Fujii, J.
Osaka J. Math.
**26** (1989), 429–446

# ON THE SMALLEST PAIRWISE SUFFICIENT SUBFIELD IN THE MAJORIZED STATISTICAL EXPERIMENT

JUNJI FUJII

(Received July 15, 1988)

## 1. Introduction

In the present paper, we discuss the question of the existence of the smallest pariwise sufficient subfield in majorized statistical experiments.

Let $\mathcal{E}=(X, \mathcal{A}, \mathcal{P})$ be a *statistical experiment,* i.e. $X$ be a set, $\mathcal{A}$ a $\sigma$-field of subsets of $X$ and $\mathcal{P}$ a family of probability measures on $\mathcal{A}$.

Assume, throughout the present paper, that there exists a "*majorizing*" measure $\mu$ on $\mathcal{A}$, with respect to which each $P$ in $\mathcal{P}$ has an $\mathcal{A}$-measurable density $dP/d\mu$. Accordingly, $\mathcal{E}$ is called a *majorized experiment.*

For each $P\in\mathcal{P}$, $S_P=\{x\in X; dP/d\mu(x)>0\}$ is called an *$\mathcal{E}$-support* of $P$. We notice that $S_P$ is uniquely determined up to a $\mathcal{P}$-null set and satisfies (1) $P(S_P)=1$, and (2) if $N\subset S_P$ and $P(N)=0$, then $N$ is $\mathcal{P}$-null (see section 2). Conversely, if each $P$ has an $S_P\in\mathcal{A}$ satisfying (1) and (2), then, not only $\mathcal{E}$ is majorized, but it has an "*equivalent majorizing measure*" $\nu$, that is, all the $\mathcal{P}$-null sets are $\nu$-null (see [4] Lemma 9.3). Consequently, every majorized experiment has an equivalent majorizing measure.

A sub $\sigma$-field $\mathcal{B}$(or simply a subfield) of $\mathcal{A}$, ahich is pairwise sufficient and contains a version of the support $S_P$ for all $P$ in $\mathcal{P}$ is called PSS (*pairwise sufficient with supports*). This is a concept in between the usual concepts of sufficiency and pairwise sufficiency. All the three concepts coincide with each other in case $\mathcal{E}$ is dominated. In each of the classes of the pairwise sufficient, PSS and the sufficient subfields, the smallest and the minimal subfields are defined as follows.

For two subfields $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, we write $\mathcal{B}\subset\mathcal{C}[\mathcal{P}]$ if $\mathcal{B}\subset\mathcal{C}\vee\mathcal{N}_{\mathcal{P}}$, the latter being the subfield generated by $\mathcal{C}$ and all the $\mathcal{P}$-null sets. If $\mathcal{B}\subset\mathcal{C}[\mathcal{P}]$ and $\mathcal{C}\subset\mathcal{B}[\mathcal{P}]$, we write $\mathcal{B}=\mathcal{C}[\mathcal{P}]$.

A pairwise sufficient (resp. PSS, sufficient) subfield $\mathcal{B}$ is called *smallest* if $\mathcal{B}\subset\mathcal{C}[\mathcal{P}]$ for every pairwise sufficient (resp. PSS, sufficient) subfield $\mathcal{C}$. A pairwise sufficient (resp. PSS, sufficient) subfield $\mathcal{B}$ is called *minimal* if for every pairwise sufficient (resp. PSS, sufficient) subfield $\mathcal{C}$ with $\mathcal{B}\subset\mathcal{C}[\mathcal{P}]$, it holds that $\mathcal{B}=\mathcal{C}[\mathcal{P}]$.

It is proved that the properties of being minimal and smallest coincide with each other for sufficiency (Burkholder [3]), but not for pairwise sufficiency ([6] and [5]).

The question as to the existence of the smallest (minimal) sufficient subfield in vairous statistical experiments has called attention of such authors as Bahadur, Pitcher, Burkholder and Hasegawa and Perlman (see [1], [8], [9], [3] and [7]). The existence of the smallest PSS subfield is shown for the majorized experiments in [6]. Similar question in pairwise sufficiency is treated here. It is known (see [6]) that the smallest pairwise sufficient subfield does not exist in the discrete experiments. Another condition for the non-existence is given in [5] for a broader class of majorized experiments.

The present paper points out that there are cases of existence as well as non-existence, by giving conditions both for existence and non-existence, the latter being an improvement on that given in the previous paper [5].

Before we study this question, a separation property of the pairwise sufficient subfields, which is essential in handling the question and seems to be important in its own right, is given in section 3. We define a $\sigma$-ring $\mathcal{S}$ as the one generated by all the pairwise likelihood ratios (see Definition 2). Then it is proved that a subfield is pairwise sufficient if and only if it separates $\mathcal{S}$, and PSS if and only if it includes $\mathcal{S}$. A similar characterization has been given in [5], but as it relies upon the concept of maximal decompositions (see section 3), the present one is both a simplification of and an improvement over it.

In Section 4, it is proved that every majorized experiment has a minimal pairwise sufficient subfield. In case $\mathcal{S}$ has an atom, it is given as the subfield generated by all other sets in $\mathcal{S}$, and otherwise it is the subfield generated by all the sets in $\mathcal{S}$, namely $\mathcal{D}=\sigma(\mathcal{P})$. Incidentally this $\mathcal{D}$ is known to be the smallest PSS subfield.

In section 5, as a natural consequence of the foregoing result, the non-existence of the smallest pairwise sufficient subfield is shown for the case which has at least two atoms. The reason is simply that there are at least two minimal subfields corresponding to the atoms and their intersection is not pairwise sufficient. Then the case with only one atom is reduced to the atomless case.

In Seetion 6, the $\mathcal{S}$-atomless case is studied from a general viewpoint. It is pointed out that we can freely choose any element of a lattice $\boldsymbol{\Sigma}$ of $\sigma$-fields to designate it as the basic $\sigma$-filed $\mathcal{A}$ in the experiment $\mathcal{E}=(X, \mathcal{A}, \mathcal{P})$, and the answer to our question is decided by the relative position of $\mathcal{A}$ in the hierarchy of $\boldsymbol{\Sigma}$. Accordingly, proofs of the existence (resp. non-existence) are given for smaller (resp. larger) elements of $\boldsymbol{\Sigma}$ in later sections. Theorem 7 gives a general criterion useful for those proofs.

In section 7, we prove first that $\mathcal{D}$ is the samllest pairwise sufficient subfield when $\mathcal{A}=\mathcal{A}^0$, the $\sigma$-field of all the sets of countable or co-countable type, which is the smallest possible element of $\boldsymbol{\Sigma}$ (Theroem 8). Further, the same conclusion is extended in Theorem 9 to the case where $\mathcal{A}$ is generated by $\mathcal{A}^0$ and a countable number of the sets of uncountable type, provided the latter sets are mutually disjoint.

In seation 8, the non-existence is proved for the case where $\mathcal{A}$ is large enough to allow an injective Borel homomorphism with certain conditions from some set in $\mathcal{S}$ to itself (Theoerm 10). This is an improvement on a result to a similar effect in [5], Theorem 9, and seems to be applied to a fairly wide class of experiments, as is illustrated in Examples.

## 2. Preliminary notions.

In an experiment $\mathcal{E}=(X, \mathcal{A}, \mathcal{P})$, we adopt the following notations. For each $P\in\mathcal{P}$, $A$ set $N\subset X$ is called *P-null* if there exists a set $A\in\mathcal{A}$ such that $N\subset A$ and $P(A)=0$. For each $P\in\mathcal{P}$, we denote by $\mathcal{N}_P$ the class of all $P$-null sets. Put $\mathcal{N}_{\mathcal{P}}=\bigcap_{P\in\mathcal{P}}\mathcal{N}_P$. Each element $N$ of $\mathcal{N}_{\mathcal{P}}$ is called a *$\mathcal{P}$-null* set, and written $N=\phi\,[\mathcal{P}]$. For two subsets $A_1$ and $A_2$ of $X$, we write $A_1\subset A_2[\mathcal{P}]$ if $A_1\backslash A_2=\phi\,[\mathcal{P}]$. Let $\mathcal{B}$ be a subfield of $\mathcal{A}$ and $\mathcal{G}$ a class of subsets of $X$. We denote by $\boldsymbol{S}(\mathcal{G})$ and $\sigma(\mathcal{G})$ the $\sigma$-ring and the $\sigma$-field generated by $\mathcal{G}$, respectively, and we put $\mathcal{B}\vee\mathcal{G}=\sigma(\mathcal{B}\cup\mathcal{G})$.

For two subfields $\mathcal{B}$, $\mathcal{C}$ of $\mathcal{A}$, we write $\mathcal{B}\subset\mathcal{C}[\mathcal{P}]$ if $\mathcal{B}\subset\mathcal{C}\vee\mathcal{N}_{\mathcal{P}}$. If $\mathcal{B}\subset\mathcal{C}$ $[\mathcal{P}]$ and $\mathcal{C}\subset\mathcal{B}[\mathcal{P}]$, we write $\mathcal{B}=\mathcal{C}[\mathcal{P}]$.

Let $\mathcal{B}$ be a subfield of $\mathcal{A}$. For two subsets $A_1, A_2\in\mathcal{A}$ with $A_1\cap A_2=\phi[\mathcal{P}]$, if there exists a set $B\in\mathcal{B}$ such that $B\supset A_1[\mathcal{P}]$ and $B\cap A_2=\phi[\mathcal{P}]$, we say that *$\mathcal{B}$ separates* $\{A_1, A_2\}$. For a subclass $\mathcal{H}$ of $\mathcal{A}$, "*$\mathcal{B}$ separates $\mathcal{H}$*" means that $\mathcal{B}$ separates $\{H_1, H_2\}$ for every pair $H_1, H_2\in\mathcal{H}$ with $H_1\cap H_2=\phi[\mathcal{P}]$.

Let $\mathcal{E}=(X, \mathcal{A}, \mathcal{P})$ be a majorized experiment. We denote by $\mathcal{S}_0$ the class of all $\mathcal{E}$-supports: $\mathcal{S}_0=\{S_P;\, P\in\mathcal{S}\}$. We notice that $\mathcal{S}_0$ is uniquely determined up to $\mathcal{N}_{\mathcal{P}}$ because so is each $S_P$. All the sets in $\mathcal{A}$ are classified into three types as follows.

Definition 1. A set $A\in\mathcal{A}$ is called a set of *countable type* if $A\subset\bigcup_{n\geq 1}S_{P_n}[\mathcal{P}]$ for a countable family $\{S_{P_n};\, n\geq 1\}$ of $\mathcal{S}_0$. (It is same as saying that $A$ is $\sigma$-finite with respect to any of the equivalent majorizing measures.) A set $A\in\mathcal{A}$ is called a set of *co-countable type* if its complement $X\backslash A$ is of countable type. If a set $A\in\mathcal{A}$ is neither of countable nor co-countable type, it is called a set of *uncountable type*.

This is the same classification as the one given in [5] (cf. [5] Theorem 5), though in a slightly different expression.

We define a $\sigma$-ring $\mathcal{S}$, which plays an important role throughout the present paper.

Definition 2 (*the $\sigma$-ring of pairwise likelihood ratios*).

For each pair $P, Q$ in $\mathcal{P}$ and $0 \leq a < 1$, we put

$A(P, Q; a) = \{x \in X; 0 < dP/d(P+Q)(x) \cdot I_{S_P \cup S_Q}(x) \leq a\}$, and define
$\mathcal{S} = \boldsymbol{S}\{A(P, Q; a); P, Q \in \mathcal{P}, a \in (0, 1]\}$.

Define further $\mathcal{S}^+ = \{S \in \mathcal{S}; S \neq \phi[\mathcal{S}]\}$.

We notice that this $\sigma$-ring $\mathcal{S}$ has the following properites. $\mathcal{S}$ is uniquely determined up to $\mathcal{N}_{\mathcal{P}}$ and it consists only of the sets of countable type, because all of its generators are of countable type. Further, $\mathcal{S}$ satisfies the Countable Chain Condition (*C. C. C.*). Namely, suppose that $S \in \mathcal{S}^+$ and $S \supset \bigcup_{\alpha \in L} S_\alpha$ for some disjoint subclass $\{S_\alpha; \alpha \in L\}$ of $\mathcal{S}^+$, then the subclass is countable. This follows from the fact that each element in $\mathcal{S}$ is $\sigma$-finite with respect to any of the equivalent majorizing measures (see [4] Lemma 3.1).

In [6], it was proved that the smallest PSS, written $\mathcal{D}$, exists in a majorized experiment and $\mathcal{D} = \sigma\{dP/d(P+Q) \cdot I_{S_P \cup S_Q}; P, Q \in \mathcal{S}\}$. As $\mathcal{S}$ is a $\sigma$-ring, $\sigma(\mathcal{S}) = \{A \in \mathcal{A}; A \in \mathcal{S}$ or $X \backslash A \in \mathcal{S}\}$. Hence it follows that $\mathcal{D} = \sigma(\mathcal{S})[\mathcal{P}]$.

In what follows we assume that $\mathcal{E}$ is undominated, just to avoid trivial complications in our presentation. It is equivalent to assuming that a set cannot be of both countable and co-countable type, or that $\mathcal{D} \supsetneqq \mathcal{S}$.

Definition 3. A set $A \in \mathcal{S}^+$ is said to be an *$\mathcal{S}$-atom* if $S \in \mathcal{S}$ and $S \subset A[\mathcal{S}]$, then either $S = \phi[\mathcal{S}]$ or $S = A[\mathcal{S}]$ holds.

## 3. A separation property of pairwise sufficient subfields.

First we state the main results of this section. The proofs are given later in this section.

**Theorem 1.** *Let $\mathcal{E} = (X, \mathcal{A}, \mathcal{P})$ be a majorized experiment, $\mathcal{S}$ the $\sigma$-ring of pairwise likelihood ratios defined in section 2 and $\mathcal{B}$ a subfield of $\mathcal{A}$. Then,*
*$\mathcal{B}$ is pairwise sufficient if and only if $\mathcal{B}$ separates $\mathcal{S}$, and*
*$\mathcal{B}$ is PSS if and only if $\mathcal{B}$ includes $\mathcal{S}[\mathcal{P}]$.*

**Corollary 1.** *Let $A \in \mathcal{A}$ and $S \in \mathcal{S}$ be sets of countable type satisfying $A \cap S = \phi[\mathcal{S}]$. If $\mathcal{B}$ is pairwise sufficient, then $\mathcal{B}$ separates $\{A, S\}$.*

There is a similar and essentially equivalent characterization in [5], which invokes the concept of a maximal decompostion. As both the characterization and the concept are required later, they are briefly sketched here.

A subclass $\mathcal{F}$ of $\mathcal{A}$ is called *a maximal decomposition* if it satisfies the following conditions.

(1) For each $F \in \mathcal{F}$, $F \neq \phi[\mathcal{P}]$ and $F \subset S_P[\mathcal{P}]$ for some $P \in \mathcal{P}$,
(2) for each pair of distinct $F, G \in \mathcal{F}$, $F \cap G = \phi[\mathcal{P}]$, and
(3) for any $A \in \mathcal{A}$, if $A \cap F = \phi$ for all $F \in \mathcal{F}$, then $A = \phi[\mathcal{P}]$.

It is proved in [5] (Lemma 2) that there exists a maximal decomposition $\mathcal{F}$ such that $\mathcal{F}\subset\sigma(\mathcal{S}_0)$ $[\mathcal{P}]$. Take such an $\mathcal{F}$ and a subfield $\mathcal{B}$ and assume that $\mathcal{B}\cap F$ is sufficient on every $F\in\mathcal{F}$.

Then ([4], Theorem 5),

*$\mathcal{B}$ is pairwise sufficient if and only if it separates $\mathcal{F}$, and,*

*$\mathcal{B}$ is PSS if and only if it includes $\mathcal{F}$.*

Thus our Theorem 1 is a much simpler characterization, as it does not require the concept of maximal decompositions or verification of sufficiency of $\mathcal{B}$ on every $F$ in $\mathcal{F}$. On the other hand, the previous result is still found useful on occasions, as in the proof of Corollary 1, because of its applicability to the wide class of maximal decompositions.

Theorem 1 is applied to the discrete experiments, the simplest type of majorized experiments, as follows.

REMARK 1. An experiment $\mathcal{E}=(X, \mathcal{A}, \mathcal{P})$ is said to be discrete if the whole space $X$ is an uncountable set, $\mathcal{A}$ is the power set of $X$, each $P\in\mathcal{P}$ is a discrete probability measure on $\mathcal{A}$, and $\mathcal{N}_{\mathcal{P}}=\{\phi\}$.

In [2], it was shown that there exists the smallest sufficient statistic $\boldsymbol{M}$. It is a partition of $X$ consisting of the sets of the following form $[x]$ $(x\in X)$.

For each $x\in X$, we put $\mathcal{P}_x=\{P\in\mathcal{P};\ P(x)>0\}$, $f_{P,Q}=dP/d(P+Q)\cdot I_{S_P\cup S_Q}$, and define

$$[x]=(\bigcap_{P\in\mathcal{P}_x} S_P)\cap(\bigcap_{P,Q\in\mathcal{P}_x}\{y\in X;\ f_{P,Q}(y)=f_{P,Q}(x)\}).$$

Clearly $x\in[x]$. $[x]$ is apparently represented as an uncountable intersection of sets in $\mathcal{S}$. However it is a subset of a countable set $S_P$ (for each $P\in\mathcal{P}_x$), and so the intersections can be expressed as countable intersections, and hence $[x]\in\mathcal{S}$.

We claim that each $[x]$ is an $\mathcal{S}$-atom.

As $[x]$ belongs to $\mathcal{S}$, if it is not an $\mathcal{S}$-atom, then there must be a point $z\in[x]$ such that $x$ and $z$ are separated by $\mathcal{S}$. That is, there exists an $A(P, Q; a)=\{x\in X;\ 0<f_{P,Q}(x)\leq a\}$, one of the generators of $\mathcal{S}$, which separates $x$ and $z$. In that event, $f_{P,Q}(x)$ and $f_{P,Q}(z)$ are different from each other. But it is impossible, as $f_{P,Q}$ is constant on $[x]$ by its definition.

Thus in any discrete experiment, the class of all $\mathcal{S}$-atoms coinsides with the smallest sufficient statistic $\boldsymbol{M}$, and every element of the $\sigma$-ring $\mathcal{S}$ can be represented as a countable union of sets in $\boldsymbol{M}$.

Consequently, in the following Corollary 2 we obtain a result given in [6] (Lemma 4.1) as a special case of Theorem 1. Note that $\mathcal{B}$ separates $\mathcal{S}$ if and only if it separates $\boldsymbol{M}$.

**Corollary 2.** ([6] Lemma 4.1) *In the discrete experiment, a subfield $\mathcal{B}$ is pairwise sufficient if and only if it separates the smallest sufficient statistic $\boldsymbol{M}$.*

Proof of Theorem 1 requires several lemmas.

Lemma 1. *Let $\mathcal{B}$ be a subfield and $\mathcal{K}$ a semi-ring ($\mathcal{K}\subset\mathcal{A}$).*
*It $\mathcal{B}$ separates $\mathcal{K}$, then $\mathcal{B}$ separates $\boldsymbol{S}(\mathcal{K})$.*

Proof. Let $\boldsymbol{R}(\mathcal{K})$ be the ring generated by the semi-ring $\mathcal{K}$. Every element in $\boldsymbol{R}(\mathcal{K})$ is represented as a finite disjoint union of elements in $\mathcal{K}$, so that $\mathcal{B}$ separates $\boldsymbol{R}(\mathcal{K})$ as well as $\mathcal{K}$. Fix $R\in\boldsymbol{R}(\mathcal{K})$. We consider the following class $\mathcal{A}(R)$. $\mathcal{A}(R)=\{A\in\mathcal{A};\ \mathcal{B}$ separates $\{A\backslash R, R\backslash A\}\}$. Note that $\mathcal{A}(R)$ is a monotone class including $\boldsymbol{R}(\mathcal{K})$. By the monotone class theorem, $\boldsymbol{S}(\mathcal{K})=\boldsymbol{S}(\boldsymbol{R}(\mathcal{K}))\subset\mathcal{A}(R)$ holds. Since $R$ is taken arbitrarily from $\boldsymbol{R}(\mathcal{K})$, $\boldsymbol{S}(\mathcal{K})\subset\mathcal{A}(R)$ for all $R\in\boldsymbol{R}(\mathcal{K})$, i.e. for every $S\in\boldsymbol{S}(\mathcal{K})$ and every $R\in\boldsymbol{R}(\mathcal{K})$, $\mathcal{B}$ separates $\{S\backslash R, R\backslash S\}$. Fix $S\in\boldsymbol{S}(\mathcal{K})$. The same argument shows that $\boldsymbol{S}(\mathcal{K})\subset\mathcal{A}(S)$ for all $S\in\boldsymbol{S}(\mathcal{K})$. Therefore it follows that for every pair $S_1$, $S_2$ in $\boldsymbol{S}(\mathcal{K})$, $\mathcal{B}$ separates $\{S_1\backslash S_2, S_2\backslash S_1\}$. This proves that $\mathcal{B}$ separates $\boldsymbol{S}(\mathcal{K})$.

**Lemma 2.** *Let $\mathcal{S}_0=\{S_P;\ P\in\mathcal{P}\}$ be $\mathcal{E}$-supports.*
*Then the following class $\mathcal{K}$ is a semi-ring including $\mathcal{S}_0$.*

$$\mathcal{K}=\{A\in\mathcal{A};\ (\exists\, n)\,(\exists\, a=(a_1,\cdots,a_n)\in\{0,1\}^n \text{ with } \sum_{i=1}^n a_i\geq 1)$$
$$(\exists\,\{P_i;\ 1\leq i\leq n\}\subset\mathcal{P})\,[A=\bigcap_{i=1}^n S_{P_i}^{a_i}]\}\ ,$$

*where* $S_{P_i}^{a_i}=S_{P_i}$ or $X\backslash S_{P_i}$ *according as* $a_i=1$ or $0$.

Proof. $\mathcal{K}$ is closed under the operation of a finite intersection, and $\mathcal{K}$ includes $\mathcal{S}_0$.

Take $K_1$ and $K_2$ from $\mathcal{K}$, which satisfy $K_1\subset K_2$. By the definition of $\mathcal{K}$, there exist $m, n, a=(a_1,\cdots,a_m,a_{m+1},\cdots,a_{m+n})\in\{0,1\}^{m+n}$ with $\sum_{i=1}^m a_i\geq 1$ and $\sum_{j=1}^{m+n} a_j\geq 1$ and $\{P_i;\ 1\leq i\leq m+n\}\subset\mathcal{P}$ such that $K_1=\bigcap_{i=1}^m S_{P_i}^{a_i}$ and $K_2=\bigcup_{j=m+1}^{m+n} S_{P_j}^{a_j}$. Put $I=\{a=(a_1,\cdots,a_{m+n})\in\{0,1\}^{m+n};\ \sum_{i=1}^{m+n} a_i\geq 1\}$.

We define a partition $\{F_a;\ a\in I\}\subset\mathcal{K}$ of $\bigcup_{i=1}^{m+n} S_{P_i}$ generated by $\{S_{P_i};\ 1\leq i\leq m+n\}$ as follows. For each $a\in I$, we put $F_a=\bigcap_{i=1}^{m+n} S_{P_i}^{a_i}$. Then there exist two subsets $I_1, I_2$ of $I$ such that $K_1=\bigcup_{a\in I_1} F_a$, $K_2=\bigcup_{a\in I_2} F_a$ and $I_1\subset I_2$. Therefore $K_2\backslash K_1=\bigcup_{a\in I_2\backslash I_1} F_a$. Thus $K_2\backslash K_1$ is a finite disjoint union of sets in $\mathcal{K}$.

**Lemma 3.** *Let $\mathcal{B}$ be a subfield of $\mathcal{A}$ and $\mathcal{K}$ the semi-ring defined in Lemma 2, If $\mathcal{B}$ is pairwise sufficient, then $\mathcal{B}$ separates $\mathcal{K}$.*

Proof. Take two disjoint sets $K_1, K_2$ from $\mathcal{K}$. As in the proof of Lemma 2, $K_1$ and $K_2$ can be represented as follows. $K_1=\bigcup_{a\in I_1} F_a$, $K_2=\bigcup_{b\in I_2} F_b$ for some $I_1$

and $I_2$ (not necessarily $I_1 \subset I_2$). Since $K_1 \cap K_2 = \phi$, $F_a \cap F_b = \phi$ for every $a \in I_1$ and $b \in I_2$. Hence it is enough to prove that $\mathcal{B}$ separates $\{F_a, F_b\}$ for every $a \in I_1$, $b \in I_2$. Note that there exists a $\mathcal{B}$-measurable version $g_{i,j}$ of $dP_j/d(P_i+P_j)$. Fix $a \in I_1$ and $b \in I_2$. It follows from $\sum_{i=1}^{m+n} a_i \geq 1$ that $a_i = 1$ for some $i$. In case $b_i = 1$, then $a_j \neq b_j$ for some $j \neq i$ as $F_a \cap F_b = \phi$. Put $B_0 = \{x \in X; g_{i,j}(x) > 0\} \in \mathcal{B}$. If $a_j = 1$ and $b_j = 0$, then $F_a \subset S_{P_i} \cap S_{P_j}$ and $F_b \subset S_{P_i} \cap (X \backslash S_{P_j})$. Therefore it follows that $B_0 \supset F_a[\mathcal{P}]$ and $B_0 \cap F_b = \phi\,[\mathcal{P}]$. Similarly if $a_j = 0$ and $b_j = 1$, then $B_0 \cap F_a = \phi\,[\mathcal{P}]$ and $B_0 \supset F_b\,[\mathcal{P}]$. In case $b_i = 0$, then $b_j\ b = 1$ for some $j \neq i$, as $\sum_{j=1}^{m+n} b_j \geq 1$. Put $B_1 = \{x \in X; g_{i,j}(x) < 1\} \in \mathcal{B}$. Then it follows that $B_1 \supset F_a\,[\mathcal{P}]$ and $B_1 \cap F_b = \phi\,[\mathcal{P}]$ as $F_a \subset S_{P_i}$ and $F_b \subset (X \backslash S_{P_i}) \cap S_{P_j}$. Thus $\mathcal{B}$ separates $\{F_a, F_b\}$.

Proof of Theorem 1. (Only if part) Let $\mathcal{B}$ be a pairwise sufficient subfield and $\mathcal{S}_0 = \{S_P; P \in \mathcal{P}\}$ $\mathcal{E}$-supports and $\mathcal{K}$ the semi-ring defined in Lemma 2. As $\mathcal{B}$ is a subfield (a fortiori a semi-ring) and $\mathcal{K}$ is a semi-ring, it follows that $\mathcal{V} = \{B \cap K; B \in \mathcal{B}, K \in \mathcal{K}\}$ is a semi-ring.

We claim that $\mathcal{B}$ separates $\mathcal{V}$.

Take $V_1$ and $V_2$ for $\mathcal{V}$ such that $V_1 \cap V_2 = \phi\,[\mathcal{P}]$. Then $V_i = B_i \cap K_i$ for some $B_i \in \mathcal{B}$ and $K_i \in \mathcal{K} (i=1, 2)$. By Lemma 3, $\mathcal{B}$ separates $\mathcal{K}$, so that there exists $B \in \mathcal{B}$ such that $B \supset K_1\,[\mathcal{P}]$ and $B \cap (K_2 \backslash K_1) = \phi\,[\mathcal{P}]$. Then $V_1 = B_1 \cap K_1 \subset B_1 \cap B\,[\mathcal{P}]$ and $V_2 \cap B_1 \cap B = B_2 \cap K_2 \cap B_1 \cap B = (B_2 \cap (K_2 \backslash K_1) \cap B_1 \cap B) \cup (B_2 \cap K_2 \cap K_1 \cap B_1 \cap B) = (V_2 \cap V_1 \cap B) = \phi\,[\mathcal{P}]$. Thus $\mathcal{B}$ separates $\{V_1, V_2\}$, and so $\mathcal{B}$ separates $\mathcal{V}$. Hence it follows from Lemma 1 that $\mathcal{B}$ separates $\boldsymbol{S}(\mathcal{V})$.

On the other hand, for every pair $P, Q \in \mathcal{P}$, there exists a $\mathcal{B}$-measurable version $g_{P,Q}$ of $dP/d(P+Q)$ by pairwise sufficiency of $\mathcal{B}$. For each generator $A(P, Q; a)$ of the $\sigma$-ring $\mathcal{S}$ defined in section 2, it follows that $A(P, Q; a) = \{x \in X; 0 < g_{P,Q}(x) \cdot I_{S_P \cup S_Q}(x) \leq a\}\,[\mathcal{P}]$, and so $A(P, Q; a) \in \boldsymbol{S}(\mathcal{V})\,[\mathcal{P}]$ for every $P, Q \in \mathcal{P}$ and $a \in (0, 1]$. Thus $\mathcal{S} \subset \boldsymbol{S}(\mathcal{V})\,[\mathcal{P}]$. This and that $\mathcal{B}$ separates $\boldsymbol{S}(\mathcal{V})$ imply that $\mathcal{B}$ separates $\mathcal{S}$.

(If part) Take $P$ and $Q$ from $\mathcal{P}$, and fix a version $f$ of $(dP/d(P+Q)) \cdot I_{S_P \cup S_Q}$. For each $n$, we define the following sets $A_k^n (1 \leq k \leq 2^n)$: $A_k^n = \{x \in X; (k-1)/2^n < f(x) \leq k/2^n\}$ $(1 \leq k \leq 2^n)$. Then it follows that for each $n$, $A_k^n$, $k = 1, \cdots, 2^n$, all belong to $\mathcal{S}$ up to $\mathcal{P}$-null sets and they are mutually disjoint. As $\mathcal{B}$ separates $\mathcal{S}$, there exists a disjoint subclass $\{B_k^n; 1 \leq k \leq 2^n\}$ of $\mathcal{B}$ such that $B_k^n \supset A_k^n\,[\mathcal{P}]$ for all $k = 1, \cdots, 2^n$. For each $n$, we define a simple function $g_n$ as follows.

$$g_n(x) = \begin{cases} k/2^n & \text{if} \quad x \in B_k^n (k = 1, \cdots, 2^n)\,, \\ 0 & \text{if} \quad x \notin \bigcup_{k=1}^{2^n} B_k^n\,. \end{cases}$$

Then $g_n$ are $\mathcal{B}$-measurable for all $n$, so that $g = \limsup g_n$ is $\mathcal{B}$-measurable. It follows from the definition of $g$ that $g = f\,[\mathcal{P}]$ on $S_P \cup S_Q$. Thus we obtained a $\mathcal{B}$-measurable version $g$ of $dP/d(P+Q)$. Therefore $\mathcal{B}$ is pairwise sufficient as $P$ and $Q$ are taken arbitrarily from $\mathcal{P}$.

Proof of Corollary 1. Take a maximal decomposition $\mathcal{F}$ satisfying $\mathcal{F}\subset\sigma(\mathcal{S}_0)\,[\mathcal{P}]$.

We assume that $S\in\mathcal{S}^+$ and consider the following class $\mathcal{G}$. $\mathcal{G}=\{F\cap S;\ F\in\mathcal{F},\, F\cap S\neq\phi\,[\mathcal{P}]\}\cup\{F\backslash S;\ F\in\mathcal{F},\, F\backslash S\neq\phi\,[\mathcal{P}]\}$

Then $\mathcal{G}$ is a maximal decomposition satisfying $\mathcal{G}\subset\mathcal{S}^+$.

The set $A$ is of countable type, i.e. $A\subset\bigcup_{n\geqq 1}S_{P_n}\,[\mathcal{P}]$ for a countable family $\{S_{P_n};\, n\geq 1\}$ of $\mathcal{S}_0$. As $\mathcal{G}$ is a maximal decomposition, for each $n\geq 1$, there exists a countable class $\mathcal{G}_n(\subset\mathcal{G})$ such that $\bigcup\mathcal{G}_n\supset S_{P_n}[\mathcal{P}]$. Put $G_n=\bigcup\mathcal{G}_n$ for each $n\geq 1$. Then $G_n\in\mathcal{S}$ for each $n\geq 1$, so that $\bigcup_{n\geqq 1}G_n\in\mathcal{S}$ and $\bigcup_{n\geqq 1}G_n\supset A\,[\mathcal{P}]$. Both $\bigcup_{n\geqq 1}G_n\backslash S$ and $S$ belong to $\mathcal{S}$, and they are mutually disjoint. As $\mathcal{B}$ is pairwise sufficient, it follows from Theorem 1 that thre there exists a set $B\in\mathcal{B}$ such that $B\supset\bigcup_{n\geqq 1}G_n\backslash S\,[\mathcal{P}]$ and $B\cap S=\phi\,[\mathcal{P}]$. Further $\bigcup_{n\geqq 1}G_n\supset A\,[\mathcal{P}]$ and $S\cap A=\phi$ imply that $\bigcup_{n\geqq 1}G_n\backslash S\supset A\,[\mathcal{P}]$. Hence it follows that $B\supset A\,[\mathcal{P}]$ and $B\cap S=\phi\,[\mathcal{P}]$. This implies that $\mathcal{B}$ separates $\{A, S\}$.

## 4. Minimal pairwise sufficient subfields.

Let $\mathcal{E}=(X, \mathcal{A}, \mathcal{P})$ be a majorized experiment, $\mathcal{S}$ the $\sigma$-ring defined in section 2, $\mathcal{D}$ the smallest PSS.

**Lemma 4.** *Let $\mathcal{B}$ be a pairwise sufficient subfield of $\mathcal{A}$ and $S$ a set in $\mathcal{S}$. If there exists a set $B\in\mathcal{B}$ of countable type such that $B\supset S\,[\mathcal{P}]$, then $S\in\mathcal{B}\,[\mathcal{P}]$.*

Proof. Note that both $S$ and $B\backslash S$ are sets of countable type. By Corollary 1, there exists a set $B_1\in\mathcal{B}$ such that $B_1\supset S\,[\mathcal{P}]$ and $B_1\cap(B\backslash S)=\phi\,[\mathcal{P}]$. Hence it follows that $S=B_1\cap B\,[\mathcal{P}]$, and so $S\in\mathcal{B}\,[\mathcal{P}]$.

**Theorem 2.** *Let $\mathcal{E}$ be a majorized experiment.*

(1) *If $\mathcal{E}$ has an $\mathcal{S}$-atoms $S^*$, then $\mathcal{D}(S^*)=\sigma\{S\in\mathcal{S};\, S\cap S^*=\phi\,[\mathcal{P}]\}(\subsetneqq\mathcal{D})$ is a minimal pairwise sufficient subfield.*

(2) *If $\mathcal{E}$ is $\mathcal{S}$-atomless, i.e. $\mathcal{E}$ has no $\mathcal{S}$-atom, then $\mathcal{D}$ is a minimal pairwise sufficient subfield.*

*Thus every majorized experiment has at least one minimal pairwise sufficient subfield.*

Proof. (1) Let $\mathcal{B}$ be a pairwise sufficient subfield satisfying $\mathcal{B}\subset\mathcal{D}(S^*)\,[\mathcal{P}]$.

Take $S\in\mathcal{S}$ with $S\cap S^*=\phi\,[\mathcal{P}]$. By the separation property of $\mathcal{B}$, there exists $B\in\mathcal{B}$ such that $B\supset S\,[\mathcal{P}]$ and $B\cap S^*=\phi\,[\mathcal{P}]$. Note that each member of $\mathscr{D}(S^*)$ is either a set in $\mathcal{S}$ disjoint from $S^*$, or a set of co-countable type in $\mathscr{D}$ including $S^*$. As $B\cap S^*=\phi\,[\mathscr{P}]$ and $\mathscr{B}\subset\mathscr{D}(S^*)\,[\mathscr{P}]$, $B$ is a set of countable type. It follows from Lemma 4 that $S\in\mathscr{B}[\mathscr{P}]$. This implies that $\mathscr{B}=\mathscr{D}(S^*)$ $[\mathscr{P}]$ as $S$ is taken arbitrarily from the generators of $\mathscr{D}(S^*)$. Hence $\mathscr{D}(S^*)$ is minimal pairwise sufficient.

(2) Take a pairwise sufficient subfield $\mathscr{B}$ satisfying $\mathscr{B}\subset\mathscr{D}[\mathscr{P}]$. As $\mathscr{D}=\sigma(\mathcal{S})$, it is enough to prove that $S\in\mathscr{B}[\mathscr{P}]$ for any $S\in\mathcal{S}$.

Take $S\in\mathcal{S}^+$. As $\mathcal{E}$ is $\mathcal{S}$-atomless, there exist $S_1$, $\bar{S}_1\in\mathcal{S}^+$ such that $S=S_1\cup\bar{S}_1$ and $S_1\cap\bar{S}_1=\phi$.

We claim that either $S_1$ or $\bar{S}_1$ belongs to $\mathscr{B}[\mathscr{P}]$.

The separation property of $\mathscr{B}$ implies that there exists a set $B\in\mathscr{B}$ such that $B\supset S_1[\mathscr{P}]$ and $B\cap\bar{S}_1=\phi\,[\mathscr{P}]$. As $\mathscr{B}\subset\mathscr{D}=\sigma(\mathcal{S})\,[\mathscr{P}]$, it follows that $B\in\mathcal{S}$ or $X\backslash B\in\mathcal{S}$. Therefore it follows from Lemma 4 that either $S_1$ or $\bar{S}_1$ belongs to $\mathscr{B}[\mathscr{P}]$. We assume that $S_1\in\mathscr{B}[\mathscr{P}]$ and apply the same argument to $\bar{S}_1=S_2\cup\bar{S}_2$ with $S_2$, $\bar{S}_2\in\mathcal{S}^+$ and $S_2\cap\bar{S}_2=\phi$. This time we can prove that $S_2\in\mathscr{B}[\mathscr{P}]$. Thus for each countable ordinal $\alpha$, we can construct a sequence of mutually disjoint sets $\{S_\beta;\beta\leq\alpha\}$ and $\bar{S}_\beta$, all belonging to $\mathcal{S}$, such that for each $\beta\leq\alpha$, $S_\beta\in\mathscr{B}[\mathscr{P}]$. If $\bar{S}_\alpha$ is not $\mathscr{P}$-null, then we repeat this procedure for $\alpha+1$. Recall that $\mathcal{S}$ satisfies C.C.C.. Hence this procedure stops at some countable ordinal $\kappa$. Therefore we have a decomposition $\{S_\beta;\beta\leq\kappa\}$ $(\subset\mathscr{B})$ of $S$ such that $S=\bigcup_{\beta\leq\kappa}S_\beta[\mathscr{P}]$. Hence $S\in\mathscr{B}[\mathscr{P}]$.

REMARK 2. In non-majorized experiments, a minimal pairwise sufficient subfield does not always exist. See an example given by R. V. Ramamoorthi and B. V. Rao (see [6] Example 4.3).

## 5. Cases with one or more $\mathcal{S}$-atoms.

We first prove the non-existence of the smallest pairwise sufficient subfield when $\mathcal{E}$ has at least two $\mathcal{S}$-atoms. In particular, $\mathcal{E}$ can be a discrete experiment.

**Theorem 3.** *Let $\mathcal{E}$ be a majorized experiment.*

*If $\mathcal{E}$ has more than one $\mathcal{S}$-atom, then the smallest pairwise sufficient subfield does not exist.*

Proof. Assume that, on the contrary, there exists the smallest pairwise sufficient subfield $\mathscr{G}$. Take two distinct $\mathcal{S}$-atoms $S_1^*$ and $S_2^*$.

Then it follows that $\mathscr{G}\subset\mathscr{D}(S_i^*)\,[\mathscr{P}]\,(i=1,2)$. By the separation property of $\mathscr{G}$, there exists a set $C\in\mathscr{G}$ such that $C\supset S_1^*[\mathscr{P}]$ and $C\cap S_2^*=\phi\,[\mathscr{P}]$. Since $C\in\mathscr{D}(S_1^*)\,[\mathscr{P}]$ and $C\supset S_1^*[\mathscr{P}]$, $C$ is necessarily of co-countable type. Similarly $C\in\mathscr{D}(S_2^*)$ and $C\cap S_2^*=\phi\,[\mathscr{P}]$ imply that $C$ is of countable type. This is a contradiction.

Next we consider the case that $\mathcal{E}$ has only one $\mathcal{S}$-atom $S^*$.

We put $X'=X\backslash S^*$. For each $P\in\mathcal{P}$ with $P(X')>0$, we put $P'(\cdot)=P(\cdot)/P(X')$, $(\cdot\in\mathcal{A}\cap X')$ and $\mathcal{P}'=\{P';P\in\mathcal{P}, P(X')>0\}$.

We define an experiment $\mathcal{E}'$ on $X'$ by $\mathcal{E}'=(X', \mathcal{A}\cap X', \mathcal{P}')$. It is also majorized as the restriction of any majorizing measure to $X'$ majorizes $\mathcal{E}'$. Further, it is easily verified that $\mathcal{S}'$, the $\sigma$-ring of pairwise likelihood ratios for $\mathcal{E}'$, coincides with $\mathcal{S}\cap X'$: $\mathcal{S}'=\mathcal{S}\cap X'=\{S\in\mathcal{S};\ S\subset X'\}$. Hence $\mathcal{E}'$ is $\mathcal{S}$-atomless in $X'$. It follows from Theorem 1 that a subfield $\mathcal{B}'$ of $\mathcal{A}\cap X'$ is pairwise sufficient if and only if $\mathcal{B}'$ separates $\mathcal{S}'$, or equivalently $\mathcal{S}$ in $X'$.

Suppose that there exists the smallest pairwise sufficient subfield for $\mathcal{E}'$. It is given by $\mathcal{D}'=\sigma(\mathcal{S}')=\{S\in\mathcal{S};\ S\subset X'\}\cup\{A\subset X';\ X'\backslash A\in\mathcal{S}\}$, because of Theorem 2 (2) and that $\mathcal{E}'$ is $\mathcal{S}$-atomless. We will show that $\mathcal{D}(S^*)$ is the smallest pairwise sufficient subfield for $\mathcal{E}$. Let $\mathcal{B}$ be a pairwise sufficient subfield in $\mathcal{E}$. As $\mathcal{B}\cap X'$ is pairwise sufficient in $\mathcal{E}'$, $\mathcal{D}'\subset\mathcal{B}\cap X'[\mathcal{P}]$. Hence for each $S\in\mathcal{S}$ with $S\cap S^*=\phi$, there exists a set $B\in\mathcal{B}$ such that $B\cap X'=S[\mathcal{P}]$. Clearly $B\in\mathcal{B}$ is of countable type and satisfies $B\supset S[\mathcal{P}]$. Hence by Lemma 4, we have that $S\in\mathcal{B}[\mathcal{P}]$, and so $\mathcal{D}(S^*)\subset\mathcal{B}[\mathcal{P}]$ as required.

Conversely, if we suppose that $\mathcal{E}$ has the smallest pairwise sufficient subfield, then by Theorem 2 (1), it must be $\mathcal{D}(S^*)$. We will show that $\mathcal{D}'$, defined above, is the smallest pairwise sufficient subfield for $\mathcal{E}'$. Take any other pairwise sufficient subfield $\mathcal{B}'$ of $\mathcal{A}\cap X'$. Define a subfield $\mathcal{B}=\{B'\cup S^*;\ B'\in\mathcal{B}'\}$ of $\mathcal{A}$. It is pairwise sufficient in $\mathcal{E}$ as $\mathcal{B}$ separates $\mathcal{S}$. Hence $\mathcal{D}(S^*)\subset\mathcal{B}[\mathcal{P}]$. So $X'\cap\mathcal{D}(S^*)\subset X'\cap\mathcal{B}[\mathcal{P}]$, and hence $\mathcal{D}'\subset\mathcal{B}'[\mathcal{P}]$.

Thus we have proved the following reduction theorem.

**Theorem 4.** *Suppose that $\mathcal{E}$ has a single $\mathcal{S}$-atoms $S^*$.*

*Then, the existence of the smallest pairwise sufficient subfield in $\mathcal{E}$ is equivalent to its existence in the $\mathcal{S}$-atomless experiment $\mathcal{E}'$ on $X\backslash S^*$.*

## 6. $\mathcal{S}$-atomless case: Hierarchy of majorized experiments.

In this section, we set out to study the question of the existence of the smallest pairwise sufficient subfield for the case without $\mathcal{S}$-atom, as the other cases are either resolved or reduced to the atomless case. In this case the question is same as that of the existence of a pairwise sufficient subfield which does not include $\mathcal{D}$, as the minimality of the latter has been proved. The possible answer to this question obviously depends upon the relative size of $\mathcal{A}$ to $\mathcal{D}$---the larger $\mathcal{A}$, the more chance it has to include such a subfield in it. Notice in this connection that while setting up an experiment $\mathcal{E}=(X, \mathcal{A}, \mathcal{P})$ we have certain freedom in choosing $\mathcal{A}$ on which $\mathcal{P}$ is defined. For, each $P$ in $\mathcal{P}$ has all its substantial nature in its values for the the subsets in $\mathcal{A}\cap S_P$, as $P$ is 0 outside $S_P$. Therefore, if we replace $\mathcal{A}$ with a $\sigma$-field $\mathcal{A}'$ on $X$ which contains all these subsets and, in addition, all the supports $S_P(P\in\mathcal{P})$, it would serve equally well as the $\sigma$-field on which all $P$'s are to be defined. That is, $\mathcal{A}'$ should satisfy the following

Condition L: (1) $\mathcal{A}' \cap S_P = \mathcal{A} \cap S_P[\mathcal{P}]$ for all $P \in \mathcal{P}$, and
(2) $\mathcal{A}' \supset \mathcal{S}_0[\mathcal{P}]$.

Take all such $\sigma$-fields and denote by $\boldsymbol{\Sigma}$ the totality of them. It is a lattice, if we identify the $\sigma$-fields which are equivalent with each other (up to $\mathcal{N}_{\mathcal{P}}$). The largest and smallest elements in $\boldsymbol{\Sigma}$ are $\bar{\mathcal{A}} = \bigcap_{P \in \mathcal{P}} \mathcal{A} \vee \mathcal{N}_P$, the weak completion of $\mathcal{A}$, and $\mathcal{A}^0$, the $\sigma$-field of the sets of countable or co-countable type, respectively as is proved in Remark 3. Hence every element $\mathcal{A}'$ of $\boldsymbol{\Sigma}$ remains within $\bar{\mathcal{A}}$, so that the measures in $\mathcal{P}$ are redefined in an obvious way on $\mathcal{A}'$ through restriction or extension, giving rise to a family $\mathcal{P}'$ on $\mathcal{A}'$.

Here is, thus, a hierarchy of experiments $\mathcal{E}(\mathcal{A}') = (X, \mathcal{A}', \mathcal{P}')$, $\mathcal{A}' \in \boldsymbol{\Sigma}$, all defined on $X$ with the families of measures (almost) same as $\mathcal{P}$. As is easily seen, all of them are majorized experiments having the same family $\mathcal{S}$ as the $\sigma$-ring of pairwise likelihood ratios, and the same family $\mathcal{D}$ as the smallest PSS. Hence a pairwise sufficient subfield $\mathcal{B}$ for $\mathcal{E}(\mathcal{A}^0)$ is also a pairwise sufficient subfield for every $\mathcal{E}(\mathcal{A}')$, $\mathcal{A}' \in \boldsymbol{\Sigma}$, provided $\mathcal{B} \subset \mathcal{A}'$. Further, all $\mathcal{A}' \in \boldsymbol{\Sigma}$ have $\bar{\mathcal{A}}$ and $\mathcal{A}^0$ in common, so that if we start from any such $\mathcal{A}'$, substitute it for $\mathcal{A}$ in Condition L and look for the totality of $\sigma$-fields satisfying the condition thus revised, then we shall arrive at the same $\boldsymbol{\Sigma}$ as before.

Thus the lattice $\boldsymbol{\Sigma}$ gives us a general scheme for viewing the problem in a wider perspective: When given $\mathcal{E} = (X, \mathcal{A}, \mathcal{P})$, find out $\mathcal{S}$ and $\boldsymbol{\Sigma}$ and ask the question of the existence of the smallest pairwise sufficient subfield for $\mathcal{A}$ in $\boldsymbol{\Sigma}$, in relation to its position in the hierarchy in $\boldsymbol{\Sigma}$. The first thing to be noticed under this scheme is of course the simple fact written in the form of the following

**Theorem 5.** *Suppose that $\mathcal{E}$ is $\mathcal{S}$-atomless and $\mathcal{A}_1$ and $\mathcal{A}_2$ are elements of $\boldsymbol{\Sigma}$.*

*If $\mathcal{A}_1 \subset \mathcal{A}_2$ and if the smallest PSS $\mathcal{D}$ is the smallest pairwise sufficient subfield in $\mathcal{E}(\mathcal{A}_2)$, then so is it in $\mathcal{E}(\mathcal{A}_1)$.*

Prof. Let $\mathcal{B}_1$ be a pairwise sufficient subfield of $\mathcal{A}_1$. Then $\mathcal{B}_1$ is also a pairwise sufficient subfield of $\mathcal{A}_2$. Hence it follows from the assumption that $\mathcal{D} \subset \mathcal{B}_1[\mathcal{P}]$, and so $\mathcal{D}$ is the smallest pairwise sufficient subfield in $\mathcal{E}(\mathcal{A}_1)$ as $\mathcal{B}_1$ is an arbitrary pairwise sufficient subfield of $\mathcal{A}_1$.

The following Theorem 6 gives us a simple criterion for existence or non-existence.

**Theorem 6.** *Suppose that $\mathcal{E}$ is $\mathcal{S}$-atomless.*

*For each $\mathcal{A} \in \boldsymbol{\Sigma}$, the following statements are equivalent.*

(1) *The smallest pairwise sufficient subfield does not exist in $\mathcal{E}(\mathcal{A})$.*

(2) *There exists a pairwise sufficient subfield $\mathcal{C}$ of $\mathcal{A}$ such that $S\in\mathcal{C}[\mathcal{P}]$ for some $S\in\mathcal{S}^+$.*

Proof. By Theorem 2 (2), the smallest pairwise sufficient subfield in $\mathcal{E}(\mathcal{A})$ must be the smallest PSS $\mathcal{D}$ whenever it exists. Hence its existence is equivalent to that $\mathcal{S}\subset\mathcal{C}$ for all pairwise sufficient subfield $\mathcal{C}$ of $\mathcal{A}$ because $\mathcal{D}=\sigma(\mathcal{S})\,[\mathcal{P}]$. This implies the equivalence between (1) and (2).

Now we consider some implications of the statement (2). Assume for the moment that it holds true. We prove that the set $S$ in (2) has a subset $T\in\mathcal{S}^+$, which satisfies the following statement (*).

(*) For any disjoint subsets $S_1$, $S_2$ of $T$ in $\mathcal{S}^+$ and any $C\in\mathcal{C}$, if $C\supset S_1[\mathcal{P}]$ and $C\cap S_2=\phi\,[\mathcal{P}]$, then it is of uncountable type.

Assume that, on the contrary, for any subset $T\in\mathcal{S}^+$ of $S$, there exist two disjoint subsets $S_1$, $S_2$ of $T$ in $\mathcal{S}^+$ and a set $C\in\mathcal{C}$ of countable or co-countable type such that $C\supset S_1[\mathcal{P}]$, $C\cap S_2=\phi\,[\mathcal{P}]$. In case the set $C$ is of countable tyep, it follows from Lemma 4 that $S_1\in\mathcal{C}[\mathcal{P}]$. Similarly if $C$ is of co-countable type, then $S_2\in\mathcal{C}[\mathcal{P}]$ holds. Hence either $S_1$ or $S_2$ belongs to $\mathcal{C}[\mathcal{P}]$. Repeating this procedure as in the proof of Theorem 2 (2), we can prove $T\in\mathcal{C}[\mathcal{P}]$. As the set $T$ is an arbitrary subset of $S$ in $\mathcal{S}^+$, it follows that $S\in\mathcal{C}[\mathcal{P}]$. This is a contradiction.

Next we prove that there exists a subset $T\in\mathcal{S}^+$ of $S$ such that $T\notin\mathcal{C}[\mathcal{P}]$ and $T$ satisfies (*). We define $T^*\in\mathcal{S}^+$ as the largest subset in $\mathcal{S}^+$ of $S$ which satisfies the property (*). Certainly the set $T^*$ can be defined as $\mathcal{S}$ satisfies C.C.C.. Moreover it follows that $T^*\notin\mathcal{B}[\mathcal{P}]$. In fact, if $S=T^*[\mathcal{P}]$, then $T^*\notin\mathcal{C}[\mathcal{P}]$ as $S\notin\mathcal{C}[\mathcal{P}]$. And if $S\setminus T^*\neq\phi\,[\mathcal{P}]$, then it follows from the definition of $T^*$ that the set $S\setminus T^*$ does not satisfy [*]. Through the same argument as in the preceding paragraph, we have $S\setminus T^*\in\mathcal{C}[\mathcal{P}]$. Therefore $T^*\notin\mathcal{C}[\mathcal{P}]$, as $S\notin\mathcal{C}[\mathcal{P}]$

Now we have almost proved the following

**Theorem 7.** *Suppose that $\mathcal{E}$ as $\mathcal{S}$-atomless.*
*For each $\mathcal{A}\in\boldsymbol{\Sigma}$, the following statesments are equivalent.*

(1) *The smallest pairwise sufficient subfield does not exist in $\mathcal{E}(\mathcal{A})$.*
(2) *There exist a pairwise sufficient subfield $\mathcal{C}$ of $\mathcal{A}$ and a set $T\in\mathcal{S}^+$, which does not belong to $\mathcal{C}$, satisfying the following properties. For any disjoint subsets $S_1$, $S_2$ of $T$ in $\mathcal{S}^+$ and any $C\in\mathcal{C}$, if $C\in\mathcal{A}^0[\mathcal{P}]$, then $S_1\setminus C\neq\phi\,[\mathcal{P}]$ or $S_2\cap C\neq\phi\,[\mathcal{P}]$.*

Proof. Note that the statement "if $C\in\mathcal{A}^0[\mathcal{P}]$, then $S_1\setminus C\neq\phi\,[\mathcal{P}]$ or $S_2\cap$

$C \neq \phi[\mathcal{P}]$" is the contraposition of (*). Hence (1) implies (2). The reverse implicatin implication is trivial.

This is used effectively for proving Theorems on the existence and non-existence in later sections.

REMARK 3. We prove that $\mathcal{A}^0$ and $\bar{\mathcal{A}}$ are the smallest and the largest elements in $\boldsymbol{\Sigma}$, respectively.

Clearly, both $\bar{\mathcal{A}}$ and $\mathcal{A}^0$ satisfy the condition L. Moreover, as $\mathcal{A}^0 \subset \mathcal{A}[\mathcal{P}]$, $S_P \cap \mathcal{A}^0 \subset S_P \cap \mathcal{A}[\mathcal{P}]$ for all $P \in \mathcal{P}$. Hence for every $\mathcal{A}' \in \boldsymbol{\Sigma}$, $S_P \cap \mathcal{A}^0 \subset \mathcal{A}'[\mathcal{P}]$ for all $P \in \mathcal{P}$. Take a set A of countable type from $\mathcal{A}^0$. There exists a countable subclass $\{S_n;\, n \geq 1\}$ of $\mathcal{S}_0$ such that $A = \bigcup_{n \geq 1} (A \cap S_n)\, [\mathcal{P}]$. Hence $A \in \mathcal{A}'$ $[\mathcal{P}]$, so that $\mathcal{A}^0 \subset \mathcal{A}'[\mathcal{P}]$ as $\mathcal{A}'$ is a $\sigma$-field and $A$ is an arbitrary set of countable type. This implies that $\mathcal{A}^0$ is the smallest element in $\boldsymbol{\Sigma}$.

Fix an element $\mathcal{A}'$ in $\boldsymbol{\Sigma}$ and take $A'$ and $P$ from $\mathcal{A}'$ and $\mathcal{P}$, respectively. As $S_P \cap \mathcal{A}' = S_P \cap \mathcal{A}[\mathcal{P}]$, there exists a set $A_P \in \mathcal{A}$ such that $(A_P \cap S_P) \,\Delta\, (A' \cap S_P) = (A_P \,\Delta\, A') \cap S_P = \phi[\mathcal{P}]$. Therefore $A' \,\Delta\, A_P = ((A' \,\Delta\, A_P) \cap S_P) \cup ((A' \,\Delta\, A_P) \cap (X \backslash S_P)) \in \mathcal{N}_P$, and so $A' \in \mathcal{A} \vee \mathcal{N}_P$ for every $P \in \mathcal{P}$ as $P$ is taken arbitrarily from $\mathcal{P}$. This implies that $A' \in \bar{\mathcal{A}}$. Hence $\bar{\mathcal{A}}$ is the largest element in $\boldsymbol{\Sigma}$.

EXAMPLE 1. Let $X$ be $\boldsymbol{R}^2$, $\mathcal{A}$ the Borel $\sigma$-field on $X$ and $\mathcal{P}$ the family of all 1-dimensional Normal distributions on the lines in $X$. For each $P \in \mathcal{P}$, the $\mathcal{E}$-support $S_P$ is a line in $X$, and so $\mathcal{E}$ is majorized (see the lines following the definition of supports in section 1), and $\mathcal{S} = \boldsymbol{S}\{$all the linear Borel subsets of $X\}$, i.e. each $S \in \mathcal{S}$ is a countable union of linear Borel subsets of $X$. Note that $\mathcal{S} = \boldsymbol{S}\{$all the linear Lebesgue measurable subsets of $X\}$ up to $\mathcal{N}_{\mathcal{P}}$. It is easily seen that $\mathcal{E}$ is $\mathcal{S}$-atomless. The smallest element $\mathcal{A}^0$ and the largest element $\bar{\mathcal{A}}$ of $\boldsymbol{\Sigma}$ are given as follows.

$\mathcal{A}^0 = \{A \subset X;$ either $A$ or $X \backslash A$ is a countable union of linear Lebesgue measurable sets$\}$, and $\bar{\mathcal{A}}$=the $\sigma$-field of all locally Lebesgue measurable sets, i.e. $A \in \bar{\mathcal{A}}$ if and only if $A \cap F$ is a linear Lebesgue measurable set for each line $F$.

EXAMPLE 2. Let $X$ be $\boldsymbol{R}^2$, $\mathcal{A}$ the $\sigma$-field of all Borel subsets of $X$, and $\mathcal{P}$ all the 1-dimensional Normal distributions on vertical lines with mean 0. Then for each $P \in \mathcal{P}$, the $\mathcal{E}$-support $S_P$ of $P$ is a vertical line, and so $\mathcal{E}$ is majorized. It is easily verified that the $\sigma$-ring $\mathcal{S} = \{$all the 1-dimensional vertical and symmetric Borel sets$\}$ and that $\mathcal{E}$ is $\mathcal{S}$-atomless. Further it follows that the smallest element $\mathcal{A}^0$ in $\boldsymbol{\Sigma}$ is the $\sigma$-field generated by all the 1-dimensional vertical Lebesgue measurable sets and that the largest element $\bar{\mathcal{A}}$ in $\boldsymbol{\Sigma}$ is the $\sigma$-field of all those subsets of $X$, whose vertical sections are Lebesgue measurable.

## 7. Conditions for existence when $\mathcal{A}$ is smaller in $\Sigma$.

First we prove the existence for $\mathcal{A}^0$, the smallest possible $\mathcal{A}$.

**Theorem 8.** *Suppose that $\mathcal{E}$ is $\mathcal{S}$-atomless.*
*Then $\mathcal{D}$ is the smallest pairwise sufficient subfield in $\mathcal{E}(\mathcal{A}^0)$.*

Proof. Assume that, on the contrary, $\mathcal{D}$ is not the smallest pairwise sufficient subfield in $\mathcal{E}(\mathcal{A}^0)$. Accordingly, take a pairwise sufficient subfield $C$ of $\mathcal{A}^0$ and a set $T\in\mathcal{S}^+$ as in Theorem 7 (2). Since the experiment $\mathcal{E}(\mathcal{A})$ is $\mathcal{S}$-atomless, there exist two disjoint sets $S_1$ and $S_2$ in $\mathcal{S}^+$ such that $T=S_1\cup S_2$ and $S_1\cap S_2=\phi$. By the separation property of $C$, there exists a set $A_1\in C$ such that $A_1\supset S_1[\mathcal{P}]$ and $A_1\cap S_2=\phi[\mathcal{P}]$. It follows from Theorem 7 (2) that the set $A_1\in C$ is a set of uncountable type. This contradicts that $C$ is a subfield of $\mathcal{A}^0$, the $\sigma$-field of all the sets of countable and co-countable type.

The following Theorem shows that $\mathcal{D}$ remains smallest pairwise sufficient when a countable number of disjoint sets of uncountable type are added to $\mathcal{A}^0$. The proof of the Theorem is given by T. Kamae.

**Theorem 9.** *Suppose that $\mathcal{E}$ is $\mathcal{S}$-atomless, and let $\{A_n;\, n\in\boldsymbol{N}\}$ be a partition of $X$ consisting of a countable number of sets of uncountable type.*
*Then the smallest PSS $\mathcal{D}$ is the smallest pairwise sufficient subfield in $\mathcal{E}(\mathcal{A}^0\vee\{A_n;\, n\in\boldsymbol{N}\})$.*

Proof. As we have taken $\mathcal{A}=\mathcal{A}^0\vee\{A_n;\, n\in\boldsymbol{N}\})$, if we take a set $B$ in $\mathcal{A}$ of uncountable type and any one of the sets $A_n$, $n\in\boldsymbol{N}$, then either $A_n\cap B$ or $A_n\backslash B$ is of countable type. In the latter event, we write $A_n<B$.

Assume that $\mathcal{D}$ is not smallest pairwise sufficient in $\mathcal{E}(\mathcal{A})$. Then there exist a set $T\in\mathcal{S}^+$ and a pairwise sufficient subfield $\mathcal{B}$ of $\mathcal{A}$, which satisfy the conditions in Theorem 7 (2).

In what follows we construct, for each $n\in\boldsymbol{N}$, two sequences of sets $\{B_{n,\alpha}\}$ in $\mathcal{B}$ and $\{T_{n,\alpha}\}$ in $\mathcal{S}$. The second index $\alpha$ ranges over all the countable ordinals and the definitions are done by inductions on $\alpha$.

As $\mathcal{E}$ is $\mathcal{S}$-atomless, $T=T_1\cup T_2$ for some disjoint sets $T_1$ and $T_2$ in $\mathcal{S}^+$. By the separation property of $\mathcal{B}$, there exists a decomposition $\{B_1, B_2\}(\subset\mathcal{B})$ of $X$ such that $B_i\supset T_i[\mathcal{P}]$, $i=1, 2$. We define $B_{n,1}=B_1$ and $T_{n,1}=T$, or $B_{n,1}=B$ and $T_{n,1}=T_2$, according as $A_n<B_1$ or $A_n<B_2$. As we have noted above, either of these two alternatives holds. By induction, we construct $B_{n,\alpha}\in\mathcal{B}$ and $T_{n,\alpha}\in\mathcal{S}$ for each countable ordinal $\alpha$ as follows.

Case 1. $\alpha$ is a successive ordinal. If $T_{n,\alpha-1}=\phi[\mathcal{P}]$, then define $B_{n,\alpha}=B_{n,\alpha-1}$ and $T_{n,\alpha}=T_{n,\alpha-1}$. If $T_{n,\alpha-1}\neq\phi[\mathcal{P}]$, then $T_{n,\alpha-1}=T_1'\cup T_2'$ for some disjoint sets $T_1'$ and $T_2'$ in $\mathcal{S}^+$. As in the preceding paragraph, $T_i'\subset B_i'[\mathcal{P}]$ for some $B_i'\in\mathcal{B}(i=1, 2)$. Define $B_{n,\alpha}=B_1'\cap B_{n,\alpha-1}$ and $T_{n,\alpha}=T_1'$, or $B_{n,\alpha}=B_2'\cap B_{n,\alpha-1}$ and $T_{n,\alpha}=T_2'$, according as $A_n<B_1'$ or $A_n<B_2'$.

Case 2. $\alpha$ is a limit ordinal. Define $B_{n,\alpha}=\bigcap_{\beta<\alpha} B_{n,\beta}$ and $T_{n,\alpha}=\bigcap_{\beta<\alpha} T_{n,\beta}$. Notice that the number of the ordinals $\beta$ such that $\beta<\alpha$ is countable, so that $B_{n,\alpha}\in\mathcal{B}$ and $T_{n,\alpha}\in\mathcal{S}$. It can happen that $B_{n,\alpha}$ and/or $T_{n,\alpha}=\phi\,[\mathcal{P}]$, while $B_{n,\beta}$ and $T_{n,\beta}$ for $\beta<\alpha$ are all $\neq\phi\,[\mathcal{P}]$.

It follows from these definitions that $B_{n,\alpha}\subset B_{n,\beta}$ and $T_{n,\alpha}\subset T_{n,\beta}$ for any countable ordinals $\alpha$, $\beta$ with $\beta<\alpha$ and that $A_n<B_{n,\alpha}$ and $B_{n,\alpha}\cap T=T_{n,\alpha}\,[\mathcal{P}]$ for every $\alpha$.

For each $n$, there exists a countable ordinal $\alpha_n$ such that $T_{n,\alpha_n}=\phi\,[\mathcal{P}]$. Because, if $T_{n,\alpha}\neq\phi\,[\mathcal{P}]$ for every countable ordinal, then the sets $T_{n,\alpha-1}\backslash T_{n,\alpha}$ for all the succesive countable ordinals are uncountable disjoint sets belonging to $\mathcal{S}^+$ included in $T$, a contradiction with the C.C.C. for $\mathcal{S}$.

Now define

$$B=X\backslash \bigcup_{n\geqq1} B_{n,\alpha_n}\in\mathcal{B} \quad\text{and}\quad S=T\backslash \bigcup_{n\geqq1} T_{n,\alpha_n}\in\mathcal{S}\,.$$

Then

$$\begin{aligned} B &= \bigcup_{n\geqq1} A_n\backslash \bigcup_{k\geqq1} B_{k,\alpha_k}\subset \bigcup_{n\geqq1}(A_n\backslash B_{n,\alpha_n}) \quad\text{and}\\ S\backslash B &= (T\backslash \bigcup_{n\geqq1} T_{n,\alpha_n})\cap(\bigcup_{n\geqq1} B_{n,\alpha_n})\subset T\cap \bigcup_{n\geqq1} B_{n,\alpha_n}\\ &= \bigcup_{n\geqq1} T_{n,\alpha_n}=\phi\,[\mathcal{P}]\,. \end{aligned}$$

Hence $B$ is a set of countable type which satisfies $S\subset B\,[\mathcal{P}]$. Therefore $S\in\mathcal{B}\,[\mathcal{P}]$ by Lemma 4. As $S=T[\mathcal{P}]$ from the definition of $S$, it follows that $T\in\mathcal{B}\,[\mathcal{P}]$, which is a contradiction.

REMARK 4. The conclusion of the foregoing Theorem does not hold true when the sets $A_n$, $n\geq1$, do not form a countable partition of $X$. See Example 3 in section 8.

## 8. Conditions for non-existence when $\mathcal{A}$ is larger in $\boldsymbol{\Sigma}$.

**Theorem 10.** *Suppoe Suppose that $\mathcal{E}$ is $\mathcal{S}$-atomless and $\mathcal{A}\in\boldsymbol{\Sigma}$. If for some $T\in\mathcal{S}^+$, there exists an injective Borel homomorphism $\xi$ defined on $T\cap\mathcal{S}$ into $\bar{\mathcal{D}}\cap\mathcal{A}$ such that for every subset $S$ of $T$ in $\mathcal{S}$,*

(1) *$\xi(S)\cap T=S$ and*

(2) *is $S\neq\phi\,[\mathcal{P}]$ and $T\backslash S\neq\phi\,[\mathcal{P}]$, then $\xi(S)\in\bar{\mathcal{D}}\cap\mathcal{A}\backslash\mathcal{D}$, then the smallest pairwise sufficient subfield does not exist.*

*(By using the term "injective Borel homomorphism", we assume that $\xi$ satisfies $\xi(\phi)=\phi$, $\xi(T\backslash S)=\xi(T)\backslash\xi(S)$ for every subset $S$ of $T$ in $\mathcal{S}$ and $\xi(\bigcup_{i\geqq1} T_i)=\bigcup_{i\geqq1}\xi(T_i)$ for every $\{T_i;\,i\geq1\}\subset T\cap\mathcal{S}$. Incidentally, the weak completion $\bar{\mathcal{D}}$ means $\bigcap_{P\in\mathcal{P}}\mathcal{D}\vee\mathcal{N}_P$).*

Proof. We prove the assertion by constructing a pairwise sufficient subfield $C_T$, which does not contain $T$, a set in $\mathcal{S}^+$.

Put $\mathcal{C}_T=\sigma(\{\xi(T\cap S);\ S\in\mathcal{S}\}\cup\{S\in\mathcal{S};\ T\cap S=\phi\})$. First we prove that $\mathcal{C}_T$ separates $\mathcal{S}$. Take a disjoint pair $S_1$, $S_2$ from $\mathcal{S}^+$.

Put $C_1=(\xi(T\cap S_1)\cup(S_1\backslash T))\backslash(S_2\backslash T)$. This belongs to $\mathcal{C}_T$, as $\xi(T\cap S)$, $S_1\backslash T$ and $S_2\backslash T$ belong to $\mathcal{C}_T$. Further it follows from the following formulas that $\mathcal{C}_T$ separates $\{S_1, S_2\}$.

$$\begin{aligned} S_1\backslash C_1 &\subset S_1\backslash(\xi(T\cap S_1)\cup(S_1\backslash T)) \\ &\subset S_1\backslash((T\cap S_1)\cup(S_1\backslash T)=\phi\,, \quad \text{and} \\ C_1\cap S_2 &= (C_1\cap(S_2\cap T))\cup(C_1\cap(S_2\backslash T)) \\ &= (\xi(T\cap S_1)\cup(S_1\backslash T))\cap(S_2\cap T) \\ &= \xi(T\cap S_1)\cap(S_2\cap T)\subset\xi(T\cap S_1)\cap\xi(S_2\cap T) \\ &= \xi(T\cap S_1\cap S_1)=\phi\,. \end{aligned}$$

Thus $\mathcal{C}_T$ separates $\mathcal{S}$, and so $\mathcal{C}_T$ is pairwise sufficient.

Next we prove that $T\notin\mathcal{C}_T$. First, we claim that for each $C\in\mathcal{C}_T$, $C\,\Delta\,\xi(C\cap T)$ is a set of countable type.

Notice that $C\cap T\in\mathcal{S}$, as $C\in\mathcal{C}_T\subset\mathcal{D}\cap\mathcal{A}$ and $T\in\mathcal{S}$. By the definition of $\mathcal{C}_T$, there exist countable subclasses $\{T_i(\subset T);\ i\geq 1\}$ and $\{S_j;\ T\cap S_j=\phi,\ j\geq 1\}$ of $\mathcal{S}$ such that $C\in\sigma\{\xi(T_i);\ i\geq 1\}\cup\{S_j;\ j\geq 1\}$. Put $W=X\backslash\bigcup_{j\geqq 1} S_j$. Then the set $C\cap W$ belongs to the $\sigma$-field $\mathcal{W}$ on $W$ generated by $\{\xi(T_i)\cap W;\ i\geq 1\}$. Hence it can be written as a union of countable intersections of these generators. That is, $C\cap W=\bigcup_{a\in L}\bigcap_{i\geqq 1}(\xi(T_i)\cap W)^{a_i}$ for some $L\subset\{a;\ a=(a_i)_{i\geq 1}\}$, where $(\xi(T_j)\cap W)^{a_i}=\xi(T_i)\cap W$ or $W\backslash\xi(T_i)$ according as $a_i=1$ or 0. Since the map $\xi$ is an injective Borel homomorphism, $C\cap W=\bigcup_{a\in L}(\xi(\bigcap_{i\geqq 1}T_i^{a_i})\cap W)$, where, similarly, $T^{a_i}=T$ or $T\backslash T_i$.

Fix a point $w\in C\cap W$. Then $w\in\xi(\bigcap_{i\geqq 1}T_i^{a_i})\cap W$ for one and only one $a\in L$. The set $\bigcap_{i\geqq 1}T_i^{a_i}(\subset T\subset W)$, corresponding to the same $a\in L$, is not empty. Take a point $x$ in it. Then $w\in C\cap W$ if and only if $x\in C\cap W$, as $w$ and $x$ can not be separated by any generator in $\mathcal{W}$. As $x\in T$, $x\in C\cap W$ if and only if $x\in C\cap T$. Further $x\in C\cap T$ if only if $w\in\xi(C\cap T)$. It follows that $C\cap W=W\cap\xi(C\cap T)$, because the point $w\in W$ is fixed arbitrarily. Thus $C\,\Delta\,\xi(C\cap T)$ is a subset of $\bigcup_{j\geqq 1}S_j$, and so it is of countable type.

So, in particular, $T\,\Delta\,\xi(T)$ has to be a set of countable type if $T\in\mathcal{C}_T$. By the condition (2), both $\xi(S)$ and $\xi(T\backslash S)$ are sets of uncountable type for every subset $S$ of $T$ in $\mathcal{S}$, which satisfies $S\neq\phi[\mathcal{P}]$ and $T\backslash S\neq\phi[\mathcal{P}]$. As $\xi$ is an injective Borel homomorphism, $\xi(T)=\xi(S)\cup\xi(T\backslash S)$, and so it is a set of either uncountable or co-countable type in $\bar{\mathcal{D}}\cap\mathcal{A}$. So $T\,\Delta\,\xi(T)$ is not of countable type, and hence $T$ cannot belong to $\mathcal{C}_T$.

Therefore the smallest pairwise sufficient subfield does not exist in $\mathcal{E}(\mathcal{A})$.

EXAMPLE 3. Consider the same $X$ and $\mathcal{P}$ as in Example 2, but replace $\mathcal{A}$ by $\mathcal{A}^0\vee\{A_r; r\in\boldsymbol{Q}^+\}$, where $A_r=(-r, r)\times\boldsymbol{R}$ and $\boldsymbol{Q}^+$ denotes all the positive rationals. We will prove, by constructing a Borel homomorphism $\xi$, that the smallest pairwise sufficient subfield does not exist in $\mathcal{E}(\mathcal{A}^0\vee\{A_r; r\in\boldsymbol{Q}^+\})$, as is claimed in Remark 4.

Fix a vertical line $T=\{0\}\times\boldsymbol{R}$. Then $T\in\mathcal{S}$, and $\mathcal{A}^0\vee\{A_r; r\in\boldsymbol{Q}^+\}$ includes all the rectangles of the form $B\times\boldsymbol{R}$ for any symmetric Borel subset $B$ of $\boldsymbol{R}$. To each symmetric Borel subset $B$ of $T$, we assign $\xi(B)=B\times\boldsymbol{R}$, a $\bar{\mathcal{D}}$-measurable set of uncountable type. Then it is easily verified that $T$ and $\xi$ satisfy the conditions in Theorem 10. Thus, by Theorem 5, for every $\mathcal{A}\in\boldsymbol{\Sigma}$ which is larger than $\mathcal{A}^0\vee\{A_r; r\in\boldsymbol{Q}^+\}$, the smallest pairwise sufficient subfield for $\mathcal{E}(\mathcal{A})$ does not exist. In particular, it does not exist in $\mathcal{E}(\bar{\mathcal{A}})$.

EXAMPLE 4. This time we modify Example 1, to show that a much smaller $\mathcal{A}$ than in Example 1 already ensures the non-existence of the smallest pairwise sufficient subfield. Take the same $X$, $\mathcal{P}$ and hence $\mathcal{S}$, and let $I$ be the open interval (0, 1) in $\boldsymbol{R}$ and $\mathcal{B}_I$ denote the Borel $\sigma$-field on $I$. Take the horizontal line segment $T=I\times\{0\}$ ($\in\mathcal{S}$), $\mathcal{K}=\{I\times B; B\in\mathcal{B}_I\}$ and $\mathcal{A}=\mathcal{A}^0\vee\mathcal{K}$. As each subset $S$ of $T$ in $\mathcal{S}$ is written as $S=B\times\{0\}$ for some $B\in\mathcal{B}_I$, we assign $\xi(S)=(I\times B)\cup S$ to it. Then $T$ and $\xi$ satisfy the conditions in Theorem 10. We could of course take $(0, \varepsilon)$ instead of $I$, however small $\varepsilon$ be.

EXAMPLE 5. This is an example of an apparently "small" $\mathcal{A}$ satisfying the condition for non-existence given in the foregoing Theorem. In fact, the sets added to $\mathcal{A}^0$ to form $\mathcal{A}$ are all ancillary on $X\setminus T$, a sufficient condition for sufficiency of $\mathcal{D}$ for $\mathcal{A}$ (see [5] Theorem 7). As it is, $\mathcal{A}$ contains enough "number" of sets to serve as the range of an injective Borel homomorphism as is envisaged in the Theorem.

For each $i\in\boldsymbol{R}$, put $X_i=\boldsymbol{R}^2$ and $\mathcal{B}_{X_i}=$the Borel $\sigma$-field of $\boldsymbol{R}^2$. Define $X=\sum_{i\in\boldsymbol{R}} X_i$, the direct sum of $X_i$'s, $\mathcal{A}^0=\sigma\{A\subset X; A\in\mathcal{B}_{X_i}$ for some $i\in\boldsymbol{R}\}$. Let $N(0, 1)\otimes N(\theta, 1)$ be the product measure on $X_0$, where $N(\theta, 1)$ is the Normal distribution on $\boldsymbol{R}$ with mean $\theta\in\boldsymbol{R}$ and variance 1. Define $P_\theta^0(A)=(N(0, 1)\otimes N(\theta, 1))(A\cap X_0)$ and $P_\theta^i(A)=(N(\theta, 1)\otimes N(0, 1))(A\cap X_i)$ for each $i\neq 0$. Put $\mathcal{K}=\{A\subset X$; For some $B\in\mathcal{B}_{\boldsymbol{R}}$, $A\cap X_i=\boldsymbol{R}\times B$ for all $i\in\boldsymbol{R}\}$ and take $\mathcal{A}=\mathcal{A}^0\vee\mathcal{H}$. The experiment $\mathcal{E}=(X, \mathcal{A}, \mathcal{P}=\{P_\theta^i; i\in\boldsymbol{R}, \theta\in\boldsymbol{R}\})$ is majorized as $X_i$ is an $\mathcal{E}$-support of $P_\theta^i$ for every $i$ and $\theta$. The $\sigma$-ring $\mathcal{S}$ of pairwise likelihood ratios coincides with the $\sigma$-ring generated by $\{\boldsymbol{R}\times B\subset X_0; B\in\mathcal{B}_{\boldsymbol{R}}\}$ and $\bigcup_{i\in\boldsymbol{R}}\{B\times\boldsymbol{R}\subset X_i; B\in\mathcal{B}_{\boldsymbol{R}}\}$. Note that $\mathcal{E}$ is $\mathcal{S}$-atomless. It is easily seen that $\mathcal{D}$ is sufficient for $\mathcal{A}$. Put $T=X_0$. We define an injective Borel homomorphism $\xi$ on $T\cap\mathcal{S}\rightarrow\bar{\mathcal{D}}\cap\mathcal{A}$ as follows. As each elements $S\in T\cap\mathcal{S}$ is written as $S=\boldsymbol{R}\times B$ for some $B\in\mathcal{B}_{\boldsymbol{R}}$, we assign $\xi(S)=\sum_{i\in\boldsymbol{R}} H_i$, where $H_i=\boldsymbol{R}\times B\subset X_i$ for all $i$, to it. Then $T$

and $\xi$ thus defined satisfy the conditions in Theorem 10.

### Acknowledgements.

The present work was done in Indian Statistical Institute, Calcutta, where the author had useful discussions from May 1986 till April 1987.

The author wishes to thank Professors J.K. Ghosh, B.V. Rao, R.V. Ramamoorthi, Haimanti Sarbadhikari and S.M. Srivastava who made very helpful suggestions in various stages. Especially, Prof. B.V. Rao communicated the author the present version of the proof of Theorem 2 (2), which is much better than the original one. The author is also grateful to Professors T. Kusama, T. Kamae, and H. Morimoto who gave useful suggestions and encouraged him for publishing the paper.

This work was supported by an exchange programme between Indian National Science Academy and Japanese Society for the Promotion of Science.

### References

[1] R.R. Bahadur: *Sufficiency and statistical decision functions,* Ann. Math. Statist., **25** (1954), 423–462.

[2] D. Basu and J.K. Ghosh : *Sufficient statistics in sampling from a finite universe,* Proc. 36th Session Int. Statist. Inst. (1967), 850–859.

[3] D.L. Burkholder: *Sufficiency in the undominated case,* Ann. Math. Statist. **32** (1961), 1191–1200.

[4] F.R. Diepenbrock: *Characterisierung einer allgemeineren Bedingung als Dominiertheit mit Hilfe von localisierten Massen,* Thesis (1971); Univ. Muenster.

[5] J. Fujii and H. Morimoto: *Sufficiency and pairwise sufficiency in majorized experiments,* Sankhya, **48** (1986), Ser. A, 315–330.

[6] J.K. Ghosh, H. Morimoto and S. Yamada: *Neyman factorization and minimality of pairwise sufficient subfields,* Ann. Statist. **9** (1981), 514–530.

[7] M. Hasegawa and M.D. Perlman: *On the existence of minimal sufficient subfield,* Ann. Statist. **3** (1974), 1371–1372.

[8] T.S. Pitcher: *Sets of measures not admitting necessary and sufficient statistics or subfields,* Ann. Math. Statist. **28** (1957), 267–268.

[9] T.S. Pitcher: *A more general property than domination for sets of probability measures,* Pacific J. Math., **15** (1965), 597–611.

Department of Mathematics
Osaka City University
Sugimoto-3, Sumiyoshi-ku
Osaka 558, Japan