of thermal elastic-plastic creep analysis to study the mechanical behaviors of welded joints during welding and stress relief annealing[1)–4)]. Here, the thermal elastic-plastic creep theory will be further developed by introducing more general forms of workhardening rule and creep law respectively, according to the following procedure.

(1) To express creep constitutive equations at multiaxial stress state in a general form.

(2) To introduce the above creep constitutive equation into "Theory of Thermal Elastic-Plastic Analysis with A More General Workhardening Rule" represented by the authors[5)].

2. Creep Strain at Multiaxial Stress State

Creep behavior of metal is usually influenced by stress, temperature and changes of its internal structure. In this chapter, such creep behavior at multiaxial stress state will be expressed in such a general form of equation as to introduce it into the thermal elastic-plastic theory shown in Ref. (5).

2.1 Creep constitutive equation in uniaxial stress state (Creep hardening rule)[6)]

Creep constitutive equation of metal is usually expressed in a differential form for creep strain rate $\dot{\epsilon}^c$ which is regarded as a state function of stress σ, temperature T and suitable internal variables s_i ($i = 1, 2, \cdots, n$) which represent changes of its internal structure as follows.

$$\dot{\epsilon}^c = p(\sigma, s_1, s_2, \cdots, s_n, T) \tag{1}$$

$$\dot{s}_i = q_i(\sigma, s_1, s_2, \cdots, s_n, T) \quad (i=1,2,\cdots,n) \tag{2}$$

As it is considered that change of creep strain rate represents the hardening of the material, which progresses with deformation, the theory which rules change of creep strain rate due to change of the internal structure is called creep hardening rule. Based on this, unsteady-state creep constitutive equation can be formulated. Creep hardening rule is classified according to the kind of physical quantities to represent internal variables in Eq. (1).

One of such examples is creep strain ϵ^c. This is one of simple measures which represent change of internal structure, progressing with deformation. In this case, the constitutive equation of creep strain rate is expressed as

$$\dot{\epsilon}^c = p(\sigma, \epsilon^c, T) \tag{3}$$

As the above equation assumes that the hardening of the material is controlled by creep strain, this rule is called

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strain hardening rule. Many detailed examinations about its appropriateness have been conducted and strain hardening rule is considered superior to most of other classical creep hardening rules. Thus, this rule is adopted in many unsteady-state creep analyses.

2.2 Creep constitutive equation in multiaxial stress state

As it is very rare that stress state is uniaxial in actual engineering creep problems, it is necessary to expand the uniaxial creep constitutive equation described briefly in the previous section so as to express the one in multiaxial stress state. This expansion can be conducted by the same way as for the plastic constitutive equation, employing similar hypotheses introduced in plasticity. They are:

(1) no change of volume due to creep deformation
(2) no influence of hydrostatic stress on creep deformation

Appropriateness of the above hypotheses is confirmed for metals experimentally.

In similar to plastic strain increment, creep strain rate is expressed as a vector $\{\dot{\epsilon}^c\}$in a multiaxial state, and the direction and magnitude of the creep strain rate $\{\dot{\epsilon}^c\}$ should be determined. Here, the direction and magnitude of creep strain rate $\{\dot{\epsilon}^c\}$ are assumed to be treated separately. For the direction, as usual, it is assumed that the flow law (creep flow law) holds like the case of plastic deformation and the creep strain rate $\{\dot{\epsilon}^c\}$ is expressed as follows.

$$\{\dot{\epsilon}^c\} = \Lambda \left\{\frac{\partial g}{\partial(\sigma-\theta_c)}\right\} \tag{4}$$

In the above equation, Λ is a positive scalar coefficient, and g is a scalar function which depends on the total histories of stresses, temperature, etc. and is called the creep potential. Like the yield surface, $g=0$ represents a closed curved surface (the creep potential surface) in the stress space, which contains the current stress point, and $\{\theta_c\}$ is a vector which indicates the position of center of the creep potential surface (see **Fig. 1**). The creep strain rate $\{\dot{\epsilon}^c\}$ is expressed as a vector outward normal to the creep potential surface at the current stress point.

Here, Eq. (4) will be rewritten in the same form as Eq. (13) for plastic strain increment shown in Ref. (5), adopting a creep potential surface which may move in the stress space.

$$\{\dot{\epsilon}^c\} = \dot{\epsilon}_1^c \ \{n_c\} \tag{5}$$

where $\dot{\epsilon}_1^c$: the magnitude of the creep strain rate $\{\dot{\epsilon}^c\}$ (that is, the length of the vector $\{\dot{\epsilon}^c\}$)

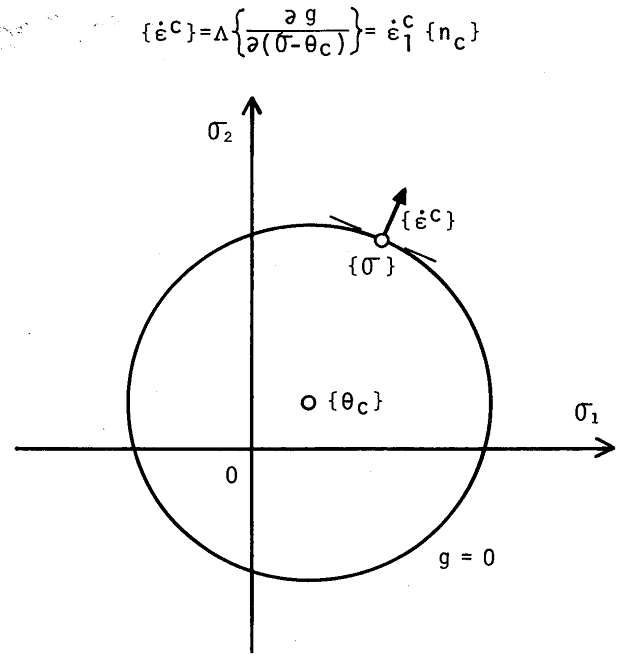

Fig. 1 Schematic Illustration of Relation between Creep Potential Surface g=0 and Creep Strain Rate $\{\dot{\epsilon}^c\}$

$\{n_c\}$: the unit vector outward normal to the creep potential surface at the current stress point

$$\{n_c\} = \left\{\frac{\partial g}{\partial(\sigma-\theta_c)}\right\} \Big/ g_1', \ g_1' = \left|\left\{\frac{\partial g}{\partial(\sigma-\theta_c)}\right\}\right| = \left(\left\{\frac{\partial g}{\partial(\sigma-\theta_c)}\right\}^T \left\{\frac{\partial g}{\partial(\sigma-\theta_c)}\right\}\right)^{\frac{1}{2}}$$

In the above equation, the magnitude $\dot{\epsilon}_1^c$ of creep strain rate $\{\dot{\epsilon}^c\}$ is expressed by the creep hardening rule (uniaxial creep constitutive equation) explained briefly in the previous section.

The characteristics of the creep potential surface depend on the creep characteristics of the metal, like the creep hardening rule. For the shape of the surface, von Mises type and Tresca type, for example, can be adopted, which are widely used for the yield surface. For translation of the creep potential surface in the stress space, Bailey's theory[7] and Orowan's theory[8], etc. may be applied to rule the translation rate $\{\dot{\theta}_c\}$ of center of the creep potential surface.

2.3 Creep strain increment

By introduction of the creep strain increment into the thermal elastic-plastic theory shown in Ref. (5), thermal elastic-plastic creep theory can be developed. A creep strain increment can be obtained by integrating the creep strain rate, described in the previous section, for each increment. The creep strain rate is usually a function of stresses, temperature, internal variables and their histories.

So, there are two procedures for the above integration. One is to integrate the creep strain rate, regarding it as constant in each time increment. Another is to integrate the rate with consideration of the changes of variables during each time increment. Here, the creep strain increment will be obtained by the former method which decreases the accuracy of the result in comparison with the latter, but the method is simple and can be applied for all creep theory (creep strain rate). That is, the creep strain increment $\{d\epsilon^c\}$ between t_i and t_{i+1} is calculated by multiplying the creep strain rate $\{\dot{\epsilon}^c\}$ at time t_i, by the time increment $dt = t_{i+1} - t_i$.

$$\{d\epsilon^c\} = \{\dot{\epsilon}^c\}\, dt \tag{6}$$

In the case where the creep strain increment is expressed as Eq. (6), it can be calculated immediately as the product of two known quantities. Therefore, even if the creep effect is taken into account in the thermal elastic-plastic theory, the theory including the creep effect does not become more complex than the thermal elastic-plastic one.

If it is necessary to obtain more accurate creep strain increment, the integration must be performed for a very small time increment or with consideration of the change of the creep strain rate. In the latter case, the method of the integration and the form of the creep strain increment derived by such accurate integration are usually different in each cases. The authors have derived the creep strain increments with consideration of the changes of the variables (stresses, creep constants) as accurately as possible for some comparatively simple creep theories (isotropic power hardening theory[1),2)], isotropic exponential hardening theory[2)], isotropic time hardening theory and isotropic strain hardening theory[3)]).

3. Basic Equations for Thermal Elastic-Plastic State

To develop the constitutive equations (the stress-total strain incremental relations) for thermal elastic-plastic creep state in the next chapter, the hypotheses and basic equations which were used for thermal elastic-plastic state and appeared in Ref. (5), will be reviewed briefly. (The same equation number as in Ref. (5) will be used, adding "P" at the head.)

(1) Thermal strain increment $\{d\epsilon^T\}$

$$\{d\epsilon^T\} = \{\alpha\}\, dT \tag{P-1}$$

where $\{\alpha\}$: instantaneous linear expansion coefficient (expansion or shrinkage due to both temperature change and transformation)

dT : temperature increment

(2) Incremental relation between stress and elastic strain

In the case where the elasticity matrix $[D^e]$ (containing the material properties) is dependent upon temperature,

$$\{d\sigma\} = [D^e_d]\,\{d\epsilon^e\} + \frac{d[D^e]}{dT}\, dT\, \{\epsilon^e\} \tag{P-6}$$

where $[D^e_d] = [D^e] + \dfrac{d[D^e]}{dT}\, dT$

(3) Workhardening rule and characteristics of yield surface (yield function)

The usual combined rule of isotropic and kinematic workhardening is adopted and it is expanded so as to include the effect of temperature changes. In this case, the yield surface is expressed as follows.

$$f(\sigma_{ij} - \theta_{ij}, \sigma_0) = 0 \tag{P-8}$$

where $\{\theta\}$: the translation vector which indicates the position of center of the yield surface in the stress space

σ_0 : the measure of the size of the yield surface

$$\sigma_0 = \sigma_0(\epsilon^p_1, T) \tag{P-10}$$

$$d\sigma_0 = \frac{\partial \sigma_0}{\partial \epsilon^p_1}\, d\epsilon^p_1 + \frac{\partial \sigma_0}{\partial T}\, dT$$

where ϵ^p_1 : the length of the locus of the plastic strain

$d\epsilon^p_1$: the length of the vector of the plastic strain increment (see Eq. (P-13))

$$\epsilon^p_1 = \int d\epsilon^p_1$$

$$\{d\theta\} = k\, d\epsilon^p_1\, \{n_\theta\} \tag{P-11}$$

where k : proportional coefficient (the value can be calculated by Eq. (P-18))

$\{n_\theta\}$: the unit vector which indicates the direction of translation increment of the yield surface

(4) Plastic strain increment $\{d\epsilon^p\}$

Plastic strain increment is expressed by separately indicating its direction and magnitude. The direction is defined in the following form by introducing the yield function (f of Eq. (P-8)) as the plastic potential.

$$\{d\epsilon^p\} = d\epsilon^p_1\, \{n\} \tag{P-13}$$

where $d\epsilon^p_1$: the magnitude of the plastic strain increment $\{d\epsilon^p\}$

$\{n\}$: the unit vector outward normal to the yield surface at the current stress point

$$\{n\} = \left\{\frac{\partial f}{\partial(\sigma-\theta)}\right\} / f_1', \quad f_1' = \left|\left\{\frac{\partial f}{\partial(\sigma-\theta)}\right\}\right|$$

$$= \left(\left\{\frac{\partial f}{\partial(\sigma-\theta)}\right\}^T \left\{\frac{\partial f}{\partial(\sigma-\theta)}\right\}\right)^{\frac{1}{2}}$$

(5) Increment of yield function df

The following condition must be satisfied in the case where the material is under loading in the plastic range.

$$df = 0 \tag{P-14}$$

(6) Workhardening modulus H

$$H = k\, n_{\theta f} - f_1'^{-1} \frac{\partial f}{\partial \sigma_0} \frac{\partial \sigma_0}{\partial \epsilon_1^p} \tag{P-18}$$

where $n_{\theta f} = \{n\}^T \{n_\theta\}$

$$d\sigma_f = H\, d\epsilon_1^p - f_1'^{-1} \frac{\partial f}{\partial \sigma_0} \frac{\partial \sigma_0}{\partial T} dT \tag{P-19}$$

where $d\sigma_f = \{n\}^T \{d\sigma\}$

4. Constitutive Equations for Thermal Elastic-Plastic Creep State

Though the interaction between plasticity and creep have been investigated, the existing various theories are still under discussion. Then, it is assumed in this study that plasticity and creep are independent phenomena and there is no interaction between them so that plastic strain and creep strain are defined separately. Accordingly, the creep theory described in chapter 2 and the theory of plasticity reviewed in chapter 3 can be adopted together with no modification.

4.1 Thermal elastic creep constitutive equation

In the case where the current stress point is inside the yield surface, that is, the material is in the elastic range, accompanying temperature changes and creep strains, the total strain increment $\{d\epsilon\}$ is represented as the summation of the thermal strain increment $\{d\epsilon^T\}$, elastic strain increment $\{d\epsilon^e\}$ and creep strain increment $\{d\epsilon^c\}$.

$$\{d\epsilon\} = \{d\epsilon^T\} + \{d\epsilon^e\} + \{d\epsilon^c\} \tag{7}$$

The constitutive equation (the stress-total strain incremental relation) for this state will be obtained by using Eq. (P-6) and transforming its right side in the following way.

(1) To replace the elastic strain increment $\{d\epsilon^e\}$ by the total strain increment $\{d\epsilon\}$ etc., using Eq. (7).

(2) To express the thermal strain increment $\{d\epsilon^T\}$ by Eq. (P-1) and the creep strain increment $\{d\epsilon^c\}$ by Eq. (6).

After the above transformations, rearrangement of the right side of Eq. (P-6) provides the following thermal elastic creep constitutive equation,

$$\{d\sigma\} = [D_d^e]\{d\epsilon\} - [D_d^e]\left\{\left(\{\alpha\} - [D_d^e]^{-1}\frac{d[D^e]}{dT}\{\epsilon^e\}\right)dT + \{\dot{\epsilon}^c\}dt\right\} \tag{8}$$

This equation is the same as for thermal elastic state shown in Ref. (5) except that the term, $-[D_d^e]\{\dot{\epsilon}^c\}\,dt$, is supplemented on the right side.

4.2 Thermal elastic-plastic creep constitutive equation

In the plastic range, the total strain increment $\{d\epsilon\}$ is expressed by the summation of the components as

$$\{d\epsilon\} = \{d\epsilon^T\} + \{d\epsilon^e\} + \{d\epsilon^p\} + \{d\epsilon^c\} \tag{9}$$

First, the relationship between the magnitude $d\epsilon_1^p$ of the plastic strain increment $\{d\epsilon^p\}$ and the total strain increment $\{d\epsilon\}$ will be derived, based on Eq. (P-14) for the loading condition in the plastic range. In the case where the yield surface and the changes of its size and position are expressed by Eq. (P-8), Eq. (P-10) and Eq. (P-11) respectively, and there is the relationship of Eq. (P-18) between the workhardening modulus and them, Eq. (P-14) is written as

$$0 = df = \left\{\frac{\partial f}{\partial(\sigma-\theta)}\right\}^T \{d(\sigma-\theta)\} + \frac{\partial f}{\partial \sigma_0} d\sigma_0$$

$$= f_1'\{n\}^T\{d\sigma\} - f_1'\, k\, n_{\theta f}\, d\epsilon_1^p + \frac{\partial f}{\partial \sigma_0}\frac{\partial \sigma_0}{\partial \epsilon_1^p} d\epsilon_1^p + \frac{\partial f}{\partial \sigma_0}\frac{\partial \sigma_0}{\partial T} dT$$

$$= f_1'\{n\}^T\{d\sigma\} - f_1' H\, d\epsilon_1^p + \frac{\partial f}{\partial \sigma_0}\frac{\partial \sigma_0}{\partial T} dT \tag{10}$$

where $f_1' = \left|\left\{\frac{\partial f}{\partial(\sigma-\theta)}\right\}\right| = \left(\left\{\frac{\partial f}{\partial(\sigma-\theta)}\right\}^T \left\{\frac{\partial f}{\partial(\sigma-\theta)}\right\}\right)^{\frac{1}{2}}$

$$\{n\} = \left\{\frac{\partial f}{\partial(\sigma-\theta)}\right\} / f_1'$$

$$n_{\theta f} = \{n\}^T\{n_\theta\}$$

The above equation will be further transformed according to the following procedure.

(1) To substitute Eq. (P-6) into the stress increment $\{d\sigma\}$ which appears in the first term of the right side of Eq. (10) and express in terms of the elastic strain

increment $\{d\epsilon^e\}$ etc..

(2) To replace the elastic strain increment $\{d\epsilon^e\}$ by the total strain increment $\{d\epsilon\}$ etc., using Eq. (9).

(3) To express the thermal strain increment $\{d\epsilon^T\}$ by Eq. (P-1), the plastic strain increment $\{d\epsilon^p\}$ by Eq. (P-13) and the creep strain increment $\{d\epsilon^c\}$ by Eq. (6).

As a result of this manipulation, Eq. (10) is expressed by two unknown quantities which are the total strain increment $\{d\epsilon\}$ and the magnitude $d\epsilon_1^p$ of the plastic strain increment $\{d\epsilon^p\}$. Rearrangement of the equation provides the relationship between $\{d\epsilon\}$ and $d\epsilon_1^p$ as

$$d\epsilon_1^p = \left[\{n\}^T [D_d^e] \{d\epsilon\} - \left[\left\{ \{n\}^T [D_d^e] \left(\{\alpha\} - [D_d^e]^{-1} \frac{d[D^e]}{dT} \{\epsilon^e\} \right) - f_1'^{-1} \frac{\partial f}{\partial \sigma_0} \frac{\partial \sigma_0}{\partial T} \right\} dT + \{n\}^T [D_d^e] \{\dot{\epsilon}^c\} dt \right] \right] \Big/ S \qquad (11)$$

where $S = \{n\}^T [D_d^e] \{n\} + H$

The above equation is the same as for thermal elastic-plastic state shown in Ref. (5) except that the term, $-\{n\}^T [D_d^e] \{\dot{\epsilon}^c\} dt$, is added in the numerator.

Next, the constitutive equation (the stress-total strain incremental relation) will be derived. Based on Eq. (P-6) which represents the relationship between the stress increment $\{d\sigma\}$ and the elastic strain increment $\{d\epsilon^e\}$, its right side will be transformed as follows.

(1) To replace the elastic strain increment $\{d\epsilon^e\}$ by the total strain increment $\{d\epsilon\}$ etc., using Eq. (9).

(2) To express the thermal strain increment $\{d\epsilon^T\}$ by Eq. (P-1), the plastic strain increment $\{d\epsilon^p\}$ by Eq. (P-13) and the creep strain increment $\{d\epsilon^c\}$ by Eq. (6). Further, replace the magnitude $d\epsilon_1^p$ of $\{d\epsilon^p\}$ by $\{d\epsilon\}$ etc., using Eq. (11).

As a result of the above calculations, only the total strain increment $\{d\epsilon\}$ remains as an unknown on the right side of Eq. (P-6), that is, Eq. (P-6) becomes the incremental equation representing the relationship between the stress increment $\{d\sigma\}$ and the total strain increment $\{d\epsilon\}$. Rearrangement of the right side and separation of the expression into terms relating to the total strain increment $\{d\epsilon\}$ and the other terms furnish the following thermal elastic-plastic creep constitutive equation.

$$\{d\sigma\} = [D_d^p] \{d\epsilon\} - \left[\left\{ [D_d^p] \left(\{\alpha\} - [D_d^e]^{-1} \frac{d[D^e]}{dT} \{\epsilon^e\} \right) + [D_d^e] \{n\} f_1'^{-1} \times \frac{\partial f}{\partial \sigma_0} \frac{\partial \sigma_0}{\partial T} \Big/ S \right\} dT + [D_d^p] \{\dot{\epsilon}^c\} dt \right] \qquad (12)$$

where $[D_d^p] = [D_d^e] - [D_d^e] \{n\} \{n\}^T [D_d^e] \big/ S$

The above equation is the same as for the thermal elastic-plastic state shown in Ref. (5) except that the term, $-[D_d^p] \{\dot{\epsilon}^c\} dt$, is added on the right side.

4.3 Application of the finite element method

When very simple creep problems such as in one dimension are dealt, the theoretical analysis may be performed by using only the constitutive equation. However, the actual problems are usually in two or three dimension and some analytical method should be employed, such as the finite element method which is a very powerful tool. When the finite element method is applied, the basic equations should be derived, introducing the constitutive equation. The procedure to derive the basic equations is the same as for thermal elastic-plastic state described in Ref. (5).

5. Concluding Remarks

In this study, the creep constitutive equation in multiaxial stress state is expressed in a general form and is introduced into "Theory of Thermal Elastic-Plastic Analysis with A More General Workhardening Rule" [5], assuming that there is no interaction between plasticity and creep. As a result, it is possible to adopt many of complex workhardening rules and creep laws. For actual analysis of creep problems except simple cases, some analytical method should be used. When the finite element method is applied, the basic theory of thermal elastic-plastic creep analysis can be formulated with the aid of the constitutive equation, taking the same procedure as for thermal elastic-plastic analysis in Ref. (5).

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