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Author(s)	Kamae, Teturo; Yu-Mei, Xue
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TWO DIMENSIONAL WORD WITH $2k$ MAXIMAL PATTERN COMPLEXITY

TETURO KAMAE and XUE YU-MEI

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1. Introduction

For an infinite 1-dimensional word $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ over a finite alphabet A, Teturo Kamae and Luca Zamboni [1] introduced the maximal pattern complexity as

$$p^*_\alpha(k) := \sup_\tau \sharp\{\alpha_{n+\tau(0)}\alpha_{n+\tau(1)}\cdots\alpha_{n+\tau(k-1)};\ n = 0,\ 1,\ 2,\ldots\}$$

where the supremum is taken over all sequences of integers $0 = \tau(0) < \tau(1) < \cdots < \tau(k-1)$ of length k, and $\sharp S$ denotes the cardinality of the set S. They proved that α is eventually periodic if and only if $p^*_\alpha(k)$ is bounded in k, while otherwise, $p^*_\alpha(k) \geq 2k$ $(k = 1,\ 2,\ldots)$.

Teturo Kamae, Rao Hui and Xue Yu-Mei [3] considered the maximal pattern complexity for 2-dimensional words defined on $\mathbb{Z}^2$ and proved that either $p^*_\alpha(k)$ is bounded in k or $p^*_\alpha(k) \geq 2k$ $(k = 1,\ 2,\ldots)$ if α satisfies a 2-dimensional recurrence condition.

In this paper, we consider the maximal pattern complexity for 2-dimensional words defined on

$$\Omega := \mathbb{N}^2 \setminus \{(0,0)\}.$$

Let $\alpha = (\alpha(x,\, y)_{(x,y)\in\Omega}) \in A^\Omega$ be a 2-dimensional word over $\mathbf{A} = \{0, 1\}$ defined on Ω. Let τ be a finite set in $\mathbb{Z}^2$ with $(0,0) \in \tau$ and $\sharp\tau = k$, which is called a *k-window*. For any $i \in \Omega$ with $i + \tau \subset \Omega$, we denote

$$\alpha[i+\tau] := (\alpha(i+j))_{j\in\tau} \in A^\tau.$$

We also denote

$$F_\tau(\alpha) := \{(\alpha[i+\tau];\ i \in \Omega \text{ with } i+\tau \subset \Omega\}$$
$$p^*_\alpha(k) := \sup\{\sharp F_\alpha(\tau);\ \tau\colon k\text{-window}\}\quad (k = 1,\ 2,\ldots).$$

DEFINITION 1. α is called *eventually 2-periodic* if there exist $p,\ q \in \mathbb{Z}_+$ and a, $b \in \mathbb{N}$ such that for any $(x,\, y) \in \Omega$, $\alpha(x,\, y) = \alpha(x+p,\, y)$ holds if $x \geq a$ and $\alpha(x,\, y) = \alpha(x,\, y+q)$ holds if $y \geq b$.

DEFINITION 2. α is called *minimal* if for any positive integer L, there exists N such that for any $(n, m) \in \Omega$ there exists $(n', m') \in \Omega$ with $|n - n'| \leq N$, $|m - m'| \leq N$ such that $\alpha(x + n', y + m') = \alpha(x, y)$ holds for any $(x, y) \in \Omega$ with $x < L$, $y < L$.

DEFINITION 3. α is called *sectionally periodic* if for any (a, b), $(p, q) \in \Omega$, the word β on $n \in \mathbb{N}$ defined by $\beta(n) = \alpha(a + np, b + nq)$ is periodic.

In this paper, we characterize the words with bounded maximal pattern complexity. We give an example of word α with $p^*_\alpha(k) = 2k$ $(k = 1, 2, \ldots)$ which is minimal and sectionally periodic.

2. Words with bounded maximal pattern complexity

Theorem 1. *α is eventually 2-periodic if and only if $p^*_\alpha(k)$ is bounded in k.*

Proof. Assume that α is eventually 2-periodic. Take $p, q \in \mathbb{Z}_+$ and $a, b \in \mathbb{N}$ such that for any $(x, y) \in \Omega$, $\alpha(x, y) = \alpha(x + p, y)$ holds if $x \geq a$ and $\alpha(x, y) = \alpha(x, y + q)$ holds if $y \geq b$.

Let τ be a k-window. Let

$$\begin{aligned}
\Omega_1 &:= \{i = (x, y) \in \Omega;\ i + \tau \subset \Omega \cap [a, \infty) \times [b, \infty)\} \\
\Omega_2 &:= \{i = (x, y) \in \Omega \setminus \Omega_1;\ i + \tau \subset \Omega \cap [a, \infty) \times [0, \infty)\} \\
\Omega_3 &:= \{i = (x, y) \in \Omega \setminus \Omega_1;\ i + \tau \subset \Omega \cap [0, \infty) \times [b, \infty)\} \\
\Omega_4 &:= \{i = (x, y) \in \Omega \setminus (\Omega_1 \cup \Omega_2 \cup \Omega_3);\quad i + \tau \subset \Omega\}.
\end{aligned}$$

For any $i = (x, y) \in \Omega_1$, we have

$$\alpha[i + (np, mq) + \tau] = \alpha[i + \tau] \quad (\forall n,\ m = 0, 1, 2, \ldots).$$

Therefore, there exist at most pq different elements among $\alpha[i + \tau]$ with $i = \Omega_1$.

For any $i = (x, y) \in \Omega_2$, we have

$$\alpha[i + (np, 0) + \tau] = \alpha[i + \tau] \quad (\forall n = 0, 1, 2, \ldots).$$

Hence, there exist at most pb different elements among $\alpha[i + \tau]$ with $i = \Omega_2$.

In the same way, there exist at most qa different elements among $\alpha[i + \tau]$ with $i = \Omega_2$. Finally, there exist at most ab elements in Ω_4.

Therefore, we have

$$\sharp F_\alpha(\tau) \leq pq + pb + qa + ab = (p + a)(q + b).$$

Thus, $p^*_\alpha(k) \leq (p + a)(q + b)$ for $k = 1, 2, \ldots$, and hence, $p^*_\alpha(k)$ is bounded in k.

Conversely, assume that $\sup_{k=1,\ 2,\dots} p^*_\alpha(k) = C < \infty$. There exist $k = 1,\ 2, \dots$ and a k-window τ such that $\sharp F_\alpha(\tau) = C$. Take a positive integer L such that τ is contained in a square of size $L \times L$. Let σ be the $(L+1)^2$-window such that

$$\sigma = \{(x, y) \in \Omega;\ 0 \le x \le L,\ 0 \le y \le L\}$$

and σ' be the $(L+2)^2$-window such that

$$\sigma' = \{(x, y) \in \Omega;\ 0 \le x \le L+1,\ 0 \le y \le L+1\}.$$

Since

$$C = \sharp F_\alpha(\tau) \le \sharp F_\alpha(\sigma) \le \sharp F_\alpha(\sigma') \le C,$$

we have $\sharp F_\alpha(\sigma) = \sharp F_\alpha(\sigma') = C$. This implies that each element $\xi \in F_\alpha(\sigma)$ has a unique extension in $F_\alpha(\sigma')$. Therefore, there exists a function $h\colon F_\alpha(\sigma) \to F_\alpha(\sigma')$ such that $h(\alpha[i+\sigma]) = \alpha[i+\sigma']$ for any $i \in \Omega$.

In particular, there exist functions $f,\ g\colon F_\alpha(\sigma) \to F_\alpha(\sigma)$ such that

$$\begin{aligned} f(\alpha[i+\sigma]) &= \alpha[i+(1,0)+\sigma] \\ g(\alpha[i+\sigma]) &= \alpha[i+(0,1)+\sigma] \end{aligned} \tag{1}$$

for any $i \in \Omega$.

Since f is a transformation on a finite set, there exist $a \in \mathbb{N}$ and a period $p \in \mathbb{Z}_+$ such that

$$f^{n+p} = f^n \tag{2}$$

any $n = a,\ a+1,\ a+2, \dots.$ Since

$$\alpha[(x, y) + \sigma] = f^x(\alpha[(0, y) + \sigma])$$

by (1), it follows from (2) that

$$\alpha[(x, y) + \sigma] = \alpha[(x+p, y) + \sigma]$$

for any $(x, y) \in \Omega$ with $x \ge a$.

In particular, we have

$$\alpha(x, y) = \alpha(x+p, y)$$

for any $(x, y) \in \Omega$ with $x \ge a$. In the same way, we have

$$\alpha(x, y) = \alpha(x, y+q)$$

for any $(x, y) \in \Omega$ with $y \ge b$. Thus, α is eventually 2-periodic. □

3. A word with 2*k* maximal pattern complexity

A window τ' is said to be an *immediate extension* of a window τ if $\tau' \supset \tau$ and $\sharp\tau' = \sharp\tau + 1$.

The following Lemma 1 is proved in [2, Theorem 3] for words defined on $\mathbb{N}$. It remains true for words defined on Ω.

Lemma 1. *Let* $\alpha \in \{0,1\}^{\Omega}$ *be such that* $p^*_\alpha(2) = 4$*. Assume that for any* 2*-window* τ *and for any immediate extension* τ' *of* τ*, it holds that* $\sharp F_\alpha(\tau') \le \sharp F_\alpha(\tau) + 2$*. Then, we have* $p^*_\alpha(k) \le 2k$ $(k = 1, 2, \ldots)$.

Define a 2-dimensional word $\alpha \in \{0,1\}^{\Omega}$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } e_2(x) = e_2(y) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

for any $(x, y) \in \Omega$, where for $x \in \mathbb{N}$, $e_2(x) = n$ if and only if $2^n \mid x$ and $2^{n+1} \nmid x$. We also define $e_2(0) = \infty$.

REMARK 1. The word α defined by (3) together with $\alpha((0,0)) = 0$ is the fixed point of the 2-dimensional substitution

$$\sigma:\ 0 \to \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \quad \text{and} \quad 1 \to \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}, \tag{4}$$

so that $\alpha = \sigma^\infty(0)$.

Theorem 2. *For* α *defined by* (3)*, we have* $p^*_\alpha(k) = 2k$ *for any* $k = 1, 2, \ldots$.

Proof. First we prove that $p^*_\alpha(k) \ge 2k$ $(k = 1, 2, \ldots)$. It is clear that $p^*_\alpha(1) = 2$. For any $k = 2, 3, \ldots$, take a k-window $\tau := \{(0,0), (1,1), \ldots, (k-1, k-1)\}$. Then, since

$$\begin{aligned} \alpha[(1,1) + \tau] &= (1, 1, \ldots, 1) \\ \alpha[(2^k - n,\ 2^{k+1} - n) + \tau] &= (1, \ldots, 1, \overset{(n)}{0}, 1, \ldots, 1) \\ &\quad (n = 0, 1, \ldots, k-1), \end{aligned}$$

$F_\alpha(\tau)$ contains $k + 1$ elements containing the letter 0 at most once.

Now, let us consider the elements in $F_\alpha(\tau)$ containing the letter 0 at least twice. They are determined by $a \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $0 \le a < 2^n$ and $a + 2^n < k$ since there exists a unique element in $F_\alpha(\tau)$ of the form

$$(1, \ldots, 1, \overset{(a)}{0}, 1, \ldots, 1, \overset{(a+2^n)}{0}, * * *)$$

which is realized as $\alpha[(2^n - a,\ 2^{n+1} - a) + \tau]$. There are exactly

$$L := \sum_{n=0}^{\lfloor \log_2 k \rfloor} \min\{2^n, k - 2^n\}$$

number of elements of this type. Since

$$\begin{aligned} L &= \sum_{n=0}^{\lfloor \log_2 k \rfloor - 1} 2^n + k - 2^{\lfloor \log_2 k \rfloor} \\ &= 2^{\lfloor \log_2 k \rfloor} - 1 + k - 2^{\lfloor \log_2 k \rfloor} = k - 1, \end{aligned}$$

we have $\sharp F_\alpha(\tau) = k + 1 + k - 1 = 2k$. Thus, $p^*_\alpha(k) \geq 2k$ $(k = 1,\ 2, \ldots)$.

To prove that $p^*_\alpha(k) \leq 2k$ $(k = 1,\ 2, \ldots)$, it is sufficient by Lemma 1 to prove that for any 2-window τ and for any immediate extension τ' of τ, it holds that

$$\sharp F_\alpha(\tau') \leq \sharp F_\alpha(\tau) + 2. \tag{5}$$

Take an arbitrary 2-window $\tau = \{(0, 0) = \tau_0,\ \tau_1\}$ and an arbitrary immediate extension $\tau' = \{(0, 0) = \tau_0,\ \tau_1,\ \tau_2\}$ of τ.

To prove (5), we divide into 3 cases according to the parity of τ_1

Case 1: $\tau_1 \in e \times e$

Case 2: $\tau_1 \in e \times o$

Case 3: $\tau_1 \in o \times o$,

where "e" stands for the set of even numbers, while "o" stands for the set of odd numbers. By symmetry, we can reduce the case $\tau_1 \in o \times e$ to Case 2.

Lemma 2. (i) *In* Case 1, $F_\alpha(\tau) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ *holds*.
(ii) *In* Case 2, $F_\alpha(\tau) = \{(0, 0), (0, 1), (1, 0)\}$ *holds*.
(iii) *In* Case 3, $F_\alpha(\tau) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ *holds*.

Proof. Let $\tau_1 = (u, v)$.
(i) Let $(u, v) \in e \times e$. For $(x, y) \in e \times o$, we have $\alpha[(x, y) + \tau] = (0, 0)$. If $u = v$, then by taking integers N and M with $e_2(u) < N < M$, we have $\alpha[(2^N, 2^M) + \tau] = (0, 1)$. If $u \neq v$, then assuming that $u < v$ without loss of generality, we have $\alpha[(v - u, 0) + \tau] = (0, 1)$. If $u \neq v$, then we have $\alpha[(2^N v - u, 2^N v - u) + \tau] = (1, 0)$ for a sufficiently large integer N. If $u = v$, then by taking integers N and M with $e_2(u) < N < M$, we have $\alpha[(2^N - u, 2^M - v) + \tau] = (1, 0)$. Finally, for $(x, y) \in o \times o$, we have $\alpha[(x, y) + \tau] = (1, 1)$.
(ii) Let $(u, v) \in e \times o$. Then, $\alpha[(2, 4) + \tau] = (0, 0)$, $\alpha[(v, u) + \tau] = (0, 1)$, $\alpha[(1, 1) + \tau] =$

$(1,0)$, while $\alpha[(x,y)+\tau] = (1,1)$ is impossible since either x and y have different parities or $x+u$ and $y+v$ have different parities.
(iii) Let $(u,v) \in o \times o$. For $(x,y) \in e \times o$, we have $\alpha[(x,y)+\tau] = (0,0)$. We also have $\alpha[(2,4)+\tau] = (0,1)$ and $\alpha[(2^N - u, 2^M - v)+\tau] = (1,0)$ for integers N and M such that $u+v < 2^N < 2^M$. Moreover, $\alpha[(2,2)+\tau] = (1,1)$. □

We divide the above 3 cases into the following 10 subcases according to the parity of τ_2

Case 1-1: $\tau_1 \in e \times e,\ \tau_2 \in e \times e$
Case 1-2: $\tau_1 \in e \times e,\ \tau_2 \in e \times o$
Case 1-3: $\tau_1 \in e \times e,\ \tau_2 \in o \times o$
Case 2-1: $\tau_1 \in e \times o,\ \tau_2 \in e \times e$
Case 2-2: $\tau_1 \in e \times o,\ \tau_2 \in e \times o$
Case 2-3: $\tau_1 \in e \times o,\ \tau_2 \in o \times e$
Case 2-4: $\tau_1 \in e \times o,\ \tau_2 \in o \times o$
Case 3-1: $\tau_1 \in o \times o,\ \tau_2 \in e \times e$
Case 3-2: $\tau_1 \in o \times o,\ \tau_2 \in e \times o$
Case 3-3: $\tau_1 \in o \times o,\ \tau_2 \in o \times o$.

Lemma 3. (i) *In* Case 1-2, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,1,1),(1,0,1),(1,1,1)\}$.
(ii) *In* Case 1-3, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,1,0),(1,0,0)\}$.
(iii) *In* Case 2-1, $F_\alpha(\tau') \subset F_\alpha(\tau) \times \{0,1\} \setminus \{(0,1,1)\}$.
(iv) *In* Case 2-2, $F_\alpha(\tau') \subset F_\alpha(\tau) \times \{0,1\} \setminus \{(1,0,1)\}$.
(v) *In* Case 2-3, $F_\alpha(\tau') \subset F_\alpha(\tau) \times \{0,1\} \setminus \{(1,0,1)\}$.
(vi) *In* Case 2-4, $F_\alpha(\tau') \subset F_\alpha(\tau) \times \{0,1\} \setminus \{(0,1,1)\}$.
(vii) *In* Case 3-1, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,0,1),(1,0,0)\}$.
(viii) *In* Case 3-2, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,1,1),(1,0,1),(1,1,1)\}$.
(ix) *In* Case 3-3, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,0,1),(0,1,0)\}$.

Proof. Let $\tau_1 = (u,v)$, $\tau_2 = (u',v')$ and $(x,y) \in \Omega$.

(i) Since either x and y have different parities or $x+u'$ and $y+v'$ have different parities, $(1,0,1)$, $(1,1,1)$ do not belong to $F_\alpha(\tau')$. Moreover, since either $x+u$ and $y+v$ have different parities or $x+u'$ and $y+v'$ have different parities, $(0,1,1)$ does not belong to $F_\alpha(\tau')$.

(ii) Note that $\alpha[(x,y)+\tau] \in \{(1,0),(0,1)\}$ implies $(x,y) \in e \times e$. Since $(x,y) \in e \times e$ implies $\alpha((x,y)+(u',v')) = 1$, $(0,1,0)$ and $(1,0,0)$ do not belong to $F_\alpha(\tau')$.

(iii)(iv)(v)(vi)(viii) They follow by applying the parity argument in the proof of (i).

(vii) It follows by the same argument as in the proof of (ii).

(ix) Note that $\alpha((x,y)+(u,v)) \neq \alpha((x,y)+(u',v'))$ implies $(x,y) \in o \times o$. Since $(x,y) \in o \times o$ implies that $\alpha((x,y)) = 1$, $(0,0,1)$, $(0,1,0)$ does not belong to $F_\alpha(\tau')$. □

Lemma 4. (i) *For any subcase except for* Case 1-1, *we have* (5).
(ii) *For any subcase except for* Case 1-1, *we have*

$$\sharp(F_\alpha(\tau') \setminus \{(0,0,0),(1,1,1)\}) \leq 4. \tag{6}$$

Proof. Clear from Lemma 2 and Lemma 3. □

Now we consider Case 1-1. Assume that $\tau_1 \in e \times e$, $\tau_2 \in e \times e$. Then, we have $\alpha[(x,y)+\tau'] = (1,1,1)$ if $(x,y) \in o \times o$ and $\alpha[(x,y)+\tau'] = (0,0,0)$ if $(x,y) \in e \times o \cup o \times e$. Hence we have

$$F_\alpha(\tau') = \{\alpha[(x,y)+\tau'];\ (x,y) \in e \times e\} \cup \{(0,0,0),(1,1,1)\}.$$

Let $\tau'/2 := \{0, \tau_1/2, \tau_2/2\}$. Since $e_2(x) = e_2(y)$ is equivalent to $e_2(2x) = e_2(2y)$, we have $\alpha[(x,y)+\tau'] = \alpha[(x/2,y/2)+\tau'/2]$ for any $(x,y) \in e \times e$. Therefore, we have

$$F_\alpha(\tau') = F_\alpha(\tau'/2) \cup \{(0,0,0),(1,1,1)\}. \tag{7}$$

If $\tau'/2$ is of Case 1-1, we can apply (7) again.

By applying (7) repeatedly, we have

$$F_\alpha(\tau') = F_\alpha(\tau'/2^e) \cup \{(0,0,0),(1,1,1)\}$$

with $\tau'/2^e$ not of Case 1-1. Then, by (ii) of Lemma 4, we have $\sharp F_\alpha(\tau') \leq 6$. Thus, we have (5) by Lemma 2, which complete the proof of Theorem 2. □

Theorem 3. *The word α defined by* (3) *is minimal and sectionally periodic.*

Proof. Take any positive integer L. Let N be a positive integer such that $L < 2^N$. Take any $(n,m) \in \Omega$. Then, there exists $(n',m') \in \Omega$ with $|n-n'| \leq 2^N$ and $|m-m'| \leq 2^N$ such that $e_2(n') \geq N$ and $e_2(m') \geq N$. Then, since $e_2(x+n') = e_2(x)$ and $e_2(y+m') = e_2(y)$ for any $(x,y) \in \Omega$ with $x < L$ and $y < L$, we have $\alpha(x+n', y+m') = \alpha(x,y)$ for any $(x,y) \in \Omega$ with $x < L$ and $y < L$. Thus, α is minimal.

Take any (a,b), $(p,q) \in \Omega$. Let β be a word on $n \in \mathbb{N}$ defined by $\beta(n) = \alpha(a+np, b+nq)$.

Let us consider the case where $a+p = 0$ or $b+q = 0$. Without loss of generality, assume $a+p = 0$. Then, we have $a = p = 0$ and $b > 0$, $q > 0$. Hence, β is periodic since $\beta(n) = 0$ $(n = 0, 1, 2, \ldots)$.

Now assume that $a+p>0$ and $b+q>0$. Let us consider the case where $aq-bp=0$. Suppose that $p=0$. Then, $a>0$ and $q>0$ since $a+p>0$ and $p+q>0$. This contradicts with $aq-bp=0$. Therefore, $p>0$. By the same reason, $q>0$. Since $q(a+np)=p(b+nq)$ for $n=0,\ 1,\ 2,\ldots$, we have $e_2(q)+e_2(a+np)=e_2(p)+e_2(b+nq)$ $(n=0,\ 1,\ 2,\ldots)$. Therefore, either $\beta(n)=1$ $(n=0,\ 1,\ 2,\ldots)$ or $\beta(n)=0$ $(n=0,\ 1,\ 2,\ldots)$ holds according as $e_2(q)=e_2(p)$ or not, and hence, β is periodic.

Now assume that $aq-bp\neq 0$. Let N be a positive integer such that $N>e_2(|aq-bp|)$. Then, since $q(a+np)-p(b+nq)=aq-bp$ $(n=0,\ 1,\ 2,\ldots)$, we have $e_2(|q(a+np)-p(b+nq)|)<N$ $(n=0,\ 1,\ 2,\ldots)$. This implies that $\min\{e_2(q(a+np)), e_2(p(b+nq))\}<N$, and hence, $\min\{e_2(a+np), e_2(b+nq)\}<N$ $(n=0,\ 1,\ 2,\ldots)$. Therefore, if $e_2(a+np)=e_2(b+nq)$, then $e_2(a+np)=e_2(b+nq)<N$ holds, and hence, we have $e_2(a+(n+2^N)p)=e_2(a+np)=e_2(b+nq)=e_2(b+(n+2^N)q)$.

If $e_2(a+np)<e_2(b+nq)$, then either $e_2(a+np)<e_2(b+nq)\le N$ or $e_2(a+np)<N\le e_2(b+nq)$ holds, and hence, we have $e_2(a+(n+2^N)p)=e_2(a+np)<\min\{e_2(b+nq), N\}\le e_2(b+(n+2^N)q)$. In the same way, if $e_2(a+np)>e_2(b+nq)$, then $e_2(a+(n+2^N)p)>e_2(b+(n+2^N)q)$.

Hence, we proved that $e_2(a+np)=e_2(b+nq)$ holds if and only if $e_2(a+(n+2^N)p)=e_2(b+(n+2^N)q)$ holds, so that $\beta(n)=\beta(n+2^N)$ $(n=0,\ 1,\ 2,\ldots)$ and β is periodic.

Thus, α is sectionally periodic. □

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Teturo Kamae
Osaka City Univ.
Osaka 558-8585, Japan
e-mail: kamae@sci.osaka-cu.ac.jp

Xue Yu-Mei
Osaka City Univ.
Osaka 558-8585, Japan
e-mail: setu@sci.osaka-cu.ac.jp