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THE INTEGRAL PONTRJAGIN HOMOLOGY OF THE BASED LOOP SPACE ON A FLAG MANIFOLD

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Abstract

The based loop space homology of a special family of homogeneous spaces, flag manifolds of connected compact Lie groups is studied. First, the rational homology of the based loop space on a complete flag manifold is calculated together with its Pontrjagin structure. Second, it is shown that the integral homology of the based loop space on a flag manifold is torsion free. This results in a description of the integral homology. In addition, the integral Pontrjagin structure is determined.

Contents

1. Introduction

A *complete flag manifold* of a compact connected Lie group G is a homogeneous space G/T, where T is a maximal torus in G. In this paper we study the integral Pontrjagin homology of the based loop space on a complete flag manifold G/T.

Compact homogeneous spaces, in particular, flag manifolds play a significant role in many areas of physics and mathematics, such as theory of characteristic classes of fibre bundles, representation theory, string topology and quantum physics. Still there are only few homogeneous spaces for which the integral homology ring of their based loop spaces is known. Some of them are classical simple Lie groups, spheres, and complex projective spaces.

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The motivation for our study comes from Borel's work [2] in which he described the family of compact homogeneous spaces whose cohomology ring is torsion free. In particular, homogeneous spaces G/U where $\operatorname{rank} G = \operatorname{rank} U$ stand out as homogeneous spaces which behave nicely under application of algebraic topological techniques. In this case Sullivan minimal model theory together with the Milnor–Moore theorem can be employed to calculate the rational homology ring of their based loop spaces. As one of the main results of our paper (see Theorem 2.1) we prove that the homology of the based loop space on a complete flag manifold is torsion free.

Furthermore, we explicitly calculate the integral Pontrjagin homology ring of the loop spaces on the complete flag manifolds of simple compact Lie groups $SU(n+1)$, $Sp(n)$, $SO(2n+1)$, $SO(2n)$, G_2, F_4 and E_6 (see Theorems 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7).

It is a classical result (see for example [3], or [5]) that the homology of the e-connected component $\Omega_0 G$ of the loop space on G is torsion free for any compact connected Lie group G. Thus by the use of rational calculations, we show that there is a split extension of algebras

$$1 \to H_*(\Omega_0 G; \mathbb{Z}) \to H_*(\Omega(G/T); \mathbb{Z}) \to H_*(T; \mathbb{Z}) \to 1$$

and describe the integral Pontrjagin ring structure on $\Omega(G/T)$ for a simple compact Lie group G.

Throughout the paper, the loop space on a topological space will mean a based loop space.

2. Torsion in the homology of loop spaces

We start by recalling some well known facts about the (co)homology of classical simple compact Lie groups and their based loop spaces (see for example [10]). It is a classical result that for any compact connected Lie group G of rank n,

$$H^*(G; \mathbb{Q}) \cong \bigwedge(z_1, \ldots, z_n), \quad H_*(\Omega_0 G; \mathbb{Q}) \cong \mathbb{Q}[b_1, \ldots, b_n]$$

where $\deg(z_i) = 2k_i - 1$ and $\deg(b_i) = 2k_i - 2$ for $1 \le i \le n$, and k_i are the exponents of the group G. For simple compact Lie groups, these exponents are established.

For $G = SU(n+1)$ or $G = Sp(n)$, the integral homology of G and ΩG is torsion free and it is given by

$$H^*(G; \mathbb{Z}) \cong \bigwedge(x_1, \ldots, x_n), \quad H_*(\Omega G; \mathbb{Z}) \cong \mathbb{Z}[y_1, \ldots, y_n]$$

where $\deg(x_i) = 2k_i - 1$, and $\deg(y_i) = 2k_i - 2$ for $1 \le i \le n$. Under the rationalisation the integral generators $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are mapped onto the rational generators $z_1, \ldots, z_n$ and $b_1, \ldots, b_n$, respectively.

For $G = SO(2n+1)$ or $G = SO(2n)$, the integral homology of G and ΩG has 2-torsion.

Borel [2, Proposition 29.1] proved that the homology of a flag manifold G/T is torsion free for the classical Lie groups G and for $G = G_2$ or F_4. Using Morse theory, it is proved in [4] that this is true for any compact connected Lie group.

Our first result states that the complete flag manifold of a compact connected Lie group behaves nicely with respect to the loop space homology functor.

Theorem 2.1. *The homology of the based loop space on the complete flag manifold of a compact connected Lie group is torsion free.*

We will first show that to prove the theorem it is enough to consider the case when G is a simple, compact Lie group.

Proposition 2.2. *The loop space on the flag manifold of a compact, connected Lie group G decomposes into a product of the loop spaces on flag manifolds of simple, compact Lie groups.*

Proof. It is a classical result (see Onishchik [12]) that a compact connected Lie group G can be decomposed into a locally direct product of connected simple normal subgroups. That is, $G = G_1 \cdots G_k$, where G_i is a simple, connected Lie group or a torus, $1 \le i \le k$, such that

$$\dim G_i \cap (G_1 \cdots G_{i-1} \cdot G_{i+1} \cdots G_k) = 0.$$

Let $\tilde{G}$ be $G_1 \times \cdots \times G_k$ and $p\colon \tilde{G} \to G$ defined by $p(g_1, \ldots, g_k) = g_1 \cdots g_k$. Since $\operatorname{Ker} p = \bigcup_{i=1}^{n} G_i \cap (G_1 \cdots \hat{G}_i \cdots G_k)$, we obtain that $\operatorname{Ker} p$ is discrete or in other words $p\colon \tilde{G} \to G$ is a covering. Thus $\operatorname{Ker} p$ is contained in the center $Z(\tilde{G})$ of $\tilde{G}$. Let $T = T_1 \times \cdots \times T_k$ be a maximal torus in $\tilde{G}$, where T_i is a maximal torus in G_i for $1 \le i \le k$. Then $\operatorname{Ker} p \subset T$ and therefore

$$G_1/T_1 \times \cdots \times G_k/T_k = \tilde{G}/T = (\tilde{G}/\operatorname{Ker} p)/T = G/T.$$

Hence

$$\Omega(G/T) \simeq \Omega(G_1/T_1) \times \cdots \times \Omega(G_k/T_k). \qquad \square$$

Proof of Theorem 2.1. Let G be a compact connected Lie group and T its maximal torus. We have that the complete flag manifold G/T for any compact connected Lie group G is homeomorphic to the complete flag manifold $\tilde{G}/T$ of its universal cover $\tilde{G}$. Therefore, we may assume G to be simply connected. For G simply connected, it is classical result (see for example [13]) that $\Omega(G/T)$ has the same homotopy type as

$\Omega(G) \times T$. To verify this notice that related to the principal fibration as topological spaces. For G a simple, compact, simply connected Lie group, it is a classical result that the integral homology of ΩG is torsion free. Now using homotopy splitting of $\Omega(G/T)$, we conclude that the homology of $\Omega(G/T)$ is torsion free in this case. The statement of the theorem now follows readily from Proposition 2.2. □

3. Rational homology

In this section we calculate the rational homology ring of the loop space on a flag manifold by looking separately at each simple Lie group.

To calculate the rational homology of the based loop space on a complete flag manifold of a classical simple Lie group we will apply Sullivan minimal model theory. Let us start by recalling the key constructions and setting the notation related to the Sullivan minimal model and rational homology of loop spaces which we are going to use in the subsequent sections.

3.1. Rational homology of loop spaces. Let M be a simply connected topological space with the rational homology of finite type. Let $\mu = (\Lambda V, d)$ be a Sullivan minimal model for M. Then $d\colon V \to \Lambda^{\geq 2} V$ can be decomposed as $d = d_1 + d_2 + \cdots$, where $d_i\colon V \to \Lambda^{\geq i+1} V$. In particular, d_1 is called the *quadratic part* of the differential d.

The homotopy Lie algebra $\mathcal{L}$ of μ is defined in the following way. Define a graded vector space L by requiring that

$$sL = \mathrm{Hom}(V, \mathbb{Q})$$

where as usual the suspension sL is defined by $(sL)_i = (L)_{i-1}$. We can define a pairing $\langle\ ;\ \rangle\colon V \times sL \to \mathbb{Q}$ by $\langle v; sx\rangle = (-1)^{\deg v} sx(v)$ and extend it to $(k+1)$-linear maps

$$\Lambda^k V \times sL \times \cdots \times sL \to \mathbb{Q}$$

by letting

$$\langle v_1 \wedge \cdots \wedge v_k; sx_k, \ldots, sx_1\rangle = \sum_{\sigma \in S_k} \epsilon_\sigma \langle v_{\sigma(1)}; sx_1\rangle \cdots \langle v_{\sigma(k)}; sx_k\rangle$$

where S_k is the symmetric group on k letters and $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \epsilon_\sigma v_1 \wedge \cdots \wedge v_k$. It is important to notice that L inherits a Lie bracket $[\ ,\ ]\colon L \times L \to L$ from d_1 uniquely determined by

$$\langle v; s[x, y]\rangle = (-1)^{\deg y + 1} \langle d_1 v; sx, sy\rangle \quad \text{for} \quad x, y \in L,\ v \in V. \tag{1}$$

Denote by $\mathcal{L}$ the Lie algebra $(L, [\ ,\ ])$.

Recall that the graded Lie algebra $L_M = (\pi_*(\Omega M) \otimes \mathbb{Q}; [\ ,\ ])$ is called the *rational homotopy Lie algebra of M*. The commutator $[\ ,\ ]$ is given by the Samelson product. There is an isomorphism between the rational homotopy Lie algebra L_M and the homotopy Lie algebra $\mathcal{L}$ of μ. Using the theorem in the Appendix of Milnor and Moore [9], it follows that

$$H_*(\Omega M; \mathbb{Q}) \cong U\mathcal{L}$$

where $U\mathcal{L}$ is the universal enveloping algebra for $\mathcal{L}$. Further on,

$$U\mathcal{L} \cong T(L)/\langle xy - (-1)^{\deg x \deg y} yx - [x, y]\rangle.$$

For a more detailed account of this construction see for example [7], Chapters 12 and 16.

As the notion of formality will be important for our calculation we recall it here.

DEFINITION 3.1. A commutative cochain algebra (A, d) satisfying $H^0(A) = \mathbb{Q}$ is *formal* if it is weakly equivalent to the cochain algebra $(H(A), 0)$.

Thus (A, d) and a path connected topological space X are formal if and only if their minimal Sullivan models can be computed directly from their cohomology algebras.

REMARK 3.2. There are some known cases of topological spaces for which a minimal model can be explicitly computed and formality proved. Some of them, that are important for us in this work, are the spaces that have so called "good cohomology" in terminology of [1]. Namely, topological space X is said to have good cohomology if

$$H^*(X; \mathbb{Q}) \cong \mathbb{Q}[u_1, \dots, u_n]/\langle P_1, \dots, P_k\rangle$$

where the polynomials $P_1, \dots, P_k$ form the regular sequence in $\mathbb{Q}[u_1, \dots, u_n]$, or in other words, the ideal $\langle P_1, \dots, P_k\rangle$ is a Borel ideal in $\mathbb{Q}[u_1, \dots, u_n]$. In this case Bousfield and Gugenheim [1] proved that the minimal model of X is given by

$$\mu(X) = \mathbb{Q}[u_1, \dots, u_n] \otimes \bigwedge(v_1, \dots, v_k)$$

where $\deg(v_i) = \deg(P_i) - 1$ for $1 \le i \le k$, and the differential d is given by

$$d(u_i) = 0, \quad d(v_j) = P_j.$$

3.2. The loop space on a complete flag manifold. In this section we calculate the rational homology of the loop space on the complete flag manifold of a simple Lie group.

Recall from Borel [2, Section 26] that the rational (as well as integral) cohomology of $SU(n+1)/T^n$ is the polynomial algebra on $n+1$ variables of degree 2 quotient out by the ideal generated by the symmetric functions in these variables

$$H^*(SU(n+1)/T^n;\mathbb{Q}) \cong \mathbb{Q}[u_1,\dots,u_{n+1}]/\langle S^+(u_1,\dots,u_{n+1})\rangle.$$

It is important to note that the ideal $\langle S^+(u_1,\dots,u_{n+1})\rangle$ is a Borel ideal. As a consequence, by Remark 3.2, $SU(n+1)/T^n$ is formal. Thus the minimal model for $SU(n+1)/T^n$ is the minimal model for the commutative differential graded algebra $(H^*(M;\mathbb{Q}), d=0)$ and it is given by $\mu = (\Lambda V, d)$, where

$$V = (u_1,\dots,u_n, v_1,\dots,v_n)$$

and $\deg(u_k)=2$, $\deg(v_k)=2k+1$ for $1\le k\le n$.

The differential d is defined by

$$d(u_k)=0,\quad d(v_k)=\sum_{i=1}^{n} u_i^{k+1} + (-1)^{k+1}\left(\sum_{i=1}^{n} u_i\right)^{k+1}. \tag{2}$$

It is easy to see that the quasi isomorphism $f\colon \mu=(\Lambda V,d)\to (H^*(M;\mathbb{Q}), d=0)$ is given by the following rule

$$u_i\mapsto u_i,\quad v_i\mapsto 0\quad\text{for}\quad 1\le i\le n.$$

Theorem 3.1. *The rational homology ring of the loop space on the flag manifold $SU(n+1)/T^n$ is*

(3)

$$\begin{aligned}&H_*(\Omega(SU(n+1)/T^n);\mathbb{Q})\\ &\cong (T(a_1,\dots,a_n)/\langle a_k^2 = a_pa_q + a_qa_p \mid 1\le k,\ p,q\le n,\ p\ne q\rangle)\otimes\mathbb{Q}[b_2,\dots,b_n]\end{aligned}$$

where the generators a_i are of degree 1 for $1\le i\le n$, and the generators b_k are of degree $2k$ for $2\le k\le n$.

Proof. The underlying vector space of the homotopy Lie algebra $\mathcal{L}$ of μ is given by

$$L = (a_1,\dots,a_n, b_1,\dots,b_n)$$

where $\deg(a_k)=1$, $\deg(b_k)=2k$ for $1\le k\le n$.

In order to define Lie brackets we need the quadratic part d_1 of the differential in the minimal model. In this case, using the differential d defined in (2), the quadratic part d_1 is given by

$$d_1(u_l)=0\quad\text{for } 1\le l\le n,\quad d_1(v_1)=2\sum_{i=1}^{n}u_i^2+2\sum_{i<j}u_iu_j,\quad d_1(v_k)=0\quad\text{for}\quad k\ne 1.$$

For dimensional reasons, we have

$$[a_k, b_l] = [b_k, b_l] = 0 \quad \text{for} \quad 1 \le k, l \le n.$$

By the defining property of the Lie bracket stated in (1), we have

$$\langle v_1; s[a_k, a_k]\rangle = \left\langle 2\sum u_i^2 + 2\sum u_i u_j; sa_k, sa_k\right\rangle = 2\langle u_k^2; sa_k, sa_k\rangle = 4$$

and

$$\langle v_1; s[a_k, a_l]\rangle = 2\langle u_k u_l; sa_k, sa_l\rangle = 2 \quad \text{for} \quad k \ne l$$

resulting in the commutators

$$[a_k, a_l] = 2b_1 \quad \text{for} \quad k \ne l,$$

and

$$[a_k, a_k] = 4b_1.$$

Therefore in the tensor algebra $T(a_1, \ldots, a_n, b_1, \ldots, b_n)$, the Lie brackets above induce the following relations

$$\begin{aligned} a_k a_l + a_l a_k &= 2b_1 && \text{for} \quad 1 \le k, l \le n,\ k \ne l, \\ a_k^2 &= 2b_1 && \text{for} \quad 1 \le k \le n, \\ a_k b_l &= b_l a_k && \text{for} \quad 1 \le k, l \le n, \\ b_k b_l &= b_l b_k && \text{for} \quad 1 \le k, l \le n. \end{aligned}$$

Thus

$$U\mathcal{L} \cong (T(a_1, \ldots, a_n)/\langle a_k^2 = a_p a_q + a_q a_p\rangle) \otimes \mathbb{Q}[b_2, \ldots, b_n]. \tag{4}$$

This proves the theorem. □

The rational cohomology rings for the flag manifolds $SO(2n+1)/T^n \cong Spin(2n+1)/T^n$, $SO(2n)/T^n \cong Spin(2n)/T^n$, and $Sp(n)/T^n$ (see for example Borel [2, Section 26]) are given by

$$H^*(SO(2n+1)/T^n; \mathbb{Q}) \cong H^*(Sp(n)/T^n; \mathbb{Q}) \cong \mathbb{Q}[u_1, \ldots, u_n]/\langle S^+(u_1^2, \ldots, u_n^2)\rangle,$$
$$H^*(SO(2n)/T^n; \mathbb{Q}) \cong \mathbb{Q}[u_1, \ldots, u_n]/\langle S^+(u_1^2, \ldots, u_n^2), u_1 \cdots u_n\rangle$$

where u_i is of degree 2 for $1 \le i \le n$.

By Remark 3.2, all the above mentioned complete flag manifolds are formal and therefore their minimal Sullivan model is the minimal model for their cohomology algebra with the trivial differential.

Proceeding in the same way as in the previous theorem, we obtain the following results.

Theorem 3.2. *The rational homology ring of the loop space on* $SO(2n+1)/T^n$ *and* $Sp(n)/T^n$ *is given by*

$$\begin{aligned} &H_*(\Omega(SO(2n+1)/T^n);\mathbb{Q}) \cong H_*(\Omega(Sp(n)/T^n);\mathbb{Q}) \\ &\cong \left(T(a_1,\dots,a_n)\Big/\left\langle \begin{matrix} a_1^2=\cdots=a_n^2, \\ a_ka_l=-a_la_k \ \text{for}\ k\neq l \end{matrix}\right\rangle\right)\otimes \mathbb{Q}[b_2,\dots,b_n] \end{aligned} \tag{5}$$

where the generators a_i *are of degree* 1 *for* $1\le i\le n$, *and the generators* b_k *are of degree* $4k-2$ *for* $2\le k\le n$.

Proof. We give just an outline of the proof as it is similar to the proof of Theorem 3.1. The minimal model for $SO(2n+1)/T^n$ is given by $\mu=(\Lambda V,d)$, where

$$V=(u_1,\dots,u_n,v_1,\dots,v_n),$$

and $\deg(u_k)=2$, $\deg(v_k)=4k-1$ for $1\le k\le n$.

The differential d is given by

$$d(u_k)=0,\quad d(v_k)=\sum_{i=1}^n u_i^{2k}\quad \text{for}\quad 1\le k\le n. \tag{6}$$

Therefore the underlying vector space of the homotopy Lie algebra $\mathcal{L}$ of μ is

$$L=(a_1,\dots,a_n,b_1,\dots,b_n)$$

where $\deg(a_k)=1$, $\deg(b_k)=4k-2$ for $1\le k\le n$, and the quadratic part d_1 of the differential d is given by

$$d_1(u_l)=0\quad\text{for}\quad 1\le l\le n,\quad d_1(v_1)=\sum_{i=1}^n u_i^2,\quad d_1(v_k)=0\quad\text{for}\quad k\ge 2.$$

The induced Lie brackets on L are equal to

$$\begin{aligned} &[a_k,b_l]=[b_k,b_l]=0 &&\text{for}\quad 1\le k,l\le n,\\ &[a_k,a_k]=2b_1 &&\text{for}\quad 1\le k\le n,\\ &[a_k,a_l]=0 &&\text{for}\quad k\neq l. \end{aligned}$$

This implies the following relations in $U\mathcal{L}$:

$$\begin{aligned} &a_k^2 = b_1 && \text{for} \quad 1 \le k \le n,\\ &a_k a_l + a_l a_k = 0 && \text{for} \quad k \ne l,\\ &a_k b_l = b_l a_k && \text{for} \quad 1 \le k, l \le n,\\ &b_k b_l = b_l b_k && \text{for} \quad 1 \le k, l \le n. \end{aligned}$$

The theorem follows now at once knowing that $H_*(\Omega(SO(2n+1)/T^n); \mathbb{Q}) \cong U\mathcal{L}$. $\square$

Theorem 3.3. *The rational homology ring of the loop space on $SO(2n)/T^n$ for $n > 2$ is given by*

$$\begin{aligned} &H_*(\Omega(SO(2n)/T^n); \mathbb{Q})\\ &\cong \left(T(a_1, \dots, a_n) \middle/ \left\langle \begin{matrix} a_1^2 = \cdots = a_n^2, \\ a_k a_l = -a_l a_k \text{ for } k \ne l \end{matrix} \right\rangle \right) \otimes \mathbb{Q}[b_2, \dots, b_{n-1}, b_n] \end{aligned}$$

where the generators a_i are of degree 1 for $1 \le i \le n$, the generators b_k are of degree $4k-2$ for $2 \le k \le n-1$, and the generator b_n is of degree $2n-2$.

Proof. To be reader friendly we outline a proof. The minimal model for $SO(2n)/T^n$ is given by $\mu = (\Lambda V, d)$, where

$$V = (u_1, \dots, u_n, v_1, \dots, v_{n-1}, v_n),$$

and $\deg(u_k) = 2$, $\deg(v_k) = 4k-1$ for $1 \le k \le n-1$ and $\deg(v_n) = 2n-1$.

The differential d is given by

$$d(u_k) = 0, \quad d(v_k) = \sum_{i=1}^{n} u_i^{2k} \quad \text{and} \quad d(v_n) = u_1 \cdots u_n. \tag{7}$$

Hence the underlying vector space of the homotopy Lie algebra $\mathcal{L}$ of μ is

$$L = (a_1, \dots, a_n, b_1, \dots, b_{n-1}, b_n)$$

where $\deg(a_k) = 1$, $\deg(b_k) = 4k-2$ for $1 \le k \le n-1$, $\deg(b_n) = 2n-2$, and the quadratic part d_1 of the differential d is given by

$$d_1(u_l) = 0 \quad \text{for} \quad 1 \le l \le n, \quad d_1(v_1) = \sum_{i=1}^{n} u_i^2, \quad d_1(v_k) = 0 \quad \text{for} \quad 2 \le k \le n.$$

The induced Lie brackets on L are equal to

$$\begin{aligned}
&[a_k, b_l] = [b_k, b_l] = 0 && \text{for} \quad 1 \le k, l \le n,\\
&[a_k, a_k] = 2b_1 && \text{for} \quad 1 \le k \le n,\\
&[a_k, a_l] = 0 && \text{for} \quad k \ne l,
\end{aligned}$$

and thus in $U\mathcal{L}$:

$$\begin{aligned}
&a_k^2 = b_1 && \text{for} \quad 1 \le k \le n,\\
&a_k a_l + a_l a_k = 0 && \text{for} \quad k \ne l,\\
&a_k b_l = b_l a_k && \text{for} \quad 1 \le k, l \le n,\\
&b_k b_l = b_l b_k && \text{for} \quad 1 \le k, l \le n.
\end{aligned}$$

Since $H_*(\Omega(SO(2n)/T^n); \mathbb{Q}) \cong U\mathcal{L}$, we have proved the theorem. □

In the theorems that follow we compute the rational homology rings of the based loop space on the complete flag manifolds of the exceptional Lie groups G_2, F_4 and E_6. We refer to [6] and [11] for the Weyl group invariant polynomials which we use for the descriptions of the rational cohomology rings of the complete flag manifolds of these groups. We want also to emphasize that the rational, as well as the integral, cohomology rings of the flag manifolds G_2/T^2, F_4/T^4 and E_6/T^6 are thoroughly discussed in [15].

Theorem 3.4. *The rational homology ring of the loop space on G_2/T^2 is given by*

$$H_*(\Omega(G_2/T^2); \mathbb{Q}) \cong (T(a_1, a_2)/\langle a_1a_2 + a_2a_1 = a_1^2 = a_2^2\rangle) \otimes \mathbb{Q}[b_5]$$

where $\deg b_5 = 10$, *and* $\deg a_1 = \deg a_2 = 1$.

Proof. Recall that

$$H^*(G_2/T^2; \mathbb{Q}) \cong \mathbb{Q}[u_1, u_2, u_3]/\langle P_1, P_2, P_6\rangle$$

where $P_1 = \sum_{i=1}^3 u_1$, $P_2 = \sum_{i=1}^3 u_i^2$, $P_6 = \sum_{i=1}^3 u_i^6$ and $\deg u_1 = \deg u_2 = \deg u_3 = 2$. Therefore the minimal model is $\Lambda V = \Lambda(u_1, u_2, v_1, v_5)$ where $\deg v_1 = 3$, $\deg v_5 = 11$, and the differential d is given by $d(u_1) = d(u_2) = 0$, $d(v_1) = 2(u_1^2 + u_2^2 + u_1u_2)$, $d(v_5) = u_1^6 + u_2^6 + (u_1 + u_2)^6$. Thus

$$d_1(u_1) = d_1(u_2) = d_1(v_5) = 0, \quad d_1(v_1) = 2(u_1^2 + u_2^2 + u_1u_2).$$

In the homotopy Lie algebra $L = (a_1, a_2, b_1, b_5)$ the induced commutator relations are given by

$$[a_i, b_j] = 0 \quad \text{for} \quad i = 1, 2,\ j = 1, 5, \quad [b_i, b_j] = 0 \quad \text{for} \quad i, j = 1, 5,$$
$$[a_1, a_2] = 2b_1, \quad [a_1, a_1] = [a_2, a_2] = 4b_1.$$

Hence the following relations in $U\mathcal{L}$ hold:

$$\begin{aligned} &a_i b_j - b_j a_i = 0 \quad \text{for} \quad i = 1, 2,\ j = 1, 5,\\ &b_1 b_5 = b_5 b_1,\\ &a_1 a_2 + a_2 a_1 = 2b_1,\\ &a_1^2 = a_2^2 = 2b_1. \end{aligned}$$

□

Theorem 3.5. *The rational homology ring of the loop space on F_4/T^4 is given by*

$$H_*(\Omega(F_4/T^4); \mathbb{Q}) \cong \left(T(a_1, a_2, a_3, a_4) \middle/ \left\langle \begin{array}{l} a_1^2 = \cdots = a_4^2, \\ a_i a_j = -a_j a_i \ \textit{for}\ i \neq j \end{array} \right\rangle \right) \otimes \mathbb{Q}[b_5, b_7, b_{11}]$$

where $\deg a_i = 1$ *for* $1 \le i \le 4$, $\deg b_5 = 10$, $\deg b_7 = 14$, *and* $\deg b_{11} = 22$.

Proof. The rational cohomology algebra of F_4/T^4 is

$$H^*(F_4/T^4; \mathbb{Q}) \cong \mathbb{Q}[u_1, u_2, u_3, u_4]/\langle P_2, P_6, P_8, P_{12}\rangle$$

where $\deg u_i = 2$ for $1 \le i \le 4$, and

$$P_k = u_1^k + u_2^k + u_3^k + u_4^k + \frac{1}{2^{k+1}}(\pm u_1 \pm u_2 \pm u_3 \pm u_4)^k$$

for $k = 2, 6, 8, 12$. For degree reasons, the only relevant generator for determining d_1 is $P_2 = 3(u_1^2 + u_2^2 + u_3^2 + u_4^2)$. Therefore we have

$$V = (u_1, u_2, u_3, u_4, v_2, v_6, v_8, v_{12}), \quad \text{where} \quad \deg v_k = 2k - 1$$

and the quadratic part of d is given by

$$d_1(u_i) = 0 \quad \text{for} \quad 1 \le i \le 4, \quad d_1(v_j) = 0 \quad \text{for} \quad j = 6, 8, 12,$$

and

$$d_1(v_2) = 3(u_1^2 + u_2^2 + u_3^2 + u_4^2).$$

This determines the homotopy Lie algebra

$$L = (a_1, a_2, a_3, a_4, b_1, b_5, b_7, b_{11})$$

where $\deg a_i = 1$, $\deg b_1 = 2$, $\deg b_5 = 10$, $\deg b_7 = 14$, and $\deg b_{11} = 22$ with the Lie brackets given by

$$\begin{aligned}
&[a_i, b_j] = [b_l, b_j] = 0 && \text{for} \quad 1 \leq j \leq 4 \quad \text{and} \quad j, l = 1, 5, 7, 11,\\
&[a_i, a_j] = 0 && \text{for} \quad i \neq j,\\
&[a_i, a_i] = 6b_1 && \text{for} \quad 1 \leq i \leq 4.
\end{aligned}$$

This implies that in $U\mathcal{L}$ for every possible i and j, a_i and b_j commute as well as b_i and b_j does. Also the additional relations in $U\mathcal{L}$ hold:

$$a_1^2 = a_2^2 = a_3^2 = a_4^2 = 3b_1, \quad \text{and} \quad a_i a_j + a_j a_i = 0 \quad \text{for} \quad i \neq j.$$

The statement of the theorem now follows directly. $\square$

Theorem 3.6. *The rational homology ring of the loop space on E_6/T^6 is given by*

$$\begin{aligned}
&H_*(\Omega(E_6/T^6); \mathbb{Q})\\
&\cong \left(T(a_1, \ldots, a_5, a) \Big/ \left\langle \begin{array}{l} a^2 = a_k^2 = a_p a_q + a_q a_p \ \textit{for} \ 1 \leq k,\ p, q \leq 5,\ p \neq q \\ a a_i = -a_i a \ \textit{for} \ 1 \leq i \leq 5 \end{array} \right\rangle \right)\\
&\quad \otimes \mathbb{Q}[b_4, b_5, b_7, b_8, b_{11}],
\end{aligned}$$

where $\deg a_i = 1$ *for* $1 \leq i \leq 5$, $\deg a = 1$, *and* $\deg b_j = 2j$ *for* $j = 4, 5, 7, 8, 11$.

Proof. The rational cohomology of E_6/T^6 is

$$H^*(E_6/T^6; \mathbb{Q}) \cong \mathbb{Q}[u_1, u_2, u_3, u_4, u_5, u_6, u]/\langle P_1, P_2, P_5, P_6, P_8, P_9, P_{12}\rangle$$

where $\deg u_i = 2$ for $1 \leq i \leq 6$, $\deg u = 2$, and

$$P_k = \sum_{i=1}^{6} (u_i \pm u)^k + \sum_{1 \leq i < j \leq 6} (-1)^k (u_i + u_j)^k$$

for $k = 2, 5, 6, 8, 9, 12$, and $P_1 = \sum_{i=1}^{6} u_i$. It follows that

$$V = (u_1, u_2, u_3, u_4, u_5, u, v_2, v_5, v_6, v_8, v_9, v_{12})$$

and d_1 is determined only by

$$P_2 = 12\left(u_1^2 + \cdots + u_5^2 + u^2 + \sum_{1 \leq i < j \leq 5} u_i u_j \right).$$

In a similar fashion as before we obtain that

$$L = (a_1, a_2, a_3, a_4, a_5, a, b_1, b_4, b_5, b_7, b_8, b_{11}),$$

where $\deg a_i = \deg a = 1$ for $1 \le i \le 5$, and $\deg b_j = 2j$ for $j = 1, 4, 5, 7, 8, 11$. The commutators are

$$[a_i, a] = [a_i, b_j] = [a, b_j] = [b_l, b_j] = 0 \quad \text{for} \quad 1 \le i \le 5 \quad \text{and} \quad j, l = 1, 4, 5, 7, 8, 11,$$
$$[a_i, a_j] = 12b_1 \quad \text{for} \quad 1 \le i,\ j \le 5,\ i \neq j, \quad \text{and} \quad [a_i, a_i] = [a, a] = 24b_1 \quad \text{for} \quad 1 \le i \le 5.$$

The last three commutator relations imply the following relations in $U\mathcal{L}$:

$$a^2 = a_i^2 = a_k a_l + a_l a_k = 12b_1 \quad \text{for} \quad 1 \le i, l, k \le 5,\ k \neq l.$$

This directly implies the statement of the theorem. $\square$

4. Integral Pontrjagin homology

In this section we study the integral Pontrjagin ring structure of $\Omega(G/T)$, where G is a simple Lie group. We make use of the rational homology calculations for $\Omega(G/T)$ from the previous section and the results from [3], [11] and [16] on integral homology of the identity component $\Omega_0 G$ of the loop space on G. Recall that $H_*(\Omega_0 G; \mathbb{Q})$ is primitively generated for a compact connected Lie group G.

4.1. The integral homology of $\Omega(SU(n+1)/T^n)$.

Theorem 4.1. *The integral Pontrjagin homology ring of the loop space on $SU(n+1)/T^n$ is*

$$H_*(\Omega(SU(n+1)/T^n); \mathbb{Z})$$
$$\cong (T(x_1, \ldots, x_n) \otimes \mathbb{Z}[y_1, \ldots, y_n]) \Big/ \left\langle \begin{array}{l} x_k^2 = x_p x_q + x_q x_p = 2y_1 \\ \textit{for } 1 \le k,\ p, q \le n,\ p \neq q \end{array} \right\rangle$$

where the generators $x_1, \ldots, x_n$ are of degree 1, and the generators y_i are of degree $2i$ for $1 \le i \le n$.

Proof. It is well known that if G is a simply connected Lie group, then $\pi_2(G/T) \cong \mathbb{Z}^{\dim T}$ and $\pi_3(G/T) \cong \mathbb{Z}$. Let

$$W\colon \pi_2(G/T) \otimes \pi_2(G/T) \to \pi_3(G/T)$$

denote the pairing given by the Whitehead product. In what follows, we identify $H_1(T, \mathbb{Z})$ with $\pi_2(G/T)$ and $H_2(\Omega G, \mathbb{Z})$ with $\pi_3(G/T)$ via natural homomorphisms. Thus since there is no torsion in homology, and using the rational homology result (3), we obtain that there is a split extension of algebras

$$1 \to H_*(\Omega SU(n+1);\mathbb{Z}) \to H_*(\Omega(SU(n+1)/T^n);\mathbb{Z}) \to H_*(T^n;\mathbb{Z}) \to 1$$

with the extension given by $[\alpha,\beta] = W(\alpha,\beta) \in H_2(\Omega SU(n+1);\mathbb{Z})$, where $\alpha,\beta \in H_1(T^n;\mathbb{Z})$.

We explain the extension of the algebra in more detail. Notice that there is a monomorphism of two split extensions of algebras

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & H_*(\Omega SU(n+1);\mathbb{Z}) \arrow[r] \arrow[d] & H_*(\Omega(SU(n+1)/T^n);\mathbb{Z}) \arrow[r] \arrow[d] & H_*(T^n;\mathbb{Z}) \arrow[r] \arrow[d] & 1 \arrow[d] \\
1 \arrow[r] & H_*(\Omega SU(n+1);\mathbb{Q}) \arrow[r] & H_*(\Omega(SU(n+1)/T^n);\mathbb{Q}) \arrow[r] & H_*(T^n;\mathbb{Q}) \arrow[r] & 1.
\end{tikzcd}$$

Denote by $\bar{c}_2,\dots,\bar{c}_{n+1}$ the universal transgressive generators in $H^*(SU(n+1);\mathbb{Z})$ which map to the symmetric polynomials $c_2 = \sum_{1\le i<j\le n+1} x_i x_j, \dots, c_{n+1} = x_1 \cdots x_n x_{n+1}$ generating $H^*(BSU(n+1);\mathbb{Z})$. The elements $x_1,\dots,x_n,x_{n+1}$ are the integral generators of $H_*(T^n;\mathbb{Z})$ and $\sum_{i=1}^{n+1} x_i = 0$. Now let $y_1,\dots,y_n$ be the integral generators of $H_*(\Omega SU(n+1);\mathbb{Z})$ obtained by the transgression of the elements from $H_*(SU(n+1);\mathbb{Z})$ which are the Poincare duals of $\bar{c}_2,\dots,\bar{c}_{n+1}$. Further, the subspace of primitive elements in $H_*(\Omega SU(n+1);\mathbb{Z})$ is spanned by the elements $\sigma_1,\dots,\sigma_n$ which can be expressed in terms of $y_1,\dots,y_n$ using the Newton formula

$$\sigma_k = \sum_{i=1}^{k-1}(-1)^{i-1}\sigma_{k-i}y_i + (-1)^{k-1}ky_k, \quad 1 \le k \le n. \tag{8}$$

The integral elements $\sigma_1,\dots,\sigma_n$ rationalise to the elements $b_1,\dots,b_n \in H_*(\Omega SU(n+1);\mathbb{Q})$. The generators $a_1,\dots,a_n$ in $H_*(T^n;\mathbb{Q})$ are the rationalised images of the integral generators $x_1,\dots,x_n$ in $H_*(T^n;\mathbb{Z})$. To decide the integral extension, we consider the rational Pontrjagin ring structure (3) of $\Omega(SU(n+1)/T^n)$. Looking at the above commutative diagram of the algebra extensions, we conclude that the integral elements

$$\begin{aligned}
&x_k x_l + x_l x_k - 2\sigma_1 && \text{for} \quad 1 \le k, l \le n,\ k \ne l,\\
&x_k^2 - 2\sigma_1 && \text{for} \quad 1 \le k \le n,\\
&x_k\sigma_l - \sigma_l x_k && \text{for} \quad 2 \le k, l \le n,\\
&\sigma_k\sigma_l - \sigma_l\sigma_k && \text{for} \quad 2 \le k, l \le n
\end{aligned}$$

from $H_*(\Omega(SU(n+1)/T^n);\mathbb{Z})$ map to zero in $H_*(\Omega(SU(n+1)/T^n);\mathbb{Q})$. As the map between the algebra extensions is a monomorphism, we conclude that these integral elements are zero. Using that there is no torsion in homology and Newton formula (8), we have

$$\begin{aligned}
&x_k x_l + x_l x_k = 2y_1 &&\text{for} \quad 1 \le k, l \le n,\ k \ne l,\\
&x_k^2 = 2y_1 &&\text{for} \quad 1 \le k \le n,\\
&x_k y_l - y_l x_k = 0 &&\text{for} \quad 2 \le k, l \le n,\\
&y_k y_l - y_l y_k = 0 &&\text{for} \quad 2 \le k, l \le n
\end{aligned}$$

which completely describes the integral Pontrjagin ring of $\Omega(SU(n+1)/T^n)$ and finishes the proof. □

4.2. The integral homology of $\Omega(Sp(n)/T^n)$.

Theorem 4.2. *The integral Pontrjagin homology ring of the based loop space on $Sp(n)/T^n$ is*

$$\begin{aligned}
&H_*(\Omega(Sp(n)/T^n);\mathbb{Z})\\
&\cong \left(T(x_1,\dots,x_n) \middle/ \left\langle \begin{matrix} x_1^2 = \cdots = x_n^2, \\ x_k x_l = -x_l x_k \ \textit{for}\ k \ne l \end{matrix} \right\rangle \right) \otimes \mathbb{Z}[y_2,\dots,y_n]
\end{aligned}$$

where the generators $x_1,\dots,x_n$ are of degree 1, and the generators y_i are of degree $4i-2$ for $2 \le i \le n$.

Proof. The proof is analogous to the proof of Theorem 4.1. Denote by $\bar{c}_1,\dots,\bar{c}_n$ the universal transgressive generators in $H^*(Sp(n);\mathbb{Z})$ which map to the generators $c_1 = \sum_{i=1}^n x_i^2,\dots,c_n = x_1^2\cdots x_n^2$ of $H^*(BSp(n);\mathbb{Z})$. Let $y_1,\dots,y_n$ be the generators in $H_*(\Omega Sp(n);\mathbb{Z})$ obtained by the transgression of the elements in $H_*(Sp(n);\mathbb{Z})$ which are the Poincare duals of $\bar{c}_1,\dots,\bar{c}_n$. Recall from [3] that the subspace of the primitive elements in $H_*(\Omega Sp(n);\mathbb{Z})$ is spanned by the elements $\sigma_1,\dots,\sigma_n$ given by

$$\sigma_k = \sum_{i=1}^{k-1} (-1)^{i-1} \sigma_{k-i} y_i + (-1)^{k-1} k y_k, \quad 1 \le k \le n. \tag{9}$$

The integral elements $\sigma_1,\dots,\sigma_n$ rationalise to the generators $b_1,\dots,b_n$ of $H_*(\Omega Sp(n);\mathbb{Q})$ given in (5). The generators $a_1,\dots,a_n$ of $H_*(T^n;\mathbb{Q})$ are the rationalised images of the integral generators $x_1,\dots,x_n$ in $H_*(T^n;\mathbb{Z})$. Therefore we conclude that in $H_*(\Omega(Sp(n)/T^n);\mathbb{Z})$ the following integral elements are zero:

$$\begin{aligned}
&x_k x_l + x_l x_k - \sigma_1 &&\text{for} \quad 1 \le k, l \le n,\ k \ne l,\\
&x_k^2 - \sigma_1 &&\text{for} \quad 1 \le k \le n,\\
&x_k \sigma_l - \sigma_l x_k &&\text{for} \quad 2 \le k, l \le n,\\
&\sigma_k \sigma_l - \sigma_l \sigma_k &&\text{for} \quad 2 \le k, l \le n.
\end{aligned}$$

Since there is no torsion in homology, going back to Newton formula (9), we obtain the same relations between $y_1, \dots, y_n$ and $x_1, \dots, x_n$ which determine the integral Pontrjagin ring structure on $\Omega(Sp(n)/T^n)$. $\square$

4.3. The integral homology of $\Omega(SO(2n)/T^n)$ and $\Omega(SO(2n+1)/T^n)$. As mentioned before, $SO(m)$ is not simply connected and the cohomology of $SO(m)$ and the homology of $\Omega SO(m)$ are not torsion free, namely, they have 2-torsion. Nevertheless, since $SO(m)/T \cong Spin(m)/T$, where T is a maximal torus, the rational homology calculations enable us to prove the following.

Theorem 4.3. *The integral Pontrjagin homology ring of the based loop space on $SO(2n+1)/T^n$ is given by*

$$H_*(\Omega(SO(2n+1)/T^n); \mathbb{Z}) \cong (T(x_1, \dots, x_n) \otimes \mathbb{Z}[y_1, \dots, y_{n-1}, 2y_n, \dots 2y_{2n-1}])/I$$

where I is generated by

$$\begin{aligned} &x_1^2 - y_1, \quad x_i^2 - x_{i+1}^2 \quad \textit{for} \quad 1 \le i \le n-1, \\ &x_k x_l + x_l x_k \quad \textit{for} \quad k \ne l, \\ &y_i^2 - 2y_{i-1}y_{i+1} + \dots \pm 2y_{2i} \quad \textit{for} \quad 1 \le i \le n-1, \end{aligned}$$

where $\deg x_i = 1$ *for* $1 \le i \le n$, $\deg y_i = 2i$ *for* $1 \le i \le 2n-1$, $\deg 2y_i = 2i$ *for* $n \le i \le 2n-1$, *and* $y_0 = 1$.

REMARK 4.1. Before proving Theorem 4.3, let us recall the ring structure of $H_*(\Omega_0 SO(2n+1); \mathbb{Z})$. It is proved in [3] that the algebra $H_*(\Omega_0 SO(2n+1); \mathbb{Z})$ is generated by the classes $y_1, \dots, y_{n-1}, 2y_n, \dots, 2y_{2n-1}$ which satisfy the relations

$$y_i^2 - 2y_{i-1}y_{i+1} + 2y_{i-2}y_{i+2} - \dots \pm 2y_{2i} = 0 \quad \text{for} \quad 1 \le i \le n-1$$

where $\deg y_i = 2i$ for $1 \le i \le n-1$, $\deg 2y_i = 2i$ for $n \le i \le 2n-1$, and $y_0 = 1$. For $[(n+1)/2] \le i \le n-1$, these relations express $2y_{2i}$ in terms of $y_1, \dots, y_{n-1}, 2y_n, \dots, 2y_{2i-1}$ and thus eliminate $2y_{2i}$ as generators. For $1 \le i \le [(n+1)/2]-1$, the relations above imply new relations on the generators y_{2i}, that is, $2y_{2i} = \pm(y_i^2 - 2y_{i-1}y_{i+1} + \dots \pm 2y_1 y_{2i-1})$. This implies that the elements y_{2i} for $1 \le i \le [(n+1)/2]-1$ are generators only in the homology of $\Omega_0 SO(2n+1)$ with coefficients where 2 is not invertible. Consider the rational elements p_k defined by the recursion formula

$$p_k - p_{k-1}y_1 + \dots \pm k y_k = 0 \quad \text{for} \quad 1 \le k \le 2n-1 \quad \text{where} \quad p_0 = 1. \tag{10}$$

The relations in $H_*(\Omega_0 SO(2n+1); \mathbb{Z})$ imply that only $p_1, p_3, \dots, p_{2n-1}$ are non zero. According to [3] the elements $p_1, p_3, \dots, p_{2[n/2]-1}, 2p_{2[n/2]+1}, \dots, 2p_{2n-1}$ span the subspace of primitive elements in $H_*(\Omega_0 SO(2n+1); \mathbb{Z})$. These elements are obtained by transgressing the elements in $H_*(SO(2n+1); \mathbb{Z})$ which are the Poincare duals of the universal transgressive generators $\bar{\sigma}_1, \dots, \bar{\sigma}_n$ in $H^*(SO(2n+1); \mathbb{Z})$. The generators $\bar{\sigma}_1, \dots, \bar{\sigma}_n$ map to the symmetric polynomials $\sigma_i(x_1^2, \dots, x_n^2)$ for $1 \le i \le n$ generating the free part in $H^*(BSO(2n+1); \mathbb{Z})$. In this way we see that $p_1, p_3, \dots, 2p_{2[n/2]+1}, \dots, 2p_{2n-1}$ rationalise to the rational generators b_i in $H_*(\Omega_0 SO(2n+1); \mathbb{Q})$ (see Theorem 3.2).

REMARK 4.2. If we denote the generators of $H_*(\Omega_0 SO(2n+1); \mathbb{Z})$ by $y_1, \dots, y_{n-1}, y_n, \dots, y_{2n-1}$, then the relations are slightly more complicated and they are given by

$$y_i^2 + 2 \sum_{k=1}^{\min\{i, n-1-i\}} (-1)^k y_{i-k} y_{i+k} + \sum_{k=n-i}^{i} (-1)^k y_{i-k} y_{i+k} = 0$$

where $1 \le i \le n-1$.

Proof. Recall that $SO(2n+1)/T^n \cong Spin(2n+1)/T^n$ implying that $\Omega(SO(2n+1)/T^n) \cong \Omega(Spin(2n+1)/T^n)$. It is known that $\Omega Spin(2n+1) \cong \Omega_0 SO(2n+1)$, see for example [10]. Consider the morphism of two extensions of algebras

$$\begin{array}{ccccc} H_*(\Omega_0 SO(2n+1); \mathbb{Z}) & \longrightarrow & H_*(\Omega(SO(2n+1)/T^n); \mathbb{Z}) & \longrightarrow & H_*(T^n; \mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ H_*(\Omega_0 SO(2n+1); \mathbb{Q}) & \longrightarrow & H_*(\Omega(SO(2n+1)/T^n); \mathbb{Q}) & \longrightarrow & H_*(T^n; \mathbb{Q}). \end{array}$$

By Remark 4.1, we have that all the generators $b_1, \dots, b_n$ of $H_*(\Omega_0 SO(2n+1); \mathbb{Q})$ are in the rationalisation of the integral elements $p_1, p_3, \dots, p_{2[n/2]-1}, 2p_{2[n/2]+1}, \dots, 2p_{2n-1}$ of $H_*(\Omega_0 SO(2n+1); \mathbb{Z})$. Since the map between two algebra extensions is a monomorphism, we conclude that in $H_*(\Omega(SO(2n+1)/T^n); \mathbb{Z})$ the following relations hold

$$\begin{array}{ll} x_k x_l + x_l x_k = p_1 & \text{for } \; 1 \le k, l \le n, \; k \ne l, \\ x_k^2 = p_1 & \text{for } \; 1 \le k \le n, \\ x_k p_{2l-1} = p_{2l-1} x_k & \text{for } \; 2 \le k, l \le n, \\ p_{2k-1} p_{2l-1} = p_{2l-1} p_{2k-1} & \text{for } \; 2 \le k, l \le n \end{array}$$

as these elements map to zero in $H_*(\Omega(SO(2n+1)/T^n); \mathbb{Q})$. Note that $p_1 = y_1$, which gives that $y_1 = x_1^2$ in $H_*(\Omega(SO(2n+1)/T^n); \mathbb{Z})$.

The fact that differs this case from the case of $SU(n+1)$ or $Sp(n)$ is that these integral elements $p_1, \dots, 2p_{2n-1}$ that map onto rational generators, do not produce all the generators in $H_*(\Omega_0 SO(2n+1); \mathbb{Z})$. Nevertheless, since there is no torsion in homology, we can also deduce from the rational homology calculations that there is a split extension of algebras

$$1 \to H_*(\Omega_0 SO(2n+1); \mathbb{Z}) \to H_*(\Omega(SO(2n+1)/T^n); \mathbb{Z}) \to H_*(T; \mathbb{Z}) \to 1.$$

We have that y_{2i-1} survive as the generators in $H_*(\Omega(SO(2n+1)/T^n); \mathbb{Z})$ for $2 \le i \le n$ using the relations coming from $H_*(\Omega_0 SO(2n+1); \mathbb{Z})$ and the fact that the integral elements $p_3, \dots, 2p_{2n-1}$ rationalise to the generators $b_2, \dots, b_n$ in $H_*(\Omega(SO(2n+1)/T^n); \mathbb{Q})$.

Therefore, in order to verify the above splitting we need to show that the generators y_{2i} for $1 \le i \le [(n+1)/2]$ in $H_*(\Omega_0 SO(2n+1);\mathbb{Z})$ survive as generators in $H_*(\Omega(SO(2n+1)/T^n);\mathbb{Z})$. We prove this by induction on i. If y_2 is not a generator in $H_*(\Omega(SO(2n+1)/T^n);\mathbb{Z})$, then it can be expressed as

$$y_2 = \alpha x_1^4 + \sum_{i=2}^{n} \beta_i x_1^3 x_i + \sum_{2\le i<j\le n} \gamma_{ij} x_1^2 x_i x_j + \sum_{1\le i<j<k<l\le n} \delta_{ijkl} x_i x_j x_k x_l,$$

where α, β_i, γ_{ij}, δ_{ijkl} are integers. On the other hand, in $H_*(\Omega_0 SO(2n+1);\mathbb{Z})$ we have that $2y_2 = y_1^2$ which translates to $2y_2 = x_1^4$ in $H_*(\Omega(SO(2n+1)/T);\mathbb{Z})$. This implies that $\beta_i = \gamma_{ij} = \delta_{ijkl} = 0$, and $2\alpha = 1$, which is impossible since α is an integer. In the same way, assuming that y_{2i} for $1 \le i \le k < [(n+1)/2]$ are generators in $H_*(\Omega(SO(2n+1)/T^n);\mathbb{Z})$, we prove that $y_{2(k+1)}$ is a generator as well. If it were not, we would have

$$y_{2(k+1)} = \alpha y_{k+1}^2 + P(x_1,\dots,x_n,y_2,\dots,y_{2k+1})$$

where $\alpha \in \mathbb{Z}$ and P is a polynomial with integer coefficients which does not contain y_{k+1}^2. On the other hand, in the relation $2y_{2(k+1)} = \pm(y_{k+1}^2 - 2y_k y_{k+2} - \cdots \pm 2y_1 y_{2k+1})$ in $H_*(\Omega_0 SO(2n+1);\mathbb{Z})$, when translating to $H_*(\Omega(SO(2n+1)/T^n);\mathbb{Z})$ we have by the inductive hypothesis that y_{k+1}^2 can not be eliminated. This implies that the coefficient α satisfies $2\alpha = \pm 1$ which is impossible.

We are left with a verification of the commutator relations in $H_*(\Omega(SO(2n+1)/T^n);\mathbb{Z})$. Since $2y_2 = x_1^4$, we have $2y_2 x_i = x_1^4 x_i = x_i x_1^4 = 2x_i y_2$, that is, $y_2 x_i = x_i y_2$. Now by induction on k, we prove that $y_k x_j = x_j y_k$ for an arbitrary y_k. For k odd, relation (10) together with the inductive hypothesis gives that x_i for $1 \le i \le n$ commutes with y_k. Let k be even. Since $\deg y_i$ is even for any i, each monomial in the polynomial $P(x_1,\dots,x_n,y_2,\dots,y_{k-1}) = 2y_k$ contains even number of generators $x_1,\dots,x_n$. Using now the inductive hypothesis, we have that every x_i commutes with P and thus with y_k. □

Theorem 4.4. *The integral Pontrjagin homology ring of the based loop space on $SO(2n)/T^n$ is given by*

$$\begin{aligned}&H_*(\Omega(SO(2n)/T^n);\mathbb{Z})\\ &\cong (T(x_1,\dots,x_n)\otimes \mathbb{Z}[y_1,\dots,y_{n-2},\,y_{n-1}+z,\,y_{n-1}-z,\,2y_n,\dots,2y_{2(n-1)}])/I\end{aligned}$$

where I is generated by

$$x_1^2 - y_1,\ x_i^2 - x_{i+1}^2 \quad \text{for} \quad 1 \le i \le n-1,$$
$$x_k x_l + x_l x_k \quad \text{for} \quad k \ne l,$$
$$y_i^2 - 2y_{i-1}y_{i+1} + 2y_{i-2}y_{i+2} - \cdots \pm 2y_{2i} \quad \text{for} \quad 1 \le i \le n-2,$$
$$(y_{n-1}+z)(y_{n-1}-z) - 2y_{n-1}y_{n+1} + \cdots \pm 2y_{2(n-1)},$$

where $\deg x_i = 1$ *for* $1 \le i \le n$, $\deg y_i = 2i$ *for* $1 \le i \le n-2$, $\deg(y_{n-1}+z) = \deg(y_{n-1}-z) = 2(n-1)$, $\deg 2y_i = 2i$ *for* $n \le i \le 2(n-1)$ *and* $y_0 = 1$.

REMARK 4.3. Recall from [3] that the algebra $H_*(\Omega_0 SO(2n);\mathbb{Z})$ is generated by the elements $y_1, \dots, y_{n-2}, y_{n-1}+z, y_{n-1}-z, 2y_n, \dots, 2y_{2(n-1)}$ which satisfy the relations

$$y_i^2 - 2y_{i-1}y_{i+1} + 2y_{i-2}y_{i+2} - \cdots \pm 2y_{2i} = 0 \quad \text{for} \quad 1 \le i \le n-2,$$
$$(y_{n-1}+z)(y_{n-1}-z) - 2y_{n-1}y_{n+1} + \cdots \pm 2y_{2(n-1)} = 0.$$

As in previous case, these relations eliminate $2y_{2i}$ as generators for $[(n+1)/2] \le i \le n-2$, while for $1 \le i \le [(n+1)/2]-1$, they induce new relations on y_{2i} implying that y_{2i} are generators only in the homology of $\Omega_0 SO(2n)$ with coefficients where 2 is not invertible. The subspace of primitive elements in $H_*(\Omega_0 SO(2n);\mathbb{Z})$ is spanned by the elements $p_1, p_3, \dots, p_{n-2}, 2z, 2p_n, \dots, 2p_{2(n-1)-1}$ for n odd and by the elements $p_1, p_3, \dots, p_{n-1}, 2z, 2p_{n+1}, \dots, 2p_{2(n-2)+1}$ for n even. These primitive generators are obtained by transgressing the elements in $H_*(SO(2n);\mathbb{Z})$ which are the Poincare duals of the universal transgressive generators $\bar{\sigma}_1, \dots, \bar{\sigma}_{n-1}, \bar{\lambda}$ in $H^*(SO(2n);\mathbb{Z})$. The generators $\bar{\sigma}_1, \dots, \bar{\sigma}_{n-1}, \bar{\lambda}$ map to the polynomials $\sigma_i(x_1^2, \dots, x_n^2)$ for $1 \le i \le n-1$ and $\lambda = x_1 \cdots x_n$ which generate the free part in $H^*(BSO(2n);\mathbb{Z})$.

Proof. The proof is analogous to the proof of Theorem 4.3. □

4.4. The integral homology of $\Omega(G_2/T^2)$.

Theorem 4.5. *The integral Pontrjagin homology ring of* $\Omega(G_2/T^2)$ *is given by*

$$\begin{aligned} &H_*(\Omega(G_2/T^2);\mathbb{Z}) \\ &\cong (T(x_1,x_2)\otimes \mathbb{Z}[y_1,y_2,y_5])/\langle x_1x_2+x_2x_1 = x_1^2 = x_2^2 = 2y_1,\ 2y_2 = x_1^4\rangle, \end{aligned}$$

where $\deg x_1 = \deg x_2 = 1$, $\deg y_2 = 4$, *and* $\deg y_5 = 10$.

REMARK 4.4. The integral homology algebra of ΩG_2 has the following form [3]:

$$H_*(\Omega G_2;\mathbb{Z}) = \mathbb{Z}[y_1, y_2, y_5]/\langle 2y_2 - y_1^2\rangle,$$

with $\deg y_1 = 2$, $\deg y_2 = 4$, and $\deg y_5 = 10$.

Proof. Consider a morphism of two extensions of algebras

$$\begin{tikzcd} H_*(\Omega G_2;\mathbb{Z}) \arrow[r] \arrow[d] & H_*(\Omega(G_2/T^2);\mathbb{Z}) \arrow[r] \arrow[d] & H_*(T^2;\mathbb{Z}) \arrow[d] \\ H_*(\Omega G_2;\mathbb{Q}) \arrow[r] & H_*(\Omega(G_2/T^2);\mathbb{Q}) \arrow[r] & H_*(T^2;\mathbb{Q}). \end{tikzcd}$$

In $H_*(\Omega G_2;\mathbb{Q})$ the generators b_1 and b_5 are the rationalisations of the integral elements y_1 and y_5 in $H_*(\Omega G_2;\mathbb{Z})$. It follows that the relations between x_1, x_2 and between y_1, y_5 in $H_*(\Omega(G_2/T^2);\mathbb{Z})$ are lifted from the relations on their rationalisations. We further show that there is a split extensions of algebras

$$1 \to H_*(\Omega G_2;\mathbb{Z}) \to H_*(\Omega(G_2/T^2);\mathbb{Z}) \to H_*(T^2;\mathbb{Z}) \to 1.$$

To deduce the splitting above, we use that there is no torsion in the corresponding homologies. We first need to show that the generator $y_2 \in H_*(\Omega G_2;\mathbb{Z})$ survives as a generator in $H_*(\Omega(G_2/T^2);\mathbb{Z})$. If it were not, we would have that $y_2 = \alpha x_1^4 + \beta x_1^3 x_2$, and that $2y_2 = x_1^4$ using the relations in $H_*(\Omega G_2;\mathbb{Z})$. This would imply that $2\alpha = 1$ which is impossible since α is an integer. Since $2y_2 = x_1^4$ and there is no torsion in homology, using already established relations, we get that y_2 commutes with other generators in $H_*(\Omega(G_2/T^2);\mathbb{Z})$. □

4.5. The integral homology of $\Omega(F_4/T^4)$.

Theorem 4.6. *The integral Pontrjagin homology ring of $\Omega(F_4/T^4)$ is given by*

$$\begin{aligned} &H_*(\Omega(F_4/T^4);\mathbb{Z}) \\ &\cong (T(x_1, x_2, x_3, x_4) \otimes \mathbb{Z}[y_1, y_2, y_3, y_5, y_7, y_{11}])/I \end{aligned}$$

where $I = \langle x_i^2 = 3y_1,\ 1 \le i \le 4,\ x_i x_j = x_j x_i,\ i \ne j,\ 2y_2 = x_1^4,\ 3y_3 = x_1^2 y_2\rangle$, *where* $\deg x_i = 1$ *for* $1 \le i \le 4$, *and* $\deg y_i = 2i$ *for* $i = 2, 3, 5, 7, 11$.

REMARK 4.5. The integral homology algebra $H_*(\Omega F_4;\mathbb{Z})$ is computed in [16] and it is given by

$$H_*(\Omega F_4;\mathbb{Z}) = \mathbb{Z}[y_1, y_2, y_3, y_5, y_7, y_{11}]/\langle y_1^2 - 2y_2, y_1 y_2 - 3y_3\rangle.$$

Proof. As in the previous cases, we first prove that there is a split extension of algebras

$$1 \to H_*(\Omega F_4;\mathbb{Z}) \to H_*(\Omega(F_4/T^4);\mathbb{Z}) \to H_*(T^4;\mathbb{Z}) \to 1.$$

Since there is no torsion in homology, the rational homology calculations for $\Omega(F_4/T^4)$ gives that it is enough to prove that y_2 and y_3 survive as generators in $H_*(\Omega(F_4/T^4);\mathbb{Z})$. If y_2 were not a generator in $H_*(\Omega(F_4/T^4);\mathbb{Z})$, we would have $y_2 = \alpha x_1^4 + \sum_{i=2}^4 \alpha_i x_1^3 x_i + \sum_{2\le i<j\le 4} \alpha_{ij} x_1^2 x_i x_j + \beta x_1 x_2 x_3 x_4$ for some $\alpha, \alpha_i, \alpha_{ij}, \beta \in \mathbb{Z}$. On the other hand, the relation $2y_2 = y_1^2$ from $H_*(\Omega F_4;\mathbb{Z})$ becomes $2y_2 = x_1^4$ in $H_*(\Omega(F_4/T^4);\mathbb{Z})$. This implies that $2\alpha = 1$ which is impossible. In the similar way we prove that y_3 is also a generator in $H_*(\Omega(F_4/T^4);\mathbb{Z})$. If it were not, we would have $y_3 = \alpha x_1^6 + \sum_{i=2}^4 \alpha_i x_1^5 x_i + \sum_{2\le i<j\le 4} x_1^4 x_i x_j + \beta x_1^3 x_2 x_3 x_4 + \delta x_1^2 y_2 + \sum_{1\le i<j\le 4} \delta_{ij} x_i x_j y_2$. From $H_*(\Omega F_4;\mathbb{Z})$, we also have that $3y_3 = x_1^2 y_2$. This together leads to $3\delta = 1$ which is impossible. □

4.6. The integral homology of $\Omega(E_6/T^6)$.

REMARK 4.6. The integral homology algebra $H_*(\Omega E_6;\mathbb{Z})$ is described in [11] and it is given by

$$H_*(\Omega E_6;\mathbb{Z}) \cong \mathbb{Z}[y_1, y_2, y_3, y_4, y_5, y_7, y_8, y_{11}]/\langle y_1^2 - 2y_2, y_1y_2 - 3y_3\rangle,$$

where $\deg y_i = 2i$ for $i = 1, 2, 3, 4, 5, 7, 8, 11$.

Using the same argument as for the previous cases, we deduce the integral Pontrjagin homology of the based loop space on E_6/T^6.

Theorem 4.7. *The integral Pontrjagin homology ring of $\Omega(E_6/T^6)$ is given by*

$$\begin{aligned}&H_*(\Omega(E_6/T^6);\mathbb{Z})\\ &\cong (T(x_1, x_2, x_3, x_4, x_5, x_6) \otimes \mathbb{Z}[y_1, y_2, y_3, y_4, y_5, y_7, y_8, y_{11}])/I\end{aligned}$$

where $I = \langle x_k^2 = x_px_q + x_qx_p = 12y_1$ for $1 \le k, p, q \le 6$, $2y_2 = x_1^4$, $3y_3 = x_1^2y_2\rangle$ and where $\deg x_i = 1$ for $1 \le i \le 6$, and $\deg y_i = 2i$ for $i = 2, 3, 4, 5, 7, 8, 11$.

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