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EXAMPLES OF PROJECTIVE MANIFOLDS NOT ADMITTING KÄHLER METRIC WITH CONSTANT SCALAR CURVATURE

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Introduction. We denote by $\boldsymbol{P}^k$ a complex projective space of dimension k. The direct product $\boldsymbol{P}^m\times\boldsymbol{P}^n$ $(m\leqq n)$ has a natural imbedding in $\boldsymbol{P}^{mn+m+n}$, called the Segre imbedding. Let $M_{m,n}$ be a non-singular hyperplane section of $\boldsymbol{P}^m\times\boldsymbol{P}^n$ in $\boldsymbol{P}^{mn+m+n}$. The first Chern class of $M_{m,n}$ is positive, (the convention here is that the first Chern class of $\boldsymbol{P}^k$ is positive). The purpose of this note is to prove the following

Theorem. *If $m<n$, $M_{m,n}$ does not admit a Kähler metric whose scalar curvature is constant. If $m=n$, $M_{m,n}$ admits a homogeneous Einstein-Kähler metric and hence its scalar curvature is constant.*

This implies immediately that $M_{m,n}$ $(m<n)$ can not be Einstein-Kähler. This fact for the cases $m=1$, $n\geqq 2$ and $m=2$, $n=3$ and 4 have been verified by Y. Sakane [3] and [4]. We show here that some modification of his idea leads to the above theorem.

1. Let (z_α), (w_β) and $(\xi_{\alpha\beta})$ be homogeneous coordinates of $\boldsymbol{P}^m$, $\boldsymbol{P}^n$ and $\boldsymbol{P}^{mn+m+n}$ respectively. The equations

$$\xi_{\alpha\beta}=z_\alpha w_\beta\,,\quad \alpha=0,\cdots,m\,,\quad \beta=0,\cdots,n$$

define the Segre imbedding, which is independent on the choice of these homogeneous coordinates. Let p_1 and p_2 denote the projections of $\boldsymbol{P}^m\times\boldsymbol{P}^n$ into the 1st and 2nd components respectively. Let H_1 and H_2 be the hyperplane bundles of $\boldsymbol{P}^m$ and $\boldsymbol{P}^n$ respectively. Then the line bundle $p_1^*H_1\otimes p_2^*H_2$ is the restriction of the hyperplane bundle over $\boldsymbol{P}^{mn+m+n}$ to the submanifold $\boldsymbol{P}^m\times\boldsymbol{P}^n$. We denote by M the intersection of a hyperplane in $\boldsymbol{P}^{mn+m+n}$ with $\boldsymbol{P}^m\times\boldsymbol{P}^n$. Thus the line bundle $\{M\}$ associated to the divisor M in $\boldsymbol{P}^m\times\boldsymbol{P}^n$ is $p_1^*H_1\otimes p_2^*H_2$.

Let $K(M)$ and $K(\boldsymbol{P}^m\times\boldsymbol{P}^n)$ be the canonical line bundles for M and for

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$\boldsymbol{P}^m \times \boldsymbol{P}^n$ respectively. Then

$$K(M) = \boldsymbol{K}(P^m \times P^n) \otimes \{M\}\,|_M\,,$$

and hence

$$K(M) = p_1^* H_1^{-m} \otimes p_2^* H_2^{-n}\,|_M\,.$$

This means that the 1st Chern class of M is positive definite.

2. We shall show that any holomorphic vector field on M extends to a holomorphic vector field on $\boldsymbol{P}^m \times \boldsymbol{P}^n$. Indeed, the following lemma asserts a stronger fact.

Put $X = \boldsymbol{P}^m \times \boldsymbol{P}^n$. Let $T(X)$ and $T(M)$ be the tangent bundles of X and M respectively. Given a holomorphic vector bundle E, we dencte by $\Omega^p(E)$ the sheaf of germs of holomorphic E-valued p-forms.

Lemma (Sakane [3], p 356). *The exact sequence*

$$0 \to \Omega^0(T(X) \otimes \{M\}^{-1}) \to \Omega^0(T(X)) \to \Omega^0(T(X)\,|_M) \to 0$$

induces the exact sequence

$$0 \to H^0(X,\, T(X)) \to H^0(M,\, T(X)\,|_M) \to 0\,.$$

Here, we present a simpler proof. The above given sequence implies the following exact sequence:

$$\begin{aligned} 0 &\to H^0(X,\, \Omega^0(T(X) \otimes \{M\}^{-1})) \to H^0(X,\, \Omega^0(T(X))) \\ &\to H^0(M,\, \Omega^0(T(X)\,|_M)) \to H^1(X,\, \Omega^0(T(X) \otimes \{M\}^{-1})) \to . \end{aligned}$$

In order to prove the lemma, it suffices to show that

$$H^q(X,\, \Omega^0(T(X) \otimes \{M\}^{-1})) = 0\,, \quad \text{for } q = 0,\, 1\,.$$

The cohomology group in question is isomorphic to

$$H^{m+n-q}(X,\, \Omega^1(K(X) \otimes \{M\}))$$

by the Serre duality. Since

$$K(X) \otimes \{M\} = p_1^* H_1^{-m} \otimes p_2^* H^{-n}\,,$$

we can apply the Künneth formula,

$$\begin{aligned} &H^{m+n-q}(X,\, \Omega^1(K(X) \otimes \{M\})) = \\ &\sum_{p'+p''=m+n-q} \{H^{p'}(P^m,\, \Omega^0(H_1^{-m})) \otimes H^{p''}(P^n,\, \Omega^1(H_2^{-n}) + \\ &\qquad H^{p'}(P^m,\, \Omega^1(H_1^{-m})) \otimes H^{p''}(P^n,\, \Omega^0(H_2^{-n})\}\,. \end{aligned}$$

As a result of the Kodaira-Nakano vanishing theorem and the Serre duality,

$$H^p(P^m, \Omega^0(H_1^{-m})) = 0 \quad \text{for all } p .$$

Thus, we see that

$$H^{m+n-q}(X, \Omega^1(K(X)\otimes\{M\})) = 0 \quad \text{for all } q ,$$

completing the proof.

3. The group of all holomorphic automorphisms of $\boldsymbol{P}^m\times\boldsymbol{P}^n$ is $PSL(m+1, \boldsymbol{C})\times PSL(n+1, \boldsymbol{C})$. By the previous lemma the group G of all holomorphic automorphisms of $\boldsymbol{P}^m\times\boldsymbol{P}^n$ which map M onto itself contains the identity connected component of the group of all holomorphic automorphisms of M.

Let $\tilde{G}$ be the inverse image of G under the projection $SL(m+1, \boldsymbol{C})\times SL(n+1, \boldsymbol{C}) \rightarrow PSL(m+1, \boldsymbol{C})\times PSL(n+1, \boldsymbol{C})$. The isomorphism $\boldsymbol{C}^{m+1}\otimes\boldsymbol{C}^{n+1} = \boldsymbol{C}^{(m+1)(n+1)}$ defines a canonical imbedding of $SL(m+1, \boldsymbol{C})\times SL(n+1, \boldsymbol{C})$ into $SL((m+1)(n+1), \boldsymbol{C})$. Thus, an element in $SL(m+1, \boldsymbol{C})\times SL(n+1, \boldsymbol{C})$ is a linear transformation of $\boldsymbol{C}^{(m+1)(n+1)}$ and maps a hyperplane onto another hyperplane, obviously.

Let H be the hyperplane in $\boldsymbol{P}^{mn+m+n}$ such that $M=(\boldsymbol{P}^m\times\boldsymbol{P}^n)\cap H$. Then, $\tilde{G}$ consists of elements $(\sigma, \tau)\in SL(m+1, \boldsymbol{C})\times SL(n+1, \boldsymbol{C})$ which leave the hyperplane H invariant. The hyperplane H is the zero of a non-zero linear form A on $\boldsymbol{C}^{m+1}\otimes\boldsymbol{C}^{n+1}$. If we put

$$a(z, w) = A(z\otimes w),\ z\in\boldsymbol{C}^{m+1},\ w\in\boldsymbol{C}^{n+1} ,$$

a is a bilinear form on $\boldsymbol{C}^{m+1}\times\boldsymbol{C}^{n+1}$. Thus, $\tilde{G}$ is the set of all $(\sigma, \tau)\in SL(m+1, \boldsymbol{C})\times SL(n+1, \boldsymbol{C})$ with the property

$$a(\sigma(z), \tau(w)) = \lambda_{\sigma,\tau}a(z\ w),\ z\in\boldsymbol{C}^{m+1},\ w\in\boldsymbol{C}^{n+1} ,$$

for some non-zero constant $\lambda_{\sigma,\tau}$.

Now, choose homogeneous coordinates on $\boldsymbol{C}^{m+1}$ and $\boldsymbol{C}^{n+1}$ so that

$$a(z, w) = z_0w_0+ \cdots +z_kw_k .$$

where $0\leqq k\leqq m(\leqq n)$. Suppose $k<m$. Take a point (z, w) with $z_0=\cdots=z_{k-1}=w_0=\cdots=w_{k-1}=0$. Then the point is in M, but is a singular point, as

$$dA|_{C^{m+1}\times C^{n+1}} = \textstyle\sum_{\alpha=0}^{k-1} w_\alpha dz_\alpha+z_\alpha dw_\alpha$$

vanishes at this point. Therefore k must be m, and

$$a(z, w) = z_0w_0+ \cdots +z_mw_m .$$

Write an $(n+1)\times(n+1)$ matrix τ in the form

$$(*) \qquad \left(\begin{array}{c|c} \tau_1 & \beta \\ \hline \gamma & \delta \end{array}\right)$$

where τ_1 is an $(m+1)\times(m+1)$ matrix. For $(\sigma, \tau)\in SL(m+1, \boldsymbol{C})\times SL(n+1, \boldsymbol{C})$,

$${}^t\sigma \left(\begin{array}{c|c} I_{m+1} & 0 \end{array}\right) \tau = \left(\begin{array}{c|c} \lambda I_{m+1} & 0 \end{array}\right)$$

if and only if $\tau_1=\lambda{}^t\sigma^{-1}$ and, $\beta=0$.

Let L be the subgroup of $SL(n+1, \boldsymbol{C})$ consisting of all matrices τ with $\beta=0$ in the above expression $(*)$. The map $(\sigma, \tau)\rightarrow\tau$ defines a covering homomorphism of $\tilde{G}$ onto L. The subset N in L consisting of all matrices of the form

$$\left(\begin{array}{c|c} I_{m+1} & 0 \\ \hline * & I_{n-m} \end{array}\right)$$

is a normal subgroup in L. The center Z of L is the set of all matrices of the form

$$\left(\begin{array}{c|c} \alpha I_{m+1} & 0 \\ \hline 0 & \beta I_{n-m} \end{array}\right)$$

with $\alpha^{m+1}\beta^{n+1}=1$. The radical of L is $N\cdot Z$ and $N\cap Z=\{\text{identity}\}$. Thus, we conclude that if $m<n$, the Lie algebra of all holomorphic vector fields is not reductive, and if $m=n$, the identity connected component of G is isomorphic to $PSL(m+1, \boldsymbol{C})$, which acts transitively on M;

$$M = SU(m+1)/S(U(1)\cdot U(1)\cdot U(m+1))\,.$$

The complex Lie algebra of all holomorphic vector fields on a compact complex manifold is known to be reductive if M admits a Kähler metric with constant scalar curvature (A. Lichnerowicz [2], see [1]). Applying this result to our case, we obtain our non-existence theorem. When $m=n$, M is a compact homogeneous space of kählerian type and admits an Einstein-Kähler metric (see M. Takeuchi [5], Corollary in p. 195).

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