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ON THE EXISTENCE OF A HARMONIC MAPPING

BY

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Introduction. Let C be a simple closed Jordan curve in the 3-dimensional Euclidean (x_1, x_2, x_3)-space. Any surface R with the contour C can be represented parametrically by a vector $\mathfrak{X}(u, v)$ with components x_1, x_2, x_3, given as functions of two parameters u, v which vary over a domain B of the (u, v)-plane bounded by a curve ∂B.

The surface R is said to be harmonic when the vector $\mathfrak{X}(u, v)$ satisfies the Laplace differential equation

$$\Delta\mathfrak{X} = \frac{\partial^2\mathfrak{X}}{\partial u^2}+\frac{\partial^2\mathfrak{X}}{\partial v^2} = 0\,,$$

or

$$\Delta x_j = 0\,, \quad j = 1, 2, 3\,.$$

The harmonic functions $x_j(u, v)$ give rise to analytic functions

$$H_j(w) = x_j+i\tilde{x}_j$$

of the complex variable $w=u+iv$, where $\tilde{x}_j(u, v)$ is conjugate harmonic to $x_j(u, v)$. Then by the Cauchy-Riemann relations we get

$$\Phi(w) = \sum_{j=1}^{3}[H'_j(w)]^2 = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial u}-i\frac{\partial x_j}{\partial v}\right)^2 = g_{11}-g_{22}-2ig_{12}\,,$$

where

$$g_{11} = |\mathfrak{X}_u|^2 = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial u}\right)^2, \quad g_{22} = |\mathfrak{X}_v|^2 = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial v}\right)^2,$$

$$g_{12} = (\mathfrak{X}_u,\ \mathfrak{X}_v) = \sum_{j=1}^{3}\frac{\partial x_j}{\partial u}\frac{\partial x_j}{\partial v}$$

is the first fundamental quantities of the surface R. Hence $\Phi(w)=g_{11}-g_{22}-2ig_{12}$ is a holomorphic function of the complex variable $w=u+iv$. A harmonic surface solves the least area problem of Plateau, only when $\Phi(w)$ vanishes identically (cf. Courant [3], pp. 96–118). (Numbers in the square brackets refer to the bibliography at the end of this paper.)

These circumstances will naturally lead to the following definition for *harmonic mapping* from one Riemann surface onto another:

Let R and R' be two topologically equivalent Riemann surfaces and let $\mathfrak{q}=f(\mathfrak{p})$ ($\mathfrak{p}\in R$, $\mathfrak{q}\in R'$) a smooth homeomorphism between them. z and w denote local coordinates at $\mathfrak{p}$ and $\mathfrak{q}$ respectively. Suppose given a conformal metric $\eta=ds_{\mathfrak{q}}^2=\rho(w)|dw|^2$ on R'; $\rho(w)>0$ is continuous in w and $\rho(w)|dw|^2$ rests invariant under any conformal transformations of the local coordinate w. The line element $ds_{\mathfrak{q}}$ at a point $\mathfrak{q}$ on R' is written in several forms:

$$\begin{aligned} ds_{\mathfrak{q}}^2 &= \rho(w)|dw|^2 = \rho(w(z))\left|\frac{\partial w}{\partial z}dz+\frac{\partial w}{\partial \bar{z}}d\bar{z}\right|^2 \\ &= \rho(w(z))\left(\left|\frac{\partial w}{\partial z}\right|^2+\left|\frac{\partial w}{\partial \bar{z}}\right|^2\right)|dz|^2+2\rho(w(z))\Re\left\{\frac{\partial w}{\partial z}\left(\overline{\frac{\partial w}{\partial \bar{z}}}\right)dz^2\right\} \\ &= g_{11}dx^2+2g_{12}dx\,dy+g_{22}dy^2 \\ &= \left(\frac{g_{11}+g_{22}}{2}\right)|dz|^2+\frac{1}{2}\Re\{(g_{11}-g_{22}-2ig_{12})dz^2\}\ , \end{aligned}$$

where

$$g_{11} = \rho(w(z))\left|\frac{\partial w}{\partial z}+\frac{\partial w}{\partial \bar{z}}\right|^2, \quad g_{22} = \rho(w(z))\left|\frac{\partial w}{\partial z}-\frac{\partial w}{\partial \bar{z}}\right|^2,$$

$$g_{12} = -2\rho(w(z))\Im\left\{\frac{\partial w}{\partial z}\left(\overline{\frac{\partial w}{\partial \bar{z}}}\right)\right\}.$$

The mapping $\mathfrak{q}=f(\mathfrak{p})$ is called *harmonic* relative to this conformal metric $\eta=\rho(w)|dw|^2$, if $[(g_{11}-g_{22})-2ig_{12}]dz^2$ is a regular analytic quadratic differential on R.

The present paper is concerned with the existence of a schlicht harmonic mapping from a compact Riemann surface R onto another such R' homotopic to a given homeomorphism. Our aim is to give a detailed proof of the existence by using the method of calculus of variations and solving an extremum problem in the theory of functions. The method employed here may be said direct in such a sense that no use is made of automorphic functions as well as in the manner in which the variational problems are treated. The basic idea in our formulation consists in establishing a *complex variant* of the Dirichlet Principle. The author is very much indebted for the motivation of this article to the suggestive work of Gerstenhaber-Rauch [6]. (The résumé of this article has been announced in [8].)

Preliminaries

Let R and R' be two closed (i.e. compact) Riemann surfaces with the same positive genus. Let f be an arbitrary topological mapping from R to R'. Let z be some local coordinate defined near a point $\mathfrak{p}_0 \in R$, w some local coordinate defined near $f(\mathfrak{p}_0) \in R'$ and $w = w(z)$ the function defined by the mapping f. We call $w(z)$ a *local realization* of f in the neighbourhood of $\mathfrak{p}_0$. Let $|z| < 1$ and $|w| < 1$ be respective local coordinate disks corresponding to the neighbourhoods $\mathfrak{U}(\mathfrak{p}_0)$ of $\mathfrak{p}_0 \in R$ and $\mathfrak{U}(f(\mathfrak{p}_0))$ of $f(\mathfrak{p}_0) \in R'$ in question. $w(z)$ is a topological mapping from $|z| < 1$ to $|w| < 1$. We agree to say that $w(z)$ has the property (Q), when it fulfills the two conditions in $|z| < 1$:

1) *$w(z)$ is L^2-derivable in the sense of Sobolev-Friedrichs* (*cf. Bers* [2], *Friedrichs* [4]), *and* 2) *absolutely continuous in 2-dimensional sense.*

This property (Q) is invariant under any conformal transformations of both local coordinates z and w. So it is meaningful to say that the mapping f has the property (Q) at a point $\mathfrak{p}_0$ on R, when a local realization $w(z)$ of f has this property in some neighbourhood of $\mathfrak{p}_0$; f is said to have the property (Q) on the whole surface R, if it is so at every point of R.

Consider the class $\mathbf{A}$ of all the homeomorphisms which are homotopic to a given topological mapping $\mathfrak{q} = \chi(\mathfrak{p})$ from R to R'. Then $\mathbf{A}$ contains at least one quasi-conformal mapping (cf. Teichmüller [10], pp. 28–34), the maximal dilatation of which we denote by K.

Suppose fixed on R' a conformal metric $\eta = ds^2 = \rho(w)|dw|^2$ where $\rho(w)$ is a positive continuous function in a local coordinate w.

Let $\bigcup_{j=1}^{k} \mathfrak{N}_j$ be a system of arbitrarily fixed local corodinate neighbourhoods covering R. Suppose that the surface R is furnished with a triangulation Σ in the following fashion: Σ is the collection of a finite number of non-overlapping singular 2-simplices S_j $(j = 1, 2, \cdots, \kappa)$ which are closed subsets of the 2-dimensional manifold R, and whose boundary 1-simplices are composed of analytic arcs, such that any point $\mathfrak{p}$ of R belongs to at least one of the 2-simplices S_j (i.e., $\mathfrak{p}$ may be in the interior, on the sides of S_j or coincides with one of the vertices of S_j) $(j = 1, 2, \cdots, \kappa)$ and that each S_j is necessarily comprised in the interior of at least one of the local coordinate neighbourhoods $\mathfrak{N}_j$ $(j = 1, 2, \cdots, k)$.

Denote by $|z| \leq 1$ and $|z'| \leq 1$ the local coordinate disks corresponding to arbitrary mutually adjacent singular simplices S_l and S_{l+1} respectively. Let $\mathfrak{p}_0$ be an arbitrary point in the singular 1-simplex $S_l \cap S_{l+1}$ with the coordinates ζ and ζ' relative to each disk considered. $S_l \cap S_{l+1}$ will appear on $|z| = 1$ as an arc $\widehat{h_1, h_2}$ and on $|z'| = 1$ as $\widehat{h'_1, h'_2}$. Suppose that a point $\mathfrak{p}$ approaches $\mathfrak{p}_0$ non-tangentially from interior of S_j $(j = l, l+1)$. Then $z(\mathfrak{p})$ in $|z| < 1$ (resp. $z'(\mathfrak{p})$ in $|z'| < 1$) approaches ζ (resp. ζ') non-tangentially. Let ω be a measurable linear differential on R and let $\omega=\tau(z)dz=\tau'(z')dz'$ in the respective local coordinates for S_l and S_{l+1} above mentioned ($l=1, 2,\cdots, \kappa-1$). We call ω to be *locally holomorphic* on R if the following conditions are fulfilled:

(i) ω is holomorphic and non-vanishing in the interior of every simplex S_j ($j=1, 2,\cdots, \kappa$),

(ii) $\begin{cases} |\tau(z)| \text{ has a non-tangential limit } |\tau(\zeta)|=\lim|\tau(z(\mathfrak{p}))| \text{ as } z(\mathfrak{p})\to\zeta(\mathfrak{p}_0) \\ |\tau'(z')| \text{ has a non-tangential limit } |\tau'(\zeta')|=\lim|\tau'(z'(\mathfrak{p}))| \text{ as } z'(\mathfrak{p})\to\zeta'(\mathfrak{p}_0) \end{cases}$

except possibly for a set of linear measure zero on $S_l\cap S_{l+1}$ ($l=1, 2,\cdots, \kappa-1$),

(iii) $|\tau(\zeta(\mathfrak{p}_0))|\,|d\zeta(\mathfrak{p}_0)|=|\tau'(\zeta'(\mathfrak{p}_0))|\,|d\zeta'(\mathfrak{p}_0)|$ for a.a. $\mathfrak{p}_0\in\bigcup_{j=1}^{\kappa}\partial S_j$,

(iv) $$\sum_{j=1}^{\kappa}\int_{\partial S_j}|\omega|\leq M,$$

M being a constant greater than 1.

Let us denote by Ω the set of all the locally holomorphic differentials $\omega=\tau(z)dz$ which are square-summable on R and normalized by the condition

$$||\omega||^2=(\omega, *\omega)=\iint_R|\tau(z)|^2|dz\wedge d\bar{z}|=1.$$

Here $*\omega$ shall denote the differential conjugate to ω and $|dz\wedge d\bar{z}|$ the absolute value of vector product $dz\wedge d\bar{z}$ of the 2-vectors dz and $d\bar{z}$.

Plainly $\Omega\neq\phi$, since one can construct a linear differential $\omega=\tau(z)dz$ on R, by Poisson integral, which is holomorphic in each $\operatorname{int} S_j$ and $|\omega|=|\tau(z)|\,|dz|$ is continuous on the whole R ($j=1, 2,\cdots, \kappa$), where $\operatorname{int} S_j$ stands for the set of all the interior points of S_j.

We begin with two lemmas for the convenience in the sequel.

Lemma 1. *The family Ω is normal in each singular simplex of Σ.*

Proof. Let $|z|<1$ be a local coordinate disk, which is a topological image of the interior of a simplex $S_j\in\Sigma$. Let $\{\omega_n\}_{n=1}^{\infty}$ be an arbitrary sequence chosen out of Ω. Set

$$T_n(z)=\int_0^z\tau_n(\zeta)d\zeta,\quad \tau_n(z)dz=\omega_n$$

for $|z|<1$. Then $\{T_n(z)\}_{n=1}^{\infty}$ is equicontinuous in $|z|<1$. For, otherwise, there exist a constant $c>0$, a pair of sequences $\{z'_\nu\}_{\nu=1}^{\infty}$, $\{z''_\nu\}_{\nu=1}^{\infty}$ in $|z|<1$ and a subsequence $\{T_{n_\nu}(z)\}_{\nu=1}^{\infty}$, such that $|z''_\nu - z'_\nu| \to 0$ as $\nu \to \infty$, while

$$|T_{n_\nu}(z''_\nu) - T_{n_\nu}(z'_\nu)| \geq c, \quad \nu = 1, 2, \cdots.$$

Let us denote by z_0 one of the accumulation points of $\{z'_\nu\}_{\nu=1}^{\infty}$. Our proof dispenses with the case when $|z_0|<1$, since it is trivial. Assume $|z_0|=1$. Draw a circular arc C_r with centre z_0 and with radius $r<1$ on $|z|\leq 1$. However small $\varepsilon>0$ may be, there can be found a pair of points z'_ν and z''_ν separated from $z=0$ by C_r, such that

$$|T_{n_\nu}(z'_\nu) - T_{n_\nu}(z''_\nu)| \geq c.$$

Hence the length $L(r)$ of the image curve $T_{n_\nu}(C_r)$ $(\varepsilon \leq r < 1)$ is not smaller than c. Thus the well-known length-area principle for holomorphic functions will lead to a contradiction:

$$c^2 \leq [L(r)]^2 \leq 2\pi r \int_0^{2\pi} |\tau_{n_\nu}(re^{i\theta})|^2 r d\theta,$$

whence

$$\int_\varepsilon^1 \frac{dr}{r} \leq \frac{2\pi}{c^2} \iint_{|z|<1} |\tau_{n_\nu}(re^{i\theta})|^2 r d\theta dr.$$

On the other hand, the right-hand member is uniformly bounded on account of the normalization.

Therefore $\{T_n(z)\}_{n=1}^{\infty}$ contains a subsequence, say again $\{T_n(z)\}_{n=1}^{\infty}$, which converges on every compact set in $|z|<1$. Accordingly $\{\tau_n(z)\}_{n=1}^{\infty}$ is uniformly convergent there. Repeating this process on every simplex S_j $(j=1, 2, \cdots, \kappa)$ we finally get a sequence of 1-forms belonging to Ω, whose coefficients constitute a uniformly convergent sequence on every compact comprised in arbitrarily fixed local coordinate disk corresponding to S_j $(j=1, 2, \cdots, \kappa)$, determining a differential holomorphic in $R - \bigcup_{j=1}^{\kappa} S_j$.

REMARK 1. It is easily seen that $\{T_n(z)\}_{n=1}^{\infty}$ is normal on $|z|\leq 1$ too.

Lemma 2. *It is possible to introduce the distance between two points on R by means of any ω belonging to Ω.*

Proof. We integrate an arbitrary holomorphic differential $\tau(z)dz=\omega$

along a path in a simplex S_j: we set $T(z)=\int_0^z \tau(\zeta)d\zeta$ for $|z|<1$. Then by Lemma 1, $T(z)$ is uniformly continuous in $|z|<1$. Hence we see that $T(z)$ is continuously prolongable up to $|z|\leq 1$. In fact, let ζ be an arbitrary point on $|z|=1$ and let $\{z_n\}_{n=1}^{\infty}$ an arbitrary sequence in $|z|<1$ tending to ζ. Given any $\varepsilon>0$, there is $\delta_1=\delta_1(\varepsilon)$ depending only on ε, such that

$$(1) \qquad |T(z)-T(z')|<\varepsilon \quad \text{so far as } |z-z'|<\delta_1$$

by uniform continuity of $T(z)$. Taking $n_0=n_0(\delta_1)$ sufficiently large, we shall have $|\zeta-z_n|<\delta_1/2$ $(n\geq n_0)$ and so

$$(2) \qquad |z_n-z_m|\leq |z_n-\zeta|+|\zeta-z_m|<\delta_1, \qquad n, m\geq n_0\,.$$

From (1), (2) it follows that $|T(z_n)-T(z_m)|<\varepsilon$ $(n, m\geq n_0)$, which implies the existence of $Z=\lim_{n\to\infty} T(z_n)$. Z depends only on ζ: let another sequence $\{z'_n\}_{n=1}^{\infty}$ tend to ζ. Then $Z'=\lim_{n\to\infty} T(z'_n)$ exists. Since the mixed sequence $z_1, z'_1, z_2, z'_2,\cdots, z_n, z'_n, z_{n+1},\cdots$ also tends to ζ, $T(z_1)$, $T(z'_1)$, $T(z_2)$, $T(z'_2)$, $\cdots, T(z_n)$, $T(z'_n)$, $T(z_{n+1}),\cdots$ has a limit, say Z''. $\{T(z_n)\}_{n=1}^{\infty}$, as a subsequence of the last convergent sequence, must have the same limit; $Z=Z''$. In analogy $Z'=Z''$. Thus we can determine the local distance on each singular simplex by means of this integral of ω, although the line elements may fail to be defined on a certain sebset of $\bigcup_{j=1}^{\kappa} \partial S_j$.

Formulation and solution of the first extremum problem

Let $\mathfrak{F}_\omega$ be the family of mappings f satisfying the following five condition (I)~(V), which we shall admit as the concurrence mapping to our first extremum problem:

(I) *f is a sense-preserving topological mapping from R to R'.*
(II) *f belongs to the given homotopy class* **A**.
(III) *f^{-1} as well as f, possess the property (Q).*

Let $w=w(z)$ be a local realization of the $f\in\mathfrak{F}_\omega$ defined in a local coordinate disk $|z|^2=x^2+y^2<1$ taking on all the values in $|w|<1$ and let S be a rectangle $\{(x, y)\,|\,a_1\leq x\leq a_2,\ b_1\leq y\leq b_2\}$ comprised in this disk. On account of the condition (III), $w(x+iy_0)$ is absolutely continuous in x on the interval (a_1, a_2) for almost all y_0 belonging to the interval (b_1, b_2)

and $w(x_0+iy)$ is absolutely continuous in y on the interval (b_1, b_2) for almost all x_0 belonging to the interval (a_1, a_2) (cf. e.g., Bers [2]). Hence $\partial w/\partial z=(\partial w/\partial x)-i(\partial w/\partial y)$ and $\partial w/\partial \bar{z}=(\partial w/\partial x)+i(\partial w/\partial y)$ are finitely defined almost everywhere on S. Then our fourth requirement is

$$\text{(IV)} \qquad \iint_R \rho(w(z))\left(\left|\frac{\partial w}{\partial z}\right|^2+\left|\frac{\partial w}{\partial \bar{z}}\right|^2\right)|dz\wedge d\bar{z}|\leq\frac{1}{2}\left(K+\frac{1}{K}\right)$$

under the normalization

$$\iint_{R'} \rho(w)|dw\wedge d\bar{w}| = 1\,,$$

where $w(z)$ is a local realization of f.

It will be noteworthy that one derives the almost everywhere total differentiability of $w(z)$ from the conditions (Q. 1) and (I) (cf. Gehring-Lehto [6]). Making use of this fact, we can show that

$$\operatorname{mes} w(S) = \iint_S \left(\left|\frac{\partial w}{\partial z}\right|^2-\left|\frac{\partial w}{\partial \bar{z}}\right|^2\right)dxdy:$$

as a matter of fact, let F be such a closed subset of S, that $w(z)$ is totally differentiable at every point of F, $\partial w/\partial z$ and $\partial w/\partial \bar{z}$ is continuous on F and that $\max\left\{\operatorname{mes} w(S-F),\ \iint_{S-F}\left(\left|\frac{\partial w}{\partial z}\right|^2+\left|\frac{\partial w}{\partial \bar{z}}\right|^2\right)dxdy\right\}$ is smaller than any given $\varepsilon>0$. Since $\operatorname{mes} w(F)=\iint_F\left(\left|\frac{\partial w}{\partial z}\right|^2-\left|\frac{\partial w}{\partial \bar{z}}\right|^2\right)dxdy$, we have

$$\operatorname{mes} w(S)-\iint_S\left(\left|\frac{\partial w}{\partial z}\right|^2-\left|\frac{\partial w}{\partial \bar{z}}\right|^2\right)dxdy\leq \operatorname{mes} w(S-F)+\iint_{S-F}\left(\left|\frac{\partial w}{\partial z}\right|^2+\left|\frac{\partial w}{\partial \bar{z}}\right|^2\right)dxdy$$
$$<2\varepsilon\,,$$

which was to be proved. The identity $(|\partial w/\partial z|^2-|\partial w/\partial \bar{z}|^2)(|\partial z/\partial w|^2-|\partial z/\partial \bar{w}|^2)=1$ holds almost everywhere in $|z|<1$ (accordingly a.e. in $|w|<1$), yielding the conformally invariant relations

$$\rho(w(z))\left(\left|\frac{\partial w}{\partial z}\right|^2-\left|\frac{\partial w}{\partial \bar{z}}\right|^2\right)|dz\wedge d\bar{z}|>0 \qquad \text{a.e. on } R$$

$$|\tau(z(w))|^2\left(\left|\frac{\partial z}{\partial w}\right|^2-\left|\frac{\partial z}{\partial \bar{w}}\right|^2\right)|dw\wedge d\bar{w}|>0 \qquad \text{a.e. on } R'.$$

Finally we postulate

(V) *For a fixed* $\omega\in\Omega$

$$\iint_R \left(\left|\frac{\partial w}{\partial z}\right|^2+\left|\frac{\partial w}{\partial \bar z}\right|^2\right)\left(\left|\frac{\partial w}{\partial z}\right|^2-\left|\frac{\partial w}{\partial \bar z}\right|^2\right)^{-1}\omega*\omega\le\frac{M}{2}\left(K+\frac{1}{K}\right).$$

Proposition 1. $\mathfrak{F}_\omega$ *is non-void.*

Proof. Let g be a quasi-conformal mapping with maximal dilatation K homotopic to χ above mentioned. g clearly fulfills the conditions (I), (II), (III) and (V). Denoting by $w(z)$ its local realization, we have

$$\iint_R \rho(w(z))\left(\left|\frac{\partial w}{\partial z}\right|^2+\left|\frac{\partial w}{\partial \bar z}\right|^2\right)|dz\wedge d\bar z|$$

$$=\iint_R \rho(w(z))\left(\left|\frac{\partial w}{\partial z}\right|^2+\left|\frac{\partial w}{\partial \bar z}\right|^2\right)\left(\left|\frac{\partial w}{\partial z}\right|^2-\left|\frac{\partial w}{\partial \bar z}\right|^2\right)^{-1}\left(\left|\frac{\partial w}{\partial z}\right|^2-\left|\frac{\partial w}{\partial \bar z}\right|^2\right)|dz\wedge d\bar z|$$

$$=\iint_{R'} \left(\left|\frac{\partial w}{\partial z}\right|^2+\left|\frac{\partial w}{\partial \bar z}\right|^2\right)\left(\left|\frac{\partial w}{\partial z}\right|^2-\left|\frac{\partial w}{\partial \bar z}\right|^2\right)^{-1}\rho(w)|dw\wedge d\bar w|\le\frac{1}{2}\left(K+\frac{1}{K}\right).$$

Hence $g\in\mathfrak{F}_\omega$, Q.E.D.

Now we can formulate our first extremum problem:

Minimize the Dirichlet integral

$$I[f]=\frac{1}{4}\iint_R \rho(w(z))\left(\left|\frac{\partial w}{\partial z}\right|^2+\left|\frac{\partial w}{\partial \bar z}\right|^2\right)|dz\wedge d\bar z|$$

within the family $\mathfrak{F}_\omega$.

Lemma 3. $\mathfrak{F}_\omega$ *is equicontinuous.*

Proof. We show, for any $\varepsilon>0$, there exists a suitable positive number $\delta_2=\delta_2(\varepsilon)$ independent of an individual $f\in\mathfrak{F}_\omega$, such that, dist $(f(\mathfrak{p}_1), f(\mathfrak{p}_2))<\varepsilon$ so far as dist $(\mathfrak{p}_1, \mathfrak{p}_2)<\delta_2$. Here the distances on R' and R are to be measured with regard to the fixed conformal metric $ds^2=\rho(w)|dw|^2$ and the metric induced by $\omega=\tau(z)dz$ respectively.

Suppose, on the contrary, there exist a positive number c_1, two sequences of points $\{\mathfrak{p}_n^{(1)}\}_{n=1}^\infty$, $\{\mathfrak{p}_n^{(2)}\}_{n=1}^\infty$ on R and a sequence $\{f_n\}_{n=1}^\infty$ belonging to $\mathfrak{F}_\omega$, such that dist $(\mathfrak{p}_n^{(1)}, \mathfrak{p}_n^{(2)})\to 0$ as $n\to\infty$, while dist $(f_n(\mathfrak{p}_n^{(1)}), f_n(\mathfrak{p}_n^{(2)}))\ge c_1$. We may assume that $\mathfrak{p}_n^{(j)}$ $(j=1, 2)$ are contained in a local coordinate neighbourhood $\mathfrak{U}$ for all $n\ge n_0$. Let $|z|<1$ be a local coordinate disk corresponding to $\mathfrak{U}$ and set $z(\mathfrak{p}_n^{(j)})=z_j$ $(j=1, 2)$ simply for some fixed n. We draw a circular disk $|z_1-z|<r_0$ which contains z_2 in its interior and is contained in $|z|<1$. A domain on R corresponding to the circular annulus $\varepsilon=|z_2-z_1|<|z-z_1|<r_0$ is mapped by $f_n(\mathfrak{p}(z))$ onto an annular domain on R'. Let $L(r)$ be the length of the image curve of $|z-z_1|=r$

by $f_n(\mathfrak{p}(z))$, which is defined for almost all values of r in the interval $[\varepsilon, r_0]$. Set

$$I(r)=\int_0^r\int_0^{2\pi}\rho(w(z))\left(\left|\frac{\partial w}{\partial z}\right|^2+\left|\frac{\partial w}{\partial \bar z}\right|^2\right)td\theta dt \qquad z=te^{i\theta},$$

where $w(z)$ is a local realization of $f_n(\mathfrak{p})$. Then we would have by the condition (III)

$$c_1^2\leq (L(r))^2=\left(\oint\sqrt{\rho(w)}\,|dw|\right)^2=\left(\int_{\theta=0}^{2\pi}\sqrt{\rho(w(z))}\left|\frac{\partial w}{\partial\theta}\right|d\theta\right)^2$$

$$\leq\left(\oint_{|z|=r}\sqrt{\rho(w(z))}\left(\left|\frac{\partial w}{\partial z}\right|+\left|\frac{\partial w}{\partial \bar z}\right|\right)|dz|\right)^2$$

$$\leq\left(\oint_{|z|=r}\rho(w(z))\left(\left|\frac{\partial w}{\partial z}\right|+\left|\frac{\partial w}{\partial \bar z}\right|\right)^2|dz|\right)\cdot\oint_{|z|=r}|dz|$$

$$\leq 2\left(\oint_{|z|=r}\rho(w(z))\left(\left|\frac{\partial w}{\partial z}\right|^2+\left|\frac{\partial w}{\partial \bar z}\right|^2\right)|dz|\right)\cdot\oint_{|z|=r}|dz|=4\pi r\frac{dI(r)}{dr}.$$

Therefore

$$\frac{dr}{r}\leq\frac{4\pi}{c_1^2}dI(r).$$

Integrating from $r=\delta$ to $r=r_0$, we obtain

$$\log\frac{r_0}{\varepsilon}\leq\frac{4\pi}{c_1^2}[I(r_0)-I(\varepsilon)]\leq\frac{M}{c_1^2}\left(K+\frac{1}{K}\right).$$

This is a contradiction, since ε can be chosen arbitrarily small.

Theorem 1. *The family* $\mathfrak{F}_\omega$ *is normal on* R.

Proof. Let $\{f_n(\mathfrak{p})\}_{n=1}^\infty$ be an arbitrary sequence, such that $f_n\in\mathfrak{F}_\omega$ $(n=1,2,\cdots)$. Since R is separable, we can choose a countable subset $E=\{\mathfrak{p}_n\}_{n=1}^\infty$ which is everywhere dense on R. On account of compactness of R', its infinite subset $\{f_n(\mathfrak{p}_1)\}_{n=1}^\infty$ accumulates at one point on R'. Hence its suitable subsequence $\{f_{1n}(\mathfrak{p}_1)\}_{n=1}^\infty$ converges. In analogous way we can choose a suitable subsequence out of $\{f_{1n}(\mathfrak{p}_2)\}_{n=1}^\infty$, which we denote by $\{f_{2n}(\mathfrak{p}_2)\}_{n=1}^\infty$. Repeating this procedure, we get a sequence $\{f_{ln}(\mathfrak{p}_l)\}_{n=1}^\infty$ convergent at $\mathfrak{p}_l$. Since $\{f_{ln}(\mathfrak{p})\}_{n=1}^\infty$ is a subsequence of $\{f_{mn}(\mathfrak{p})\}_{n=1}^\infty$ provided $l>m$,

$\{f_{ln}(\mathfrak{p})\}_{n=1}^{\infty}$ converges at $\mathfrak{p}=\mathfrak{p}_j$ $(j=1, 2,\cdots, l)$. If we consider the *diagonal sequence* $\{f_{nn}(\mathfrak{p})\}_{n=1}^{\infty}$. it is obviously a subsequence of $\{f_{ln}(\mathfrak{p})\}_{n=1}^{\infty}$. Therefore it converges at $\mathfrak{p}=\mathfrak{p}_j$ $(j=1, 2,\cdots l)$. Here, since l may be arbitrarily large, we see that $\{f_{nn}(\mathfrak{p})\}_{n=1}^{\infty}$ converges at every point of the set E. With this in mind, we shall show that the subsequence $\{f_{nn}(\mathfrak{p})\}_{n=1}^{\infty}$ of $\{f_n(\mathfrak{p})\}_{n=1}^{\infty}$ is uniformly convergent in the whole surface R.

In virtue of equicontinuity of $\mathfrak{F}_{\omega}$, there exists a positive number δ_3, such that for any $\varepsilon>0$ the inequalities

$$\operatorname{dist}(f_{nn}(\mathfrak{p}),\ f_{nn}(\mathfrak{p}'))<\varepsilon/3\,, \quad \operatorname{dist}(f_{mm}(\mathfrak{p}),\ f_{mm}(\mathfrak{p}'))<\varepsilon/3$$

hold, whenever $\operatorname{dist}(\mathfrak{p}, \mathfrak{p}')<\delta_3$. We can choose a finite subset $\{\mathfrak{p}_j\}_{j=1}^{l}$ of E, such that for any point $\mathfrak{p}$ on R

$$\inf_{1\leqq j\leqq l} \operatorname{dist}(\mathfrak{p}, \mathfrak{p}_j) = \operatorname{dist}(\mathfrak{p}, \mathfrak{p}_{j_0})<\delta_3\,.$$

Hence we have

$$\operatorname{dist}(f_{nn}(\mathfrak{p}),\ f_{nn}(\mathfrak{p}_{j_0}))<\varepsilon/3\,, \quad \operatorname{dist}(f_{mm}(\mathfrak{p}),\ f_{mm}(\mathfrak{p}_{j_0}))<\varepsilon/3\,.$$

Since $\{f_{nn}(\mathfrak{p})\}_{n=1}^{\infty}$ converges at $\mathfrak{p}_j$ $(j=1, 2,\cdots, l)$, we can choose $n_0=n_0(\varepsilon)$ large enough, so that

$$\operatorname{dist}(f_{nn}(\mathfrak{p}_j),\ f_{mm}(\mathfrak{p}_j))<\varepsilon/3 \qquad j = 1, 2,\cdots, l$$

as far as $n, m\geq n_0(\varepsilon)$. Therefore, if $n, m\geq n_0(\varepsilon)$, we have

$$\begin{aligned}&\operatorname{dist}(f_{nn}(\mathfrak{p}),\ f_{mm}(\mathfrak{p}))\\ &\leq \operatorname{dist}(f_{nn}(\mathfrak{p}),\ f_{nn}(\mathfrak{p}_{j_0}))+\operatorname{dist}(f_{nn}(\mathfrak{p}_{j_0}),\ f_{mm}(\mathfrak{p}_{j_0}))+\operatorname{dist}(f_{mm}(\mathfrak{p}_{j_0}),\ f_{mm}(\mathfrak{p}))<\varepsilon.\end{aligned}$$

Setting $\mathfrak{q}_n=f_{nn}(\mathfrak{p})$, we see that $\{\mathfrak{q}_n\}_{n=1}^{\infty}$ form a Cauchy sequence on R'. In virtue of compactness of R', a point $\mathfrak{q}$ is uniquely determined, so that

$$\lim_{n\to\infty} \operatorname{dist}(\mathfrak{q}, \mathfrak{q}_n) = 0\,.$$

Let the correspondence $\mathfrak{p}\to\mathfrak{q}$ be denoted by f_0. Then, for any $\varepsilon>0$

$$\operatorname{dist}(f_0(\mathfrak{p}),\ f_{nn}(\mathfrak{p}))<\varepsilon \qquad \text{so far as } n\geq n_0(\varepsilon)$$

and $n_0(\varepsilon)$ is clearly independent of choice of $\mathfrak{p}\in R$. It implies that the subsequence $\{f_{nn}(\mathfrak{p})\}_{n=1}^{\infty}$ of $\{f_n(\mathfrak{p})\}_{n=1}^{\infty}$ converges uniformly on R towards $f_0(\mathfrak{p})$, Q.E.D.

Now it is asked whether the functional $I[f]$ can be minimized within the family $\mathfrak{F}_\omega$. First of all, we can form a minimizing sequence $\{f_n\}_{n=1}^{\infty}$ for this extremum problem:

$$\lim_{n\to\infty} I[f_n] = \inf_{\mathfrak{F}_\omega} I[f] = I_\omega \geq 0, \qquad f_n \in \mathfrak{F}_\omega .$$

By Theorem 1, the minimizing sequence contains a subsequence, say $\{f_n\}_{n=1}^{\infty}$ again for brevity, uniformly convergent on R. Let the limiting map be denoted by $f_\infty(\mathfrak{p})$. Then $f_\infty(\mathfrak{p})$ is, by definition, one-valued as well as continuous.

Proposition 2. *$f_\infty(\mathfrak{p})$ is schlicht.*

Proof. Suppose, on the contrary, there exist a pair of points $\mathfrak{p}_1$ and $\mathfrak{p}_2$ on R, such that $f_\infty(\mathfrak{p}_1)=f_\infty(\mathfrak{p}_2)=\mathfrak{q}$. Then there could be found, for each n, a continuum $\mathfrak{c}_n \subset R$, such that $\mathfrak{c}_n \ni \mathfrak{p}_1, \mathfrak{p}_2$ and the diameter of the closed set $f_n(\mathfrak{c}_n)$ tends to zero as $n\to\infty$. Let Γ be a simple closed Jordan curve enclosing $\mathfrak{p}_1$ in its interior, which is comprised in a local coordinate neighbourhood of $\mathfrak{p}_1$ and separates $\mathfrak{p}_2$ from $\mathfrak{p}_1$, Then $\mathfrak{c}_n \cap \Gamma$ is always non-empty, on which we take an arbitrary point $\mathfrak{p}^{(n)}$. Let z be some local coordinate about $\mathfrak{p}_1$, w some local coordinate about $f_n(\mathfrak{p}_1)$ and $z=z_n(w)$ the local realization of f_n^{-1} in the neighbourhood of $f_n(\mathfrak{p}_1)$. Setting $w(f_n(\mathfrak{p}^{(n)}))=w'$, we may assume without loss of generality that $w(f_n(\mathfrak{p}_1))=0$. When n is sufficiently large, we see clearly $|w'|<\varepsilon$, while dist $(z_n(w'), z_n(0))\geq c_2>0$. Draw a circle $|w|=r$ $(\varepsilon\leq r\leq r_0)$ in the local coordinate disk $|w|<1$, where $\log(r_0/\varepsilon)\geq \pi M(K^2+1)/c^2K$. Putting

$$J_n(r) = \int_0^r\int_0^{2\pi} |\tau(z_n(w))|^2\left(\left|\frac{\partial z_n}{\partial w}\right|^2+\left|\frac{\partial z_n}{\partial \overline{w}}\right|^2\right)|dw\wedge d\overline{w}|,$$

we have just as in the proof of Lemma 3

$$c_2^2 \leq \left(\oint |\tau(z)|\,|dz|\right)^2 \leq \left(\oint_{|w|=r} |\tau(z_n(w))|\left(\left|\frac{\partial z_n}{\partial w}\right|+\left|\frac{\partial z_n}{\partial \overline{w}}\right|\right)|dw|\right)^2$$

$$\leq 2\left[\oint_{|w|=r} |\tau(z_n(w))|^2\left(\left|\frac{\partial z_n}{\partial w}\right|^2+\left|\frac{\partial z_n}{\partial \overline{w}}\right|^2\right)|dw|\right]\oint_{|w|=r}|dw| = 4\pi r\frac{dJ_n(r)}{dr}$$

for every r in $[\varepsilon, r_0]$. Hence we get successively

$$\frac{dr}{r} \leq \frac{4\pi}{c_2^2}dJ_n(r),$$

$$\frac{\pi M(K^2+1)}{c_2^2K} \leq \log\frac{r_0}{\varepsilon} \leq \frac{4\pi}{c_2^2}[J_n(r_0)-J_n(\varepsilon)] < \frac{4\pi J_n(r_0)}{c_2^2}.$$

The last inequality contradicts the condition (V): indeed, let $w_n(z)$ be the inverse function of $z_n(w)$. Then, by the condition (III)

$$|dw\wedge d\bar{w}| = \left(\left|\frac{\partial w_n}{\partial z}\right|^2-\left|\frac{\partial w_n}{\partial \bar{z}}\right|^2\right)|dz\wedge d\bar{z}| = \frac{|dz\wedge d\bar{z}|}{\left|\frac{\partial z_n}{\partial w}\right|^2-\left|\frac{\partial z_n}{\partial \bar{w}}\right|^2},$$

whence

$$J_n(r) = \iint_R \left(\left|\frac{\partial z_n}{\partial w}\right|^2+\left|\frac{\partial z_n}{\partial \bar{w}}\right|^2\right)\left(\left|\frac{\partial z_n}{\partial w}\right|^2-\left|\frac{\partial z_n}{\partial \bar{w}}\right|^2\right)^{-1}|\tau(z)|^2|dz\wedge d\bar{z}|$$

$$= \iint_R \left(\left|\frac{\partial w_n}{\partial z}\right|^2+\left|\frac{\partial w_n}{\partial \bar{z}}\right|^2\right)\left(\left|\frac{\partial w_n}{\partial z}\right|^2-\left|\frac{\partial w_n}{\partial \bar{z}}\right|^2\right)^{-1}|\tau(z)|^2|dz\wedge d\bar{z}| \leq \frac{M}{2}\left(K+\frac{1}{K}\right).$$

Lemma 4. *The sequence* $\{f_n^{-1}\}_{n=1}^{\infty}$ *converges uniformly on* R'.

Proof. Let $\{\mathfrak{q}_n\}_{n=1}^{\infty}$ be a convergent sequence of points on R', such that $\lim_{n\to\infty}\mathfrak{q}_n=\mathfrak{q}_\infty$. Then the sequence of points $\{\mathfrak{p}_n\}_{n=1}^{\infty}$ satisfying the relation $f_n(\mathfrak{p}_n)=\mathfrak{q}_n$ $(n=1, 2,\cdots)$ accumulates at some points, one of which we denote by $\mathfrak{p}_\infty$. A suitable subsequence $\{\mathfrak{p}_{n_\nu}\}_{\nu=1}^{\infty}$ will converge to $\mathfrak{p}_\infty$. If ν is sufficiently large for any given $\varepsilon>0$,

$$\begin{aligned}&\mathrm{dist}\,(f_{n_\nu}(\mathfrak{p}_{n_\nu}),\ f_\infty(\mathfrak{p}_\infty))\\ &\leq \mathrm{dist}\,(f_{n_\nu}(\mathfrak{p}_{n_\nu}),\ f_{n_\nu}(\mathfrak{p}_\infty))+\mathrm{dist}\,(f_{n_\nu}(\mathfrak{p}_\infty),\ f_\infty(\mathfrak{p}_\infty))<\varepsilon,\end{aligned}$$

that is,

$$f_\infty(\mathfrak{p}_\infty) = \lim_{\nu\to\infty} f_{n_\nu}(\mathfrak{p}_{n_\nu}) = \mathfrak{q}_\infty.$$

This implies that any accumulation point of the sequence $\{\mathfrak{p}_n\}_{n=1}^{\infty}$ coincides with $f_\infty^{-1}(\mathfrak{q}_\infty)$; the original sequence $\{\mathfrak{p}_n\}_{n=1}^{\infty}$ accumulates only at a single point $f_\infty^{-1}(\mathfrak{q}_\infty)$, since f_∞ is schlicht. Hence

$$\lim_{n\to\infty} f_n^{-1}(\mathfrak{q}_n) = f_\infty^{-1}(\mathfrak{q}_\infty),$$

in particular

$$\lim_{n\to\infty} f_n^{-1}(\mathfrak{q}_\infty) = f_\infty^{-1}(\mathfrak{q}_\infty).$$

Since $\mathfrak{q}_\infty$ was arbitrary, it follows that for any $\mathfrak{q}\in R'$

$$\lim_{n\to\infty} f_n^{-1}(\mathfrak{q}) = f_\infty^{-1}(\mathfrak{q}).$$

If the last convergence were not uniform, there would exist a positive number ε, a subsequence $\{f_{n_m}^{-1}(\mathfrak{q})\}_{m=1}^{\infty}$ of $\{f_n^{-1}(\mathfrak{q})\}_{m=1}^{\infty}$ and a sequence $\{\mathfrak{q}'_m\}_{m=1}^{\infty}$

on R' converging to $\mathfrak{q}'$, such that

$$\operatorname{dist}(f_{nm}^{-1}(\mathfrak{q}'),\ f_{\infty}^{-1}(\mathfrak{q}')) \geq \varepsilon > 0 ,$$

which is contrary to the above relation

$$\lim_{n\to\infty} f_{nm}^{-1}(\mathfrak{q}'_m) = f^{-1}(\mathfrak{q}'_\infty) .$$

Proposition 3. f_∞^{-1} *is continuous.*

Proof. $f_\infty^{-1}(\mathfrak{q})$ is the uniform limit of the sequence of continuous maps $f_n^{-1}(\mathfrak{q})$.

Proposition 4A. f_∞ *is L^2-derivable and measurable (i.e., absolutely continuous in 2-dimensional sense) on R.*

Proof. Let $w=w_\nu(z)$ be a local realization of $f_\nu(\mathfrak{p})$ near a fixed point $\mathfrak{p}_0$ $(\nu=n, \infty)$. There exists a sufficiently large number n_0 and a suitable neighbourhood $\mathfrak{U}(\mathfrak{p}_0)$, such that for any $n\geq n_0$ $f_n(\mathfrak{U}(\mathfrak{p}_0))$ is contained in a fixed local coordinate neighbourhood of $f_\infty(\mathfrak{p}_0)$. Let us denote by $|z|<1$ the local coordinate disk corresponding to $\mathfrak{U}(\mathfrak{p}_0)$, by S the square $|x|<1/2$, $|y|<1/2$ and by $\varphi_0(z)$ a continuously differentiable function vanishing outside S. Then we have by Green's formula

$$0 = \int_{\partial S} [w_n(z)-w_{n+m}(z)]\varphi_0(z)dx = \iint_S \frac{\partial}{\partial y}\{[w_n(z)-w_{n+m}(z)]\varphi_0(z)\}\, dxdy ,$$

where m is an arbitrary positive integer. Hence

$$\iint_S \left(\frac{\partial w_n}{\partial y}-\frac{\partial w_{n+m}}{\partial y}\right)\varphi_0(z)dxdy = -\iint_S [w_n(z)-w_{n+m}(z)]\frac{\partial \varphi_0}{\partial y}dxdy .$$

Since the right hand integral tends to zero as $n\to\infty$, the definite integrals

$$\iint_S \frac{\partial w_n}{\partial y}\varphi_0(z)dxdy \qquad n = 1, 2,\cdots$$

constitute a Cauchy seqnence. Thus the existence of $\lim_{n\to\infty}\iint_S (\partial w/\partial y)$ $\varphi_0(z)dxdy$ is concluded.

This limiting value is obviously a linear functional in φ_0. So we shall denote if by $\Lambda(\varphi_0)$. If n_0 is large enough comparatively with $1/\varepsilon$, it follows from Schwarz's inequality that for $n\geq n_0(\varepsilon)$

$$[|\Lambda(\varphi_0)-\varepsilon]^2 \leq \iint_S \left|\frac{\partial w_n}{\partial y}\right|^2 dxdy \iint_S |\varphi_0(z)|^2 dxdy \leq A \iint_S |\varphi_0(z)|^2 dxdy ,$$

where A is a constant independent of n. Let $\varepsilon\to 0$. Then the classical theorem due to F. Riesz allows us to extend the domain of arguments to the class of square-summable functions $\varphi(z)$ and to write

$$\Lambda(\varphi) = \iint_S w_{\infty(y)}\varphi(z)dxdy$$

with a function $w_{\infty(y)}(z)$, which is of summable square and uniquely determined almost everywhere in S (cf. Riesz [7]). It follows that

$$\iint_S w_\infty(z)\frac{\partial\varphi_0}{\partial y}dxdy = \lim_{n\to\infty}\iint_S w_n(z)\frac{\partial\varphi_0}{\partial y}dxdy = -\lim_{n\to\infty}\iint_S \frac{\partial w_n}{\partial y}\varphi_0(z)dxdy$$

$$= -\Lambda(\varphi_0) = -\iint_S w_{\infty(y)}\varphi_0(z)dxdy\,.$$

Since S was arbitrarily taken in $|z|<1$, we may write according to the definition by Sobolev-Friedrichs

$$\frac{\partial w_\infty}{\partial y} = w_{\infty(y)}(z) \quad \text{in the } L^2\text{–sense.}$$

Analogously we have

$$\frac{\partial w_\infty}{\partial x} = w_{\infty(x)}(z) \quad \text{in the } L^2\text{–sense,}$$

where $w_{\infty(x)}$ is the weak limit of $\{\partial w_n/\partial x\}_{n=1}^{\infty}$ with respect to $\varphi_0(z)$:

$$\iint_S w_{\infty(x)}(z)\,\varphi_0(z)dxdy = \lim_{n\to\infty}\iint_S \frac{\partial w_n}{\partial x}\varphi_0(z)dxdy\,.$$

As for the demonstration of measurability of the mapping $w_\infty(z)$, we refer to the skillful device due to Ahlfors [1] and give here an expository version of it.

Since $w_n(z)$ is topological in $|z|<1$, the image of any Borel set X under each mapping $w_n(z)$ is again a Borel set and $\operatorname{mes} w_n(X)$ is a non-negative additive set-function defined for all Borel sets in $|z|<1$. Therefore, by Lebesgue's theorem, the ratio $\operatorname{mes} w_n(X)/\operatorname{mes} X$ tends to a finite value, when X shrinks to a point z except possibly for a set of measure zero. We can see by making use of the total differentiability of $w_n(z)$ that this limiting value equals $|\partial w_n/\partial z|^2-|\partial w_n/\partial\bar{z}|^2$. From the absolute continuity of $\operatorname{mes} w_n(X)$ follows the relation

$$\operatorname{mes} w_n(X) = \iint_X (|\partial w_n/\partial z|^2-|\partial w_n/\partial\bar{z}|^2)dxdy\,.$$

If we take the disk $|z|\leq r<1$ for X in particular and denote its image under every w_ν by E_ν ($\nu=\infty, 1, 2, \cdots$), we see readily

$$\limsup_{n\to\infty} E_n \subseteq E_\infty \subseteq \liminf_{n\to\infty} E_n ,$$

which, together with the well known inequalities

$$\text{mes}\,(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} \text{mes}\, E_n \leq \limsup_{n\to\infty} \text{mes}\, E_n \leq \text{mes}\,(\limsup_{n\to\infty} E_n)$$

yields

$$\lim_{n\to\infty} \text{mes}\, E_n = \text{mes}\, E_\infty . \tag{3}$$

Consider the contour integral

$$A_{mn}(r) = \frac{1}{2i} \oint_{|z|=r} [w_m(z)-w_n(z)]\, d\overline{[w_m(z)-w_n(z)]} .$$

On account of the condition (III), this integral can be defined for almost all r in the interval (0, 1). By Green's formula

$$A_{mn}(r) = \iint_{|z|\leq r} \left[\left| \frac{\partial w_m}{\partial z} - \frac{\partial w_n}{\partial z} \right|^2 - \left| \frac{\partial w_m}{\partial \bar z} - \frac{\partial w_n}{\partial \bar z} \right|^2 \right] dxdy .$$

Given $\varepsilon>0$, we can choose $n_0(\varepsilon)$ so large that $|w_m(z)-w_n(z)|<\varepsilon$ uniformly on $0<|z|=r<1$, if $n, m \geq n_0(\varepsilon)$. Then

$$A_{mn}(r) \leq \varepsilon \oint \left(\left|\frac{\partial w_m}{\partial z}\right| + \left|\frac{\partial w_n}{\partial z}\right| + \left|\frac{\partial w_m}{\partial \bar z}\right| + \left|\frac{\partial w_n}{\partial \bar z}\right| \right) d \arg z$$

is valid uuiformly for a.a. $r<1$. From the condition (IV) and the Schwarz's inequality follows

$$\lim_{m,n\to\infty} \int_0^1 [A_{mn}(r)]^2 dr = 0 . \tag{4}$$

On the other hand

$$A_{mn}(r) = \iint_{|z|\leq r} \left(\left|\frac{\partial w_m}{\partial z}\right|^2 - \left|\frac{\partial w_m}{\partial z}\right|^2 \right) dxdy + \iint_{|z|\leq r} \left(\left|\frac{\partial w_n}{\partial \bar z}\right|^2 - \left|\frac{\partial w_n}{\partial \bar z}\right|^2 \right) dxdy$$
$$-2\Re\left\{ \iint_{|z|\leq r} \left[\frac{\partial w_m}{\partial z} \left(\overline{\frac{\partial w_n}{\partial z}}\right) - \frac{\partial w_m}{\partial \bar z} \left(\overline{\frac{\partial w_n}{\partial \bar z}}\right) \right] dxdy .\right.$$

Letting first $m\to\infty$ then $n\to\infty$, we obtain by (3) and the weak convergence of the complex derivatives

$$\lim_{n\to\infty} \lim_{m\to\infty} A_{mn}(r) = 2\,\text{mes}\, E_\infty - 2 \iint_{|z|\leq r} \left(\left|\frac{\partial w_\infty}{\partial z}\right|^2 - \left|\frac{\partial w_\infty}{\partial \bar z}\right|^2 \right) dxdy . \tag{5}$$

By making use of the uniform boundedness of $A_{mn}(r)$ on (0, 1), we substitute (5) into (4) after application of Lebesgue's theorem. Then we see that the right-hand side in (5) equals zero for almost every r in (0, 1). Thus it has been concluded that

$$\text{mes}\, E_\infty = \iint_{|z|\leq r} \left(\left| \frac{\partial w_\infty}{\partial z} \right|^2 - \left| \frac{\partial w_\infty}{\partial \bar{z}} \right|^2 \right) dxdy \tag{6}$$

for a.a. r and by continuity (6) holds good for all r on the interval [0, 1]. This reasoning is valid not only for the disk $|z| \leq r$ but also for arbitrary disks contained in $|z| < 1$. The density of the set-function is defined almost everywhere and equal to $|\partial w_\infty/\partial z|^2 - |\partial w_\infty/\partial \bar{z}|^2$. Integral expression analogous to (6) on an arbitrary Borel set implies the measurability of the mapping $w_\infty(z)$.

Since the point $\mathfrak{p}_0$ was arbitrarily taken on R, we complete the proof.

Proposition 4B. *f_∞^{-1} is L^2-derivable and measurable (i.e., absolutely continuous in 2-dimensional sense) on R'.*

Proof. By Lemma 4, the sequence $\{f_n^{-1}\}_{n=1}^{\infty}$ converges to f_∞^{-1} uniformly on R'. Hence the proof for L^2-derivability follows the same line as that in Proposition 4A.

Let $w(z)$ be a local realization of f_∞ around a point $\mathfrak{p} \in R$. Let E_0 be the subset of R, where any local realization of f_∞ ceases to be totally differentiable. Since mes $E_0 = 0$ (cf. Gehring-Lehto [5]), $f_\infty(E_0)$ is also of measure zero by the preceding proposition, whence it is seen that both $w(z)$ and $z(w)$ are totally differentiable in respective variables at almost every point on the surfaces. Concerning the local realization $w(z)$ of f_∞ near such a point $\mathfrak{p}$, we set

$$w(z) = u(x+iy) + iv(x+iy)\,, \quad \Delta z = \Delta x + i\Delta y\,.$$
$$w(z+\Delta z) - w(z) = \Delta w = \Delta u + i\Delta v\,.$$

Then

$$\Delta x = x_u(u, v)\Delta u + x_v(u, v)\Delta v + o(|\Delta u| + |\Delta v|)\,,$$
$$\Delta y = y_u(u, v)\Delta u + y_v(u, v)\Delta v + o(|\Delta u| + |\Delta v|)\,.$$

In particular, putting $\Delta v = 0$, we have

$$\Delta x = x_u(u, v)\Delta u + o(\Delta u), \quad \Delta y = y_u(u, v)\Delta u + o(\Delta u)\,,$$

whence

$$\begin{aligned}\Delta u &= u_x(x, y)\Delta x + u_y(x, y)\Delta y + o(|\Delta x| + |\Delta y|) \\ &= u_x(x, y)[x_u(u, v)\Delta u + o(\Delta u)] + u_y(x, y)[y_u(u, v)\Delta u + o(\Delta u)] \\ &\qquad + o(|\Delta x| + |\Delta y|)\,.\end{aligned}$$

Letting $\Delta u \to 0$, we see

$$1 = u_x(x, y) \cdot x_u(u, v) + u_y(x, y) \cdot y_u(u, v),$$

and analogously

$$0 = v_x(x, y) \cdot x_u(u, v) + v_y(x, y) \cdot y_u(u, v).$$

Therefore

$$\begin{aligned} x_u(u, v) &= \frac{v_y(x, y)}{u_x(x, y)v_y(x, y) - u_y(x, y)v_x(x, y)}, \\ y_u(u, v) &= -\frac{v_x(x, y)}{u_x(x, y)v_y(x, y) - u_y(x, y)v_x(x, y)}, \end{aligned} \tag{7}$$

and similarly

$$\begin{aligned} x_v(u, v) &= -\frac{u_y(x, y)}{u_x(x, y)v_y(x, y) - u_y(x, y)v_x(x, y)}, \\ y_v(u, v) &= \frac{u_x(x, y)}{u_x(x, y)v_y(x, y) - u_y(x, y)v_x(x, y)}. \end{aligned} \tag{8}$$

The equivalents to (7), (8) are

$$\frac{\partial z}{\partial w} = \frac{\overline{\left(\frac{\partial w}{\partial z}\right)}}{\left|\frac{\partial w}{\partial z}\right|^2 - \left|\frac{\partial w}{\partial \bar{z}}\right|^2}, \qquad \frac{\partial z}{\partial \bar{w}} = \frac{-\frac{\partial w}{\partial \bar{z}}}{\left|\frac{\partial w}{\partial z}\right|^2 - \left|\frac{\partial w}{\partial \bar{z}}\right|^2},$$

which shows that, $|\partial w/\partial z|^2 - |\partial w/\partial \bar{z}|^2 \neq 0$ at all points where $w(z)$ as well as its inverse function $z(w)$ are totally differentiable. In other words, f_∞ never transforms any set of positive measure on R to a set of measure zero on R', which implies the absolute continuity of f_∞.

Proposition 5. *f_∞ is homotopic to χ.*

Proof. Let $J_n = [1-2^{1-n}, 1-2^{-n}]$ denote the closed interval ($n = 1, 2, \cdots$). Since $f_n \approx f_{n+1}$, there exists such a function $f(\mathfrak{p}, t)$ continuous on the product set $R \times J_n$, that $f(\mathfrak{p}, 1-2^{1-n}) = f_n(\mathfrak{p})$, $f(\mathfrak{p}, 1-2^{-n}) = f_{n+1}(\mathfrak{p})$. Moreover it is possible to take $f(\mathfrak{p}, t)$, so that

$$\begin{aligned} &\sup_R \operatorname*{osc}_{J_n} f(\mathfrak{p}, t) \\ &\quad = \sup_R \sup_{J_n} \operatorname{dist}(f(\mathfrak{p}, t_1), f(\mathfrak{p}, t_2)) \leq \sup_R \operatorname{dist}(f_n(\mathfrak{p}), f_{n+1}(\mathfrak{p})). \end{aligned} \tag{9}$$

To this end we shall have only to define as follows: $f(\mathfrak{p}, t)$ is the point on a uniquely determined geodesic arc (e.g., with respect to the Poincaré metric) connecting $f_j(\mathfrak{p})$ ($j = n, n+1$), such that $\operatorname{dist}(f_n(\mathfrak{p}), f(\mathfrak{p}, t))/\operatorname{dist}(f_n(\mathfrak{p}), f_{n+1}(\mathfrak{p})) = t$.

We first show that $f(\mathfrak{p}, t)$ defined in $R\times \bigcup_{\nu=1}^{\infty} J_\nu$ is continuous on $R\times \bigcup_{\nu=1}^{m} J_\nu$ for every m. It is evidently true for $m=1$. Suppose the assertion holds for $m=n$. Then the analogous conclusion is also valid for $m=n+1$. Because, in general

$$\begin{aligned} & \operatorname{dist}(f(\mathfrak{p}_1, 1-2^{-n}-\delta), f(\mathfrak{p}_2, 1-2^{-n}+\delta)) \\ & \leq \operatorname{dist}(f(\mathfrak{p}_1, 1-2^{-n}-\delta), f(\mathfrak{p}_1, 1-2^{-n})) \\ & + \operatorname{dist}(f(\mathfrak{p}_1, 1-2^{-n}), f(\mathfrak{p}_1, 1-2^{-n}+\delta)) \\ & \qquad + \operatorname{dist}(f(\mathfrak{p}_1, 1-2^{-n}+\delta), f(\mathfrak{p}_2, 1-2^{-n}+\delta)). \end{aligned} \tag{10}$$

Since $f(\mathfrak{p}, t)$ is uniformly continuous on $R\times \bigcup_{\nu=1}^{m} J_\nu$ as well as on $R\times J_{m+1}$, we can find suitable positive numbers $\delta_j=\delta_j(\varepsilon)$ for any $\varepsilon>0$ $(j=4, 5)$, such that the inequality

$$\operatorname{dist}(f(\mathfrak{p}, t), \quad f(\mathfrak{p}', t'))<\varepsilon \tag{11}$$

holds in the two cases:

(1°) when $0\leq t, t'\leq 1-2^{-n}$, $\operatorname{dist}(\mathfrak{p}, \mathfrak{p}')+|t-t'|<\delta_4$,

(2°) when $1-2^{-n}\leq t, t'\leq 1-2^{1-n}$, $\operatorname{dist}(\mathfrak{p}, \mathfrak{p}')+|t-t'|<\delta_5$.

Take $\mathfrak{p}_1$, $\mathfrak{p}_2$ and δ_6, so that $\operatorname{dist}(\mathfrak{p}_1, \mathfrak{p}_2)+\delta_6<\min(\delta_4, \delta_5)$. Then by (10), (11)

$$\operatorname{dist}(f(\mathfrak{p}_1, 1-2^{-n}-\delta_6), f(\mathfrak{p}_2, 1-2^{-n}+\delta_6)<3\varepsilon.$$

Hence $f(\mathfrak{p}, t)$ is continuous on $R\times \bigcup_{\nu=1}^{m+1} J_\nu$. We can conclude inductively that $f(\mathfrak{p}, t)$ is defined and continuous on R [0, 1).

Next, if we set $f(\mathfrak{p}, 1)=f_\infty(\mathfrak{p})$, $f(\mathfrak{p}, t)$ is continuously prolonged up to $R\times[0, 1]$. Actually, we have

$$\begin{aligned} & \operatorname{dist}(f(\mathfrak{p}, 1), f(\mathfrak{p}', t)) \\ & \leq \operatorname{dist}(f(\mathfrak{p}, 1), f(\mathfrak{p}', 1))+\operatorname{dist}(f(\mathfrak{p}', 1), f(\mathfrak{p}', t)) \\ & \leq \operatorname{dist}(f(\mathfrak{p}, 1), f(\mathfrak{p}', 1))+\operatorname{dist}(f_\infty(\mathfrak{p}'), f_n(\mathfrak{p}'))+\operatorname{dist}(f_n(\mathfrak{p}'), f(\mathfrak{p}', t)). \end{aligned}$$

Since $\{f_n\}_{n=1}^{\infty}$ converges to f_∞ uniformly on R, we can choose $n_0(\varepsilon)$, so that $\operatorname{dist}(f_\infty(\mathfrak{p}'), f_n(\mathfrak{p}'))<\varepsilon/3$ so far as $n\geq n_0(\varepsilon)$. If t belongs to J_n, we have in virtue of (9)

$$\operatorname{dist}(f_n(\mathfrak{p}'), f(\mathfrak{p}', t))<\varepsilon/3 \qquad \text{for any } \mathfrak{p}'\in R.$$

Taking $\mathfrak{p}, \mathfrak{p}'$ so near, that the relation $\operatorname{dist}(f(\mathfrak{p}, 1), f(\mathfrak{p}', 1))<\varepsilon/3$ holds, we obtain finally

$$\operatorname{dist}(f(\mathfrak{p}, 1),\ f(\mathfrak{p}', t)) < \varepsilon .$$

Thus we see $f_\infty(\mathfrak{p}) \approx f_1(\mathfrak{p})$, which was to be proved.

Proposition 6. *f_∞ is a sense-preserving map.*

Proof. Let $w_n(z)$ be a local realization near $\mathfrak{p}_0$ ($|z|<1$; $n=\infty, 1, 2, \cdots$), such that $z=0$ corresponds to $\mathfrak{p}_0$ and $w_\infty(0)=0$. If n is large enough, the image of a fixed annulus $0<a\leq|z|\leq b<1$ under the mapping $w_n(z)$ is bounded away from zero. Suitable choice of a branch enables us to define a one-valued function $\operatorname{Log} w_\infty(z) - \operatorname{Log} w_n(z)$ on $a\leq|z|\leq b$, which tends to zero uniformly there. We have by Green's formula and Riesz's theorem

$$\lim_{n\to\infty} \iint_{a\leq|z|\leq b} \frac{z}{w_n}\frac{\partial w_n}{\partial z}|dz\wedge d\bar z| = \iint_{a\leq|z|\leq b} \frac{z}{w_\infty}\frac{\partial w_\infty}{\partial z}|dz\wedge d\bar z| ,$$

$$\lim_{n\to\infty} \iint_{a\leq|z|\leq b} \frac{z}{w_n}\frac{\partial w_n}{\partial \bar z}|dz\wedge d\bar z| = \iint_{a\leq|z|\leq b} \frac{z}{w_\infty}\frac{\partial w_\infty}{\partial \bar z}|dz\wedge d\bar z| ,$$

whence follows

$$\lim_{n\to\infty} \iint_{a\leq|z|\leq b} \frac{\partial \arg w_n}{\partial \arg z}|dz\wedge d\bar z| = \iint_{a\leq|z|\leq b} \frac{\partial \arg w_\infty}{\partial \arg z}|dz\wedge d\bar z| .$$

Since $w_n(z)$ is sense-preserving, it is valid that

$$\oint_{|z|=r} \frac{\partial \arg w_n}{\partial \arg z} d\arg z = 2\pi$$

on almost all circumferences $|z|=r$ where $\arg w$ is absolutely continuous function in $\arg z$. Thus we get

$$\iint_{a\leq|z|\leq b} \frac{\partial \arg w_n}{\partial \arg z}|dz\wedge d\bar z| = 2\pi(b^2-a^2) ,$$

accordingly

$$\iint_{z\leq|a|\leq b} \frac{\partial \arg w_\infty}{\partial \arg z}|dz\wedge d\bar z| = 2\pi(b^2-a^2) ,$$

and

$$\lim_{\Delta c\to 0} \frac{1}{\Delta c} \iint_{c\leq|z|\leq c+\Delta c} \frac{\partial \arg w_\infty}{\partial \arg z}|dz\wedge d\bar z| = 4\pi c \qquad \text{for every } c\in[a, b] .$$

It follows that

$$2\pi = \oint_{|z|=c} \frac{\partial \arg w_\infty}{\partial \arg z} d\arg z = \oint_{|z|=c} \frac{\partial \arg w_n}{\partial \arg z} d\arg z \qquad \text{for every } c\in[a, b] .$$

This shows that the assigned orientation on $|z|=c$ is preserved by $w_\infty(z)$ as well as by $w_n(z)$. f_∞ is a sense-preserving map, since it is topological. Q.E.D.

Owing to the Propositions 4A and 4B, the dilatation $D_{f_\infty}(\mathfrak{p})$ of the mapping $f_\infty(\mathfrak{p})$ is finitely defined at almost every point of R.

Proposition 7.

$$\iint_R \left[D_{f_\infty}(z)+\frac{1}{D_{f_\infty}(z)}\right]|\tau(z)|^2|dz\wedge d\bar z|\leq M\left(K+\frac{1}{K}\right).$$

Proof. Let $w=w_\nu(z)$, $z=z_\nu(w)$ be local realizations of $f_\nu(\mathfrak{p})$, $f_\nu^{-1}(\mathfrak{q})$ respectively; set

$$\beta_\nu = \tau(z_\nu(w))\left(\frac{\partial z_\nu}{\partial w}dw+\frac{\partial z_\nu}{\partial \bar w}d\bar w\right) \qquad \nu = n,\ \infty\ .$$

Then

$$\frac{1}{2}\iint_R \left[D_{f_\nu}(z)+\frac{1}{D_{f_\nu}(z)}\right]|\tau(z)|^2|dz\wedge d\bar z|$$

$$(12)\qquad =\iint_R \left(\left|\frac{\partial w_\nu}{\partial z}\right|^2+\left|\frac{\partial w_\nu}{\partial \bar z}\right|^2\right)\left(\left|\frac{\partial w_\nu}{\partial z}\right|^2-\left|\frac{\partial w_\nu}{\partial \bar z}\right|^2\right)|\tau(z)|^2|dz\wedge d\bar z|$$

$$=\iint_R \left(\left|\frac{\partial z_\nu}{\partial w}\right|^2+\left|\frac{\partial z_\nu}{\partial \bar w}\right|^2\right)|\tau(z_\nu(w))|^2|dw\wedge d\bar w| = \|\beta_\nu\|^2 .$$

Let $\{e_j\}_{j=1}^{k'}$ be a smooth *partition of unity* relative to the finite covering $\{\mathfrak{N}'_j\}_{j=1}^{k'}$ of R' and let Ψ a smooth differential of the first degree on R'. If we write $\Psi=\psi(w)dw+\tilde\psi(w)d\bar w$, we see

$$(\beta_\infty-\beta_n,\ \Psi)$$

$$=\iint_R \left\{\left[\tau(z_\infty(w))\frac{\partial z_\infty}{\partial w}-\tau(z_n(w))\frac{\partial z_n}{\partial w}\right]\overline{\psi(w)}+\left[\tau(z_\infty(w))\frac{\partial z_\infty}{\partial \bar w}\right.\right.$$

$$\left.\left.-\tau(z_n(w))\frac{\partial z_n}{\partial \bar w}\right]\overline{\tilde\psi(w)}\right\}dw\wedge d\bar w$$

$$=\sum_{j=1}^{k'}\iint_{\mathfrak{N}'_j} e_j(w)\left\{\left[\tau(z_\infty(w))\frac{\partial z_\infty}{\partial w}-\tau(z_n(w))\frac{\partial z_n}{\partial w}\right]\overline{\psi(w)}+\left[\tau(z_\infty(w))\frac{\partial z_\infty}{\partial w}\right.\right.$$

$$\left.\left.-\tau(z_n(w))\frac{\partial z_n}{\partial \bar w}\right]\overline{\tilde\psi(w)}\right\}dw\wedge d\bar w$$

$\to 0$ as $n\to\infty$, since in each local coordinate disk the complex derivatives of $z_n(w)$ tend weakly to the corresponding one of $z_\infty(w)$. It turns out that

$$|(\beta_\infty-\beta_n,\ \beta_\infty)|\leq|(\beta_\infty-\beta_n,\ \Psi)|+|(\beta_\infty-\beta_n,\ \beta_\infty-\Psi)|$$
$$\leq|(\beta_\infty-\beta_n,\ \Psi)|+||\beta_\infty-\beta_n||\cdot||\beta_\infty-\Psi||\ .$$

Since the smooth differentials are dense in the space of the square-summable differentials on R', $||\beta_\infty-\Psi||$ can be made smaller than any $\varepsilon>0$, while

$$0\leq||\beta_\infty-\beta_n||^2=||\beta_\infty||^2+||\beta_n||^2-2\Re\{(\beta_\infty,\ \beta_n)\}\ .$$

Therefore

$$||\beta_\infty||^2\leq\liminf_{n\to\infty}||\beta_n||^2\ , \tag{13}$$

which, together with (12) yields

$$\iint_R\left[D_{f_\infty}(z)+\frac{1}{D_{f_\infty}(z)}\right]|\tau(z)|^2|dz\wedge d\bar z|\leq\liminf_{n\to\infty}\iint_R\left[D_{f_n}(z)+\frac{1}{D_{f_n}(z)}\right]$$
$$|\tau(z)|^2|dz\wedge d\bar z|$$
$$\leq M\left(K+\frac{1}{K}\right).\qquad \text{Q.E.D.}$$

The Propositions 2, 3, 4A, 4B, 5, 6 and 7 imply $f_\infty\in\mathfrak{F}_\omega$. It remains to show that f_∞ is extremal for our problem.

Now set

$$\alpha_\nu=\sqrt{\rho(w)}\,dw=\sqrt{\rho(w_\nu(z))}\left(\frac{\partial w_\nu}{\partial z}dz+\frac{\partial w_\nu}{\partial\bar z}d\bar z\right)$$

with local realizations $w_\nu(z)$ of $f_\nu(\mathfrak{p})$ ($\nu=n,\ \infty$). Then, in the same manner as in the proof of Proposition 7, for an arbitrary smooth differential Ψ of the first degree on R', we have

$$\lim_{n\to\infty}(\alpha_\infty-\alpha_n,\ \Psi)=0\ ,$$

whence

$$\lim_{n\to\infty}(\alpha_\infty-\alpha_n,\ \alpha_\infty)=0\ .$$

From

$$0\leq||\alpha_\infty-\alpha_n||^2=||\alpha_\infty||^2+||\alpha_n||^2-2\Re\{(\alpha_\infty,\ \alpha_n)\}$$

follows

$$||\alpha_\infty||^2\leq\liminf_{n\to\infty}||\alpha_n||^2=I_\omega\ ,$$

while

$$I_\omega\leq||\alpha_\infty||^2\ .$$

Therefore

$$||\alpha_\infty||^2=\frac{1}{4}\iint_R\rho(w_\infty(z))\left(\left|\frac{\partial w_\infty}{\partial z}\right|^2+\left|\frac{\partial w_\infty}{\partial\bar z}\right|^2\right)|dz\wedge d\bar z|=I_\omega\ .$$

What we have just demonstrated is that to any $\omega \in \Omega$ there corresponds at least one extremal map f_∞. So we shall denote it henceforth by f_ω in place of f_∞. Our results can be summarized concisely in

Proposition 8. *There exists at least one homeomorphism $f_\omega \in \mathfrak{F}_\omega$, such that $I[f_\omega] = \inf_{f \in \mathfrak{F}_\omega} I[f]$.*

Formulation and solution of the second extremum problem

Consider the family $\mathfrak{F}_\Omega = \{f_\omega, \omega \in \Omega\}$ consisting of all the solutions of our first problem. Clearly $\mathfrak{F}_\Omega \neq \phi$. Now we shall settle the second extremum problem as follows:

Minimize the functional

$$I[f_\omega] = \frac{1}{4} \iint_R \rho(w_\omega(z)) \left(\left| \frac{\partial w_\omega}{\partial z} \right|^2 + \left| \frac{\partial w_\omega}{\partial \bar{z}} \right|^2 \right) |dz \wedge d\bar{z}|$$

within the family $\mathfrak{F}_\Omega$, $w_\omega(z)$ being a local realization of f_ω.

By definition we can find a minimizing sequence $\{f_{\omega_n}\}_{n=1}^\infty$ such that

$$\lim_{n \to \infty} I[f_{\omega_n}] = \inf_{\mathfrak{F}_\Omega} I[f_\omega].$$

Lemma 1′. *$\mathfrak{F}_\Omega$ is equicontinuous.*

Proof. Otherwise, we would have, for some sequences $\{\mathfrak{p}_n^{(1)}\}_{n=1}^\infty$, $\{\mathfrak{p}_n^{(2)}\}_{n=1}^\infty$ on R and $f_{\omega_n} \in \mathfrak{F}_\Omega$, such that $\mathfrak{p}_n^{(2)}$ belongs to an arbitrary neighbourhood $\mathfrak{U}(\mathfrak{p}_n^{(1)})$ from some number n_0 on, that the relations dist $(f_{\omega_n}(\mathfrak{p}_n^{(1)}), f_{\omega_n}(\mathfrak{p}_n^{(2)})) \geq a' > 0$ $(n = 1, 2, \cdots)$ with some constant a'. Suitable subsequences of $\{\mathfrak{p}_n^{(j)}\}_{n=1}^\infty$, say again $\{\mathfrak{p}_n^{(j)}\}_{n=1}^\infty$ $(j = 1, 2)$, will converge to a point $\mathfrak{p}$ on R. Then its local coordinate neighbourhood $\mathfrak{N}_l$ contains $\mathfrak{p}_n^{(j)}$ $(n \geq n_0)$. Let $|z| < 1$ be a local coordinate disk corresponding to $\mathfrak{N}_l$ and let $w_{\omega_n}(z)$ a local realization of f_{ω_n}. We would find in the same way as in the proof of Lemma 3 that the integral

$$\int_0^r \int_0^{2\pi} \rho(w_{\omega_n}(z)) \left(\left| \frac{\partial w_{\omega_n}}{\partial z} \right|^2 + \left| \frac{\partial w_{\omega_n}}{\partial \bar{z}} \right|^2 \right) t d\theta dt \qquad z = te^{i\theta}$$

increases indefinitely, when n grows towards infinity. It is contrary to the uniform boundedness of $I[f_{\omega_n}]$.

Hence we get immediately

Theorem 1′. *The family $\mathfrak{F}_\Omega$ is normal on R.*

The minimizing sequence $\{f_{\omega_n}\}_{n=1}^{\infty}$ contains at least one subsequence uniformly convergent on R. For brevity we denote it by the same notation: $\{f_{\omega_n}(\mathfrak{p})\}_{n=1}^{\infty}$ converges to a mapping $\mathfrak{q}=g_\infty(\mathfrak{p})$ ($\mathfrak{p}\in R$, $\mathfrak{q}\in R'$), which is clearly one-valued and continuous.

Lemma 5. *The family Ω is compact on R.*

Proof. Take any infinite sequence $\{\omega_n\}_{n=1}^{\infty}$ out of Ω. Let $|z|\leq 1$ (resp. $|z'|\leq 1$) be the homeomorphic image of S_j (resp. S_{j+1}) under a coordinate map $z=z(\mathfrak{p})$ (resp. $z'=z'(\mathfrak{p})$). Set

$$\omega_n = \begin{cases} \tau_n(z)dz & \text{in} \quad |z|<1, \\ \tau_n'(z')dz' & \text{in} \quad |z'|<1, \end{cases} \qquad T_n(z) = \int_0^z \tau_n(\zeta)d\zeta \qquad |z|\leq 1.$$

Then by the Remark to Lemma 1, $\{T_n(z)\}_{n=1}^{\infty}$ contains a suitable subsequence $\{T_{n_\nu}(z)\}_{\nu=1}^{\infty}$ converging uniformly on $|z|\leq 1$. Write

$$T_\infty(z) = \lim_{\nu\to\infty} T_{n_\nu}(z) \text{ on } |z|\leq 1, \quad \tau_\infty(z) = dT_\infty(z)/dz \text{ in } |z|<1.$$

Then $\{\tau_{n_\nu}(z)\}_{\nu=1}^{\infty}$ converges uniformly to $\tau_\infty(z)$ on $|z|\leq r<1$, whence

$$\int_0^{2\pi} |\tau_\infty(re^{i\theta})|rd\theta = \lim_{\nu\to\infty}\int_0^{2\pi} |\tau_{n_\nu}(re^{i\theta})|rd\theta \leq M' < M$$

by the condition imposed on Ω. Since $r<1$ is arbitrary, $\tau_\infty(z)$ has the Fatou property in $|z|<1$. The boundary values $\tau_\infty(e^{i\theta})$ can be defined for almost all $\theta\in(0, 2\pi)$. Since the total variation $V[T_{n_\nu}(e^{i\theta});\ 0, 2\pi]$ of the function $T_{n_\nu}(e^{i\theta})$ over the interval $(0, 2\pi)$ ($\nu=1, 2,\cdots$) is uniformly bounded, we get by Helly's theorem that

$$(14)\quad \lim_{\nu\to\infty}\int_0^{2\pi} |\tau_{n_\nu}(e^{i\theta})|d\theta = \lim_{\nu\to\infty} V[T_{n_\nu}(e^{i\theta});\ 0, 2\pi] = V[T_\infty(e^{i\theta});\ 0, 2\pi].$$

On the other hand, on account of the monotonicity of concentrically circumferential integrals and the uniform convergence, we have

$$\int_0^{2\pi} |\tau_\infty(e^{i\theta})|d\theta \geq \int_0^{2\pi} |\tau_\infty(re^{i\theta})|rd\theta = \lim_{\nu\to\infty}\int_0^{2\pi} |\tau_{n_\nu}(re^{i\theta})|rd\theta$$

for every $r<1$, whence

$$\lim_{\nu\to\infty}\int_0^{2\pi} |\tau_{n_\nu}(e^{i\theta})|\,d\theta \leq \int_0^{2\pi} |\tau_\infty(e^{i\theta})|\,d\theta\,.$$

This inequality together with (14) afford

$$(15)\qquad \int_0^{2\pi} |\tau_\infty(e^{i\theta})|\,d\theta = \lim_{\nu\to\infty}\int_0^{2\pi} |\tau_{n_\nu}(e^{i\theta})|\,d\theta = V[T_\infty(e^{i\theta})\,;\ 0, 2\pi]\,,$$

which implies the absolute continuity of the absolute variation $V[T_\infty(e^{i\theta})\,;\ 0, \Theta]$ for every $\Theta\in(0, 2\pi)$. Taking into consideration the uniform convergence of $\{T_{n_\nu}(e^{i\theta})\}_{\nu=1}^{\infty}$ towards $T_\infty(e^{i\theta})$ on every subinterval (θ_1, θ_2) of $(0, 2\pi)$, we obtain

$$T_\infty(e^{i\theta_2}) - T_\infty(e^{i\theta_1}) = \int_{\theta_1}^{\theta_2} \tau_\infty(e^{i\theta})ie^{i\theta}d\theta$$

and from (15)

$$(16)\qquad \int_{\theta_1}^{\theta_2} |\tau_\infty(e^{i\theta})|\,d\theta = \lim_{\nu\to\infty}\int_{\theta_1}^{\theta_2} |\tau_{n_\nu}(e^{i\theta})|\,d\theta\,.$$

Consider the case in which $S_j\cap S_{j+1}$ contains a 1-simplex. Then setting $\widehat{\zeta_1, \zeta_2}=z(\gamma)$ and $\widehat{\zeta_1', \zeta_2'}=z'(\gamma)$ for any arc γ contained in $S_j\cap S_{j+1}$, we can see by (16) that concerning the integration along γ

$$\int_{\zeta_1}^{\zeta_2}|\tau_\infty(\zeta)|\,|d\zeta| = \lim_{\nu\to\infty}\int_{\zeta_1}^{\zeta_2} |\tau_{n_\nu}(\zeta)|\,|d\zeta| = \lim_{\nu\to\infty}\int_{\zeta_1'}^{\zeta_2'} |\tau_{n_\nu}'(\zeta')|\,|d\zeta'| = \int_{\zeta_1'}^{\zeta_2'} |\tau_\infty'(\zeta')|\,|d\zeta'|\,.$$

The assertion $|\tau_\infty(\zeta)|\,|d\zeta| = |\tau_\infty'(\zeta')|\,|d\zeta'|$ for a.a. $\mathfrak{p}\in S_j\cap S_{j+1}$ $(j=1, 2, \cdots, \kappa-1)$ follows at once, where $\zeta=z(\mathfrak{p})$ and $\zeta'=z'(\mathfrak{p})$. Thus we have had the linear differential $\omega_\infty=\tau_\infty(z)dz$ holomorphic in $R-\bigcup_{j=1}^{\kappa}\partial S_j$, and $|\omega_\infty| = |\tau_\infty(z)|\,|dz|$ is well-defined at almost all points on $\bigcup_{j=1}^{\kappa}\partial S_j$ and clearly $\sum_{j=1}^{\kappa}\int_{\partial S_j}|\omega_\infty|\leq M.$

Next, let $|z|=r$ be the circle on which $\tau_\infty(z)\neq 0$. If $|\zeta|<r$, we have

$$\log|\tau_{n_\nu}(\zeta)| = \frac{1}{2\pi}\int_0^{2\pi}(\log|\tau_{n_\nu}(z)|)\frac{r^2-|\zeta|^2}{|z-\zeta|^2}d\arg z\,.$$

The right-hand integral tends to $(1/2\pi)\int_0^{2\pi}(\log|\tau_\infty(z)|)[(r^2-|\zeta|^2)/|z-\zeta|^2]$ $d\arg z$ as $\nu\to\infty$, while $\tau_{n_\nu}(\zeta)$ converges to $\tau_\infty(\zeta)$ uniformly in $|\zeta|<r$.

Hence

$$|\tau_\infty(\zeta)| = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}(\log|\tau_\infty(z)|)\frac{r^2-|\zeta|^2}{|z-\zeta|^2}d\arg z\right)\neq 0\,, \qquad |\zeta|<r\,.$$

Since r is taken arbitrarily near 1, it follows that $\tau_\infty(\zeta)\neq 0$ in $|\zeta|<1$. One concludes that ω_∞ belongs to Ω.

Lemma 6. *Let $\mathfrak{p}_j$ $(j=1, 2)$ be arbitrary points on a singular simplex S_l and let Γ the class of all rectifiable simple arcs which connects the two points $\mathfrak{p}_j$ on S_l. Then for any $\gamma\in\Gamma$ and for any $\omega=\tau(z)dz\in\Omega$, the quantity*

$$\int_\gamma|\omega| = \int_\gamma|\tau(z)||dz|$$

is bounded away from zero.

Proof. Suppose on the contrary

$$\inf_{\substack{\gamma\in\Gamma\\ \omega\in\Omega}}\int_\gamma|\omega| = 0\,.$$

Then there would exist $\omega_n=\tau_n(z)dz\in\Omega$ and $\gamma_n\in\Gamma$ $(n=1, 2,\cdots)$, such that

$$\lim_{n\to\infty}\int_{\gamma_n}|\omega_n| = 0\,. \tag{17}$$

Let z be a local coordinate defined in $|z|<1$ corresponding to the interior to S_l, such that $z(\mathfrak{p}_0)=0$, $\mathfrak{p}_0$ being a fixed interior point of S_l. Set

$$T_n(z) = \int_0^z\tau_n(\zeta)d\zeta \quad\text{for}\quad |z|<1\,.$$

Then, according to the Remark after Lemma 1, $T_n(z)$ can be prolonged continuously onto $|z|\leq 1$ and the sequence $\{T_n(z)\}_{n=1}^\infty$ contains at least one subsequence, say $\{T_{n_\nu}(z)\}_{\nu=1}^\infty$, converging uniformly on $|z|\leq 1$. Put $T_\infty(z) = \lim_{\nu\to\infty} T_{n_\nu}(z)$. Then

$$|T_\infty(z_2)-T_\infty(z_1)| = \lim_{\nu\to\infty}|T_{n_\nu}(z_2)-T_{n_\nu}(z_1)|\leq\lim_{\nu\to\infty}\int_{\gamma_{n_\nu}}|\tau_{n_\nu}(z)||dz|\,,$$
$$z_j = z(\mathfrak{p}_j)\,,\qquad j=1,2\,.$$

The right-hand side is equal to zero by (17). Hence $T_\infty(z_1)=T_\infty(z_2)$. $\tau_\infty(z) =dT_\infty/dz$ is defined and possesses the Fatou property in $|z|<1$. γ_n are analytic arcs which are uniformly bounded. So by Vitali's theorem there exists a suitable path $\gamma_\infty\in\Gamma$ from $\mathfrak{p}_1$ to $\mathfrak{p}_2$, such that

$$\int_{\gamma_\infty} |\tau_\infty(z)|\,|dz| = 0\,. \tag{18}$$

If we take on γ_∞ any point $\mathfrak{p}$, we have by (18) $T_\infty(z(\mathfrak{p}))=T_\infty(z_1)=T_\infty(z_2)$. Therefore $T_\infty(z)=\text{const.}$ on $|z|\leq 1$, whence $\omega_\infty=\tau_\infty(z)dz=0$ on $|z|=1$. Since ω_∞ belongs to Ω, it follows from the requirement concerning Ω that $\omega_\infty\equiv 0$ on every simplex adjacent to S_l, accordingly on the whole R, which is absurd.

Proposition 9. $g_\infty(\mathfrak{p})$ *is schlicht.*

Proof. Suppose, on the contrary, $g_\infty(\mathfrak{p}_1)=g_\infty(\mathfrak{p}_2)=\mathfrak{q}$ for some $\mathfrak{p}_1, \mathfrak{p}_2\in R$ $(\mathfrak{p}_1\neq\mathfrak{p}_2)$, $\mathfrak{q}\in R'$. Let $z=z_{\omega_n}(w)$ $(|w|<1)$ be a local realization of $f_{\omega_n}^{-1}$ about $f_{\omega_n}(\mathfrak{p}_1)$, such that $f_{\omega_n}(\mathfrak{p}_1)$ corresponds to $w=0$. Set

$$A_n(r)=\int_0^r\int_0^{2\pi} |\tau_n(z_{\omega_n}(w))|^2\left(\left|\frac{\partial z_{\omega_n}}{\partial w}\right|^2+\left|\frac{\partial z_{\omega_n}}{\partial \overline{w}}\right|^2\right) t\,d\theta\, dt \qquad w=te^{i\theta}\,.$$

Then clearly

$$A_n(r)\leq\frac{M}{4}\left(K+\frac{1}{K}\right).$$

We can find, by Lemma 6, a positive c independent of t and n, such that

$$c\leq\int_0^{2\pi} |\tau_n(z_{\omega_n}(te^{i\theta}))|\,|dz_{\omega_n}(te^{i\theta})|\,.$$

Hence follows the inequality

$$c^2\leq 4\pi t\; dA_n(t)/dt\,,$$

leading to a contradiction.

Proposition 10. $g_\infty(\mathfrak{p})$ *is a homeomorphism between* R *and* R'.

Proof. This is an immediate consequence from the continuity of $g_\infty(\mathfrak{p})$ and Proposition 9.

Proposition 11A. $g_\infty(\mathfrak{p})$ *is L^2-derivable and absolutely continuous in 2-dimensional sense on* R.

Proposition 11B. $g_\infty^{-1}(\mathfrak{q})$ *is L^2-derivable and absolutely continuous in 2-dimensional sense on* R'.

The proofs are similar to that of Proposition 4A (resp. Proposition 4B), since g_∞ (resp. g_∞^{-1}) is the uniform limit of $\{f_{\omega_n}\}_{n=1}^\infty$ (resp. $\{f_{\omega_n}^{-1}\}_{n=1}^\infty$).

Proposition 12. *g_∞ is homotopic to χ.*

Proposition 13. *g_∞ is a sense-preserving map.*

The proofs are similar to that of Propositions 5 and 6, since g_∞ is the uniform limit of $\{f_{\omega_n}\}_{n=1}^{\infty}$.

The class Ω of the locally holomorphic differentials ω is compact by Lemma 5, so the sequence $\{\omega_n\}_{n=1}^{\infty}$ contains at least one subsequence uniformly convergent in $R-\bigcup_{j=1}^{\kappa}\partial S_j$, the limit of which we denote by $\omega_\infty = \tau_\infty(z)dz$; it belongs again to Ω. Let $z_{\omega_n}(w)$, $Z_\infty(w)$ be local realizations of $f_{\omega_n}^{-1}(\mathfrak{q})$, $g_\infty^{-1}(\mathfrak{q})$ respectively. Then, using the similar reasoning to that in the proof of Proposition 7, we see

$$\iint_{R'}\left(\left|\frac{\partial Z_\infty}{\partial w}\right|^2+\left|\frac{\partial Z_\infty}{\partial \bar{w}}\right|^2\right)|\tau_\infty(Z_\infty(w))|^2|dw\wedge d\bar{w}|$$

$$\leq \liminf_{n\to\infty}\iint_{R'}\left(\left|\frac{\partial z_{\omega_n}}{\partial w}\right|^2+\left|\frac{\partial z_{\omega_n}}{\partial \bar{w}}\right|^2\right)|\tau_n(z_{\omega_n}(w))|^2|dw\wedge d\bar{w}|\leq\frac{M}{2}\left(K+\frac{1}{K}\right).$$

Thus we have obtained, concerning the dilatation D_{g_∞} of the mapping g_∞ the following

Proposition 14.

$$\iint_R\left[D_{g_\infty}(\mathfrak{p})+\frac{1}{D_{g_\infty}(\mathfrak{p})}\right]\omega_\infty * \omega_\infty\leq M\left(K+\frac{1}{K}\right).$$

The Propositions 9, 10, 11A, 11B, 12, 13 and 14 imply the fact that g_∞ belongs to $\mathfrak{F}_{\omega_\infty}$. We are now going to compare it with f_{ω_∞}, defined as one of the extremal maps in $\mathfrak{F}_{\omega_\infty}$; since $\{f_{\omega_n}\}_{n=1}^{\infty}$ converges to g_∞ uniformly on R, we have by the procedure often employed (cf. Proposition 7)

$$(19)\qquad \frac{1}{2}\left(K+\frac{1}{K}\right)\geq\lim_{n\to\infty}\iint_R\rho(w_{\omega_n}(z))\left(\left|\frac{\partial w_{\omega_n}}{\partial z}\right|^2+\left|\frac{\partial w_{\omega_n}}{\partial \bar{z}}\right|^2\right)|dz\wedge d\bar{z}|$$

$$\geq\iint_R\rho(W_\infty(z))\left(\left|\frac{\partial W_\infty}{\partial z}\right|^2+\left|\frac{\partial W_\infty}{\partial \bar{z}}\right|^2\right)|dz\wedge d\bar{z}|\;,$$

where $w_{\omega_n}(z)$ and $W_\infty(z)$ stand for local realizations of f_{ω_n} and g_∞ respectively. The second member in (19) is, by definition of $\{f_{\omega_n}\}_{n=1}^{\infty}$, equal to

$$I = \inf_{\omega\in\Omega} \iint_R \rho(w_\omega(z))\left(\left|\frac{\partial w_\omega}{\partial z}\right|^2 + \left|\frac{\partial w_\omega}{\partial \bar{z}}\right|^2\right) |dz\wedge d\bar{z}| .$$

We get therefore

$$\iint_R \rho(w_\omega(z))\left(\left|\frac{\partial w_{\omega_\infty}}{\partial z}\right|^2 + \left|\frac{\partial w_{\omega_\infty}}{\partial \bar{z}}\right|^2\right) |dz\wedge d\bar{z}| \leq \iint_R \rho(W_\infty(z))\left(\left|\frac{\partial W_\infty}{\partial z}\right|^2 + \left|\frac{\partial W_\infty}{\partial \bar{z}}\right|^2\right) |dz\wedge d\bar{z}| \leq I,$$

whence we have the identity

$$\iint_R \rho(w_\omega(z))\left(\left|\frac{\partial w_{\omega_\infty}}{\partial z}\right|^2 + \left|\frac{\partial w_{\omega_\infty}}{\partial \bar{z}}\right|^2\right) |dz\wedge d\bar{z}^2| = I,$$

with a local realization w_{ω_∞} of f_{ω_∞}. The above relation describes the fact that the mapping f_{ω_∞} just solves the second extremum problem. This extremum f_{ω_∞}, however, as well as the families Ω and $\mathfrak{F}_\Omega$, still depend on the constant M. So we write Ω_M, $\omega_M=\tau_M(z)dz$, $f_M=f_{\omega_M}$ and $w_M(z)$ in place of Ω, ω_∞, f_{ω_∞} and $w_{\omega_\infty}(z)$ respectively henceforth in the present section to study the behaviour of $I[f_M]$, when M is considerably large. It will turn out below that, although the quantity $I[f_M]$ never increases as M grows, it is not steadily decreasing for $M\to\infty$; there are two values of M, say, $M_1<M_2$ such that $I[f_{M_1}]=I[f_{M_2}]$. Hence f_{M_1} solves automatically our second extremum problem for $M=M_2$.

Proposition 15. *If M is sufficiently large, there exists at least one locally holomorphic differential* $\tilde{\omega}_M\in\Omega_M$, *such that*

$$\iint_R \left[Df_M(\mathfrak{p}) + \frac{1}{Df_M(\mathfrak{p})}\right]\tilde{\omega}_M * \tilde{\omega}_M < M\left(K+\frac{1}{K}\right). \tag{20}$$

Proof. We see trivially that the left-hand side of (20) cannot be greater than $M\left(K+\frac{1}{K}\right)$. If the inequality sign enters in this circumstance for some large value of M, the conclusion is immediate by posing $\tilde{\omega}_M=\omega_M$. So we may assume

$$\iint_R \left[Df_M(\mathfrak{p}) + \frac{1}{Df_M(\mathfrak{p})}\right]\omega_M * \omega_M = M\left(K+\frac{1}{K}\right) \quad \text{for every } M. \tag{21}$$

Under these hypotheses our purpose is to derive $\tilde{\omega}_M$ from ω_M through some approximation process for sufficiently large M. In spite of an extreme naïvety of the underlying plan, the author has regrettedly failed to avoid a certain cumbersomeness in the details of proof.

Let us set

$$E^{+}=\left\{\mathfrak{p}\,|\,Df_M(\mathfrak{p})+\frac{1}{Df_M(\mathfrak{p})}\geq M\left(K+\frac{1}{K}\right)\right\},$$
$$E^{-}=\left\{\mathfrak{p}\,|\,Df_M(\mathfrak{p})+\frac{1}{Df_M(\mathfrak{p})}\leq K+\frac{1}{K}\right\}.$$

In view of the conditions (IV), (V) and (21), both E^+ and E^- are sure to be of positive measure. In order to dispense with unnecessary lengthiness of the context, we shall deal with the case when there exists at least one simplex $S_l\in\Sigma$, such that $\operatorname{mes}(E^+\cap S_l)\operatorname{mes}(E^-\cap S_l)>0$. The proof for the other case follows almost the similar line.

Given an arbitrarily small $\varepsilon>0$, let F be a closed subset of $E^-\cap S_l$, such that

$$\iint_{(E^-\cap S_l)-F}\left[Df_M(\mathfrak{p})+\frac{1}{Df_M(\mathfrak{p})}\right]\omega_M*\omega_M<\varepsilon/M. \tag{22}$$

Let B_z be a simply connected subdomain of a local coordinate disk $U_z=\{z\,|\,|z|<1\}$ corresponding to $\operatorname{int}S_l$ with piecewise analytic boundary which encloses $z(F)$, such that

$$\iint_{B_z-z(F)}\left[Df_M(z)+\frac{1}{Df_M(z)}\right]|\tau_M(z)|^2|dz\wedge d\bar z|<\varepsilon/M. \tag{23}$$

Suppose, for the time being, that $\bar B_z$ contains a subarc $\gamma_z\subset\partial U_z$ of length $2\delta>0$, whose middle point we denote by z_0. Let $a<a'$ be two positive numbers smaller than 1 and let $\tilde B_z$ a Jordan subregion confined in B_z. How to fit them shall be found soon below. Employing them, we are going to define two domains Δ_z, Δ'_z and the positive quantity $\bar c$:

$$\Delta_z=\{z\,|\,|\arg z-\arg z_0|<\delta\}\cap\{z\,|\,|z|>a'\},$$
$$\Delta'_z=\Delta_z\cup\{z\,|\,a<|z|<a'\},$$
$$\bar c=\iint_{|z|<1}|\tau_M(z)|^2|dz\wedge d\bar z|\Big/\iint_{\tilde B_z}|\tau_M(z)|^2|dz\wedge d\bar z|.$$

There exists a polynomial $\varphi(\zeta)$ which satisfies the inequalities

$$\left|\frac{1}{2}\log c-\varphi(\zeta)\right|\leq(\varepsilon/M)^2\qquad\text{on}\quad|\zeta|\leq a,$$
$$\left|3\log\frac{1}{M}-\varphi(\zeta)\right|\leq(\varepsilon/M)^2\qquad\text{for}\ \zeta\in U_\zeta-\Delta'_\zeta\ \text{belonging to}\ |\zeta|\geq a' \tag{24}$$

(cf. Walsh [11], p. 15). Set $A_0=\max|\exp\varphi(z)|$. Through an adequate choice of a, a' and $\tilde B_z$ it is possible to find such a linear transformation

$\zeta(z)$ of U_z onto itself, that $\zeta(\tilde{B}_z)$ is contained in $|\zeta|<a$, and that

$$a'<\min_{U_z-B_z}|\zeta(z)|<\max_{U_z-B_z}|\zeta(z)|<1,\ \zeta(U_z-B_z)\cap\Delta'_z=\phi\,,$$

$$\iint_{\Delta'_\zeta}\left[Df_M(\zeta)+\frac{1}{Df_M(\zeta)}\right]|\sigma_M(\zeta)|^2|d\zeta\wedge d\bar{\zeta}|<\varepsilon/A_0^2M^2\,, \tag{25}$$

$$\int_{\partial U_\zeta\cap\Delta_\zeta}|\sigma_M(\zeta)|\,|d\zeta|<\frac{1}{A_0}\oint_{|z|=1}|\tau_M(z)|\,|dz|\,,$$

where $\sigma_M(\zeta)d\zeta=\tau_M(z)dz$. Put

$$P^{(1)}(\zeta)=\sigma_M(\zeta)\exp\varphi(\zeta)\,,\qquad |\zeta|<1\,.$$

Then by (24), (25)

$$\iint_{|\zeta|<1}|P^{(1)}(\zeta)|^2|d\zeta\wedge d\bar{\zeta}|\geq(1-2(\varepsilon/M)^2)\iint_{|z|<1}|\tau_M(z)|^2|dz\wedge d\bar{z}|\,, \tag{26}$$

$$\oint_{|\zeta|=1}|P^{(1)}(\zeta)|\,|d\zeta|<(1+e^{(\varepsilon/M)^2}/M^3)\oint_{|z|=1}|\tau_M(z)|\,|dz|\,. \tag{26$'$}$$

In possession of (22), (24), (25) we assert that there exists a positive constant c_0, such that

$$\begin{aligned}&\iint_{|\zeta|<1}\left[Df_M(\zeta)+\frac{1}{Df_M(\zeta)}\right]|P^{(1)}(\zeta)|^2|d\zeta\wedge d\bar{\zeta}|\\ &\leq\iint_{|z|<1}\left[Df_M(z)+\frac{1}{Df_M(z)}\right]|\tau_M(z)|\,|^2dz\wedge d\bar{z}|-c_0\,.\end{aligned} \tag{26$''$}$$

Even in case in which B_z lies in U_z entirely apart from ∂U_z, one is sure to succeed in deriving the same results (26) etc. by means of linear transformations and an analytic prolongation across the boundary.

Next set

$$P(z)=P^{(1)}(z)\left[\iint_{|z|<1}|\tau_M(z)|^2|dz\wedge d\bar{z}|\Big/\iint_{|z|<1}|P^{(1)}(z)|^2|dz\wedge d\bar{z}|\right].$$

Then

$$\iint_{|z|<1}|P(z)|^2|dz\wedge d\bar{z}|=\iint_{|z|<1}|\tau_M(z)|^2|dz\wedge d\bar{z}|\,, \tag{27}$$

$$\limsup_{r\to1}\oint_{|z|=r}|P(z)|\,|dz|\leq((1+e^{(\varepsilon/M)^2}/M^3)/(1-2(\varepsilon/M)^2))\oint_{|z|=1}|\tau_M(z)|\,|dz|, \tag{27$'$}$$

$$(27'') \qquad \iint_{|z|<1} \left[D_{f_M}(z) + \frac{1}{D_{f_M}(z)} \right] |P(z)|^2 |dz \wedge d\bar{z}|$$
$$\leq \iint_{|z|<1} \left[D_{f_M}(z) + \frac{1}{D_{f_M}(z)} \right] |\tau_M(z)|^2 |dz \wedge d\bar{z}| - c_0' .$$

with the constant $c_0' \geq c_0/(1-2(\varepsilon/M)^2)$.

Let S_{lj} be the simplex adjacent to S_l and let s_j a small subarc of ∂S_{l_i} adjacent to $S_{li} \cap S_l$, such that $\int_{s_j} |\omega_M| = a_0 > 0$ $(j=1, 2, 3)$. We take suitably a simply connected subregion S_{lj}^* of S_{l_i}, partly bounded by $s_j \cup (S_{lj} \cap S_l)$, so that

$$(28) \qquad \begin{aligned} &\iint_{S_{lj}^*} \omega_M * \omega_M < \delta' \iint_{S_{lj}} \omega_M * \omega_M , \\ &\iint_{S_{lj}^*} \left[D_{f_M}(\mathfrak{p}) + \frac{1}{D_{f_M}(\mathfrak{p})} \right] \omega_M * \omega_M < \frac{\varepsilon}{\max(A_0^2, 1)} . \end{aligned}$$

Let $|z'| \leq 1$ be a local coordinate disk corresponding to S_{lj}. The above composed linear differential $|P(z)| |dz|$ induces on the circular arc $z'(S_{lj} \cap S_l)$ the summable linear differential $|P'(z')| |dz'|$ which is identical with $|P(z)| |dz|$. In view of the concavity of the logarithm

$$\frac{1}{b-a} \int_a^b \log u(x) dx \leq \log \left\{ \frac{1}{b-a} \int_a^b u(x) \, dx \right\} \quad \text{for summable } u(x) > 0 \text{ on } (a, b),$$

we assign the boundary value

$$u'(\zeta') = \begin{cases} \log |P'(\zeta')| - \log |\tau_M'(\zeta')| & \text{on } \zeta'(S_{lj} \cap S_l) \\ -\varepsilon & \text{on } \zeta'(s_j) \\ 0 & \text{elsewhere on } |\zeta'| = 1 \end{cases}$$

to define the Poisson integral

$$u'(z') = \frac{1}{2\pi} \int_0^{2\pi} u'(\zeta') \frac{1-|z'|^2}{|\zeta'-z'|^2} d \arg \zeta'$$

in $|z'| < 1$. After an adequate choice of S_{lj}^* and z', we may have

$$\text{(Euclidean) dist.} \; (z'(\partial S_{li}^*), \; z'(s_j) \cup z'(S_{lj} \cap S_l)) \geq \delta > 0 ,$$

$$\int_{z'(S_{lj} \cap S_l)} (|\log |P'(z')|| + |\log |\tau(z_M')||) |dz'| \leq \delta^3, \quad \int_{z'(s_j)} |dz'| \leq \delta^3 ,$$

where $|\tau_M'(z')| |dz'| = |\tau_M(z)| |dz|$. Then

$$(29) \qquad u'(z') \leq \frac{\delta}{2\pi}(1+\varepsilon) \qquad \text{in } z'(S_{lj}-S_{lj}^*).$$

With the harmonic conjugate $v'(z')$ to $u'(z')$ we put

$$P'(z') = \exp\ (\log|\tau'_M(z')| + u'(z') + iv'(z')).$$

Then

$$|P'(z')| = \begin{cases} |P'(z')| & \text{on } z'(S_{lj}\cap S_l) \\ e^{-\varepsilon}|\tau'_M(z')| & \text{on } z'(s_j) \\ |\tau'_M(z')| & \text{elsewhere on } |z'| = 1. \end{cases}$$

It follows from (28), (29) that

$$(1-\delta')\exp\left(-\frac{\delta}{\pi}(1+\varepsilon)\right)\iint_{S_{lj}^*}|\tau'_M(z')|^2|dz'\wedge d\bar{z}'| \leq \iint_{S_{lj}^*}|P'(z')|^2|dz'\wedge d\bar{z}'|,$$

$$\iint_{S_{lj}^*}\left[Df_M(z')+\frac{1}{Df_M(z')}\right]|P'(z')|^2|dz'\wedge d\bar{z}'|$$
$$\leq \max(A_0^2,\ 1)\iint_{S_{lj}^*}\left[Df_M(z')+\frac{1}{Df_M(z')}\right]|\tau'_M(z')|^2|dz'\wedge d\bar{z}'| < \varepsilon.$$

If we select δ, δ', in particular, so that

$$(30) \qquad (1-\delta')\exp\left(-\frac{\delta}{\pi}(1+\varepsilon)\right) \geq 1-(\varepsilon/M)^2,\ \ \delta(1+\varepsilon)/\pi < \varepsilon/M,$$

we get

$$(31) \qquad \iint_{S_{lj}}|P'(z')|^2|dz'\wedge d\bar{z}'| \geq (1-(\varepsilon/M)^2)\iint_{S_{lj}}|\tau'_M(z')|^2|dz\wedge d\bar{z}'|,$$

while

$$(31') \qquad \int_{s_j}|P'(z')|\,|dz'| - e^{-\varepsilon}\int_{s_j}|\tau'_M(z')|\,|dz'| \leq \int_{s_j}|\tau'_M(z')|\,|dz'| - (\varepsilon a_0/2) \qquad (a_0>0),$$

$$(31'') \qquad \iint_{S_{lj}}\left[Df_M(z')+\frac{1}{Df_M(z')}\right]|P'(z')|^2|dz'\wedge d\bar{z}'|$$
$$\leq \exp\left(\frac{\delta}{\pi}(1+\varepsilon)\right)\iint_{S_{lj}}\left[Df_M(z')+\frac{1}{Df_M(z')}\right]|\tau'_M(z')|^2|dz'\wedge d\bar{z}'| + \varepsilon$$
$$\leq \iint_{S_{lj}}\left[Df_M(z')+\frac{1}{Df_M(z')}\right]|\tau'_M(z')|^2|dz'\wedge d\bar{z}'| + 3\varepsilon.$$

Let S''_{lj} $(\neq S_l)\in\Sigma$ be the possible simplex adjacent to S_{lj} and let S_{lj}^{**} its simply connected subregion, such that $\partial S_{lj}^{**}\supset s_j$ and that

(32) $$\iint_{S^{**}_{lj}} \omega_M * \omega_M \leq \delta' \iint_{S^*_{lj}} \omega_M * \omega_M, \quad \iint_{S^{**}_{lj}} \left[D_{f_M}(\mathfrak{p}) + \frac{1}{D_{f_M}(\mathfrak{p})} \right] \omega_M * \omega_M \leq \varepsilon .$$

If we take a local coordinate disk $|z''| \leq 1$ corresponding to S''_{lj}, we have the summable linear differential $|P''(z'')|\,|dz''| = |P'(z')|\,|dz'|$, $|\tau''_M(z'')|\,|dz''| = |\tau'_M(z')|\,|dz'|$ for almost all points on $S''_{lj} \cap S_{lj}$. It is possible to adopt S^{**}_{lj} and z'' so as to be subject to the conditions:

$$\int_{z''(s_j)} |\log |P''(z'')||\,|dz''| \leq \delta^3 ,$$

$$\text{(Euclidean) dist. } (z''(s_j),\ z''(\partial S^{**}_{lj} \cap \operatorname{int} S''_{lj})) \geq \delta .$$

If δ, δ' are chosen so as to fulfill the requirement (30), the straightforward computation yields

(33) $$\iint_{S''_{lj}} |P''(z'')|^2 |dz'' \wedge d\bar{z}''| \geq (1 - (\varepsilon/M)^2) \iint_{S''_{lj}} |\tau''_M(z'')|^2 |dz'' \wedge d\bar{z}''|$$

(33″) $$\iint_{S''_{lj}} \left[D_{f_M}(z'') + \frac{1}{D_{f_M}(z'')} \right] |P''(z'')|^2 |dz'' \wedge d\bar{z}''|$$
$$\leq \iint_{S''_{lj}} \left[D_{f_M}(z'') + \frac{1}{D_{f_M}(z'')} \right] |\tau''_M(z'')|^2 |dz'' \wedge d\bar{z}''| + 3\varepsilon .$$

The differential

$$\tilde{\omega} = \begin{cases} P(z)dz & \text{for } \mathfrak{p}(z) \text{ in } \operatorname{int} S_l, \\ P'(z')dz' & \text{for } \mathfrak{p}(z') \text{ in } \bigcup_{j=1}^{3} \operatorname{int} S_{lj}, \\ P''(z'')dz'' & \text{for } \mathfrak{p}(z'') \text{ in } \bigcup_{j=1}^{3} \operatorname{int} S''_{lj}, \\ \omega_M & \text{in } R - S_l - \bigcup_{j=1}^{3} S_{lj} - \bigcup_{j=1}^{3} S''_{lj} \end{cases}$$

satisfies not only the local holomorphy conditions (i), (ii), (iii), (iv), but also

$$\iint_R \left[D_{f_M}(\mathfrak{p}) + \frac{1}{D_{f_M}(\mathfrak{p})} \right] \tilde{\omega} * \tilde{\omega} \leq M\left(K + \frac{1}{K}\right) - c''_0$$

and

$$|1 - \|\tilde{\omega}\|^2| \leq b(\varepsilon/M)^2$$

for some positive constants c''_0 and b (independent of ε and M) on account of (27), (31), (33), (27″), (31″), (33″). Let it be normalized into $\tilde{\omega}_M = \tilde{\omega}/\|\tilde{\omega}\|$: we have $\|\tilde{\omega}_M\| = 1$ and the inequality

$$\sum_{j=1}^{K}\int_{\partial S_j}|\tilde{\omega}_M|\leq\frac{1}{\sqrt{1-b(\varepsilon/M)^2}}\sum_{j=1}^{K}\int_{\partial S_j}|\tilde{\omega}|<\frac{M-\frac{a_0\varepsilon}{4}}{1-b(\varepsilon/M)^2}=\frac{M\left(1-\frac{a_0}{4}(\varepsilon/M)\right)}{1-b(\varepsilon/M)^2}<M$$

is valid in virtue of (27′), (31′) for sufficiently large M, which enables us to conclude that

$$\tilde{\omega}_M\in\Omega_M \qquad \text{for} \quad M\gg 1\,.$$

Finally it can be verified that

$$\iint_R\left[Df_M(\mathfrak{p})+\frac{1}{Df_M(\mathfrak{p})}\right]\omega_M*\omega_M:\iint_R\left[Df_M(\mathfrak{p})+\frac{1}{Df_M(\mathfrak{p})}\right]\tilde{\omega}_M*\tilde{\omega}_M$$

$$\geq(1-b(\varepsilon/M)^2)\iint_R\left[Df_M(\mathfrak{p})+\frac{1}{Df_M(\mathfrak{p})}\right]\omega_M*\omega_M:\iint_R\left[Df_M(\mathfrak{p})+\frac{1}{Df_M(\mathfrak{p})}\right]\tilde{\omega}*\tilde{\omega}$$

$$\geq(1-b(\varepsilon/M)^2)M\left(K+\frac{1}{K}\right):\left(M\left(K+\frac{1}{K}\right)-c_0''\right).$$

When M is large enough, the last term will exceed 1, which proves the assertion. Q.E.D.

Existence of a harmonic mapping

Proposition 15 shows the fact that the map f_M belongs to the family $\mathfrak{F}_{\tilde{\omega}_M}$, if M is sufficiently great. This mapping, by definition, extremizes the functional $I[f]$ in the wider family $\mathfrak{F}_{\Omega_M}$; much more does in $\mathfrak{F}_{\tilde{\omega}_M}$.

Now we are in a position to get the necessary condition for the map $f=f_M$ to minimize the Dirichlet functional within the family $\mathfrak{F}_{\tilde{\omega}_M}$. To this end, we set

$$\alpha = p\,dz+q\,d\bar{z} = \rho(w(z))\left(\frac{\partial w}{\partial z}dz+\frac{\partial w}{\partial\bar{z}}d\bar{z}\right)$$

with a local realization $w(z)$ of $f(\mathfrak{p})$.

Let F be a compact set comprised in a local coordinate neighbourhood $\mathfrak{N}_j$ $(j=1,2,\cdots,k)$ of R and let $|z|<1$ the corresponding local coordinate disk. Take a complex-valued continuously differentiable function $\lambda(z)$ defined in $|z|<1$ with the compact carrier $z(F)$, and compose, with a complex constant ε, a mapping $z'=z+\varepsilon\lambda(z)$ of $|z|<1$ onto itself. It is topological as soon as $|\varepsilon|$ is small enough. If we write $\alpha=p'\,dz'+q'd\bar{z}'$, then

$$\alpha = \left(p'\frac{\partial z'}{\partial z}+q'\frac{\partial\bar{z}'}{\partial z}\right)dz+\left(p'\frac{\partial z'}{\partial\bar{z}}+q'\frac{\partial\bar{z}'}{\partial\bar{z}}\right)d\bar{z}\,,$$

whence follows

$$p' = \frac{p\dfrac{\partial \bar{z}'}{\partial \bar{z}} - q\dfrac{\partial \bar{z}'}{\partial z}}{\left|\dfrac{\partial z'}{\partial z}\right|^2 - \left|\dfrac{\partial z'}{\partial \bar{z}}\right|^2}, \quad q' = \frac{-p\dfrac{\partial z'}{\partial \bar{z}} + q\dfrac{\partial z'}{\partial z}}{\left|\dfrac{\partial z'}{\partial z}\right|^2 - \left|\dfrac{\partial z'}{\partial \bar{z}}\right|^2}.$$

Since $dz'^2 = (1+\varepsilon\lambda_z)^2 dz^2 + \varepsilon^2\lambda_{\bar{z}}^2 d\bar{z} + 2\varepsilon(1+\varepsilon\lambda_{\bar{z}})\lambda_{\bar{z}}|dz|^2$, it is easy to verify that

$$\begin{aligned} ds^2 &= |p'dz' + q'd\bar{z}'|^2 = (|p'|^2 + |q'|^2)|dz'|^2 + 2\Re\{p'\bar{q}'dz'^2\} \\ &= (|p'|^2 + |q'|^2)|dz'|^2 + 2\Re\{p'\bar{q}'(1+\varepsilon\lambda_z)2\varepsilon\lambda_{\bar{z}}|dz|^2\} \\ &\qquad + 2\Re\{p'\bar{q}'[(1+\varepsilon\lambda_z)^2dz^2 + \varepsilon^2\lambda_{\bar{z}}^2 d\bar{z}^2]\}. \end{aligned}$$

Set $w_\varepsilon(z) = w(z+\varepsilon\lambda(z))$. Then $w_\varepsilon(z)$ becomes a local realization of some homeomorphism f_ε, which is considered to be generated from f by means of a certain local deformation. f_ε satisfies the conditions (I), (II), (III), (IV) and (V) too, with $\omega = \tilde{\omega}_M$, owing to Proposition 15 and the continuity of the functional concerned. Therefore f_ε belongs to $\mathfrak{F}_{\tilde{\omega}_M}$ again. As we compute readily

$$\begin{aligned} & I[f_\varepsilon] - I[f] \\ &= \iint_R (|p'(z')|^2 + |q'(z')|^2)|dz' \wedge d\bar{z}'| + 4\Re\Big\{\iint_R \varepsilon p'(z')\overline{q'(z')}[1+\varepsilon\lambda_z(z)]\lambda_{\bar{z}}(z) \\ &\qquad\qquad |dz\wedge d\bar{z}|\Big\} \\ &\qquad - \iint_R (|p(z)|^2 + |q(z)|^2)|dz\wedge d\bar{z}| \\ &= 4\Re\Big\{\varepsilon\iint_{\mathfrak{N}_j} \lambda_{\bar{z}}(z)[1+\varepsilon\lambda_z(z)]G(z)\,|dz\wedge d\bar{z}|\Big\}, \end{aligned}$$

where

$$G(z) = \frac{p(z)\overline{q(z)}[1+\overline{\varepsilon\lambda_z(z)}]^2 - \varepsilon\lambda_{\bar{z}}(z)[1+\overline{\varepsilon\lambda_z(z)}][|p(z)|^2 + |q(z)|^2] + p(z)\overline{q(z)}[\overline{\varepsilon\lambda_{\bar{z}}(z)}]^2}{[|1+\varepsilon\lambda_z(z)|^2 - |\varepsilon\lambda_{\bar{z}}(z)|^2]}$$

Letting $\varepsilon \to 0$ with the conditions $\lim_{\varepsilon\to 0} \arg\varepsilon = 0$ or $\pi/2$, we have

$$\lim_{\varepsilon\to 0} \frac{I[f_\varepsilon] - I[f]}{\varepsilon} = \begin{cases} 4\displaystyle\iint_{\mathfrak{N}_j} \Re\{p(z)\overline{q(z)}\lambda_{\bar{z}}(z)\}\,|dz\wedge d\bar{z}| \\ 4\displaystyle\iint_{\mathfrak{N}_j} \Im\{p(z)\overline{q(z)}\lambda_{\bar{z}}(z)\}\,|dz\wedge d\bar{z}|. \end{cases}$$

From the extremality of $I[f]$ within the family $\mathfrak{F}_{\tilde{\omega}_M}$ it is concluded that

(34) $$\iint_{\mathfrak{N}_j} \lambda_{\bar{z}}(z)p(z)\overline{q(z)}|dz\wedge d\bar{z}| = 0\,.$$

For the purpose of demonstrating our assertion, we shall utilize a kind of smoothing operator due to G. Springer [9], which works upon square-summable functions and makes them smooth. It brings us some simplification; for an arbitrary square-summable function $v(z)$ the smoothing operator is defined as follows:

$$\mathfrak{M}_r v(z) = \int_0^r\int_0^{2\pi} v(z+te^{i\theta})Y_r(te^{i\theta})\,td\theta dt\,,$$

where

$$Y_r(te^{i\theta}) = \begin{cases} 3(r^2-t^2)/\pi r^6 & t>r \\ 0 & t\geq r\,. \end{cases}$$

Take r sufficiently small, so that $|z|<1-r$ may comprise $z(F)$. Set $\xi(z)=p(z)\overline{q(z)}$. Then $\mathfrak{M}_r\xi(z)$ is defined and summable in $|z|<1-r$. By Fubini's theorem we have

(35) $$\iint_{|z|<1-r} [\mathfrak{M}_r\xi(z))]\lambda_{\bar{z}}(z)|dz\wedge d\bar{z}| = \iint_{|z|<1-r} \xi(z)[\mathfrak{M}_r\lambda_{\bar{z}}(z)]|dz\wedge d\bar{z}|$$
$$= \iint_{|z|<1-r} \xi(z)\frac{\partial}{\partial\bar{z}}[\mathfrak{M}_r\lambda(z)]|dz\wedge d\bar{z}|\,.$$

$\mathfrak{M}_r\lambda(z)$ is continuously differentiable and has a compact carrier comprised in $|z|<1-r$. On account of the orthogonality condition (34) the right-hand side in (35) vanishes, whence

$$\iint_{|z|<1-r} [\mathfrak{M}_r\xi(z)]\lambda_{\bar{z}}(z)|dz\wedge d\bar{z}| = 0\,.$$

Since $\mathfrak{M}_r\xi(z)$ is smooth, we obtain by Green's formula

$$\iint_{|z|<1-r} \frac{\partial}{\partial\bar{z}}[\mathfrak{M}_r\xi(z)]\lambda(z)|dz\wedge d\bar{z}| = 0\,.$$

Arbitrariness of $\lambda(z)$ enables us to conclude that $\partial\mathfrak{M}_r\xi(z)/\partial\bar{z}$ vanishes identically on $|z|\leq 1-r$, i.e., $\mathfrak{M}_r\xi(z)$ is holomorphic in $|z|<1-r$. Let us put

$$\mathfrak{M}_r\xi(z) = a_0+a_1z+\cdots \qquad \text{for } |z|<1-r\,.$$

$\mathfrak{M}_{r_2}[\mathfrak{M}_{r_1}\xi(z)]$ is defined in $|z|<1-r_1-r_2$ and we have

$$\mathfrak{M}_{r_2}[\mathfrak{M}_{r_1}\xi(z)] = \int_0^{r_2}\int_0^{2\pi} \mathfrak{M}_{r_1}\xi(z+te^{i\theta})Y_{r_2}(te^{i\theta})\,td\theta dt$$

$$= \int_0^{r_2}\int_0^{2\pi} [a_0+a_1(z+te^{i\theta})+\cdots]Y_{r_2}(te^{i\theta})\,td\theta dt = a_0+a_1z+\cdots$$

$$= \mathfrak{M}_{r_1}\xi(z)\,.$$

Hence by Fubini's theorem

$$\mathfrak{M}_{r_1}\xi(z) = \mathfrak{M}_{r_2}[\mathfrak{M}_{r_1}\xi(z)] = \mathfrak{M}_{r_1}[\mathfrak{M}_{r_2}\xi(z)] = \mathfrak{M}_{r_2}\xi(z)\,,$$

which implies the independency of $\mathfrak{M}_r\xi(z)$ on r. Next we shall show $\lim_{r\to 0}\mathfrak{M}_r\xi(z)=\xi(z)$ for almost all z in $|z|<1-r$. Given any $\varepsilon>0$, we can choose a positive number δ, such that $\iint_X |\xi(z)|\,|dz\wedge d\bar{z}|<\varepsilon$ for every measurable subset X of $\mathfrak{N}_j$ provided $\operatorname{mes} z(X)<\delta$. There exists a closed set E_z comprised in $|z|<1-r$, such that $\xi(z)$ is continuous on E_z and $\pi(1-r)^2-\operatorname{mes} E_z<\delta/2$. Hence, if r is sufficiently small, $|\xi(z+\zeta)-\xi(z)|<\varepsilon$ for z, $z+\zeta\in E_z, |\zeta|<r$. When we denote by $E_z^{(\zeta)}$ the translated set of E_z by a vector ζ and define the set-function $\mu(E_z)=\iint_{E_z} |\xi(z+\zeta)-\xi(z)|\,|dz\wedge d\bar{z}|$, we have $\pi(1-r)^2-\operatorname{mes}(E_z\cap E_z^{(\zeta)})\leq[\pi(1-r)^2-\operatorname{mes} E_z]+[\pi(1-r)^2-\operatorname{mes} E_z^{(\zeta)}]-2[\pi(1-r)^2-\operatorname{mes} E_z]<\delta$. So $\mu\{z\mid |z|<1-r\}\leq\mu(E_z\cap E_z^{(\zeta)})+\mu[\{z\mid |z|<1-r\}-(E_z\cap E_z^{(\zeta)})]\leq\pi(1-r)^2\varepsilon+2\varepsilon$. Therefore

$$\iint_{|z|<1-r} |\mathfrak{M}_r\xi(z)-\xi(z)|\,|dz\wedge d\bar{z}|$$

$$\leq\iint_{|\zeta|<r}\Big[\iint_{|z|<1-r} |\xi(z+\zeta)-\xi(z)|\,|dz\wedge d\bar{z}|\Big]Y_r(\zeta)\,|d\zeta\wedge d\bar{\zeta}|$$

so far as r is small enough. This implies

$$\lim_{r\to 0}\iint_{|z|<1-r'} |\mathfrak{M}_r\xi(z)-\xi(z)|\,|dz\wedge d\bar{z}| = 0 \qquad \text{for every fixed } r'<1\,.$$

By Fatou's lemma, we can choose a suitable sequence $\{r_n\}_{n=1}^{\infty}$ of radii, such that

$$\iint_{|z|<1-r'} \lim_{n\to\infty}|\mathfrak{M}_{r_n}\xi(z)-\xi(z)|\,|dz\wedge d\bar{z}| = 0\,, \qquad \lim_{n\to\infty} r_n = 0\,.$$

Therefore $\lim_{n\to\infty}[\mathfrak{M}_{r_n}\xi(z)-\xi(z)]=0$ for almost all z in $|z|<1-r'$. That is,

$\lim_{n\to\infty} \mathfrak{M}_{r_n}\xi(z)$ exists and equals $\xi(z)$ at almost every point in $|z|<1-r'$, while $\mathfrak{M}_{r_n}\xi(z)$ is a holomorphic function independent of n; $\xi(z)$ is equal to a holomorphic function $\mathfrak{M}_r\xi(z)$ almost everywhere in $|z|<1-r'$. Letting $r'\to 1$, we conclude that $\xi(z)$ can be identified with a holomorphic function in $|z|<1$ through a suitable modification of its definition on a set of measure zero. Since $\mathfrak{N}_j$ may be any one of the finite collection of coordinate neighbourhoods on R, we see consequently that $p(z)\overline{q(z)}dz^2$ is an analytic quadratic differential on R. Thus we have proved

Theorem 2. *There exists at least one topological mapping from R to R' which belongs to the given homotopy class and is harmonic relative to the given conformal metric on R'.*

EXAMPLE. Consider the simplest case in which R and R' are both tori (η, **A** being naturally fixed). Then the harmonic map of R onto R' is unique except the conformal mappings of R onto itself.

Suppose there exist two such mappings, say, f and $\tilde{f}$. Let $\mathfrak{q}$ be an arbitrary point on R' and α_f, $\alpha_{\tilde{f}}$ the holomorphic quadratic differentials on R associated with $f, \tilde{f}$ respectively. Since the dimension of the vector space of analytic quadratic differentials on R is one and $\tilde{f}$ is homotopic to f, the nets on R woven by their trajectories and orthogonal trajectories $\mathfrak{I}\{\alpha_f\}=0$ and $\mathfrak{I}\{\alpha_{\tilde{f}}\}=0$ coincide with each other and it is also the case with $\mathfrak{R}\{\alpha_f\}=0$ and $\mathfrak{R}\{\alpha_{\tilde{f}}\}=0$. Further we can see without so much difficulty that these families of analytic curves correspond to each other by the mapping $f^{-1}(\mathfrak{q})\to\tilde{f}^{-1}(\mathfrak{q})$ and that composite map $f^{-1}\tilde{f}$ reduces to a conformal transformation of R onto itself.

REMARK 2. To the limited knowledge of the present author, the question whether or not the uniqueness in Theorem 2 holds good seems to be still unanswered for the general case. So at the end of the paper we should like to pose the

PROBLEM. We restrict ourselves to the pair (R, R') of (topologically equivalent compact Riemann) surfaces with genera superior to 1. Then, is the harmonic mapping relative to the given conformal metric on R' unique within the preassigned homotopy class?

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Added in Proof. Correction. In my preceding paper [8], please read line 6 at page 206 as follows:

(iv) ω is holomorphic and free from zeros in the interior of every simplex S_j $(j=1, 2,\cdots, \kappa)$.